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Growth Estimates and Asymptotic Stability for a Class of Differential–Delay Equation Having Time-Varying Delay

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In this paper the author develops growth estimates for the solutions of differential-delay equations having time-varying delays. Time-varying differential-delay equations have a long history, and the literature on systems defined by such equations is vast. Within this area, the special case of systems in which the delay is itself a function of time has a much smaller literature, and results are more rare [1–3, 5, 8].

The specific topic of interest in this paper will be the differential-delay equation (†) $\dot{x}(t) = A_0 x(t) + A_1 x(t - h(t))$, where A_0, A_1 are fixed members of $\mathrm{R}^{n \times n}$, and $h(t)$ is an absolutely continuous function having domain $[0, \infty)$ and range contained in a bounded subset of $[0, \infty)$. We will assume that $h(t)$ takes its values in the set $H_\gamma = \{h \in [0, \infty)\colon f_h(s) = |sI - A_0 - e^{-sh} A_1|$ is nonzero for each complex s having $\mathrm{Re}(s) \geqslant \gamma\}$, and derive growth bounds for the solutions $x(t)$ of the system (†). Particularly, we give bounds for the functions $|x(t)|^2/e^{2\gamma t}$ in terms of the behavior of $|h'(t)|$ and of the average values $a(t) = (1/t) \int_0^t |h'(\tau)|\, d\tau$.

In order to clarify the discussion we now fix the notations used throughout this paper, and recall those aspects of the *autonomous* system (∗) $\dot{x}(t) = A_0 x(t) + A_1 x(t-h)$ which will be useful as a point of reference in our investigation. To begin, we let $H = [0, \infty)$, and for each $h \in H$, we let σ_h denote the delay operator having duration h. A simple and useful way to form the characteristic function $f_h(s)$ for the system (∗) is to first write $p(s, \sigma_h) = |sI - A_0 - A_1 \sigma_h| = s^n + a_{n-1}(\sigma_h)\, s^{n-1} + \cdots + a_0(\sigma_h)$, where for $k = 0, ..., n-1$, each $a_k(\sigma_h) \in \mathrm{R}[\sigma_h]$, the ring of real polynomials in the operator σ_h. We then have $f_h(s) = p(s, e^{-sh})$. The utility of this formula is clearly seen in the following lemma.

LEMMA 1.1. *Let* $A_0, A_1 \in \mathrm{R}^{n \times n}$, *let* $\tilde{h} > 0$, *and let* $\gamma \in \mathrm{R}$. *Then there exists*

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$\omega_0 = \omega_0(\bar{h}, \gamma) > 0$ *such that for all* $h \in [0, \bar{h}]$, *the characteristic function* $f_h(s) = |sI - A_0 - e^{-sh} A_1|$ *has no zeros in* $\{|s| \geqslant \omega_0,\ \mathrm{Re}(s) \geqslant \gamma\}$.

Proof. For $z \in \mathbb{C}$, write $p(s, z) = |sI - A_0 - A_1 z| = s^n + a_{n-1}(z)\, s^{n-1} + \cdots + a_0(z)$. Then $p(s, z) = s^n[1 + (a_{n-1}(z)/s) + \cdots + (a_0(z)/s^n)]$, and thus $p(s, z)/s^n \to 1$ uniformly for $|z| \leqslant e^{\bar{h}|\gamma|}$ as $|s| \to \infty$. Since $|e^{-sh}| \leqslant e^{\bar{h}|\gamma|}$ for $\mathrm{Re}(s) \geqslant \gamma$, $h \in [0, \bar{h}]$, we thus see that $f_h(s)/s^n \to 1$ uniformly for $(h, s) \in [0, \bar{h}] \times \{\mathrm{Re}(s) \geqslant \gamma\}$ as $|s| \to \infty$, and the lemma is now apparent.

Q.E.D.

It is well known that for any fixed $h_0 > 0$, the characteristic function $f_{h_0}(s)$ has at most a finite number of zeros to the right of any vertical line in $\mathbb{C}$. Thus, if $f_{h_0}(s)$ has no zeros in $\{\mathrm{Re}(s) \geqslant \gamma\}$, then there exists $\varepsilon_0 > 0$ for which $f_{h_0}(s)$ has no zeros in $\{\mathrm{Re}(s) \geqslant \gamma - \varepsilon_0\}$. One can now note that the mapping $(h, s) \to |sI - A_0 - A_1 e^{-sh}|$ is continuous over $H \times \mathbb{C}$, and set $\omega_0 = \omega_0(\bar{h}, \gamma - \varepsilon_0)$, where $\omega_0(\cdot, \cdot)$ is as in Lemma 1.1 and $\bar{h} > h_0$. It is then seen from continuity of the above mapping that for $m = \inf\{|f_{h_0}(s)|: |s| \leqslant \omega_0,\ \mathrm{Re}(s) \geqslant \gamma - \varepsilon_0\}$, one has $m > 0$. Again using continuity of the above mapping, we see that there is $r \in (0, \bar{h} - h_0)$ such that $|f_h(s) - f_{h_0}(s)| \geqslant m/2$ for each (h, s) lying in the compact subset $(H \cap \{|h - h_0| \leqslant r\}) \times \{|s| \leqslant \omega_0,\ \mathrm{Re}(s) \geqslant \gamma - \varepsilon_0\}$ of $H \times \mathbb{C}$. Thus $|f_h(s)| > 0$ for $h \in H$, $|h - h_0| \leqslant r$, $\mathrm{Re}(s) \geqslant \gamma - \varepsilon_0$, and one arrives at the following lemma.

LEMMA 1.2. *Let* $h_0 \in H$, *and suppose* $f_{h_0}(s)$ *has no zeros in* $\{\mathrm{Re}(s) \geqslant \gamma\}$. *Then there exist a real number* $\varepsilon_0 > 0$ *and a relatively open set* $U = H \cap \{|h - h_0| < r\}$ *such that if* $h \in U$, *then* $f_h(s)$ *has no zeros in* $\{\mathrm{Re}(s) \geqslant \gamma - \varepsilon_0\}$.

By applying Lemma 1.2 and a compactness argument, one immediately obtains Lemma 1.3 below.

LEMMA 1.3. *If* D *is any compact subset of* H_γ, *then there exist* $\varepsilon_0 > 0$ *and a relatively open subset* U *of* H *having* $H_{\gamma - \varepsilon_0} \supset U \supset D$.

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The purpose of this section is to present the basic facts for a matrix function found necessary in the construction and use of a Lyapunov functional introduced in Section 3.

We begin by taking any two matrices $A_0, A_1 \in \mathbb{R}^{n \times n}$. We take any *fixed* $\gamma \in \mathbb{R}$, and for each $h \in H$, we define the matrix functions $T_h(s) = (s + \gamma) I - A_0 - e^{-(s+\gamma)h} A_1$, $M_h(s) = T_h(s)^{-1}$, and the scalar function $g_h(s) = |T_h(s)|$. It will frequently be useful to denote $T_h(s)$ by $T(h, s)$ and $M_h(s)$ by $M(h, s)$. Finally, we recall the subset H_γ of H defined as $H_\gamma = \{h \in H\colon f_h(s) = |sI - A_0 - A_1 e^{-sh}|$ is nonzero for each complex s having $\text{Re}(s) \geqslant \gamma\}$, and note that $f_h(s)$ is the characteristic function for the system $(*)$ $\dot{x}(t) = A_0 x(t) + A_1 x(t-h)$.

Noting that $g_h(s) = f_h(s+\gamma)$, one sees that $H_\gamma = \{h \in H\colon g_h(s)$ is nonzero for each complex s having $\text{Re}(s) \geqslant 0\}$, and $H_{\gamma-\varepsilon_0} = \{h \in H\colon g_h(s)$ is nonzero for each complex s having $\text{Re}(s) \geqslant -\varepsilon_0\}$. From Lemma 1.3 one now sees that if D is any compact subset of H_γ, then there exist $\varepsilon_0 > 0$ and a relatively open subset U of H such that both (1) $U \supset D$, and (2) if $h \in U$, then $g_h(s)$ is nonzero for each complex s having $\text{Re}(s) \geqslant -\varepsilon_0$. Noting that $T_h(s) = sI - (A_0 - \gamma I) - (e^{-\gamma h} A_1)\, e^{-sh}$, we see that $g_h(s)$ is the characteristic function for the system $(*_\gamma)$ $\dot{y}(t) = (A_0 - \gamma I)\, y(t) + (e^{-\gamma h} A_1)\, y(t-h)$. To complete this commentary linking the systems $(*)$ and $(*_\gamma)$, we note that if $x(t)$ is the solution to the differential equation $(*)$ $\dot{x}(t) = A_0 x(t) + A_1 x(t-h)$ having initial data $\phi \in C[-h, 0]$, then the function $y(t) = e^{-\gamma t} x(t)$ is the solution to the differential equation $(*_\gamma)$ $\dot{y}(t) = (A_0 - \gamma I)\, y(t) + (e^{-\gamma h} A_1)\, y(t-h)$ having initial data $\psi(u)$, where $\psi(u) = e^{-\gamma u}\phi(u)$ for $-h \leqslant u \leqslant 0$.

The simple formula $T_h(s) = (s+\gamma)(I - (1/(s+\gamma))\, A_0 - (1/(s+\gamma))\, A_1 e^{-(s+\gamma)h})$, valid for $s \in \mathrm{C} - \{-\gamma\}$, immediately yields $M_h(s) = (s+\gamma)^{-1}\, (I - (1/(s+\gamma))\, A_0 - (1/(s+\gamma))\, A_1 e^{-(s+\gamma)h})^{-1}$, valid throughout $\{s \neq -\gamma,\ g_h(s) \neq 0\}$. Setting $F(h, i\omega) = (I - (1/(i\omega+\gamma))\, A_0 - (1/(i\omega+\gamma))\, A_1 e^{-(i\omega+\gamma)h})^{-1}$, we immediately see for any $\bar{h} > 0$ that $F(h, i\omega) \to I$ uniformly over $[0, \bar{h}]$ as $|\omega| \to \infty$. Given any symmetric matrix $W > 0$, noting that $M_h(i\omega) = (1/(i\omega+\gamma))\, F(h, i\omega)$, we write $(M_h)^*\,(i\omega)\, W M_h(i\omega) = (1/(\omega^2 + \gamma^2))\, F^*(h, i\omega)\, WF(h, i\omega)$. We thus see that if $g_h(i\omega)$ is nonzero for each $\omega \in \mathrm{R}$, then each of the entries of the matrix $(M_h)^*\,(i\omega)\, W M_h(i\omega)$ will be absolutely integrable over the unbounded interval $(-\infty, \infty)$. For each $h \in H_\gamma$, we now form the matrix $Q(h, \alpha)$, defined for every $\alpha \in \mathrm{R}$ as

$$Q(h, \alpha) = \frac{1}{2\pi} \int_{-\infty}^{\infty} (M_h)^*\,(i\omega)\, W M_h(i\omega)\, e^{-i\omega\alpha}\, d\omega.$$

There are several basic formulas which simplify the analysis of the matrix function $Q(h, \alpha)$. First among these is the formula for $f_h(s)$ given in Section 1. Recall that there we wrote $f_h(s) = p(s, e^{-sh})$, where $p(s, \sigma_h) = |sI - A_0 - A_1 \sigma_h| = s^n + a_{n-1}(\sigma_h)\, s^{n-1} + \cdots + a_0(\sigma_h)$, and σ_h is the delay operator of duration h. Using this formula and noting that $g_h(s) = f_h(s+\gamma)$, it is seen that if K is any compact subset of C, and h_0 is any member of H, then $g_h(s) \to g_{h_0}(s)$ uniformly for $s \in K$ as $h \to h_0$.

Another formula useful for analyzing the matrix $Q(h, \alpha)$ is the formula for $M_h(s)$ in terms of the matrix adjugate to $T_h(s)$, i.e., $M_h(s) = T_h(s)^{-1} =$

$(1/g_h(s))(\text{adj } T_h(s))$ for each complex s having $g_h(s) \neq 0$. From this formula it is seen that, given any $h_0 \in H$, if $g_{h_0}(s)$ is nonzero for every complex s lying in some compact subset K of C, then $M_h(s)$ is defined throughout K for each h lying in some relatively open neighborhood $U = H \cap \{|h - h_0| < r\}$, and in fact $M_h(s) \to M_{h_0}(s)$ uniformly for $s \in K$ as $h \to h_0$. With these comments as background, we can now present the first of several lemmas dealing with the matrix $Q(h, \alpha)$ found useful in Section 3.

LEMMA 2.1. *Let $h_0 \in H_\gamma$. Then there exist $\varepsilon_0 > 0$ and a neighborhood $U = H \cap \{|h - h_0| < r\}$, contained in $H_{\gamma - \varepsilon_0}$, for which $Q(h, \alpha)$ is defined and continuous throughout $U \times R$. Furthermore, $Q(h, \alpha) \to Q(h_0, \alpha)$ uniformly throughout* R *as $h \to h_0$.*

Proof. Existence: For any $h_0 \in H_\gamma$, we know from Lemma 1.2 that there is a neighborhood $U = H \cap \{|h - h_0| < r\}$ with $H_{\gamma - \varepsilon_0} \supset U$. For any $h \in U$, since $h \in H_\gamma$, we know that $Q(h, \alpha)$ is defined for each $\alpha \in$ R.

Continuity: Let U be as immediately above, and for each $h \in U$, set $R(h, \omega) = (1/2\pi)(M_h)^*(i\omega)\, WM_h(i\omega)$, and $\tilde{R}(h, \omega) = R(h, \omega) - R(h_0, \omega)$. For any fixed $\omega_1 > 0$, we write

$$Q(h, \alpha) - Q(h_0, \alpha) = \int_{|\omega| \leqslant \omega_1} \tilde{R}(h, \omega)\, e^{-i\omega\alpha}\, d\omega + \int_{|\omega| \geqslant \omega_1} \tilde{R}(h, \omega)\, e^{-i\omega\alpha}\, d\omega.$$

Since $h_0 \in H_\gamma$, we know from the comments preceding this lemma that for any $\omega_1 > 0$, $\tilde{R}(h, \omega) \to 0$ uniformly for $|\omega| \leqslant \omega_1$ as $h \to h_0$. Thus, for any $\omega_1 > 0$, we have

$$\int_{|\omega| \leqslant \omega_1} \|\tilde{R}(h, \omega)\|\, d\omega \to 0 \qquad \text{as} \quad h \to h_0.$$

Again referring to the comments preceding this lemma, we write $2\pi\tilde{R}(h, \omega) = (1/(\omega^2 + \gamma^2)) \cdot [F^*(h, i\omega)\, WF(h, i\omega) - F^*(h_0, i\omega)\, WF(h_0, i\omega)]$, and recall that $F(h, i\omega) \to I$ uniformly over bounded subsets $[0, \tilde{h}]$ of H as $|\omega| \to \infty$. We thus see that

$$\int_{|\omega| \geqslant \omega_1} \|\tilde{R}(h, \omega)\|\, d\omega \to 0$$

uniformly over bounded subsets $[0, \tilde{h}]$ of H as $\omega_1 \to \infty$.

Now writing

$$\|Q(h, \alpha) - Q(h_0, \alpha)\| \leqslant \int_{|\omega| \leqslant \omega_1} \|\tilde{R}(h, \omega)\|\, d\omega + \int_{|\omega| \geqslant \omega_1} \|\tilde{R}(h, \omega)\|\, d\omega,$$

and recalling that for any $\omega_1 > 0$,

$$\int_{|\omega| \leqslant \omega_1} \|\tilde{R}(h, \omega)\| \, d\omega \to 0 \qquad \text{as} \quad h \to h_0,$$

we conclude that $\|Q(h, \alpha) - Q(h_0, \alpha)\| \to 0$ uniformly for $\alpha \in \mathrm{R}$ as $h \to h_0$.

If we now set $S(h, \omega, \alpha) = R(h, \omega) e^{-i\omega\alpha}$, and for fixed $\alpha_0 \in \mathrm{R}$ we set $\tilde{S}(h, \omega, \alpha) = S(h, \omega, \alpha) - S(h_0, \omega, \alpha_0)$, then by applying an argument similar to the above, using the functions $S(h, \omega, \alpha)$ and $\tilde{S}(h, \omega, \alpha)$, one will find that $\|Q(h, \alpha) - Q(h_0, \alpha_0)\| \to 0$ as $(h, \alpha) \to (h_0, \alpha_0)$. Since the original choice of $h_0 \in H$ was arbitrary, the proof is complete. Q.E.D.

The next lemma, which is actually a basic fact of real analysis, is included for use in the two lemmas which immediately follow this lemma. The proof is derived from standard real analysis techniques, and is not given here.

LEMMA 2.2. *Let U_1 be any real interval, and let U_2 be either of $(-\infty, \infty)$, $[\tau, \infty)$, where τ is any member of* R. *Let $f: U_1 \times U_2 \to C^{n \times n}$, where both f and $D_1 f$ are continuous throughout $U_1 \times U_2$. For any $\beta \geqslant 0$, set $J(\beta) = U_2 \cap \{|x_2| \geqslant \beta\}$, and now suppose that there exist $\beta_0 > 0$ and a real function $\phi: J(\beta_0) \to [0, \infty)$ having the properties* (a), (b) *written below*:

(a) $\int_{J(\beta_0)} \phi(x_2) \, dx_2 < \infty$

(b) *for each $(x_1, x_2) \in U_1 \times U_2$ having $|x_2| \geqslant \beta_0$, both $\|f(x_1, x_2)\| \leqslant \phi(x_2)$, and $\|D_1 f(x_1, x_2)\| \leqslant \phi(x_2)$.*

Then for $F(x_1) = \int_{U_2} f(x_1, x_2) \, dx_2$, we know that $F(x_1)$ is defined and finite for all $x_1 \in U_1$, and in fact the derivative $F'(x_1)$ exists and is continuous throughout U_1, with $F'(x_1) = \int_{U_2} D_1 f(x_1, x_2) \, dx_2$.

In the next lemma, we again examine the behavior of the function $R(h, \omega) = (1/2\pi) M^*(h, i\omega) WM(h, i\omega)$. Here we employ two formulas for $(\partial R/\partial h)(h, \omega)$ to prove existence and continuity of $(\partial Q/\partial h)(h, \alpha)$. To obtain the first of these formulas, we recall that $M(h, i\omega) = T^{-1}(h, i\omega)$, where $T(h, i\omega) = (i\omega + \gamma) I - A_0 - e^{-(i\omega + \gamma)h} A_1$. Setting $N_1(h, \omega) = -(((i\omega + \gamma)/2\pi) e^{-(i\omega + \gamma)h}) M^*(h, i\omega) WM(h, i\omega) A_1 M(h, i\omega)$, we can give the first of the two formulas for $(\partial R/\partial h)(h, \omega)$:

$$\begin{aligned} 2\pi \frac{\partial R}{\partial h}(h, \omega) &= \left(\frac{\partial M}{\partial h}(h, i\omega)\right)^* WM(h, i\omega) + M^*(h, i\omega) W \left(\frac{\partial M}{\partial h}(h, i\omega)\right) \\ &= (-M(h, i\omega) A_1 M(h, i\omega)(i\omega + \gamma) e^{-(i\omega + \gamma)h})^* WM(h, i\omega) \\ &\quad + M^*(h, i\omega) W(-M(h, i\omega) A_1 M(h, i\omega)(i\omega + \gamma) e^{-(i\omega + \gamma)h}) \\ &= 2\pi[N_1^*(h, \omega) + N_1(h, \omega)], \end{aligned}$$

i.e.,

$$\frac{\partial R}{\partial h}(h, \omega) = N_1^*(h, \omega) + N_1(h, \omega)$$

for any (h, ω) having $|T(h, i\omega)| \neq 0$.

We now return to the formula $R(h, \omega) = (1/2\pi(\omega^2 + \gamma^2)) F^*(h, i\omega) WF(h, i\omega)$, where $F(h, i\omega) = (I - (1/(i\omega+\gamma)) A_0 - (1/(i\omega+\gamma)) A_1 e^{-(i\omega+\gamma)h})^{-1}$. Setting $N_2(h, \omega) = -(1/2\pi) e^{-(i\omega+\gamma)h} F^*(h, i\omega) WF(h, i\omega) A_1 F(h, i\omega)$, we can give the second of the formulas for $(\partial R/\partial h)(h, \omega)$,

$$\begin{aligned}
&2\pi(\omega^2 + \gamma^2) \frac{\partial R}{\partial h}(h, \omega) \\
&= \left(\frac{\partial F}{\partial h}(h, i\omega)\right)^* WF(h, i\omega) + F^*(h, i\omega)\, W\left(\frac{\partial F}{\partial h}(h, i\omega)\right) \\
&= (-F(h, i\omega) A_1 F(h, i\omega) e^{-(i\omega+\gamma)h})^* WF(h, i\omega) \\
&\quad + F^*(h, i\omega) W(-F(h, i\omega) A_1 F(h, i\omega) e^{-(i\omega+\gamma)h}) \\
&= 2\pi[N_2^*(h, \omega) + N_2(h, \omega)],
\end{aligned}$$

i.e.,

$$\frac{\partial R}{\partial h}(h, \omega) = \frac{1}{\omega^2 + \gamma^2}[N_2^*(h, \omega) + N_2(h, \omega)]$$

for (h, ω) having $(\omega, \gamma) \neq (0, 0)$, $|T(h, i\omega)| \neq 0$.

LEMMA 2.3. *Let* $h_0 \in H_\gamma$. *Then there exist* $\varepsilon_0 > 0$ *and* $U = H \cap \{|h - h_0| < r\}$, *contained in* $H_{\gamma - \varepsilon_0}$, *for which* $(\partial Q/\partial h)(h, \alpha)$ *is defined and continuous throughout* $U \times \mathrm{R}$, *with* $(\partial Q/\partial h)(h, \alpha) = \int_{-\infty}^{\infty} (\partial R/\partial h)(h, \omega)\, e^{-i\omega\alpha}\, d\omega$. *Furthermore,* $(\partial Q/\partial h)(h, \alpha) \to (\partial Q/\partial h)(h_0, \alpha)$ *uniformly throughout* R *as* $h \to h_0$.

Proof. Let $h_0 \in H_\gamma$, and let U be the neighborhood of Lemma 2.1, i.e., $U = H \cap \{|h - h_0| < r\}$, with $H_{\gamma - \varepsilon_0} \supset U$, and with $Q(h, \alpha)$ defined and continuous throughout $U \times \mathrm{R}$. Noting the formulas $2\pi R(h, \omega) = M^*(h, i\omega) WM(h, i\omega)$, and $(\partial R/\partial h)(h, \omega) = N_1^*(h, \omega) + N_1(h, \omega)$, both valid throughout $U \times \mathrm{R}$, with $-2\pi N_1(h, \omega) = (i\omega + \gamma) e^{-(i\omega+\gamma)h} M^*(h, i\omega) WM(h, i\omega) A_1 M(h, i\omega)$, we see that both $R(h, \omega)$, $(\partial R/\partial h)(h, \omega)$ are continuous throughout $U \times \mathrm{R}$.

Now note the formulas $2\pi R(h, \omega) = (1/(\omega^2 + \gamma^2)) \cdot F^*(h, i\omega) WF(h, i\omega)$, and $(\partial R/\partial h)(h, \omega) = (1/(\omega^2 + \gamma^2)) \cdot [N_2^*(h, \omega) + N_2(h, \omega)]$, both valid for $h \in U$, $(\omega, \gamma) \neq (0, 0)$, where $-2\pi N_2(h, \omega) = e^{-(i\omega+\gamma)h} F^*(h, i\omega)$

$WF(h, i\omega) A_1 F(h, i\omega)$. Since U is bounded, we know that $F(h, i\omega) \to I$ uniformly for $h \in U$ as $|\omega| \to \infty$. Now set $c = \sup\{e^{-\gamma h}: h \in U\}$, and set $K = \max\{\|W\|/2\pi + 1, (c\|A_1\| \cdot \|W\|)/2\pi + 1\}$. Then there is some $\beta_0 > 0$ such that for $|\omega| \geq \beta_0$ and $h \in U$, both $\|R(h, \omega)\| \leq K\omega^{-2}$, and $\|(\partial R/\partial h)(h, \omega)\| \leq K\omega^{-2}$. Writing $Q(h, \alpha) = \int_{-\infty}^{\infty} R(h, \omega) e^{-i\omega\alpha} d\omega$, we can apply Lemma 2.2 with $U_1 = U$, $U_2 = (-\infty, \infty)$, and $\phi(\omega) = K\omega^{-2}$ for $|\omega| \geq \beta_0$, and conclude that for each $\alpha \in \mathsf{R}$, $(\partial Q/\partial h)(h, \alpha)$ exists and is continuous in h throughout U, with $(\partial Q/\partial h)(h, \alpha) = \int_{-\infty}^{\infty} (\partial R/\partial h)(h, \omega) e^{-i\omega\alpha} d\omega$.

Having established this formula for $(\partial Q/\partial h)(h, \alpha)$ throughout $U \times \mathsf{R}$, we can now return to the formulas $(\partial R/\partial h)(h, \omega) = N_1^*(h, \omega) + N_1(h, \omega)$, and $(\partial R/\partial h)(h, \omega) = (1/(\omega^2 + \gamma^2)) \cdot [N_2^*(h, \omega) + N_2(h, \omega)]$, and employ techniques comparable to those used in Lemma 2.1. One can then prove that $(\partial Q/\partial h)(h, \alpha) \to (\partial Q/\partial h)(h_0, \alpha)$ uniformly throughout R as $h \to h_0$. Similarly, it can be shown for each $\alpha_0 \in \mathsf{R}$ that $(\partial Q/\partial h)(h, \alpha) \to (\partial Q/\partial h)(h_0, \alpha_0)$ as $(h, \alpha) \to (h_0, \alpha_0)$. Since the choice of $h_0 \in H_\gamma$ was arbitrary, we conclude that $(\partial Q/\partial h)(h, \alpha)$ is continuous throughout $U \times \mathsf{R}$.
Q.E.D.

Before introducing a formula for dealing with $Q(h, \alpha)$ for use throughout this paper, we first recall some basic facts from Fourier transform theory [9]. Consider any matrix function $f(t)$ having domain $(-\infty, \infty)$ and range in $\mathsf{C}^{n \times n}$. If $f \in L^2(-\infty, \infty)$, the Fourier transform $\hat{f}$ of f is defined as $\hat{f}(\omega) = (1/\sqrt{2\pi}) \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt$. The function $\hat{f}$ will lie in $L^2(-\infty, \infty)$, and in fact $f(t) = (1/\sqrt{2\pi}) \int_{-\infty}^{\infty} \hat{f}(\omega) e^{i\omega t} d\omega$. We let $\mathscr{F}$ denote the operator $\mathscr{F}: L^2(-\infty, \infty) \to L^2(-\infty, \infty)$ defined by $\mathscr{F}(f) = \hat{f}$, and we let $\mathscr{F}^{-1}$ denote the inverse operator. Defining the convolution of any two members f, g of $L^2(-\infty, \infty)$ by $(f * g)(t) = \int_{-\infty}^{\infty} f(u) g(t-u) du$, we recall that $\mathscr{F}(f * g) = (\sqrt{2\pi})(\hat{f})(\hat{g})$ provided $f, g \in L^1(-\infty, \infty)$. Finally, we shall use the formula $\int_{-\infty}^{\infty} f^T(u) g(u-t) du = \int_{-\infty}^{\infty} (\hat{f})^* (\omega) \hat{g}(\omega) e^{-i\omega t} d\omega$, valid for $f, g \in (L^1 \cap L^2)(-\infty, \infty)$ if f, g have range in $\mathsf{R}^{n \times n}$. This formula is readily derived by noting first that $(\hat{f})^* = \hat{f}_0$, where $f_0(t) = f^T(-t)$, by next writing

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}_0(\omega) \hat{g}(\omega) e^{-i\omega t} d\omega = (\mathscr{F}^{-1}\{(\hat{f}_0)(\hat{g})\})(-t) = \frac{1}{\sqrt{2\pi}} (f_0 * g)(-t),$$

and by then using the variable substitution $\tilde{u} = -u$ in the expression $(f_0 * g)(-t) = \int_{-\infty}^{\infty} f^T(-u) g(-t-u) du$.

The next formula, which proves extremely useful as a characterization of the matrix $Q(h, \alpha)$, arises from the link between $M_h(s)$ and its inverse Laplace transform. To explore this further, for any $h \in H_\gamma$, we denote the inverse Laplace transform of $M_h(s)$ by $Y_h(t)$, or, when h is fixed and understood from the context, merely by $Y(t)$. Then $Y(\cdot)$ is the solution to the differential–delay equation $\dot{Y}(t) = Y(t)(A_0 - \gamma I) + Y(t-h) A_1 e^{-\gamma h}$ having initial data $Y(u) = \Psi(u)$ for $-h \leqslant u \leqslant 0$, where $\Psi(u) = 0$ if $-h \leqslant u < 0$, and $\Psi(0) = I$. Noting that $Y(t) = 0$ for $t < 0$, we write $M_h(s) = \int_0^\infty Y(t) e^{-st} dt = \int_{-\infty}^\infty Y(t) e^{-st} dt$. Since $h \in H_\gamma$, we know there exist $C \geqslant 1$, $\varepsilon_0 > 0$ such that $\|Y(t)\| \leqslant Ce^{-(\varepsilon_0)t}$ for all $t \geqslant 0$. Thus $Y(\cdot) \in (L^1 \cap L^2)(-\infty, \infty)$, and with $\hat{Y} = \mathscr{F}\{Y\}$, we have $\hat{Y}(\omega) = (1/\sqrt{2\pi}) M_h(i\omega)$. If we now set $f = g = (\sqrt{W}) Y$ in the previously given formula $\int_{-\infty}^\infty f^T(u) g(u-\alpha) du = \int_{-\infty}^\infty (\hat{f})^* (\omega) \hat{g}(\omega) e^{-i\omega\alpha} d\omega$, we obtain the important formula $\int_0^\infty Y^T(u) WY(u-\alpha) du = (1/2\pi) \int_{-\infty}^\infty (M_h)^* (i\omega) WM_h(i\omega) e^{-i\omega\alpha} d\omega$, i.e., $Q(h, \alpha) = \int_0^\infty Y^T(t) WY(t-\alpha) dt$.

From this formula it is obvious that $Q(h, \alpha) \in \mathsf{R}^{n \times n}$ for $h \in H_\gamma$, $\alpha \in \mathsf{R}$. Noting the definition $Q(h, \alpha) = (1/2\pi) \int_{-\infty}^\infty (M_h)^* (i\omega) WM_h(i\omega) e^{-i\omega\alpha} d\omega$, one easily sees that $Q^*(h, \alpha) = Q(h, -\alpha)$. Thus, for $h \in H_\gamma$, $\alpha \in \mathsf{R}$, we have $Q^T(h, \alpha) = Q(h, -\alpha)$. Based also on the formula for $Q(h, \alpha)$ just derived, one may now refer to Infante and Castelan [6], or to Datko [4], and deduce the formulas for $(\partial Q/\partial\alpha)(h, \alpha)$ given in the following lemma. Alternatively, one may note the formula just derived for $Q(h, \alpha)$, and for z in some interval containing 0, one can write $(Q(h, \alpha + z) - Q(h, \alpha))/z = \int_{-\infty}^\infty Y^T(t) W((Y(t-\alpha-z) - Y(t-\alpha))/z) dt$. After some elementary analysis, one can directly apply Lemma 2.2, and differentiate the integral for $Q(h, \alpha)$ with respect to α.

LEMMA 2.4. *Let $h_0 \in H_\gamma$. Then there exist $\varepsilon_0 > 0$ and a neighborhood $U = H \cap \{|h - h_0| < r\}$, contained in $H_{\gamma - \varepsilon_0}$, for which $(\partial Q/\partial\alpha)(h, \alpha)$ is defined and continuous throughout $U \times (\mathsf{R} - \{0\})$. In fact, the following formulas hold for $(\partial Q/\partial\alpha)(h, \alpha)$:*

(a) $(\partial Q/\partial\alpha)(h, \alpha) = -Q(h, \alpha)(A_0 - \gamma I) - Q(h, \alpha + h) A_1 e^{-\gamma h}$ *for* $(h, \alpha) \in U \times (-\infty, 0)$

(b) $(\partial Q/\partial\alpha)(h, \alpha) = (A_0 - \gamma I)^T Q^T(h, -\alpha) + e^{-\gamma h}(A_1)^T Q^T(h, -\alpha + h)$ *for* $(h, \alpha) \in U \times (0, \infty)$.

COROLLARY 2.4. *Let V_1 be any compact subset of H_γ, and let V_2 be any bounded subset of $\mathsf{R} - \{0\}$. Then $\|(\partial Q/\partial\alpha)(h, \alpha)\|$ is bounded over $V_1 \times V_2$.*

Proof. Choose $\alpha_1, \alpha_2 \in \mathsf{R}^+$ such that $[-\alpha_1, 0) \cup (0, \alpha_2] \supset V_2$. Set $P_1(h, \alpha) = -Q(h, \alpha)(A_0 - \gamma I) - Q(h, \alpha + h) A_1 e^{-\gamma h}$, and note that for $h \in H_\gamma$, $\alpha < 0$, we have $(\partial Q/\partial\alpha)(h, \alpha) = P_1(h, \alpha)$. Noting continuity of $P_1(h, \alpha)$ and compactness of V_1, we see that $\|P_1(h, \alpha)\|$ is bounded over $V_1 \times [-\alpha_1, 0]$, and hence $\|(\partial Q/\partial\alpha)(h, \alpha)\|$ is bounded over $V_1 \times [-\alpha_1, 0)$. Setting $P_2(h, \alpha) = (A_0 - \gamma I)^T Q^T(h, -\alpha) + e^{-\gamma h}(A_1)^T Q^T(h, -\alpha + h)$, and noting that $(\partial Q/\partial\alpha)(h, \alpha) = P_2(h, \alpha)$ for $h \in H_\gamma$, $\alpha > 0$, we similarly find that

$\|(\partial Q/\partial\alpha)(h, \alpha)\|$ is bounded over $V_1 \times (0, \alpha_2]$. We now conclude that $\|(\partial Q/\partial\alpha)(h, \alpha)\|$ is bounded over $V_1 \times V_2$. Q.E.D.

It is worth noting that for any $h \in H_\gamma$, the matrix function $Q(h, \cdot)$ is differentiable from the right at $\alpha = 0$, with right derivative given by the formula

$$\left(\frac{\partial Q}{\partial\alpha}(h, \alpha)\right]_{\alpha=0+}\Bigg) = -I - Q(h, 0)(A_0 - \gamma I) - Q(h, h)\, A_1 e^{-\gamma h}.$$

In fact, the matrix function $Q(h, \cdot)$ is also differentiable from the left at $\alpha = 0$, with left derivative given by

$$\left(\frac{\partial Q}{\partial\alpha}(h, \alpha)\right]_{\alpha=0-}\Bigg) = -Q(h, 0)(A_0 - \gamma I) - Q(h, h)\, A_1 e^{-\gamma h}.$$

Since these formulas will not be used in any of the lemmas or theorems in this paper, the proofs of these formulas will not be given here. The interested reader will find several more formulas for $(\partial Q/\partial\alpha)(h, \alpha)$ in a paper of Datko on autonomous differential–delay equations in Hilbert space [4].

3

In this section we introduce a Lyapunov functional for the autonomous differential–delay system $(*)$ $\dot{x}(t) = A_0 x(t) + A_1 x(t-h)$. In the form that we shall use, this functional was first presented by Infante and Castelan [6], who gave a simple differential inequality relating the derivative of the functional along trajectories of the system $(*)$ to the value of the functional itself. Here, by expressing this functional in terms of a Lyapunov functional given in a later paper of Infante [7], we are able to somewhat simplify the analysis given by the above authors. The point of view we adhere to will emphasize those aspects of the functional which make it possible to adapt our analysis to the time-varying system $(\dagger)$ $\dot{x}(t) = A_0 x(t) + A_1 x(t - h(t))$ in the following section.

We begin by taking any two matrices $A_0, A_1 \in \mathsf{R}^{n \times n}$. We then let $\gamma \in \mathsf{R}$, $h \geqslant 0$ be any *fixed* real numbers such that $h \in H_\gamma$, i.e., such that $f_h(s) = |sI - A_0 - e^{-sh} A_1|$ has no zeros in $\{\text{Re}(s) \geqslant \gamma\}$. For any $\phi \in C[-h, 0]$, we let $x(t) = x(\phi, t)$ denote the solution, defined for $0 \leqslant t < \infty$, to the differential–delay equation $(*)$ $\dot{x}(t) = A_0 x(t) + A_1 x(t-h)$ having initial data ϕ on $[-h, 0]$. For each $t \geqslant 0$, as in Section 2, we define $y(\phi, t)$ by $y(\phi, t) = e^{-\gamma t} x(\phi, t)$, and note that $y(\phi, t)$ is the solution, defined for $0 \leqslant t < \infty$, to the differential equation $(*_\gamma)$ $\dot{y}(t) = (A_0 - \gamma I)\, y(t) + e^{-\gamma h} A_1\, y(t-h)$ having initial data ψ, where $\psi(u) = e^{-\gamma u}\phi(u)$ for $-h \leqslant u \leqslant 0$.

The characteristic function $g_h(s)$ for the system $(*_\gamma)$ is given by $g_h(s) = |sI - (A_0 - \gamma I) - (e^{-\gamma h} A_1)\, e^{-sh}|$, i.e., $g_h(s) = f_h(s + \gamma)$. It is important to recall here that there exist $\varepsilon_0 > 0$, $C \geqslant 1$ such that both (1) $g_h(s)$ has no zeros in $\{\mathrm{Re}(s) \geqslant -\varepsilon_0\}$, and (2) $\| Y(t) \| \leqslant Ce^{-(\varepsilon_0)t}$ for all $t \geqslant 0$, where $Y(t)$ is the inverse Laplace transform of $M_h(s)$. Noting that for $t \geqslant 0$, $y(\phi, t)$ has the representation $y(\phi, t) = Y(t)\, \psi(0) + \int_{-h}^{0} e^{-\gamma h} A_1\, Y(t - h - u)\, \psi(u)\, du$, one can apply the Cauchy–Schwartz inequality, and after some elementary calculations, one will see that

$$|y(\phi, t)| \leqslant Ce^{-(\varepsilon_0)t} \cdot \left[|\phi(0)| + \|A_1\| \cdot h^{1/2} e^{|\gamma - \varepsilon_0| h} \times \left(\int_{-h}^{0} |\phi(u)|^2\, du \right)^{1/2} \right] \qquad \text{for all} \quad t \geqslant 0.$$

We now consider the valuable positive functional $n(h, \cdot)$, defined for $\phi \in C[-h, 0]$ by $n(h, \phi)^2 = \phi^T(0)\, \phi(0) + \int_{-h}^{0} \phi^T(u)\, \phi(u)\, du$. Setting $C' = C \cdot (1 + \|A_1\| \cdot h^{1/2} e^{|\gamma - \varepsilon_0| h})$, we see that $C' \geqslant 1$ and $|y(\phi, t)| \leqslant C' n(h, \phi)\, e^{-(\varepsilon_0)t}$ for all $t \geqslant 0$. From this it is clear that $y(\phi, \cdot) \in L^2(0, \infty)$. Given any symmetric matrix $W > 0$, we here introduce the functional $V_0(\cdot) = V_0(h, \cdot)$, defined for each $\phi \in C[-h, 0]$ by $V_0(h, \phi) = \int_0^\infty y^T(\phi, t)\, Wy(\phi, t)\, dt$.

In order to analyze the behavior of $V_0(\cdot)$ along the trajectories of the system $(*)$ $\dot{x}(t) = A_0 x(t) + A_1 x(t-h)$, we first take any trajectory $x(\cdot)$ of the system $(*)$, with corresponding sections $x_t \in C[-h, 0]$ defined for $t \geqslant 0$ by $x_t(u) = x(t+u)$ for $-h \leqslant u \leqslant 0$. For $t, \tau \geqslant 0$, we can set $\phi = x_t$ in $y(\phi, \tau)$. Since the function $z(\tau) = x(t + \tau)$ is the solution to the differential–delay equation $(*)$ $\dot{z}(t) = A_0 z(t) + A_1 z(t-h)$ having initial data $z_0 = x_t \in C[-h, 0]$, we see that $y(x_t, \tau) = z(\tau)\, e^{-\gamma\tau} = x(t + \tau)\, e^{-\gamma\tau}$ for $\tau \geqslant 0$. This yields

$$V_0(x_t) = \int_0^\infty y^T(x_t, \tau)\, Wy(x_t, \tau)\, d\tau = \int_0^\infty x^T(t + \tau)\, Wx(t + \tau)\, e^{-2\gamma\tau}\, d\tau.$$

Setting $\tau' = t + \tau$, we obtain

$$V_0(x_t) = \int_t^\infty x^T(\tau')\, Wx(\tau')\, e^{-2\gamma(\tau' - t)}\, d\tau',$$

i.e.,

$$V_0(x_t) = e^{2\gamma t} \cdot \int_t^\infty x^T(\tau)\, Wx(\tau)\, e^{-2\gamma\tau}\, d\tau.$$

Noting the continuity of the integrand in this expression for $V_0(x_t)$, we write

$$\frac{dV_0(x_t)}{dt} = 2\gamma V_0(x_t) - e^{2\gamma t}(x^T(t)\, Wx(t)\, e^{-2\gamma t}) = 2\gamma V_0(x_t) - x^T(t)\, Wx(t),$$

i.e.,

$$\dot{V}_0(x_t) = 2\gamma V_0(x_t) - x^T(t)\, Wx(t).$$

Given any symmetric matrices $M, R, W > 0$, we now introduce the functional $V(\cdot) = V(h, \cdot)$, defined for each $\phi \in C[-h, 0]$ by

$$V(h, \phi) = \phi^T(0)\, M\phi(0) + e^{-\gamma h} \cdot \int_{-h}^{0} \phi^T(u)\, R\phi(u)\, e^{-2\gamma u}\, du + V_0(h, \phi).$$

By expressing the functional $V(h, \cdot)$ in terms of $y(\phi, t)$ and $\psi(u) = e^{-\gamma u}\phi(u)$, we can give this functional a more natural appearance as follows:

$$V(h, \phi) = \psi^T(0)\, M\psi(0) + e^{-\gamma h} \cdot \int_{-h}^{0} \psi^T(u)\, R\psi(u)\, du$$
$$+ \int_0^{\infty} y^T(\phi, t)\, Wy(\phi, t)\, dt.$$

It is routine to establish an inequality in which a constant multiple of $n(h, \phi)^2$ is majorized by $V(h, \phi)$. In fact, letting $\lambda_m(\cdot)$ denote the minimum eigenvalue of a symmetric matrix, we set $c_1 = c_1(h) = \min(\lambda_m(M), e^{-\gamma h}\lambda_m(R) \cdot \inf\{e^{-2\gamma u}\colon\ -h \leqslant u \leqslant 0\})$. Noting that $V_0(\cdot) \geqslant 0$, we see for each $\phi \in C[-h, 0]$ that $c_1 \cdot (\phi^T(0)\, \phi(0) + \int_{-h}^{0} \phi^T(u)\, \phi(u)\, du) \leqslant V(h, \phi)$, i.e., $c_1(h) \cdot n(h, \phi)^2 \leqslant V(h, \phi)$. Finally, noting that $\min(1, e^{2\gamma h}) = \inf\{e^{-2\gamma u}\colon -h \leqslant u \leqslant 0\}$, it is easily seen that $c_1(h) = \min(\lambda_m(M), e^{-(|\gamma|)h} \cdot \lambda_m(R))$. As in Infante and Castelan [6], we have shown:

> Let $c_1 = c_1(h) = \min(\lambda_m(M), e^{-(|\gamma|)h} \cdot \lambda_m(R))$. Then for each $\phi \in C[-h, 0]$, we have $c_1(h) \cdot n(h, \phi)^2 \leqslant V(h, \phi)$.

Any method of expressing $V_0(\phi)$ directly in terms of ϕ, without referring to the function $y(\phi, t)$, would immediately yield a direct expression for $V(\phi)$. Such an expression is possible [6, 7]. In fact, we can first recall the notation $\hat{y}(\omega) = \hat{y}(\phi, \omega)$ for $\mathscr{F}\{y(\phi, t)\}$, the Fourier transform of the function $y(t) = y(\phi, t)$. Recalling that $y(\phi, \cdot) \in L^2(0, \infty)$ for each $\phi \in C[-h, 0]$, we can apply Parseval's equality to see that

$$V_0(\phi) = \int_0^{\infty} y^T(\phi, t)\, Wy(\phi, t)\, dt = \int_{-\infty}^{\infty} (\hat{y})^*\, (\phi, \omega)\, W\hat{y}(\phi, \omega)\, d\omega.$$

Letting $\xi(s)=\xi(\phi, s)$ denote the Laplace transform of the function $y(t)=y(\phi, t)$, we note that $\hat{y}(\omega)=\xi(i\omega)/\sqrt{2\pi}$. Hence, $2\pi V_0(\phi)=\int_{-\infty}^{\infty} \xi^*(\phi, i\omega)\, W\xi(\phi, i\omega)\, d\omega$, and it is relevant here to recall the differential equation for $y(\cdot)$,

$$\begin{aligned} \dot{y}(t) &= (A_0-\gamma I)\, y(t)+e^{-\gamma h}A_1 y(t-h), \\ y(u) &= \psi(u)=e^{-\gamma u}\phi(u) \qquad \text{for} \quad -h\leqslant u\leqslant 0. \end{aligned} \tag{$*_y$}$$

Taking Laplace transforms of both sides of $(*_y)$, we obtain

$$\begin{aligned} s\xi(s)-\psi(0) &= (A_0-\gamma I)\, \xi(s)+e^{-sh}(A_1e^{-\gamma h})\, \xi(s) \\ &\quad + e^{-sh}(A_1 e^{-\gamma h})\cdot \int_{-h}^{0} \psi(u)\, e^{-su}\, du. \end{aligned}$$

Noting that $M_h(s)=(sI-(A_0-\gamma I)-(e^{-\gamma h}A_1)\, e^{-sh})^{-1}$, we simplify this to obtain

$$\xi(s)=M_h(s)\, \psi(0)+M_h(s)(A_1e^{-\gamma h})\cdot \int_{-h}^{0} \psi(u)\, e^{-s(u+h)}\, du.$$

Setting $I_h(\omega)=\int_{-h}^{0} \psi(u)\, e^{-i\omega(u+h)}\, du$, we have $\xi(i\omega)=M_h(i\omega)\, \psi(0)+M_h(i\omega)(A_1e^{-\gamma h})\, I_h(\omega)$. Expanding the expression $\xi^*(i\omega)\, W\xi(i\omega)$ and applying some elementary analysis, one uses Fubini's Theorem, and finds that

$$\begin{aligned} V_0(\phi) &= \psi^T(0)\, Q(h, 0)\, \psi(0)+2\psi^T(0)\cdot \int_{-h}^{0} Q(h, u+h)\, e^{-\gamma h}\, A_1\psi(u)\, du \\ &\quad + \int_{-h}^{0}\int_{-h}^{0} \psi^T(u)(e^{-\gamma h}A_1)^T\, Q(h, v-u)(A_1e^{-\gamma h})\, \psi(v)\, dv\, du, \end{aligned}$$

where $Q(h, \alpha)$ is the matrix which was the subject of Section 2. From this one immediately sees, as established in Infante [7], that

$$\begin{aligned} V_0(h, \phi) &= \phi^T(0)\, Q(h, 0)\, \phi(0)+2e^{-\gamma h}\phi^T(0)\cdot \int_{-h}^{0} Q(h, u+h)\, A_1\phi(u)\, e^{-\gamma u}\, du \\ &\quad + e^{-2\gamma h}\cdot \int_{-h}^{0}\int_{-h}^{0} \phi^T(u)(A_1)^T\, Q(h, v-u)\, A_1\phi(v)\, e^{-\gamma(u+v)}\, dv\, du. \end{aligned}$$

We have written the functional V_0 directly in terms of ϕ. To express the functional V in a similar way, we write

$$V(h, \phi)=\phi^T(0)\, M\phi(0)+e^{-\gamma h}\cdot \int_{-h}^{0} \phi^T(u)\, R\phi(u)\, e^{-2\gamma u}\, du+V_0(h, \phi),$$

i.e.,

$$V(h,\phi)=\phi^T(0)\,M\phi(0)+e^{-\gamma h}\cdot\int_{-h}^{0}\phi^T(u)\,R\phi(u)\,e^{-2\gamma u}\,du$$
$$+\phi^T(0)\,Q(h,0)\,\phi(0)+2e^{-\gamma h}\phi^T(0)\cdot\int_{-h}^{0}Q(h,u+h)\,A_1\phi(u)\,e^{-\gamma u}\,du$$
$$+e^{-2\gamma h}\cdot\int_{-h}^{0}\int_{-h}^{0}\phi^T(u)(A_1)^T\,Q(h,v-u)\,A_1\phi(v)\,e^{-\gamma(u+v)}\,dv\,du.$$

Using the above formula it is easily seen that for any $h_1\geqslant h$, the functional $V(h,\cdot)$ can be extended to the space $C[-h_1,0]$. In fact, for any $\Phi\in C[-h_1,0]$, we first define $\phi=r(\Phi)$ as the restriction of Φ to the interval $[-h,0]$. We now naturally define $V(h,\Phi)$ as $V(h,\phi)$. In the same sense, for any $h_1\geqslant h$ and $\Phi\in C[-h_1,0]$, we set $n(h,\Phi)=n(h,\phi)$, where $\phi=r(\Phi)$. This simple point has been included for its value in the following section.

In order to calculate $\dot V(x_t)$, the time derivative of $V(\cdot)$ along solutions $x(\cdot)$ of the differential equation $(*)$ $\dot x(t)=A_0x(t)+A_1x(t-h)$, we first write

$$V(x_t)-V_0(x_t)=x^T(t)\,Mx(t)+e^{-\gamma h}\cdot\int_{-h}^{0}x^T(t+u)\,Rx(t+u)\,e^{-2\gamma u}\,du.$$

Now setting $x(t)=e^{\gamma t}y(t)$ in the expression $x^T(t)\,Mx(t)$, and using the variable substitution $u'=t+u$ for the above integral, we obtain the following expression for $V-V_0$:

$$V(x_t)-V_0(x_t)=e^{2\gamma t}y^T(t)\,My(t)$$
$$+e^{2\gamma t}e^{-\gamma h}\cdot\int_{t-h}^{t}x^T(u')\,Rx(u')\,e^{-2\gamma u'}\,du'.$$

After a long, direct computation one can obtain

$$\frac{dV(x_t)-V_0(x_t)}{dt}$$
$$=2\gamma\cdot\left[x^T(t)\,Mx(t)+e^{-\gamma h}\cdot\int_{-h}^{0}x^T(t+u)\,Rx(t+u)\,e^{-2\gamma u}\,du\right]$$
$$+2\cdot[x^T(t)(A_0^T-\gamma I)+x^T(t-h)\,A_1^T]\,Mx(t)$$
$$+e^{-\gamma h}x^T(t)\,Rx(t)-e^{\gamma h}x^T(t-h)\,Rx(t-h).$$

It is convenient to write this as

$$\dot V(x_t) - \dot V_0(x_t) = 2\gamma \cdot [V(x_t) - V_0(x_t)] \\ + 2x^T(t)[(A_0^T - \gamma I)M + e^{-\gamma h}R]\,x(t) - e^{-\gamma h} \\ \cdot (x^T(t) - e^{\gamma h}x^T(t-h)) \begin{pmatrix} R & MA_1 \\ A_1^T M & R \end{pmatrix} \begin{pmatrix} x(t) \\ -e^{\gamma h}x(t-h) \end{pmatrix}.$$

Finally, noting that $\dot V_0(x_t) = 2\gamma V_0(x_t) - x^T(t)\,Wx(t)$, we obtain

$$\dot V(x_t) = 2\gamma V(x_t) - x^T(t)[W - 2(A_0^T - \gamma I)M - 2e^{-\gamma h}R]\,x(t) \\ - e^{-\gamma h} \cdot (x^T(t) - e^{\gamma h}x^T(t-h)) \begin{pmatrix} R & MA_1 \\ A_1^T M & R \end{pmatrix} \begin{pmatrix} x(t) \\ -e^{\gamma h}x(t-h) \end{pmatrix}.$$

Using this formula for $\dot V(x_t)$, it can be seen that by properly choosing the symmetric, positive matrices M, R, and W, one will have $\dot V(\phi) - 2\gamma V(\phi) \leqslant 0$ for all $\phi \in C[-h, 0]$. Thus the function $V(x_t)$ will satisfy $\dot V(x_t) \leqslant 2\gamma V(x_t)$ for all $t \geqslant 0$, and recalling elementary analysis, we conclude that for any solution $x(\cdot)$ of the differential equation $(*)$ $\dot x(t) = A_0 x(t) + A_1 x(t-h)$, one will have $V(x_t) \leqslant V(x_0)\,e^{2\gamma t}$ for all $t \geqslant 0$.

4

In this section we examine the behavior of differential–delay systems having time-varying delays. For this purpose we use a time-varying modification of the Lyapunov functional constructed in the preceding section for the autonomous system $(*)$ $\dot x(t) = A_0 x(t) + A_1 x(t-h)$. We shall present an analysis of the behavior of this time-varying functional along the trajectories of the system $(\dagger)$ $\dot x(t) = A_0 x(t) + A_1 x(t - h(t))$. Although this analysis will be quite laborious, the end result will be a new approach to growth estimates for differential equations having time-varying delays $h(t)$.

Before proceeding to the lemmas and theorems of this section, it is appropriate to note the basic facts of existence and uniqueness of solutions for the types of systems we will be considering. In this section we will be considering differential–delay equations of the form $(\dagger)$ $\dot x(t) = A_0 x(t) + A_1 x(t - h(t))$, where $h(\cdot)$ is a bounded, absolutely continuous function having domain $[0, \infty)$ and range contained in the set $H = [0, \infty)$. It follows from basic results in the theory of functional differential equations [5] that for any $\bar h \geqslant \sup\{h(t): t \geqslant 0\}$ and for any initial data $\chi \in C[-\bar h, 0]$, there is a unique function $x: [-\bar h, \infty) \to \mathbb{R}^n$ having $x(u) = \chi(u)$ for

$-\bar{h} \leqslant u \leqslant 0$, which is C^1 over the interval $[0, \infty)$, and which satisfies the differential equation (†) for all $t \geqslant 0$.

We are now ready to modify the functional $V(h, \phi)$ for use in the case where the delay is a function of time. We begin by noting that an absolutely continuous function is differentiable a.e. in Lebesgue measure. Now taking any *fixed* $\gamma \in \mathrm{R}$ and a compact subset D of H_γ, we define $S_\gamma(D)$ as the class of all bounded, absolutely continuous functions $h(t)$ with domain $[0, \infty)$ and range in H, having the property (i) below:

(i) $h(t) \in D$ for all $t \geqslant 0$.

For constants μ_1, μ_2 with $\mu_1 < 0 < \mu_2$, the subset $S_\gamma(D, \mu_1, \mu_2)$ of $S_\gamma(D)$ is now defined as the class of those members $h(\cdot)$ of $S_\gamma(D)$ having the property (ii) below:

(ii) $\mu_1 \leqslant h'(t) \leqslant \mu_2$ a.e. in Lebesgue measure over $[0, \infty)$.

The constant sup D will be denoted by $\bar{d}$, and finally, for any $h(\cdot) \in S_\gamma(D)$, we define the time-varying functional $G(t, \phi)$, for $t \geqslant 0$ and $\phi \in C[-\bar{d}, 0]$, as $G(t, \phi) = V(h(t), \phi)$.

It is important to particularly note here a basic consequence of Lemma 1.3. Since D is assumed to be a compact subset of H_γ, we know that there is a relatively open subset U of H having $H_\gamma \supset U \supset D$.

To assist in a technical detail occurring in our analysis of the behavior of the functional G along trajectories of the system (†) $\dot{x}(t) = A_0 x(t) + A_1 x(t - h(t))$, we now give the following lemma. This lemma will be useful in finding an expression for $\dot{G}$. Since this lemma is a basic fact of real analysis, the proof will not be given here. However, it should not be assumed that the proof is routine.

LEMMA 4.1. *Let $f\colon (a, b) \times [c - \delta, d] \to \mathrm{R}^{n \times n}$ be continuous, where δ is a positive real number. Set $U = (a, b) \times \{c - \delta \leqslant x_2 \leqslant d,\ x_2 \neq c\}$, and suppose first that $D_1 f(x_1, x_2)$ is defined and continuous on U, and also that there is a constant M such that $\|D_1 f(x_1, x_2)\| \leqslant M$ for $(x_1, x_2) \in U$. Now, for each $x_1 \in (a, b)$, define $F(x_1) = \int_c^d f(x_1, x_2)\, dx_2$. Then the derivative $F'(x_1)$ exists for each $x_1 \in (a, b)$ and $F'(x_1) = \int_c^d D_1 f(x_1, x_2)\, dx_2$.*

We now give a formula for $\dot{G}(x_t)$, the time derivative of the function $G(t, x_t)$ along solutions of the differential equation (†) $\dot{x}(t) = A_0 x(t) + A_1 x(t - h(t))$.

LEMMA 4.2. *Let D be any compact subset of H_γ, and let $h(\cdot)$ be any member of the class $S_\gamma(D)$. Consider the following functionals $E(\eta, \eta', \phi)$ and $F(\eta, \phi)$, defined for $\eta \in D$, $\eta' \in \mathrm{R}$, and $\phi \in C[-\bar{d}, 0]$:*

$$E(\eta, \eta', \phi) = -\phi^T(0)[W - 2(A_0^T - \gamma I) M - 2e^{-\gamma\eta} R]\,\phi(0)$$
$$- e^{-\gamma\eta} \cdot [\phi^T(0) - e^{\gamma\eta}\phi^T(-\eta)]$$
$$\cdot \begin{pmatrix} R & (M + \eta' Q(\eta, 0))\, A_1 \\ A_1^T(M + \eta' Q^T(\eta, 0)) & (1 + \eta')\, R \end{pmatrix} \begin{pmatrix} \phi(0) \\ -e^{\gamma\eta}\phi(-\eta) \end{pmatrix};$$

$$F(\eta, \phi) = \sum_1^8 F_i(\eta, \phi),$$

where

$$F_1(\eta, \phi) = \phi^T(0)\, D_1 Q(\eta, 0)\, \phi(0),$$

$$F_2(\eta, \phi) = -\gamma e^{-\gamma\eta} \cdot \int_{-\eta}^{0} \phi^T(u)\, R\phi(u)\, e^{-2\gamma u}\, du,$$

$$F_3(\eta, \phi) = -2\gamma\phi^T(0) \cdot \int_{-\eta}^{0} Q(m, u + \eta)\, e^{-\gamma\eta} A_1 \phi(u)\, e^{-\gamma u}\, du,$$

$$F_4(\eta, \phi) = -2\gamma \cdot \int_{-\eta}^{0} \int_{-\eta}^{0} \phi^T(u)\, e^{-\gamma\eta} A_1^T Q(\eta, v - u)\, A_1 e^{-\gamma\eta}$$
$$\cdot \phi(v)\, e^{-\gamma(u+v)}\, dv\, du,$$

$$F_5(\eta, \phi) = 2e^{-\gamma\eta}\phi^T(0) \cdot \int_{-\eta}^{0} [(A_0^T - \gamma I)\, Q(\eta, u + \eta)$$
$$+ e^{-\gamma\eta} A_1^T Q(\eta, u)]\, A_1 \phi(u)\, e^{-\gamma u}\, du,$$

$$F_6(\eta, \phi) = 2e^{-\gamma\eta}\phi^T(0) \cdot \int_{-\eta}^{0} D_1 Q(\eta, u + \eta)\, A_1 \phi(u)\, e^{-\gamma u}\, du,$$

$$F_7(\eta, \phi) = 2e^{-\gamma\eta}\phi^T(-\eta)\, A_1^T \cdot \int_{-\eta}^{0} Q(\eta, u + \eta)\, A_1 \phi(u)\, e^{-\gamma u}\, du,$$

$$F_8(\eta, \phi) = \int_{-\eta}^{0} \int_{-\eta}^{0} \phi^T(u)\, e^{-\gamma\eta} A_1^T D_1 Q(\eta, v - u)\, A_1 e^{-\gamma\eta}$$
$$\cdot \phi(v)\, e^{-\gamma(u+v)}\, dv\, du.$$

Now let $x(\cdot)$ be any solution of the differential equation (†) $\dot{x}(t) = A_0 x(t) + A_1 x(t - h(t))$, *and consider the function $G(t, x_t)$. For each nonnegative value of t at which $h(\cdot)$ is differentiable, the following formula holds for $\dot{G}(x_t) = dG(t, x_t)/dt$. Particularly, the formula holds a.e. in Lebesgue measure for $t \geqslant 0$:*

$$\dot{G}(x_t) = 2\gamma G(t, x_t) + E(h(t), h'(t), x_t) + h'(t) \cdot F(h(t), x_t).$$

Proof. We begin by explicitly displaying the functional $G(t, x_t)$ as follows:

$$G(t, x_t) = x^T(t)[M + Q(h(t), 0)]\, x(t)$$
$$+ e^{-\gamma h(t)} \cdot \int_{-h(t)}^{0} x^T(t+u)\, Rx(t+u)\, e^{-2\gamma u}\, du$$
$$+ 2e^{-\gamma h(t)} x^T(t) \cdot \int_{-h(t)}^{0} Q(h(t), u + h(t))\, A_1 x(t+u)\, e^{-\gamma u}\, du$$
$$+ e^{-2\gamma h(t)} \cdot \int_{-h(t)}^{0} \int_{-h(t)}^{0} x^T(t+u)\, A_1^T Q(h(t), v-u)$$
$$\cdot A_1 x(t+v)\, e^{-\gamma(u+v)} dv\, du.$$

Now setting $y(t) = e^{-\gamma t} x(t)$ for $t \geqslant 0$, and using the variable substitution $u' = t + u$, $v' = t + v$ for the above integrals, we obtain the identity for G written below:

$$G(t, x_t) - e^{2\gamma t} y^T(t)[M + Q(h(t), 0)]\, y(t)$$
$$= e^{2\gamma t} \cdot \Bigg[e^{-\gamma h(t)} \cdot \int_{t-h(t)}^{t} x^T(u')\, Rx(u')\, e^{-2\gamma u'}\, du' + 2e^{-\gamma(t+h(t))} x^T(t)$$
$$\cdot \int_{t-h(t)}^{t} Q(h(t), u' + h(t) - t)\, A_1 x(u')\, e^{-\gamma u'}\, du' + e^{-2\gamma h(t)}$$
$$\cdot \int_{t-h(t)}^{t} \int_{t-h(t)}^{t} x^T(u')\, A_1^T Q(h(t), v' - u')\, A_1 x(v')\, e^{-\gamma(u'+v')}\, dv'\, du' \Bigg].$$

Referring to Lemma 2.3 on the existence and continuity of $D_1 Q(h, \alpha)$, it is seen that if $h(\cdot)$ is differentiable at t, then the function $y^T(t)[M + Q(h(t), 0)]\, y(t)$ is differentiable at t. Now set

$$j_1(\alpha_1, \alpha_2) = \int_{\alpha_1}^{\alpha_2} x^T(u')\, Rx(u')\, e^{-2\gamma u'}\, du',$$

$$j_2(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = \int_{\alpha_1}^{\alpha_2} Q(\alpha_3, u' + \alpha_4)\, A_1 x(u')\, e^{-\gamma u'}\, du',$$

$$j_3(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5) = \int_{\alpha_1}^{\alpha_2} \int_{\alpha_3}^{\alpha_4} x^T(u')\, A_1^T Q(\alpha_5, v' - u')$$
$$\cdot A_1 x(v')\, e^{-\gamma(u'+v')} dv'\, du',$$

and note that the integrals occurring on the right side of the above equation can be expressed, respectively, as $J_1(t) = j_1(t - h(t), t)$, $J_2(t) =$

$j_2(t-h(t), t, h(t), h(t)-t)$, and $J_3(t)=j_3(t-h(t), t, t-h(t), t, h(t))$. Noting continuity of $x(\cdot)$ and of $Q(\cdot,\cdot)$, and again referring to Lemma 2.3, we see that the matrix functions $J_1(\cdot)$, $J_3(\cdot)$ are differentiable at t if $h(\cdot)$ is, and it is then routine to calculate their derivatives. The technical detail mentioned prior to Lemma 4.1 occurs in finding the derivative of the matrix function $J_2(\cdot)$. Here we recall Lemma 2.4 and Corollary 2.4, and then apply Lemma 4.1 to calculate $D_4 j_2(t-h(t), t, h(t), h(t)-t)$. One then makes routine use of the chain rule to calculate the derivative of the matrix function $J_2(\cdot)$ at all values of t having $h(\cdot)$ differentiable at t. Finally, a straightforward but very long computation will now yield the formula

$$\frac{dG(t, x_t)}{dt} = 2\gamma G(t, x_t) + E(h(t), h'(t), x_t) + h'(t) \cdot F(h(t), x_t). \quad \text{Q.E.D.}$$

It will prove valuable to carefully examine the functionals E and F. For this purpose we make the simplifying assumption that each of the matrices M, R, and W is a constant multiple of the identity, i.e., $M = k_M I$, $R = k_R I$, and $W = k_W I$, with k_M, k_R, $k_W > 0$. We can then set $\tilde{Q}(\eta, \alpha) = (1/2\pi) \int_{-\infty}^{\infty} M^*(\eta, i\omega) M(\eta, i\omega)\, e^{-i\omega\alpha}\, d\omega$ for $\eta \in H_\gamma$ and $\alpha \in \mathrm{R}$, and write $Q(\eta, \alpha) = k_W \cdot \tilde{Q}(\eta, \alpha)$. If we note Lemmas 2.1 and 2.3, we now see that each of the quantities $\tilde{q}_1$, $\tilde{q}_2$, and $\tilde{q}$ defined below is finite:

$$\tilde{q}_1 = \sup\{\|\tilde{Q}(\eta, \alpha)\| : \eta \in D,\ -\tilde{d} \leqslant \alpha \leqslant \tilde{d}\},$$
$$\tilde{q}_2 = \sup\{\|D_1 \tilde{Q}(\eta, \alpha)\| : \eta \in D,\ -\tilde{d} \leqslant \alpha \leqslant \tilde{d}\},$$
$$\tilde{q} = \max(\tilde{q}_1, \tilde{q}_2).$$

We are now prepared to examine the functionals E and F. To begin, we expand the expression for E given in the above lemma, thus obtaining the expression for E written below:

$$\begin{aligned} E(\eta, \eta', \phi) = {} & -k_W \phi^T(0)\, \phi(0) - (e^{\gamma\eta})(1+\eta')\, k_R \phi^T(-\eta)\, \phi(-\eta) \\ & + \phi^T(0)[2k_M(A_0^T - \gamma I) + (e^{-\gamma\eta})\, k_R I]\, \phi(0) \\ & + 2\phi^T(0)[k_M I + (\eta')\, k_W \tilde{Q}(\eta, 0)]\, A_1 \phi(-\eta). \end{aligned}$$

Using straightforward applications of the Cauchy–Schwartz inequality, we readily arrive at the inequality for E below, valid for $\eta \in D$, $\eta' \in \mathrm{R}$, $\phi \in C[-\tilde{d}, 0]$:

$$\begin{aligned} E(\eta, \eta', \phi) \leqslant {} & -k_W \cdot |\phi(0)|^2 - (e^{\gamma\eta})(1+\eta')\, k_R \cdot |\phi(-\eta)|^2 \\ & + [2\|A_0^T - \gamma I\|\, k_M + (e^{-\gamma\eta})\, k_R] \cdot |\phi(0)|^2 \\ & + 2\|A_1\| \cdot (k_M + |\eta'|\, \tilde{q} k_W)(|\phi(0)|)(|\phi(-\eta)|). \end{aligned}$$

We now turn to the functional $F=\sum_1^8 F_i$. Here we separately analyze the cases $\gamma<0$ and $\gamma>0$, and find that for $-\eta\leqslant u,\ v\leqslant 0$, one has $e^{-\gamma\eta}e^{-2\gamma u}\leqslant e^{\eta|\gamma|}$, $e^{-2\gamma\eta}e^{-\gamma(u+v)}\leqslant e^{2\eta|\gamma|}$, and $e^{-\gamma\eta}e^{-\gamma u}\leqslant e^{\eta|\gamma|}$. Motivated by the Cauchy–Schwartz inequality, we now define the quantities $b_1, ..., b_8$:

$$b_1=\tilde{q};\qquad b_2=|\gamma|\ e^{\tilde{d}|\gamma|};$$

$$b_3=2|\gamma|\ e^{\tilde{d}|\gamma|}\|A_1\|\ (\tilde{q})(\tilde{d})^{1/2};\qquad b_4=2|\gamma|\ e^{2|\gamma|\tilde{d}}\|A_1\|^2\ (\tilde{q})(\tilde{d});$$

$$b_5=2e^{\tilde{d}|\gamma|}\|A_1\|(\|A_0^T-\gamma I\|+e^{\tilde{d}|\gamma|}\|A_1\|)(\tilde{q})(\tilde{d})^{1/2};$$

$$b_6=2e^{\tilde{d}|\gamma|}\|A_1\|(\tilde{q})(\tilde{d})^{1/2};\qquad b_7=2e^{\tilde{d}|\gamma|}\|A_1\|^2\ (\tilde{q})(\tilde{d})^{1/2};$$

$$b_8=e^{2|\gamma|\tilde{d}}\|A_1\|^2\ (\tilde{q})(\tilde{d}).$$

Recalling the formula $Q(\eta,\alpha)=k_W\cdot\tilde{Q}(\eta,\alpha)$, and also recalling the functional $n(\eta,\phi)=[\phi^T(0)\,\phi(0)+\int_{-\eta}^0\phi^T(u)\,\phi(u)\,du]^{1/2}$, we see from the Cauchy–Schwartz inequality that $\int_{-\eta}^0|\phi(u)|\,du\leqslant\eta^{1/2}\cdot n(\eta,\phi)$. We can now use straightforward applications of the Cauchy–Schwartz inequality in each of the expressions for $F_i(\eta,\phi)$ given in the above lemma, with $i=1, ..., 8$. Upon so doing, we obtain the following inequalities, valid for $\eta\in D$, $\phi\in C[-\tilde{d},0]$:

$$|F_1(\eta,\phi)|\leqslant\tilde{q}k_W\cdot n(\eta,\phi)^2=b_1k_W\cdot n(\eta,\phi)^2;$$

$$|F_2(\eta,\phi)|\leqslant|\gamma|\ e^{\eta|\gamma|}k_R\cdot n(\eta,\phi)^2\leqslant b_2k_R\cdot n(\eta,\phi)^2;$$

$$|F_3(\eta,\phi)|\leqslant 2|\gamma|\ e^{\eta|\gamma|}\|A_1\|\ \tilde{q}\eta^{1/2}k_W\cdot n(\eta,\phi)^2\leqslant b_3k_W\cdot n(\eta,\phi)^2;$$

$$\begin{aligned}|F_4(\eta,\phi)|&\leqslant 2|\gamma|\ e^{2\eta|\gamma|}\|A_1\|^2\ \tilde{q}\cdot k_W\int_{-\eta}^0\int_{-\eta}^0(|\phi(u)|)(|\phi(v)|)\,dv\,du\\&=2|\gamma|\ e^{2\eta|\gamma|}\|A_1\|^2\ \tilde{q}\cdot k_W\left(\int_{-\eta}^0|\phi(u)|\,du\right)\left(\int_{-\eta}^0|\phi(v)|\,dv\right)\\&\leqslant 2|\gamma|\ e^{2\eta|\gamma|}\|A_1\|^2\ \tilde{q}\eta k_W\cdot n(\eta,\phi)^2,\end{aligned}$$

and thus

$$|F_4(\eta,\phi)|\leqslant b_4k_W\cdot n(\eta,\phi)^2;$$

$$\begin{aligned}|F_5(\eta,\phi)|&\leqslant 2e^{\eta|\gamma|}\|A_1\|\ (\|A_0^T-\gamma I\|+e^{\eta|\gamma|}\|A_1\|)\\&\qquad\cdot\tilde{q}\eta^{1/2}k_W\cdot n(\eta,\phi)^2\leqslant b_5k_W\cdot n(\eta,\phi)^2;\end{aligned}$$

$$\begin{aligned}|F_6(\eta,\phi)|&\leqslant 2e^{\eta|\gamma|}\|A_1\|\ \tilde{q}\eta^{1/2}k_W\cdot n(\eta,\phi)^2\\&\leqslant b_6k_W\cdot n(\eta,\phi)^2;\end{aligned}$$

$$\begin{aligned}|F_7(\eta,\phi)|&\leqslant 2e^{\eta|\gamma|}\|A_1\|^2\ \tilde{q}\eta^{1/2}k_W(|\phi(-\eta)|)\,n(\eta,\phi)\\&\leqslant b_7k_W\cdot(|\phi(-\eta)|)\,n(\eta,\phi);\end{aligned}$$

$$\begin{aligned}|F_8(\eta, \phi)| &\leqslant e^{2|\gamma|\eta} \|A_1\|^2 \, \tilde{q} \cdot k_W \int_{-\eta}^{0} \int_{-\eta}^{0} (|\phi(u)|)(|\phi(v)|) \, dv \, du \\ &= e^{2|\gamma|\eta} \|A_1\|^2 \, \tilde{q} \cdot k_W \left(\int_{-\eta}^{0} |\phi(u)| \, du \right) \left(\int_{-\eta}^{0} |\phi(v)| \, dv \right) \\ &\leqslant e^{2|\gamma|\eta} \|A_1\|^2 \, \tilde{q} \eta k_W \cdot n(\eta, \phi)^2,\end{aligned}$$

and thus

$$|F_8(\eta, \phi)| \leqslant b_8 k_W \cdot n(\eta, \phi)^2.$$

Expressing this concisely, we can write the following inequalities, valid for $\eta \in D$, $\phi \in C[-\tilde{d}, 0]$:

$$\begin{aligned}&|F_2(\eta, \phi)| \leqslant b_2 k_R \cdot n(\eta, \phi)^2; \\ &|F_7(\eta, \phi)| \leqslant b_7 k_W \cdot (|\phi(-\eta)|) \, n(\eta, \phi); \\ &|F_i(\eta, \phi)| \leqslant b_i k_W \cdot n(\eta, \phi)^2 \qquad \text{for} \quad i = 1, 3, 4, 5, 6, 8.\end{aligned}$$

It will be of special interest, in analyzing the functional $E(\eta, \eta', \phi) + (\eta') \, F(\eta, \phi)$, to examine the functional $E + (\eta') \, F_7$ in light of the preceding inequality for $F_7(\eta, \phi)$. Referring to the above inequality for $E(\eta, \eta', \phi)$, we now separately examine the cases $|\phi(-\eta)| \leqslant |\phi(0)|$, $|\phi(0)| < |\phi(-\eta)| \leqslant n(\eta, \phi)$, and $n(\eta, \phi) < |\phi(-\eta)|$. If $|\phi(-\eta)| \leqslant |\phi(0)|$, then

$$\begin{aligned}&E(\eta, \eta', \phi) + (\eta') \, F_7(\eta, \phi) \\ &\quad \leqslant [-k_W + 2\|A_0^T - \gamma I\| \, k_M + (e^{-\gamma\eta}) \, k_R \\ &\qquad + 2\|A_1\| \cdot (k_M + |\eta'| \, \tilde{q} k_W)] \cdot |\phi(0)|^2 \\ &\qquad - (e^{\gamma\eta})(1 + \eta') \, k_R \cdot |\phi(-\eta)|^2 + |\eta'| \, b_7 k_W \cdot n(\eta, \phi)^2;\end{aligned}$$

if $|\phi(0)| < |\phi(-\eta)| \leqslant n(\eta, \phi)$, then

$$\begin{aligned}&E(\eta, \eta', \phi) + (\eta') \, F_7(\eta, \phi) \\ &\quad \leqslant [-k_W + 2\|A_0^T - \gamma I\| \, k_M + (e^{-\gamma\eta}) \, k_R] \cdot |\phi(0)|^2 \\ &\qquad + [-(e^{\gamma\eta})(1 + \eta') \, k_R + 2\|A_1\| \cdot (k_M + |\eta'| \, \tilde{q} k_W)] \\ &\qquad \cdot |\phi(-\eta)|^2 + |\eta'| \, b_7 k_W \cdot n(\eta, \phi)^2;\end{aligned}$$

if $n(\eta, \phi) < |\phi(-\eta)|$, then

$$\begin{aligned}E(\eta, \eta', \phi) + (\eta') \, F_7(\eta, \phi) &\leqslant [-k_W + 2\|A_0^T - \gamma I\| \, k_M + (e^{-\gamma\eta}) \, k_R] \cdot |\phi(0)|^2 \\ &\quad + [-(e^{\gamma\eta})(1 + \eta') \, k_R + 2\|A_1\| \cdot (k_M + |\eta'| \, \tilde{q} k_W) \\ &\quad + |\eta'| \, b_7 k_W] \cdot |\phi(-\eta)|^2.\end{aligned}$$

Examining these inequalities, we can conclude that for all $\eta \in D$, $\eta' \in \mathrm{R}$, $\phi \in C[-\tilde{d}, 0]$, one has

$$\begin{aligned} E(\eta, \eta', \phi) + (\eta')\, F_7(\eta, \phi) &\leqslant [(-1 + 2\|A_1\| \cdot |\eta'|\, \tilde{q})\, k_W + (e^{-\gamma\eta})\, k_R \\ &\quad + 2(\|A_0^T - \gamma I\| + \|A_1\|)\, k_M] \cdot |\phi(0)|^2 + [-(e^{\gamma\eta})(1 + \eta')\, k_R \\ &\quad + (2\|A_1\|\, \tilde{q} + b_7)\, |\eta'|\, k_W + 2\|A_1\|\, k_M] \cdot |\phi(-\eta)|^2 \\ &\quad + |\eta'|\, b_7 k_W \cdot n(\eta, \phi)^2. \end{aligned}$$

After a careful inspection of this inequality, an elementary analysis will show the following:

There exist constants μ_1, μ_2, with $-1 < \mu_1 < 0 < \mu_2$, and a choice of constants k_M, k_R, $k_W > 0$, such that for $\eta \in D$, $\mu_1 \leqslant \eta' \leqslant \mu_2$, $\phi \in C[-\tilde{d}, 0]$, one has both

(a) $(-1 + 2\|A_1\| \cdot |\eta'|\, \tilde{q})\, k_W + (e^{-\gamma\eta})\, k_R + 2(\|A_0^T - \gamma I\| + \|A_1\|)\, k_M < 0,$

and

(b) $-(e^{\gamma\eta})(1 + \eta')\, k_R + (2\|A_1\|\, \tilde{q} + b_7)\, |\eta'|\, k_W + 2\|A_1\|\, k_M < 0.$

From this we immediately see, with such a choice of k_M, k_R, k_W, that

$$E(\eta, \eta', \phi) + (\eta')\, F_7(\eta, \phi) \leqslant |\eta'|\, b_7 k_W \cdot n(\eta, \phi)^2.$$

It is elementary to show that the above constants k_M, k_R, k_W can be chosen with $k_R \geqslant k_M \cdot e^{\tilde{d}|\gamma|}$. In fact, if we now recall the constant $c_1 = c_1(\eta)$ from Section 3, defined by $c_1(\eta) = \min(k_M, k_R \cdot e^{-\eta|\gamma|})$, we see that $c_1(\eta) = k_M$ if $\eta \in D$ and $k_R \geqslant k_M \cdot e^{\tilde{d}|\gamma|}$. Noting that $c_1(\eta) \cdot n(\eta, \phi)^2 \leqslant V(\eta, \phi)$, we then have $k_M \cdot n(\eta, \phi)^2 \leqslant V(\eta, \phi)$ for each $\eta \in D$. Recalling the inequalities $|F_2(\eta, \phi)| \leqslant b_2 k_R \cdot n(\eta, \phi)^2$, and $|F_i(\eta, \phi)| \leqslant b_i k_W \cdot n(\eta, \phi)^2$ for $i = 1, 3, 4, 5, 6, 8$, we can now write the following inequalities with the above choice of constants k_M, k_R, k_W:

$$|F_2(\eta, \phi)| \leqslant b_2 \frac{k_R}{k_M}\, V(\eta, \phi);$$

$$|F_i(\eta, \phi)| \leqslant b_i \frac{k_W}{k_M}\, V(\eta, \phi) \qquad \text{for} \quad i = 1, 3, 4, 5, 6, 8;$$

$$E(\eta, \eta', \phi) + (\eta')\, F_7(\eta, \phi) \leqslant |\eta'| \cdot b_7 \frac{k_W}{k_M}\, V(\eta, \phi).$$

From these inequalities we can write

$$E(\eta, \eta', \phi) + (\eta') F(\eta, \phi) = E(\eta, \eta', \phi) + (\eta') \sum_{1}^{8} F_i(\eta, \phi)$$

$$= E(\eta, \eta', \phi) + (\eta') F_7(\eta, \phi) + (\eta') \sum_{i \neq 7} F_i(\eta, \phi)$$

$$\leqslant |\eta'| \frac{1}{k_M} \left[b_1 k_W + b_2 k_R + \sum_{i=3}^{8} b_i k_W \right] V(\eta, \phi),$$

i.e.,

$$E(\eta, \eta', \phi) + (\eta') F(\eta, \phi) \leqslant |\eta'| \frac{1}{k_M} \left[b_1 k_W + b_2 k_R + \sum_{i=3}^{8} b_i k_W \right] V(\eta, \phi)$$

for $\eta \in D$, $\mu_1 \leqslant \eta' \leqslant \mu_2$, $\phi \in C[-\bar{d}, 0]$.

Setting $B = (1/k_M)[b_2 k_R + (b_1 + \sum_{i=3}^{8} b_i) k_W]$, we summarize this laborious analysis in the theorem below.

THEOREM 4.1. *Let D be a compact subset of H_γ. Then there exist constants μ_1, μ_2 having $-1 < \mu_1 < 0 < \mu_2$, and constants B, k_M, k_R, $k_W > 0$, such that for $\eta \in D$, $\mu_1 \leqslant \eta' \leqslant \mu_2$, $\phi \in C[-\bar{d}, 0]$, one has $E(\eta, \eta', \phi) + (\eta') F(\eta, \phi) \leqslant B|\eta'| \, V(\eta, \phi)$.*

Noting Lemma 4.2 and the above theorem, we can finally address the question of growth estimates for systems of the form (†) $\dot{x}(t) = A_0 x(t) + A_1 x(t - h(t))$. In order to simplify the proof of our basic growth estimate, it is convenient to recall the following lemma of real analysis.

LEMMA 4.3. *Let $u\colon [0, \infty) \to [0, \infty)$ be absolutely continuous, and let $v\colon [0, \infty) \to \mathsf{R}$ be continuous. If $u'(t) \leqslant u(t)\, v(t)$ a.e. for $t \geqslant 0$, then in fact $u(t) \leqslant u(0) \cdot \exp(\int_0^t v(\tau)\, d\tau)$ for all $t \geqslant 0$.*

Proof. Considering first the case where $0 \notin \text{range}(u(\cdot))$, we have $u'(t)/u(t) \leqslant v(t)$ a.e. for $t \geqslant 0$, and we note for $g(t) = \ln(u(t))$ that g is absolutely continuous over bounded intervals, with $g'(t) = u'(t)/u(t)$ a.e. Now integrating both sides of the inequality $u'(\tau)/u(\tau) \leqslant v(\tau)$ over $[0, t]$, and simplifying, we obtain the lemma in the case that $0 \notin \text{range}(u(\cdot))$. In the case that $0 \in \text{range}(u(\cdot))$, we let $\alpha = \inf\{t \geqslant 0\colon u(t) = 0\}$. If there were any t with $t > \alpha$, $u(t) > 0$, we could set $\alpha_t = \sup\{t'\colon \alpha \leqslant t' < t,\ u(t') = 0\}$. Applying the lemma in the case that $0 \notin \text{range}(u(\cdot))$, we find for $\alpha_t < \tau < t$ that $u(t) \leqslant u(\tau) \cdot \exp(\int_\tau^t v)$, so that $u(t) \leqslant u(\tau) \cdot \exp(\int_0^t |v|)$. Letting $\tau \downarrow \alpha_t$, we find that $u(t) = 0$. Concluding that $u(t) = 0$ for $t \geqslant \alpha$, we see that the lemma is true in the case that $0 \in \text{range}(u(\cdot))$. Q.E.D.

THEOREM 4.2. *Let D be a compact subset of H_γ, and take μ_1, μ_2, k_M, k_R, k_W, and B as in Theorem 4.1. Let $h(\cdot)$ be any member of $S_\gamma(D, \mu_1, \mu_2)$, and consider the differential–delay equation* (†) $\dot{x}(t) = A_0 x(t) + A_1 x(t - h(t))$. *For each solution $x(\cdot)$ of the system* (†), *form the function $G(t, x_t) = V(h(t), x_t)$, defined for $t \geqslant 0$. Then for each t at which $h(\cdot)$ is differentiable, and particularly a.e., the inequality* (a) *below holds for $\dot{G}(x_t) = dG(t, x_t)/dt$. Furthermore, the inequality for $G(t, x_t)$ in* (b) *holds for* all $t \geqslant 0$:

(a) $\dot{G}(x_t) \leqslant 2\gamma G(t, x_t) + B|h'(t)|\, G(t, x_t)$

(b) $G(t, x_t) \leqslant G(0, x_0)\, e^{f(t)}$, *where* $f(t) = 2\gamma t + B \int_0^t |h'(\tau)|\, d\tau$.

Proof. The inequality in (a) is an immediate consequence of Lemma 4.2 and Theorem 4.1.

To derive the inequality in (b), we first recall that $Q(\cdot,\cdot)$ is continuous, $x(\cdot)$ is continuously differentiable, and $h(\cdot)$ is absolutely continuous, and examine the first of the formulas for $G(t, x_t)$ given in the proof of Lemma 4.2. From an inspection of this formula one can show that for any solution $x(\cdot)$ of the system (†) and any $t > 0$, the function $G(\tau, x_\tau)$ is absolutely continuous over the interval $[0, t]$. The inequality in (b) now follows from the inequality in (a) and the lemma immediately above.

Q.E.D.

If we note the inequality $k_M \cdot |\phi(0)|^2 \leqslant V(\eta, \phi)$, we immediately see for each solution $x(\cdot)$ of (†) that $k_M \cdot |x(t)|^2 \leqslant G(t, x_t)$ for $t \geqslant 0$, and the inequality (b) thus yields the inequality below:

$$k_M \cdot |x(t)|^2 \leqslant G(0, x_0)\, e^{f(t)} \qquad \text{for} \quad t \geqslant 0, \text{ where } f(t) \text{ is as in (b) above.}$$

It is interesting to note that the inequality (b) above can be expressed in terms of the average value of the magnitude of $h'(\tau)$ over the interval $[0, t]$. In fact, if one writes $a(t) = (1/t) \int_0^t |h'(\tau)|\, d\tau$ for $t > 0$, then one immediately obtains $2\gamma t + B \int_0^t |h'(\tau)|\, d\tau = t(2\gamma + Ba(t))$, and thus one writes the inequality (b) as in (b′) below:

(b′) $G(t, x_t) \leqslant G(0, x_0)\, e^{t(2\gamma + Ba(t))}$ for all $t > 0$.

The next two theorems follow in a direct way from the above theorem. The first of these expresses the inequality in Theorem 4.2(b) in terms of an asymptotic upper bound on the value of $a(t)$ if such an upper bound exists.

THEOREM 4.3. *Let D be a compact subset of H_γ, and again take constants as in Theorem 4.1. Let $h(\cdot)$ be any member of $S_\gamma(D, \mu_1, \mu_2)$, and suppose there exists $\zeta > 0$ with* $\limsup_{t \to \infty} a(t) < \zeta$, *where $a(t) = (1/t) \int_0^t |h'(\tau)|\, d\tau$. Then there is $v > 0$ such that for any solution $x(\cdot)$*

of the differential equation (†) $\dot{x}(t)=A_0x(t)+A_1x(t-h(t))$, *one has* $G(t,x_t)\leqslant G(0,x_0)\,e^{t(2\gamma+B\zeta)}$ *for all* $t\geqslant v$.

Proof. If $\limsup_{t\to\infty} a(t)<\zeta$, then there is $v>0$ with $a(t)<\zeta$ for all $t\geqslant v$. The theorem now follows immediately from the inequality (b′) above. Q.E.D.

COROLLARY 4.3. *Let D be a compact subset of* H_γ, *with constants as in Theorem* 4.1. *Then for each* $\varepsilon>0$, *there exists* $\zeta=\zeta(\varepsilon)$ *for which the following holds*:

If $h(\cdot)$ *is any member of* $S_\gamma(D,\mu_1,\mu_2)$ *having* $\limsup_{t\to\infty} a(t)<\zeta$, *then there is* $v>0$ *such that for every solution* $x(\cdot)$ *of the differential equation* (†), *one has* $G(t,x_t)\leqslant G(0,x_0)\,e^{t(2\gamma+\varepsilon)}$ *for all* $t\geqslant v$.

Proof. Take $\zeta=\varepsilon/B$ in the above theorem. Q.E.D.

We particularly note that in the case $\gamma<0$, the above corollary states that there exists $\zeta>0$ such that if $\limsup_{t\to\infty} a(t)<\zeta$, then the system (†) is exponentially asymptotically stable. In the next theorem we display a simplification occurring in the special case that $|h'(t)|$ has finite integral over $[0,\infty)$.

THEOREM 4.4. *Let D be a compact subset of* H_γ, *and take constants as in Theorem* 4.1. *If* $h(\cdot)$ *is any member of* $S_\gamma(D,\mu_1,\mu_2)$ *having* $\int_0^\infty |h'(\tau)|\,d\tau<\infty$, *then for* $\kappa=\int_0^\infty |h'(\tau)|\,d\tau$ *and* $\tilde{B}=e^{B\kappa}$, *the inequality in Theorem* 4.2(b) *can be sharpened to the inequality below*:

$$G(t,x_t)\leqslant \tilde{B}\cdot G(0,x_0)\,e^{2\gamma t}\qquad \textit{for all}\quad t\geqslant 0.$$

Proof. This follows immediately from Theorem 4.2. Q.E.D.

It is interesting to investigate the important special case that $\gamma=0$. Here we remove the hypothesis that $h(\cdot)$ is a member of $S_\gamma(D,\mu_1,\mu_2)$, and instead merely stipulate that $h(\cdot)$ is absolutely continuous, with range($h(\cdot)$) contained in some compact subset D of H_0, i.e., we stipulate that $h(\cdot)$ is a member of $S_0(D)$ for some compact subset D of H_0. In this case, we again apply Lemma 4.2, and we thus have a formula for $\dot{G}(x_t)=dG(t,x_t)/dt$ at all nonnegative values of t at which $h(\cdot)$ is differentiable, and particularly a.e. in Lebesgue measure. Since $\gamma=0$, we immediately see that $2\gamma G(t,x_t)=0$ for all $t\geqslant 0$. If we now set $M=0=R$ and $W=I$, and expand the resulting expression for $\dot{G}(x_t)$, we then obtain the following expression for the time derivative of $G(t,x_t)$ along solutions of the differential–delay equation (†) $\dot{x}(t)=A_0x(t)+A_1x(t-h(t))$:

$$\dot{G}(x_t)=-x^T(t)\,x(t)+h'(t)[L_1(h(t),x_t)+L_2(h(t),x_t)],$$

where $L(\eta, \phi) = L_1(\eta, \phi) + L_2(\eta, \phi)$ is defined, for $\eta \in D$, $\phi \in C[-\bar{d}, 0]$, by

$$L_1(\eta, \phi) = 2\phi^T(-\eta)\, A_1^T \left[Q(\eta, 0)\, \phi(0) + \int_{-\eta}^{0} Q(\eta, u+\eta)\, A_1 \phi(u)\, du \right],$$

and

$$L_2(\eta, \phi) = \phi^T(0)\, D_1 Q(\eta, 0)\, \phi(0) + 2\phi^T(0) \int_{-\eta}^{0} [A_0^T Q(\eta, u+\eta)$$
$$+ A_1^T Q(\eta, u) + D_1 Q(\eta, u+\eta)]\, A_1 \phi(u)\, du$$
$$+ \int_{-\eta}^{0} \int_{-\eta}^{0} \phi^T(u)\, A_1^T D_1 Q(\eta, v-u)\, A_1 \phi(v)\, dv\, du.$$

Now recalling the constant $\tilde{q}$ defined after Lemma 4.2, noting since $W = I$ here that $Q(\cdot, \cdot) = \tilde{Q}(\cdot, \cdot)$, and again motivated by the Cauchy–Schwartz inequality, we define the following quantities b_i', $i = 1, ..., 5$:

$$b_1' = 2\tilde{q}\|A_1\|; \qquad b_2' = 2(\tilde{q})(\bar{d})\|A_1\|^2; \qquad b_3' = \tilde{q};$$
$$b_4' = 2(\tilde{q})(\bar{d})\|A_1\|(\|A_0\| + \|A_1\| + 1); \qquad b_5' = (\tilde{q})(\|A_1\|\,\bar{d})^2.$$

If we now apply the Cauchy–Schwartz inequality to the functionals $L_1(\eta, \phi)$ and $L_2(\eta, \phi)$, we readily obtain the following lemma.

LEMMA 4.4. *Let D be any compact subset of H_0, and let $h(\cdot)$ be any member of $S_0(D)$. Set $M = 0 = R$ and $W = I$ in the functional $V(\eta, \phi)$, and consider the differential–delay equation (†) $\dot{x}(t) = A_0 x(t) + A_1 x(t - h(t))$. For any solution $x(\cdot)$ of the system (†) and any $t \geq 0$, let $|x_t|$ denote $|x_t| = \sup\{|x(t+u)|: -h(t) \leq u \leq 0\}$. Then for each t at which $h(\cdot)$ is differentiable, and particularly a.e., the following inequality holds for $\dot{G}(x_t)$, the time derivative of the function $G(t, x_t)$:*

$$\dot{G}(x_t) \leq -|x(t)|^2 + |h'(t)| \cdot [|x(t - h(t))|\, (b_1'|x(t)| + b_2'|x_t|)$$
$$+ |x(t)|\, (b_3'|x(t)| + b_4'|x_t|) + b_5'|x_t|^2].$$

The above lemma will be useful in showing that for $a(t)$, the average magnitude of $h'(\cdot)$ over the interval $[0, t]$, any bounded solutions of the differential–delay equation (†) will *adhere*, in a specific sense that will be defined, to arbitrarily small values over time provided $a(t)$ at least *attains* arbitrarily small values over time. As a corollary, we will find for bounded solutions $x(t)$ of (†) that $\liminf_{t \to \infty} |x(t)| = 0$ if $\liminf_{t \to \infty} a(t) = 0$. Before giving a theorem on this topic, it is convenient to introduce the notation

$|E|$ to denote the Lebesgue measure of a Lebesgue measurable subset E of R.

THEOREM 4.5. *Let D be any compact subset of H_0, and again let $h(\cdot)$ be any member of $S_0(D)$. Let $a(t)=(1/t)\int_0^t |h'(\tau)|\,d\tau$ for $t>0$, and now consider the differential–delay system* (†) $\dot{x}(t)=A_0x(t)+A_1x(t-h(t))$. *Suppose* $\liminf_{t\to\infty} a(t)=0$. *Then for each* bounded *solution $x(\cdot)$ of the system* (†) *and for any $\varepsilon>0$, $\beta\in(0,1)$, there is a sequence $\{t_i\}$ having $t_i\uparrow\infty$ and $|\{\tau: |x(\tau)|\geqslant\varepsilon,\ 0\leqslant\tau\leqslant t_i\}|<\beta t_i$.*

Proof. Let $x(\cdot)$ be any bounded solution of the system (†), with $|x(t)|\leqslant\chi$ for all $t\geqslant 0$. Noting Lemma 4.4, we set $B'=\sum_1^5 b_i'$, and immediately obtain the following inequality for $\dot{G}(x_t)$, holding a.e. for $t\geqslant 0$:

$$\dot{G}(x_t)\leqslant -x^T(t)\,x(t)+B'\chi^2|h'(t)|.$$

Just as in Theorem 4.2, we can refer to the expression for $G(t, x_t)$ given in Lemma 4.2, and find that for each $t>0$, the function $G(\tau, x_\tau)$ is absolutely continuous over the interval $[0, t]$. For $t>0$, we now write the above inequality in the variable τ, and integrate both sides of this inequality over the interval $[0, t]$, obtaining the inequality below for $G(t, x_t)$, holding for all $t\geqslant 0$:

$$G(t, x_t)\leqslant -\int_0^t x^T(\tau)\,x(\tau)\,d\tau+B'\chi^2\int_0^t |h'(\tau)|\,d\tau+G(0, x_0).$$

For $\varepsilon>0$, $t>0$, we now define the set $E_{0,t}=E_{0,t}(\varepsilon)$ as $E_{0,t}(\varepsilon)=\{\tau: |x(\tau)|\geqslant\varepsilon,\ 0\leqslant\tau\leqslant t\}$, and we denote the Lebesgue measure of this set by $|E_{0,t}|$. From the above inequality for $G(t, x_t)$, we immediately see that

$$G(t, x_t)\leqslant -\varepsilon^2|E_{0,t}|+B'\chi^2\int_0^t |h'(\tau)|\,d\tau+G(0, x_0)\qquad\text{for}\quad t>0.$$

Given any fixed $\varepsilon>0$, $\beta\in(0,1)$, we examine the sets $E_{0,t}(\varepsilon)$ for $t>0$. If we had some $t_0>0$ with $|E_{0,t}|\geqslant\beta t$ for all $t>t_0$, then we would have

$$G(t, x_t)\leqslant -\varepsilon^2\beta t+B'\chi^2\int_0^t |h'(\tau)|\,d\tau+G(0, x_0)\qquad\text{for all}\quad t>t_0,$$

i.e.,

$$G(t, x_t)\leqslant t\cdot\left(-\varepsilon^2\beta+B'\chi^2a(t)+\frac{G(0, x_0)}{t}\right)\qquad\text{for all}\quad t>t_0.$$

Since $\liminf_{t\to\infty} a(t) = 0$, there would be some $t^+ > t_0$ with $-\varepsilon^2\beta + B'\chi^2 a(t^+) + G(0, x_0)/t^+ < 0$. Hence, we would have $G(t^+, x_{t^+}) < 0$. Since G is nonnegative, we see that it is not the case that $|E_{0,t}| \geqslant \beta t$ for all $t > t_0$, i.e., there is $t' > t_0$ with $|E_{0,t'}(\varepsilon)| < \beta t'$. We have shown that for any $t_0 > 0$, there exists $t' > t_0$ with $|E_{0,t'}(\varepsilon)| < \beta t'$. We now conclude that there is a sequence $\{t_i\}$ having $t_i \uparrow \infty$ and $|E_{0,t_i}(\varepsilon)| < \beta t_i$. Q.E.D.

COROLLARY 4.5. *Let D be any compact subset of H_0, and let $h(\cdot)$ be any member of $S_0(D)$. If* $\liminf_{t\to\infty} a(t) = 0$, *then for each* bounded *solution $x(\cdot)$ of the system* (†) $\dot{x}(t) = A_0 x(t) + A_1 x(t - h(t))$, *we have* $\liminf_{t\to\infty} |x(t)| = 0$.

Proof. This follows immediately from the above theorem. Q.E.D.

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