The Peetre K-Functional and the Riesz Summability Operator for the Fourier–Legendre Expansions

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The Peetre K-functionals and the generalized Riesz summability operators are introduced. The convergence and boundedness of the Riesz operators are discussed. The equivalent relationships of the Peetre K-functionals and the Riesz operators are established.

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1. INTRODUCTION AND NOTATIONS

Let $L_p[-1, 1]$, $1 \leq p < \infty$, denote the spaces of the Lebesgue integrable functions on $[-1, 1]$, and let $C[-1, 1]$ denote the space of the continuous functions on $[-1, 1]$, with the norms

$$
\|f\|_p := \left\{ \int_{-1}^{1} |f(x)|^p \, dx \right\}^{1/p}, \quad \text{for } f \in L_p[-1, 1], \quad \text{and}
$$

$$
\|f\|_\infty := \sup_{-1 \leq x \leq 1}|f(x)|, \quad \text{for } f \in C[-1, 1],
$$

respectively. In the following, $L_p[-1, 1]$ will always be one of the spaces $L_p[-1, 1]$ for $1 \leq p < \infty$, or $C[-1, 1]$ for $p = \infty$. Let $\Pi_n$ be the class of polynomials of degree $\leq n$. The best polynomial approximant of degree $n$ of $f \in L_p[-1, 1]$ is defined by

$$
E_n(f)_p := \inf \{\|f - p_n\|_p : p_n \in \Pi_n \}.
$$

Z. Ditzian and V. Totik [4, Chap. 7] constructed a polynomial $p_n \in \Pi_n$ satisfying

$$
\|f - p_n\|_p \leq K_{r, \varphi}(f, n^{-r})_p, \quad (1.1)
$$

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where the Peetre K-functional $K_{r, \varphi}(f, n^{-r})_p$ with weight $\varphi(x) = \sqrt{1-x^2}$ is defined by

$$K_{r, \varphi}(f, t^r)_p := \inf \left\{ \| f - g \|_p + t^r \| \varphi' \|_p : g \in \mathcal{C}[-1, 1] \right\}. \quad (1.2)$$

Result (1.1) implies that

$$E_n(f)_p \leq K_{r, \varphi}(f, n^{-r})_p.$$ 

The Peetre K-functional is a very useful tool for estimating the rate of convergence of linear operators. Recently, Z. Ditzian and K. Ivanov [3] and V. Totik [6, 7], etc., considered some strong converse inequalities of approximation by linear operators. Their results show that the order for approximation by some linear operators is completely characterized by the corresponding K-functional, which is equivalent to the some moduli of smoothness. For example, for Bernstein operators

$$B_n(f, x) := \sum_{k=0}^{n} b_{n,k}(x), \quad b_{n,k}(x) = \binom{n}{k} \lambda^k (1-x)^{n-k},$$

V. Totik [7] has proved that

$$\| f - B_n(f) \|_{C[0,1]} \approx K_{2, \varphi}(f, n^{-1})_\infty, \quad (1.3)$$

where the weight function $\varphi(x) = \sqrt{x(1-x)}$, and $A \approx B$ means there exists a positive constant “$\text{const}$” such that $(1/\text{const}) A \leq B \leq \text{const} A$. In this paper, we denote “$\text{const}$” an absolute positive constant which is dependent only on the parameters indicated by the index.

For Bernstein–Durrmeyer operators

$$M_n(f, x) := \frac{1}{n+1} \sum_{k=0}^{n} b_{n,k}(x) \int_0^1 b_{n,k}(y) f(y) \, dy,$$

W. Chen et al. [1] proved that

$$\| (M_n - I)' f \|_{L_p[0,1]} \approx K_{2r}(f, n^{-r})_p, \quad (1.4)$$

where $I$ is the identity and the Peetre K-functional is defined by

$$K_{2r}(f, n^{-2r})_p := \inf \left\{ \| f - g \|_{L_p[0,1]} + n^{-2r} \| P_1(D)' g \|_{L_p[0,1]} : g \in \mathcal{C}^{2r}[0, 1] \right\},$$

and the differential operator $P_1(D) := (d/dx) \cdot (1-x) (d/dx)$.

In his paper, Z. Ditzian [2] considered the Riesz summability operators $R_n$ for Fourier–Legendre expansions. Let $P_2(x)$ be the Legendre polynomials,
and let differential operator $P(D) := (d/dx) (1 - x^2)(d/dx)$. The formal Fourier–Legendre expansion of $f \in L_1[-1, 1]$ is given by

$$f(x) \sim \sum_{n=0}^{\infty} f^n(n) P_n(x),$$  \hspace{1cm} (1.5)

where $f^n(n)$ is the Legendre transform of $f$ defined by

$$f^n(n) := \langle f, P_n \rangle = \int_{-1}^{1} f(y) P_n(y) dy, \quad n = 0, 1, ... .$$

For series (1.5) Z. Ditzian [2] defined the Riesz operators

$$R_n(f, x) := \sum_{k=0}^{n} \left(1 - \frac{k(k+1)}{m(n+1)} \right) f^n(k) P_k(x),$$

and proved that

$$\| (R_n - I)^{r} f \|_p \approx K_{2r}(f, n^{-2r})_p,$$  \hspace{1cm} (1.6)

where $I$ is the identity and the Peetre K-functional is given by

$$K_{2r}(f, r^2)_p := \inf \{ \| f - g \|_p + r^2 \| P(D)^r g \|_p : g \in C^{2r}[ -1, 1] \}. \hspace{1cm} (1.7)$$

Combining (1.4) with (1.7) yields the inequality

$$E_n(f)_p \leq \text{const} \| (M_n - I)^{r} f \|_p, \quad 1 \leq p \leq \infty, \ r \geq 1.$$

The aim of the paper is to consider the generalized Riesz summability operators

$$R_{n,r}(f, x) := \sum_{k=0}^{n} \left(1 - \frac{k(k+1)}{m(n+1)} \right)^{r/2} f^n(k) P_k(x), \quad r \geq 1, \ n = 0, 1, ... $$

(1.8)

We will establish the equivalence

$$\| R_{n,r} f - f \|_p \approx K(f, n^{-r}; L_p, W^r_p). \hspace{1cm} (1.9)$$

The definition of the Peetre K-functional $K(f, n^{-r}; L_p, W^r_p)$ will be given in Section 2.

Remark. The equivalence result (1.9) is different from that of (1.6) in two respects. We deal with $R_{n,r} - I$ instead of the power of the operators $(R_n - I)^{r}$ and the $r$ in our definition is not restricted to be natural numbers. The proof follows closely that of [2], presented by Z. Ditzian.
2. A PEETRE K-FUNCTIONAL FOR FOURIER-LEGENDRE EXPANSIONS

Let $P(D) = (d/dx)(1 - x^2)(d/dx)$. The Legendre polynomials $P_k(x)$ are given by

$$P(D) P_k(x) = -k(k + 1) P_k(x),$$

and satisfy the orthonormality condition

$$\langle P_k, P_m \rangle = \int_{-1}^{1} P_k(x) P_m(x) \, dx = \delta_{k,m}.$$


We deduce the fractional derivative for the expansion (1.5). Let $P^r := (P(D))^r$ be the power of the operator $P(D)$ given by the relation

$$P^r P_k(x) = -k(k + 1)^r P_k(x). \quad (2.1)$$

Let $W^r_p := \{ f \in L_p[-1, 1] : \exists g \in L_p[-1, 1] \forall k \in \mathbb{N}_0, \, g^\wedge(k) = -k(k + 1)^r f^\wedge(k) \}$.

Therefore if $f \in W^r_p$ has the formal Fourier-Legendre expansion (1.5)

$$f(x) \sim \sum_{k=0}^{\infty} f^\wedge(k) P_k(x),$$

then $\mathcal{D}^r f \in L_p[-1, 1]$ and has the following formal Fourier-Legendre expansion

$$\mathcal{D}^r f(x) \sim \sum_{k=0}^{\infty} \left[ -k(k + 1)^r \right] f^\wedge(k) P_k(x).$$

The Peetre K-functional between $L_p[-1, 1]$ and $W^r_p$ is then defined by

$$K(f, r'; L_p, W^r_p) := \inf \{ \| f - g \|_p + r' \| \mathcal{D}^r g \|_p, \, g \in W^r_p \}. \quad (2.2)$$

Since $W^r_p$ is dense in $L_p[-1, 1]$ we have $K(f, r'; L_p, W^r_p) \to 0$ as $t \to 0$. It is suitable to measure the rate of convergence of the generalized Riesz summability operators by the Peetre K-functional $K(f, r'; L_p, W^r_p)$.

3. EQUIVALENCE RESULTS

The following equivalence relation is a strong converse inequality in the sense of [3].
Theorem 1. Let $f \in L_p[-1, 1], 1 \leq p \leq \infty$, and let $R_n(f, x)$ and $K(f, n^{-1}; L_p, W_p^r)$ be defined by (1.8) and (2.2), respectively. We have the equivalence relation

$$\|R_n, f - f\|_p \approx K(f, n^{-1}; L_p, W_p^r). \quad (3.1)$$

In order to prove the theorem we give the following three lemmas.

Lemma 1. Let $1 \leq p \leq \infty$, and let $R_n$ be defined by (1.8). Then $R_n, f$ is of type $(p, p)$, i.e.,

$$\|R_n, f\|_p \leq \text{const}_p, \|f\|_p, \quad f \in L_p[-1, 1]. \quad (3.2)$$

Proof. If $r = 2$, Z. Ditzian [2] gave a proof for (3.2). If $r \neq 2$, we can deduce (3.2) from theorem 3.9 in [8] by using multiplier theory. Let

$$J_k f(x) = f^k(k) P_k(x).$$

Then $\{J_k\}_{n=0}^\infty$ is a total, fundamental system of mutually orthogonal projections satisfying

$$\|(C, 1)_n f\|_p \leq \text{const}_p, \|f\|_p, \quad f \in L_p[-1, 1],$$

where $(C, 1)_n (f, x)$ is the Cesàro means

$$(C, 1)_n (f, x) := \sum_{k=0}^n \left(1 - \frac{k}{n}\right) f^k(k) P_k(x).$$

In order to verify (3.2), we use Theorem 3.9 of [8] and choose $j = 1$, $\Phi(t) = \Psi(t) = t(t + 1)$ and $e(x) = 1 - x^{r/2}$ for $0 \leq x \leq 1$ or $e(x) = 0$ for $x > 1$. We have to show that $e(x)$ satisfies $\int_0^x x^2 |de^r(x)| < \infty$. This is easy to check. In fact, we have

$$\int_0^x x^2 |de^r(x)| = \frac{1}{4} r |r - 2| \int_0^1 x^{r/2} dx = \frac{r |r - 2|}{2(r + 2)} < \infty.$$ 

Therefore all the conditions in Theorem 3.9 of [8] are satisfied. Then $\{1 - (k(k + 1)/n(n + 1))^{r/2}\}_{n=0}^\infty$ is a family of uniformly bounded multipliers on $L_p[-1, 1]$. This completes the proof of (3.2).

As a corollary of Lemma 1, we have $\lim_{n \to \infty} \|R_n, f - f\|_p = 0$ for all $f \in L_p[-1, 1]$. That is to say $\{R_n\}$ is an approximation process on $L_p[-1, 1]$. 

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Similarly to [2], we obtain the following lemma which gives the relationships between the Riesz summability operators \( R_n \), and the differential operator \( D \).

**Lemma 2.** Let \( f \in L_p[-1, 1] \), \( 1 \leq p \leq \infty \), and let \( R_n f \) be defined by (1.8). Then

\[
(n(n+1))^{\nu^2} R_n f = D R_n f.
\]  
(3.3)

**Proof.** We first note that for \( f \in L_p[-1, 1] \) there holds

\[
R_n f = \frac{1}{(n(n+1))^{\nu^2}} \sum_{k=0}^{n} \left( 1 - \left( \frac{k(k+1)}{n(n+1)} \right)^{\nu^2} \right) (k(k+1))^{\nu^2} f^\wedge(k) P_k
\]  
(3.4)
for \( 0 \leq k \leq n \). By the definition of \( D \) we have

\[
D P_k(x) = -(k+1)^{\nu^2} P_k(x).
\]

It follows that

\[
D R_n f = \sum_{k=0}^{n} \left( 1 - \left( \frac{k(k+1)}{n(n+1)} \right)^{\nu^2} \right) (k(k+1))^{\nu^2} f^\wedge(k) P_k(x).
\]

Combining this equation with (3.4) we get

\[
(n(n+1))^{\nu^2} R_n f = D R_n f.
\]

Lemma 2 is proved.

For a given function in \( W_\nu^p \), we have the Jackson-type inequality by following an idea of Ditzian [2].

**Lemma 3.** Let \( f \in W_\nu^p \), \( 1 \leq p \leq \infty \), and let \( R_n f \) be defined by (1.8). Then

\[
\| R_n f - f \|_p \leq \frac{\text{const}_{\nu, p}}{n^\nu} \| D f \|_p.
\]  
(3.5)

**Proof.** For \( f \in L_p[-1, 1] \) we have from (3.3) in Lemma 2

\[
R_n^2 f - R_n f = \frac{1}{(n(n+1))^{\nu^2}} D R_n f.
\]
By direct calculations, we know that $\mathcal{D}'$ and $R_{n,r}$ commute, that is,
\begin{equation}
\mathcal{D}' R_{n,r} f = R_{n,r} \mathcal{D}' f, \quad f \in W_p^r,
\tag{3.6}
\end{equation}
and for all $n, m$,
\begin{equation*}
R_{m,r} R_{n,r} f = R_{n,r} R_{m,r} f, \quad f \in L_p([-1, 1]).
\end{equation*}
Furthermore
\begin{equation*}
R_{m,r}^2 f - R_{m+1,r} R_{m,r} f = \frac{(m+2)r^2 - m't^2}{m(m+1)(m+2)r^2} \mathcal{D}' R_{m,r} f,
\end{equation*}
and
\begin{equation*}
R_{m+1,r}^2 f - R_{m,r} R_{m+1,r} f = \frac{(m+2)r^2 - m't^2}{m(m+1)(m+2)r^2} \mathcal{D}' R_{m+1,r} f.
\end{equation*}
Note that $(m+2)r^2 - m't^2 \approx rm^{(r/2) - 1}$ as $m \to \infty$. It follows that
\begin{equation*}
\| R_{m,r}^2 f - R_{m+1,r}^2 f \|_p \leq \frac{\text{const}_p}{m^{r+1}} (\| \mathcal{D}' R_{m,r} f \|_p + \| \mathcal{D}' R_{m+1,r} f \|_p).
\end{equation*}
Hence Lemma 1 and (3.6) yield for $f \in W_p^r$ that
\begin{equation*}
\| R_{m,r}^2 f - R_{m+1,r}^2 f \|_p \leq \frac{\text{const}_p}{m^{r+1}} (\| \mathcal{D}' f \|_p).
\end{equation*}
Lemma 1 and (3.6) also imply $\| R_{m,r}^2 f - f \|_p \to 0$ as $n \to \infty$, we have
\begin{equation*}
\| R_{m,r}^2 f - f \|_p \leq \sum_{m=n}^{\infty} \| R_{m,r}^2 f - R_{m+1,r}^2 f \|_p.
\end{equation*}
We finally get Jackson's estimate for $f \in W_p^r$
\begin{equation*}
\| R_{m,r} f - f \|_p \leq \| R_{m,r}^2 f - R_{m,r} f \|_p + \sum_{m=n}^{\infty} \| R_{m,r}^2 f - R_{m+1,r}^2 f \|_p \leq \text{const}_p \left( \frac{1}{m(n+1)} r^{1/2} + \sum_{m=n}^{\infty} \frac{1}{m^{r+1}} \right) \| \mathcal{D}' f \|_p \leq \frac{\text{const}_p}{n^{r/2}} \| \mathcal{D}' f \|_p.
\end{equation*}
Lemma 3 is proved.
Proof of Theorem 1. Let $f \in L_{p}[−1, 1]$. Choose $g \in W_{p}'$ such that
\[ \|f - g\|_{p} + n^{-r} \|\mathcal{D}'g\|_{p} \leq 2K(f, n^{-r}; L_{p}, W_{p}'). \]
We get
\[ \|R_{n,r}f - f\|_{p} \leq \|R_{n,r}(f - g) - (f - g)\|_{p} + \|R_{n,r}g - g\|_{p} \leq \text{const}_{p,r} K(f, n^{-r}; L_{p}, W_{p}') + \|R_{n,r}g - g\|_{p}. \]
By making use of Lemma 3, we have
\[ \|R_{n,r}g - g\|_{p} \leq \text{const}_{p,r} K(f, n^{-r}; L_{p}, W_{p}'). \]
Combining the inequalities above we get
\[ \|R_{n,r}f - f\|_{p} \leq \text{const}_{p,r} K(f, n^{-r}; L_{p}, W_{p}'). \]
To prove the converse result, by making use of Lemmas 2 and 3 we have
\[ \|\mathcal{D}'R_{n,r}f\|_{p} \leq \text{const}_{p,r} n^{-r} \|R_{n,r}f - f\|_{p}, \quad f \in L_{p}[−1, 1]. \]
It follows from the definition of K-functional that
\[ K(f, n^{-r}; L_{p}, W_{p}') \leq \|f - R_{n,r}f\|_{p} + n^{-r} \|\mathcal{D}'R_{n,r}f\|_{p} \leq \text{const}_{p,r} \|R_{n,r}f - f\|_{p}. \]
The proof of Theorem 1 is complete.

From Lemma 2 and the proof of Theorem 1 we deduce that
\[ \|R_{n,r}f - f\|_{p} + n^{-r} \|\mathcal{D}'R_{n,r}f\|_{p} \approx K(f, n^{-r}; L_{p}, W_{p}'). \]
This equivalence relationship shows that the $R_{n,r}f$ can serve as a realization of the K-functional $K(f, n^{-r}; L_{p}, W_{p}')$.

We now present the relationships between the best polynomial approximant and the generalized Riesz summability operators.

**Theorem 2.** Let $f \in L_{p}[−1, 1], 1 \leq p \leq \infty$, and let $R_{n,r}f$ be defined by (1.8). Then
\[ E_{n}(f)_{p} \leq \|R_{n,r}f - f\|_{p}. \]
conversely,
\[
\| R_n f - f \|_p \leq \frac{\text{const}_{p, r}}{n^r} \sum_{0 \leq k \leq n} (k + 1)^{r-1} E_k(f)_p.
\]

**Proof.** The first inequality is obvious. Concerning the second one, we have to show the following Bernstein type inequality
\[
\| \mathcal{D} Q_n \|_p \leq \text{const}_{p, r} n^r \| Q_n \|_p,
\]
where \( Q_n \) is a polynomial of order \( n \).

In fact, if \( Q_n \) is a polynomial of order \( n \), we can write \( Q_n \) as
\[
Q_n(x) = \sum_{k=0}^{n} Q_n^\wedge(k) P_k(x).
\]

By the definition of \( \mathcal{D} \), we get
\[
\mathcal{D} Q_n(x) = -\sum_{k=0}^{n} (k(k+1))^{r/2} Q_n^\wedge(k) P_k(x).
\]

Then the Bernstein type inequality is of the form
\[
\left\| \sum_{k=0}^{n} (k(k+1))^{r/2} Q_n^\wedge(k) P_k \right\|_p \leq \text{const}_{p, r} \left\| \sum_{k=0}^{n} Q_n^\wedge(k) P_k \right\|_p.
\]

This is Corollary 5.15 of [8]. The proof of Theorem 2 is complete.

By this theorem and Theorem 1 we have

**Theorem 3.** Let \( f \in L_p[-1, 1], 1 \leq p \leq \infty \), and let \( R_n, f \) be defined by (1.8). Then
\[
E_n(f)_p \leq \text{const}_{p, r} K(f, n^{-r}; L_p, W_p^r).
\]

conversely,
\[
K(f, n^{-r}; L_p, W_p^r) \leq \frac{\text{const}_{p, r}}{n^r} \sum_{0 \leq k \leq n} (k + 1)^{r-1} E_k(f)_p.
\]

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