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Flock generalized quadrangles and tetradic sets of elliptic quadrics of PG(3, q)☆

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Abstract

A flock of a quadratic cone of PG(3, q) is a partition of the non-vertex points into plane sections. It was shown by Thas in 1987 that to such flocks correspond generalized quadrangles of order (q^2, q), previously constructed algebraically by Kantor (q odd) and Payne (q even). In 1999, Thas gave a geometrical construction of the generalized quadrangle from the flock via a particular set of elliptic quadrics in PG(3, q). In this paper we characterise these sets of elliptic quadrics by a simple property, construct the generalized quadrangle synthetically from the properties of the set and strengthen the main theorem of Thas 1999.

MSC: 51E20; 51E12

Keywords: Flock; Generalized quadrangle; Elliptic quadric

1. Introduction

An *oval* of PG(2, q) is a set of $q+1$ points of PG(2, q) no three of which are collinear. Let ℓ be a line of PG(2, q), then ℓ is incident with zero, one or two points of an oval and is accordingly called an *external* line, a *tangent* or a *secant* to the oval.

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A *cap* of $\mathrm{PG}(3,q)$ is a set of points of $\mathrm{PG}(3,q)$ no three of which are collinear. A line of $\mathrm{PG}(3,q)$ will be called external, tangent or secant to a cap according to whether it contains zero, one or two points of the cap. An *ovoid* of $\mathrm{PG}(3,q)$ is a cap of size q^2+1 such that the tangents at a point form a plane, called the *tangent plane* at the point. Every plane not tangent to an ovoid meets the ovoid in an oval. If $q>2$, then a cap of $\mathrm{PG}(3,q)$ of maximal size is an ovoid. Every ovoid of $\mathrm{PG}(3,q)$, q odd, is a non-degenerate elliptic quadric of $\mathrm{PG}(3,q)$. For q even, $q=2^h$, the two known isomorphism classes of ovoids are the non-degenerate elliptic quadrics, which exist for all $h \geqslant 1$, and the Tits ovoids which exist for h odd, $h \geqslant 3$. (See [5–7] for details and references for the above.)

Let (∞, π_∞) be an incident point-plane pair of $\mathrm{PG}(3,q)$. If X, Y, Z, W are four distinct points of $\mathrm{PG}(3,q) \setminus \pi_\infty$, then we say that $\{X, Y, Z, W\}$ is a *tetrad* with respect to (∞, π_∞) if $\{\infty, X, Y, Z, W\}$ is a cap of $\mathrm{PG}(3,q)$ such that there exists a plane of $\mathrm{PG}(3,q)$ containing ∞ and exactly three of X, Y, Z, W. If ∞ and π_∞ are understood, then we will refer to $\{X, Y, Z, W\}$ as a *tetrad*.

A *tetradic* set of ovoids with respect to (∞, π_∞) is a set of ovoids of $\mathrm{PG}(3,q)$ each element of which contains ∞, has tangent plane π_∞ at ∞ and such that every tetrad with respect to (∞, π_∞) is contained in a unique ovoid of the set. If all the ovoids are elliptic quadrics, then we call the set a tetradic set of elliptic quadrics.

In this paper we shall investigate tetradic sets of elliptic quadrics of $\mathrm{PG}(3,q)$ and their connection to generalized quadrangles of order (q^2,q) constructed from a flock of a quadratic cone in $\mathrm{PG}(3,q)$. In particular, by considering work of Thas [17], we will show that a tetradic set of elliptic quadrics of $\mathrm{PG}(3,q)$ gives rise to a generalized quadrangle of order (q,q^2).

2. Flocks of Laguerre planes

A *Laguerre plane* is an incidence structure of points, lines and circles with the properties that every point lies on a unique line; a line and a circle meet in a unique point; any three pairwise non-collinear points lie on a unique circle; and, given a circle C and non-collinear points P and Q with P on C and Q not on C, there is a unique circle D on Q which meets C in exactly P. Given a finite Laguerre plane, there is an integer $n>1$ called the *order* of the plane such that there are n^2+n points, $n+1$ lines and n^3 circles, every line is incident with n points, every circle is incident with $n+1$ points, every point is incident with n^2 circles, and every pair of non-collinear points lies on n circles.

Given a Laguerre plane L and a point P of the plane, the *derived affine plane* L_P is the incidence structure with points the points of L not collinear with P, lines the circles of L incident with P and the lines of L not on P and the natural incidence relation. The structure L_P is an affine plane. If L has order n, then L_P has order n.

Let $\mathcal{K}$ be a quadratic cone in $\mathrm{PG}(3,q)$ with vertex V. The incidence structure with points the points of $\mathcal{K}$ other than V, lines the generators of $\mathcal{K}$, circles the plane sections of $\mathcal{K}$ not containing V and the natural incidence relation, is a Laguerre plane of order q. These Laguerre planes are characterised amongst all Laguerre planes by satisfying the configuration of Miquel [18,3, pp. 245–246] and hence are called *Miquelian*. General references on Laguerre planes are [1,4,3,14].

A flock $\mathcal{F}$ of a Laguerre plane L is a set of circles of L partitioning the points of L. If L has order n, then $\mathcal{F}$ contains n circles. Of particular interest will be the flocks of the Miqeulian Laguerre plane arising from a quadratic cone $\mathcal{K}$ in $\mathrm{PG}(3, q)$. Such a flock will also be called a *flock* of the quadratic cone $\mathcal{K}$. For more details on flocks of Laguerre planes see [8].

3. Generalized quadrangles with property (G)

A (finite) *generalized quadrangle* (GQ) is an incidence structure $\mathcal{S} = (\mathcal{P}, \mathcal{B}, \mathrm{I})$ in which $\mathcal{P}$ and $\mathcal{B}$ are disjoint (non-empty) sets of objects called *points* and *lines*, respectively, and for which $\mathrm{I} \subseteq (\mathcal{P} \times \mathcal{B}) \cup (\mathcal{B} \times \mathcal{P})$ is a symmetric point-line incidence relation satisfying the following axioms:

(i) Each point is incident with $1 + t$ lines ($t \geqslant 1$) and two distinct points are incident with at most one line.
(ii) Each line is incident with $1 + s$ points ($s \geqslant 1$) and two distinct lines are incident with at most one point.
(iii) If X is a point and ℓ is a line not incident with X, then there is a unique pair $(Y, m) \in \mathcal{P} \times \mathcal{B}$ for which $X \,\mathrm{I}\, m \,\mathrm{I}\, Y \,\mathrm{I}\, \ell$.

For a comprehensive introduction to GQs see [13]. The integers s and t are the *parameters* of the GQ and $\mathcal{S}$ is said to have *order* (s, t). If $s = t$, then $\mathcal{S}$ is said to have order s. If $\mathcal{S}$ has order (s, t), then it follows that $|\mathcal{P}| = (s + 1)(st + 1)$ and $|\mathcal{B}| = (t + 1)(st + 1)$ [13, 1.2.1]. If $\mathcal{S} = (\mathcal{P}, \mathcal{B}, \mathrm{I})$ is a GQ of order (s, t), then the incidence structure $\mathcal{S}^* = (\mathcal{B}, \mathcal{P}, \mathrm{I})$ is a GQ of order (t, s) called the *dual* of $\mathcal{S}$.

Given two (not necessarily distinct) points X, X' of $\mathcal{S}$, we write $X \sim X'$ and say that X and X' are *collinear*, provided there is some line ℓ for which $X \,\mathrm{I}\, \ell \,\mathrm{I}\, X'$. For $X \in \mathcal{P}$ put $X^\perp = \{X' \in \mathcal{P} : X \sim X'\}$. If $A \subset \mathcal{P}$, then we define $A^\perp = \cap\{X^\perp : X \in A\}$ and $A^{\perp\perp} = (A^\perp)^\perp$.

If $s^2 = t > 1$, then by a result of Bose and Shrikhande [2] we have $|\{X, Y, Z\}^\perp| = s+1$ for any triple $\{X, Y, Z\}$ of pairwise non-collinear points (called a *triad*). We say that $\{X, Y, Z\}$ is 3-*regular* provided $|\{X, Y, Z\}^{\perp\perp}| = s + 1$. The point X is 3-*regular* if and only if each triad $\{X, Y, Z\}$ is 3-regular.

Let $\mathcal{S} = (\mathcal{P}, \mathcal{B}, \mathrm{I})$ be a GQ of order (s, s^2), $s \neq 1$. Let X_1, Y_1 be distinct collinear points. We say that the pair $\{X_1, Y_1\}$ has *Property* (G), or that $\mathcal{S}$ has *Property* (G) *at* $\{X_1, Y_1\}$, if every triad $\{X_1, X_2, X_3\}$ of points, with $Y_1 \in \{X_1, X_2, X_3\}^\perp$, is 3-regular. The GQ $\mathcal{S}$ has *Property* (G) *at the line* ℓ, or the line ℓ has *Property* (G), if each pair of points $\{X, Y\}$, $X \neq Y$ and $X \,\mathrm{I}\, \ell \,\mathrm{I}\, Y$, has Property (G). If (X, ℓ) is a flag, then we say that $\mathcal{S}$ has *Property* (G) *at* (X, ℓ) or that (X, ℓ) has Property (G), if every pair (X, Y), $X \neq Y$ and $Y \,\mathrm{I}\, \ell$ has Property (G).

Suppose that $\mathcal{S} = (\mathcal{P}, \mathcal{B}, \mathrm{I})$ is a GQ of order (q, q^2) satisfying Property (G) at the pair of points (X, Y). We now review a construction of $\mathrm{AG}(3, q)$ from $\mathcal{S}$, X and Y due to Payne and Thas (see [16]).

We consider the following incidence structure $\mathcal{S}_{XY} = (\mathcal{P}_{XY}, \mathcal{B}_{XY}, \mathrm{I}_{XY})$:

(i) $\mathcal{P}_{XY} = X^\perp \setminus \{X, Y\}^{\perp\perp}$.

(ii) Elements of $\mathcal{B}_{XY}$ are of two types: (a) the sets $\{Y, Z, U\}^{\perp\perp} \setminus \{Y\}$, with $\{Y, Z, U\}$ a triad with $X \in \{Y, Z, U\}^{\perp}$, and (b) the sets $\{X, W\}^{\perp} \setminus \{X\}$, with $X \sim W \not\sim Y$.
(iii) I_{XY} is containment.

Then we have the following result:

Theorem 1 (*Payne and Thas, see [16]*). *The incidence structure $\mathcal{S}_{XY}$ is the design of points and lines of the affine space* $\mathrm{AG}(3, q)$. *In particular, q is a prime power.*

The planes of the affine space $\mathcal{S}_{XY} = \mathrm{AG}(3, q)$ are of two types:

(a) *The sets $\{X, Z\}^{\perp} \setminus \{Y\}$, with $X \not\sim Z$ and $Y \in \{X, Z\}^{\perp}$, and*
(b) *each set which is the union of all elements of type* (b) *of $\mathcal{B}_{XY}$ containing a point of some line m of type* (a) *of $\mathcal{B}_{XY}$.*

This construction leads us to an equivalent formulation of Property (G) at a pair of points.

Theorem 2. *Let $\mathcal{S} = (\mathcal{P}, \mathcal{B}, \mathrm{I})$ be a GQ of order (s, s^2) and $X, Y \in \mathcal{P}$ with $X \sim Y$. Then $\mathcal{S}$ satisfies Property* (*G*) *at $\{X, Y\}$ if and only if the incidence structure*

Points: $X^{\perp} \setminus \langle X, Y\rangle$,
Planes: $Y^{\perp} \setminus \langle X, Y\rangle$,
Incidence: *Collinearity in $\mathcal{S}$*,

is the point-plane incidence structure of $\mathrm{PG}(3, s)$ *with an incident point-plane pair removed.*

Let $\overline{\mathcal{S}_{XY}}$ be the projective completion of $\mathcal{S}_{XY}$ with plane at infinity π_∞. In [17] Thas gives the following interpretation of the GQ $\mathcal{S}$ in $\overline{\mathcal{S}_{XY}}$. The q^2 lines of type (b) of $\mathcal{S}_{XY}$ are parallel, so they define a point ∞ of $\overline{\mathcal{S}_{XY}}$. If we now consider any $Z \in \mathcal{P}$ with $X \not\sim Z \not\sim Y$ and U the point of $\ell = \langle X, Y\rangle$ such that $Z \sim U$, then $\mathcal{V} = \{X, Z\}^{\perp} \setminus \{U\}$ is a set of q^2 points. Clearly each line of $\overline{\mathcal{S}_{XY}}$ on ∞ meets $\mathcal{V}$ in exactly one point. Further, if U_1, U_2, U_3 are points of $\mathcal{V}$ collinear in $\overline{\mathcal{S}_{XY}}$, then it must be that $Y \in \{U_1, U_2, U_3\}^{\perp\perp}$ and so $Z \sim Y$, a contradiction since X, Y, Z is a triangle. It follows from this that $\mathcal{V} \cup \{\infty\}$ is an ovoid of $\overline{\mathcal{S}_{XY}}$ with tangent plane π_∞ at ∞. We will denote this ovoid by $\mathcal{O}_Z$.

Thas also determined the intersections of these ovoids. Consider two distinct points $Z_1, Z_2 \in \mathcal{P}$ with Z_1, Z_2 collinear with points $U_1, U_2 \mathrm{I} \ell$, respectively, with $U_1, U_2 \neq X, Y$. If $Z_1 \sim Z_2$ and $U_1 = U_2$, then $\mathcal{O}_{Z_1} \cap \mathcal{O}_{Z_2} = \{\infty\}$, since any larger intersection yields a triangle in $\mathcal{S}$.

If $Z_1 \sim Z_2$ and $U_1 \neq U_2$, then $\mathcal{O}_{Z_1} \cap \mathcal{O}_{Z_2} = \{\infty, R\}$ where R is the point of the line $\langle Z_1, Z_2\rangle$ in $X^{\perp}$. Further the point of $\langle Z_1, Z_2\rangle$ in $Y^{\perp}$ corresponds, in $\overline{\mathcal{S}_{XY}}$ to a plane which is tangent at R to both $\mathcal{O}_{Z_1}$ and $\mathcal{O}_{Z_2}$.

If $Z_1 \not\sim Z_2$ and $U_1 = U_2$, then $\mathcal{O}_{Z_1} \cap \mathcal{O}_{Z_2} = (\{X, Z_1, Z_2\}^{\perp} \setminus \{U_1\}) \cup \{\infty\}$, an intersection of size $q + 1$.

For the last case, if $Z_1 \not\sim Z_2$ and $U_1 \neq U_2$, then $\mathcal{O}_{Z_1} \cap \mathcal{O}_{Z_2} = \{X, Z_1, Z_2\}^{\perp} \cup \{\infty\}$, an intersection of size $q + 2$.

If m is a line of $\mathcal{S}$ such that $m \mathrm{I} U \mathrm{I} \ell$ and $U \neq X, Y$, then let the set ovoids of $\mathrm{PG}(3,q) = \overline{\mathcal{S}_{XY}}$ corresponding to points of $m \setminus \{U\}$ be denoted $\mathcal{R}$. The set $\mathcal{R}$ is a set of q ovoids of $\mathrm{PG}(3,q)$ meeting pairwise in a fixed point and with the same tangent plane at that point. We will call such a set $\mathcal{R}$ a *rosette* of ovoids, the fixed point of intersection the *base point* of the rosette and the common tangent plane at the base point the *base plane* of the rosette. The elements of a rosette partition the points of $\mathrm{PG}(3,q)$ not on the base plane.

If m is a line of $\mathcal{S}$ such that m and ℓ are non-concurrent, then let the set of ovoids of $\mathrm{PG}(3,q) = \overline{\mathcal{S}_{XY}}$ corresponding to points of $m \setminus (X^{\perp} \cup Y^{\perp})$ be denoted $\mathcal{T}$. The set $\mathcal{T}$ is a set of $q-1$ ovoids of $\mathrm{PG}(3,q)$ meeting pairwise in exactly two fixed points and sharing the tangent planes at those two fixed points. We will call such a set $\mathcal{T}$ a *transversal* of ovoids. These two common points are called the *base points* of the transversal and the two common tangent planes are called the *base planes* of the transversal.

Let $\mathcal{F}$ be a flock of a quadratic cone in $\mathrm{PG}(3,q)$. In [15] Thas showed that to $\mathcal{F}$ there corresponds a GQ of order (q^2,q) (which is often called a *flock GQ* in the literature) previously constructed via group coset geometry methods by Kantor [9] in the q odd case; and Payne [11] in the q even case. In [12] Payne showed that the dual of this GQ satisfies Property (G) at a line, for both q odd and even.

Suppose that $\mathcal{S} = (\mathcal{P}, \mathcal{B}, \mathrm{I})$ is a dual flock GQ of order (q,q^2), arising from the flock $\mathcal{F}$, satisfying Property (G) at the line $[\infty]$ and $X, Y \,\mathrm{I}\, [\infty]$, $X \neq Y$. In [17] Thas constructed a set of elliptic quadric ovoids of $\mathrm{PG}(3,q)$ from $\mathcal{F}$ which was then verified to be the set of ovoids $\{\mathcal{O}_Z : Z \in \mathcal{P} \setminus (X^{\perp} \cup Y^{\perp})\}$ of $\overline{\mathcal{S}_{XY}} = \mathrm{PG}(3,q)$. As a result Thas gave a geometric description of the dual flock GQs valid for both q odd and even (previously Knarr [10] had given a description valid for only q odd).

The main theorem of [17] is the following result:

Theorem 3 (*Thas [17, Main Theorem]*). *Let $\mathcal{S} = (\mathcal{P}, \mathcal{B}, \mathrm{I})$ be a GQ of order (q,q^2), $q > 1$, and assume that $\mathcal{S}$ satisfies Property (G) at the flag (X,ℓ). If q is odd then $\mathcal{S}$ is the dual of a flock GQ. If q is even and all ovoids $\mathcal{O}_Z$ are elliptic quadrics, then we have the same conclusion.*

In this paper we will show that a tetradic set of elliptic quadrics of $\mathrm{PG}(3,q)$ gives rise to a GQ of order (q,q^2) which must be the dual of a flock GQ. As a consequence we will weaken the hypothesis of Theorem 3 to assume only Property (G) at a pair of collinear points.

4. Tetradic sets of elliptic quadrics

In this section we prove a number of properties of a tetradic set of elliptic quadrics in $\mathrm{PG}(3,q)$.

Let (∞, π_∞) be an incident point-plane pair of $\mathrm{PG}(3,q)$ and let Θ be a tetradic set of elliptic quadrics with respect to (∞, π_∞).

Lemma 4. *The size of Θ is $q^3(q-1)$.*

Proof. The number of tetrads with respect to (∞, π_∞) is $q^6(q-1)^3(q-2)(q+1)/6$, each contained in a unique elliptic quadric of Θ. Each ovoid of Θ contains $(q^2+q)\binom{q}{3}(q^2-q) = q^3(q+1)(q-1)^2(q-2)/6$ tetrads from which it follows that $|\Theta| = q^3(q-1)$. □

Example 5. Let $\mathcal{O}$ be an elliptic quadric with tangent plane π_∞ at ∞ and let $\Omega = \{\mathcal{O}^g | g \in \mathrm{PGL}(4,q)$ with centre $\infty\}$. Then Ω is a tetradic set of elliptic quadrics.

Proof. We first note that $|\Omega| = q^3(q-1)$ and that the intersection of two elements of Ω cannot contain a tetrad with respect to (∞, π_∞). Hence the $q^3(q+1)(q-1)^2(q-2)/6$ tetrads contained in $\mathcal{O}$ give rise, under the action of collineations with centre ∞, to $q^6(q+1)(q-1)^3(q-2)/6$ distinct tetrads. Since this is all such tetrads it follows that every tetrad is contained in a unique element of Ω. □

Remark 6. We note that the above construction works in more generality if we replace the elliptic quadric by an ovoid $\mathcal{O}$ of $\mathrm{PG}(3,q)$.

Lemma 7. *Let G denote the group of q^3 elations of* $\mathrm{PG}(3,q)$ *which have ∞ as centre. For $\mathcal{O}_1, \mathcal{O}_2 \in \Theta$ define $\mathcal{O}_1 \bowtie \mathcal{O}_2$ if there is an element g of G such that $\mathcal{O}_1^g = \mathcal{O}_2$. Then $\bowtie$ is an equivalence relation on Θ dividing the $q^3(q-1)$ elliptic quadrics of Θ into $q-1$ classes of size q^3.*

Proof. Let $\{X, Y, Z, \infty\}$ be a 4-cap of $\mathrm{PG}(3,q)$ with $\infty \in \pi = \langle X, Y, Z\rangle$. Then there is a unique conic $\mathcal{C}$ containing X, Y, Z, ∞ and with tangent line $\pi \cap \pi_\infty$. Hence any elliptic quadric of Θ containing X, Y, Z must also contain $\mathcal{C}$. If W is any point of $\mathrm{PG}(3,q) \setminus (\pi \cup \pi_\infty)$, then $\{X, Y, Z, W\}$ is a tetrad. Hence the elliptic quadrics of Θ containing $\mathcal{C}$ must partition the points of $\mathrm{PG}(3,q) \setminus (\pi \cup \pi_\infty)$ and so there are exactly q such elliptic quadrics.

We now investigate the possibilities for such a set.

Without loss of generality, choose $\infty = (0,1,0,0)$, $\pi_\infty : X_0 = 0$ and $\mathcal{O} \in \Theta$ to be $\mathcal{O} = \{(1, f(s,t), s, t) : s, t \in \mathrm{GF}(q)\} \cup \{\infty\}$, where f is an irreducible quadratic form over $\mathrm{GF}(q)$. If q is odd we will take $f(s,t) = s^2 - \eta t^2$ where η is a fixed non-square of $\mathrm{GF}(q)$, and if q is even take $f(s,t) = s^2 + st + \rho t^2$ where ρ is a fixed element of $\mathrm{GF}(q)$ with $\mathrm{Tr}(\rho) = 1$. Let $\mathcal{C} \subset \mathcal{O}$ be the conic $\{(1, s^2, s, 0) : s \in \mathrm{GF}(q)\} \cup \{\infty\}$. Now we search for other elliptic quadrics containing $\mathcal{C}$ and meeting $\mathcal{O}$ in exactly $\mathcal{C}$. If $\mathcal{O}'$ is a such an elliptic quadric, then there exists a homography ϕ of $\mathrm{PG}(3,q)$ mapping $\mathcal{O}$ to $\mathcal{O}'$. Further, since the group of $\mathcal{O}$ in $\mathrm{PGL}(4,q)$ is 3-transitive on the points of $\mathcal{O}$, we may assume that ϕ fixes $\infty = (0,1,0,0)$, $(1,0,0,0)$ and $(1,1,1,0)$. Hence ϕ also fixes $\pi_\infty : X_0 = 0$, the plane with equation $X_3 = 0$ and the conic $\mathcal{C}$.

Firstly, fixing $(1,0,0,0)$, $(0,1,0,0)$, $(1,1,1,0)$ and the planes π_∞ and $X_3 = 0$, it follows that ϕ must have the form

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & a & 1-a & b \\ 0 & 0 & 1 & c \\ 0 & 0 & 0 & d \end{pmatrix} \quad \text{with } a, b, c, d \in \mathrm{GF}(q) \quad \text{and } a, d \neq 0.$$

Now ϕ also fixes $\mathcal{C}$ and so $(1, s^2, s, 0)^\phi = (1, as^2 + s(1-a), s, 0) \in \mathcal{C}$ for all $s \in \mathrm{GF}(q)$. Hence $a = 1$ and ϕ has the form

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & b \\ 0 & 0 & 1 & c \\ 0 & 0 & 0 & d \end{pmatrix}, b, c, d \in \mathrm{GF}(q) \quad \text{and } d \neq 0.$$

Now if q is odd, then $\mathcal{O}' = \{(1, s^2 - \eta t^2 + bt, s + ct, dt) : s, t \in \mathrm{GF}(q)\} \cup \{\infty\}$ and since $\mathcal{O} \cap \mathcal{O}' = \mathcal{C}$ it follows that the equation $s^2 - \eta t^2 + bt = (s + ct)^2 - \eta(dt)^2$ has no solutions for $t \neq 0$. That is $\eta(d^2 - 1)t - c^2 t - 2cs + b = 0$ has no solution for $t \neq 0$. If $c \neq 0$, then there is a solution for every $t \neq 0$, so $c = 0$ and our condition now becomes that the equation $\eta(d^2 - 1)t + b = 0$ has no solution for $t \neq 0$. Hence, either $d = \pm 1$ and $b \neq 0$, or $d \neq \pm 1$ and $b = 0$. For $c = 0$ and fixed b the choice of d equal to 1 or -1 gives the same ovoid $\mathcal{O}'$, so the cases reduce to $d = 1$ and $b \neq 0$ or $d \neq \pm 1$ and $b = 0$.

In the first case we obtain the $q - 1$ images of $\mathcal{O}$ under non-trivial elations of $\mathrm{PG}(3, q)$ with centre ∞ and axis $X_3 = 0$.

In the second case we obtain $(q - 3)/2$ distinct images under homologies with centre $(0, 0, 0, 1)$ and axis $X_3 = 0$.

Recall that there are exactly q ovoids of Θ containing $\mathcal{C}$ and that they meet pairwise in $\mathcal{C}$. Hence all $q - 1$ ovoids (distinct from $\mathcal{O}$) must come from the first case, or we have an example of an ovoid from the first case meeting an ovoid from the second case in exactly $\mathcal{C}$. Specifically, in this latter case we would have an ovoid of the form $\{(1, s^2 - \eta t^2 + bt, s, t) : s, t \in \mathrm{GF}(q)\} \cup \{\infty\}$ intersecting an ovoid of the form $\{(1, u^2 - \eta v^2, u, dv) : u, v \in \mathrm{GF}(q)\} \cup \{\infty\}$, $d \neq \pm 1$, in exactly $\mathcal{C}$. That is, the equation $u^2 - \eta d^2 v^2 + bdv = u^2 - \eta t^2$ has no solution with $v \neq 0$, which is the case if and only if $v[\eta(1 - d^2)v + bd] = 0$ has no solution with $v \neq 0$. This is a contradiction since $1 - d^2$ and bd are non zero. Hence any elliptic quadric of Θ containing $\mathcal{C}$ must be the image of $\mathcal{O}$ under an elation with centre ∞ and axis $X_3 = 0$.

If q is even $\mathcal{O}' = \{(1, s^2 + st + \rho t^2 + bt, s + ct, dt) : s, t \in \mathrm{GF}(q)\} \cup \{\infty\}$ and we require that the equation $s^2 + st + \rho t^2 + bt = (s + ct)^2 + (s + ct)dt + \rho d^2 t^2$ has no solution with $t \neq 0$. That is, $(\rho + c^2 + cd + \rho d^2)t + (d + 1)s + b = 0$ has no solution with $t \neq 0$. Hence $d = 1$, and either $c^2 + c = 0$ and $b \neq 0$ or $c^2 + c \neq 0$ and $b = 0$. Since for the choice $d = 1$, $c = 1$, $b \neq 0$ the homography ϕ fixes $\mathcal{O}$, the first case above is equivalent to $c = 0$, $b \neq 0$, the non-trivial elations with centre ∞ and axis $X_3 = 0$.

In the second case choosing $c = \alpha \neq 0, 1$ or $c = \alpha + 1$, gives the same $\mathcal{O}'$ and we have $(q - 2)/2$ images of $\mathcal{O}$ under homologies with centre $(0, 0, 1, 0)$ and axis $X_3 = 0$. As in the q odd case we check if an elliptic quadric of the form $\{(1, s^2 + st + \rho t^2 + bt, s, t) : s, t \in \mathrm{GF}(q)\} \cup \{\infty\}$ with $b \neq 0$ can intersect an elliptic quadric of the form $\{(1, u^2 + uv + \rho v^2, u + cv, v) : u, v \in \mathrm{GF}(q)\} \cup \{\infty\}$, $c \neq 0, 1$, in exactly $\mathcal{C}$. That is, there are no solutions to $(c^2 + c)v + b = 0$ for $v \neq 0$. This is impossible since $c^2 + c \neq 0$ and $b \neq 0$.

Now since the homography group of $\mathcal{O}$ is transitive on secant planes of $\mathcal{O}$, it follows that every image of $\mathcal{O}$ under an elation with centre ∞ and axis a secant plane of $\mathcal{O}$ is contained in Θ. This is also true for any element of Θ and hence all elations with centre ∞ fix Θ, defining equivalence classes on Θ. □

Let $\mathcal{O}$ be an elliptic quadric of Θ and $[\mathcal{O}]$ its equivalence class. Then we note that there is a unique rosette of elliptic quadrics contained in $[\mathcal{O}]$ with base point ∞, base plane π_∞ and containing $\mathcal{O}$ generated by the action of the elations with centre ∞ and axis π_∞ on $\mathcal{O}$. The remaining $q^3 - q$ elliptic quadrics in $[\mathcal{O}]$ share a common conic with $\mathcal{O}$. Hence two elliptic quadrics in $[\mathcal{O}]$ either intersect in the point ∞ or in a conic containing ∞. We also note that the q^3 elliptic quadrics in $[\mathcal{O}]$ divide into q^2 disjoint rosettes with base point ∞.

Lemma 8. *Let X, Y, Z be three distinct, non-collinear points of $\mathrm{PG}(3,q)\backslash\pi_\infty$ not coplanar with ∞. Then there are exactly $q-1$ elliptic quadrics of Θ containing $\{X, Y, Z\}$, one from each equivalence class.*

Proof. Let $\pi = \langle X, Y, \infty\rangle$ and let ℓ be a line of π incident with ∞, but not with X nor Y. Let $W = \ell \cap \langle X, Y\rangle$. If $A \in \ell \setminus \{W, \infty\}$, then there is a unique elliptic quadric of Θ containing the tetrad $\{X, Y, Z, A\}$. There are $q-1$ such points A and hence $q-1$ such elliptic quadrics, since any elliptic quadric of Θ must contain a point of $\ell \setminus \{\infty\}$. As two equivalent elliptic quadrics must intersect in either the single point ∞ or in a conic containing ∞, it follows that the $q-1$ elliptic quadrics on $\{X, Y, Z, \infty\}$ are in distinct equivalence classes. □

Lemma 9. *Let $\mathcal{O}_1$ and $\mathcal{O}_2$ be two inequivalent elliptic quadrics of Θ, then $|\mathcal{O}_1 \cap \mathcal{O}_2| \leqslant q+2$.*

Proof. The elliptic quadrics $\mathcal{O}_1$ and $\mathcal{O}_2$ may not intersect in a conic on ∞ and hence no plane on ∞ contains more than two points of $(\mathcal{O}_1 \cap \mathcal{O}_2) \setminus \{\infty\}$. Thus the lines of $\mathrm{PG}(3,q)$ spanned by ∞ and points of $(\mathcal{O}_1 \cap \mathcal{O}_2) \setminus \{\infty\}$ form an arc in the quotient space $\mathrm{PG}(3,q)/\infty$. If q is odd then such an arc has size at most $q+1$ and hence $|\mathcal{O}_1 \cap \mathcal{O}_2| \leqslant q+2$. If q is even, then the arc has size at most $q+2$ and so $|\mathcal{O}_1 \cap \mathcal{O}_2| \leqslant q+3$.

So now suppose that q is even and $|\mathcal{O}_1 \cap \mathcal{O}_2| = q+3$. In this case every plane distinct from π_∞ and containing ∞ contains exactly three points of $\mathcal{O}_1 \cap \mathcal{O}_2$. Since q is even we have that $\mathcal{O}_1$ and $\mathcal{O}_2$ both define symplectic polarities. Further, since these polarities share the polar point-plane pair (∞, π_∞) there exists a plane π containing ∞ and distinct from π_∞ such that π has the same pole N under both polarities. Now the conics $\pi \cap \mathcal{O}_1$ and $\pi \cap \mathcal{O}_2$ share exactly three points and also have the same nucleus N. However, two conics sharing three points and with the same nucleus must be identical and so we have a contradiction. Hence when q is even $|\mathcal{O}_1 \cap \mathcal{O}_2| \neq q+3$ and so $|\mathcal{O}_1 \cap \mathcal{O}_2| \leqslant q+2$. □

Lemma 10. *Let $\mathcal{O}_1$ and $\mathcal{O}_2$ be inequivalent elliptic quadrics of Θ with $|\mathcal{O}_1 \cap \mathcal{O}_2| \geqslant 3$. Then $|\mathcal{O}_1 \cap \mathcal{O}_2| = q+2$.*

Proof. Consider fixed $X, Y \in (\mathcal{O}_1 \cap \mathcal{O}_2) \setminus \{\infty\}$, $X \neq Y$ and let $[\mathcal{O}_2]$ be the equivalence class of $\mathcal{O}_2$. There are $q^2 - q$ triples $\{X, Y, Z\}$ such that $X, Y, Z \subset \mathcal{O}_1$ and $\infty \notin \langle X, Y, Z\rangle$. By Lemma 8 each such triple is contained in a unique element of $[\mathcal{O}_2]$.

We know that $X, Y \in \mathcal{O}_2$ and that any other elliptic quadric $\mathcal{O}_2' \in [\mathcal{O}_2]$ with $X, Y \in \mathcal{O}_2'$ meets $\mathcal{O}_2$ in points contained in a plane on ∞, which must be $\langle X, Y, \infty\rangle$. Further, it must be that $\mathcal{O}_2 \cap \mathcal{O}_2' = \mathcal{O}_2 \cap \langle X, Y, \infty\rangle$. There are exactly q elliptic quadrics of $[\mathcal{O}_2]$ containing $\mathcal{O}_2 \cap \langle X, Y, \infty\rangle$.

Now we count pairs $(\mathcal{O}_2', \{X, Y, Z\})$ where $\mathcal{O}_2' \in [\mathcal{O}_2]$ and X, Y, Z are distinct points of $\mathcal{O}_2'$. From above we know that the count is in fact $q^2 - q$. However, we also have that $|\mathcal{O}_1 \cap \mathcal{O}_2'| \leqslant q + 2$ and so there are at most $q - 1$ such triples $\{X, Y, Z\}$ and q such $\mathcal{O}_2'$. Hence the count is bounded above by $q(q - 1) = q^2 - q$. It follows that $|\mathcal{O}_1 \cap \mathcal{O}_2'| = q + 2$ and certainly $|\mathcal{O}_1 \cap \mathcal{O}_2| = q + 2$. □

Corollary 11. *If $\mathcal{O}_1$ and $\mathcal{O}_2$ are inequivalent elliptic quadrics of Θ, then $|\mathcal{O}_1 \cap \mathcal{O}_2| = 2$ or $q + 2$.*

Proof. We show that $|\mathcal{O}_1 \cap \mathcal{O}_2| = 1$ is impossible. Consider the rosette $\mathcal{R}$ of elliptic quadrics in $[\mathcal{O}_2]$ containing $\mathcal{O}_2$. The elliptic quadric $\mathcal{O}_1$ intersects each of the q elliptic quadrics of the rosette in 1, 2 or $q + 2$ points. The elements of $\mathcal{R}$ also partition the points of $\mathrm{PG}(3, q) \setminus \pi_\infty$ and so the q^2 points of $\mathcal{O}_1 \setminus \{\infty\}$ are partitioned into q sets of size 0, 1 or $q + 1$. This can only be the done with one set of size 1 and $q - 1$ of size $q + 1$. □

Lemma 12. *Let $\mathcal{O}$ be an elliptic quadric of Θ and E an equivalence class of Θ such that $\mathcal{O} \notin E$. If $X \in \mathcal{O} \setminus \{\infty\}$, then there is a unique $\mathcal{O}' \in E$ such that $\mathcal{O} \cap \mathcal{O}' = \{X, \infty\}$.*

Proof. For fixed $X \in \mathcal{O} \setminus \{\infty\}$ the number of pairs (Y, Z) with $Y, Z \in \mathcal{O}$ and $Y \neq Z$ such that $\infty \notin \langle X, Y, Z\rangle$ is $(q^2 - 1)(q^2 - q)$.

By Lemma 8, each triple $\{X, Y, Z\}$ is contained in a unique element of E. Further, each of the q^2 rosettes of E contains a unique elliptic quadric on the point X. If such an elliptic quadric $\mathcal{O}'$ intersects $\mathcal{O}$ in $q+2$ points, then we have $q(q-1)$ pairs (Y, Z) with $Y, Z \in \mathcal{O} \cap \mathcal{O}'$ and $Y \neq Z$ such that $\infty \notin \langle X, Y, Z\rangle$. If, on the other hand, $|\mathcal{O} \cap \mathcal{O}'| = 2$, then there are no such pairs (Y, Z).

Hence it follows that there are $q^2 - 1$ elliptic quadrics of E containing X and meeting $\mathcal{O}$ in $q + 2$ points and a unique elliptic quadric of E containing X and meeting $\mathcal{O}$ in 2 points. □

Lemma 13. *Let $\mathcal{O}$ be an elliptic quadric of Θ and $X \in \mathcal{O} \setminus \{\infty\}$. The $q - 2$ elliptic quadrics meeting $\mathcal{O}$ in exactly $\{X, \infty\}$ also meet pairwise in exactly $\{X, \infty\}$ and have a common tangent plane at X, thus together with $\mathcal{O}$ form a transversal.*

Proof. Let π_X be the tangent plane to $\mathcal{O}$ at X. For $Y \in \mathrm{PG}(3, q) \setminus (\pi_\infty \cup \pi_X \cup \langle \infty, X\rangle \cup \mathcal{O})$ count pairs $(Z, \mathcal{O}')$ with $Z \in \mathcal{O} \setminus \{X, \infty\}$, $\mathcal{O}' \in \Theta$ and $\{X, Y, Z, \infty\} \subset \mathcal{O}'$. Suppose that $Z \in \mathcal{O} \setminus \langle X, Y, \infty\rangle$, then there are $q^2 - q$ choices for Z and $\{X, Y, Z\}$ is contained in $q - 1$ elliptic quadrics, giving $(q^2 - q)(q - 1)$ pairs.

Now suppose that $Z \in \mathcal{O} \cap \langle X, Y, \infty\rangle$ and let $\mathcal{C} = \mathcal{O} \cap \langle X, Y, \infty\rangle$. Note that $Y \notin \mathcal{C}$ and since $Y \notin \pi_X$ it follows that $\langle X, Y\rangle$ is not tangent to $\mathcal{C}$ and so meets $\mathcal{C}$ in a point of $\mathcal{C} \setminus \{X\}$. Similarly $\langle Y, \infty\rangle$ meets $\mathcal{C}$ in a second point, leaving $q - 3$ possible choices for Z. For each such choice of Z the points X, Y, Z define a unique conic $\mathcal{C}'$ in $\langle X, Y, \infty\rangle$ containing ∞ and with tangent $\pi_\infty \cap \langle X, Y, \infty\rangle$. There are q elliptic quadrics of Θ containing $\mathcal{C}'$ giving $q(q - 3)$ pairs $(Z, \mathcal{O}')$ with $Z \in \mathcal{O} \cap \langle X, Y, \infty\rangle$.

So in total we have $(q^2-q)(q-1)+q(q-3)=q(q-2)(q+1)$ pairs $(Z,\mathcal{O}')$.

Counting these pairs in a second way we consider the number of elliptic quadrics of Θ containing $\{X,Y\}$. In $\langle\infty,X,Y\rangle$ there are $q-1$ conics on X,Y,∞ with tangent $\langle\infty,X,Y\rangle\cap\pi_\infty$ which means there are $q(q-1)$ elliptic quadrics containing $\{X,Y\}$, q in each class. If such an elliptic quadric is in the same class as $\mathcal{O}$, then we know that the possible intersections sizes with $\mathcal{O}$ are 1 and $q+1$, and so they must all be $q+1$, since the intersection is at least $\{\infty,X\}$. This gives $q(q-1)$ pairs $(Z,\mathcal{O}')$. There are $q(q-2)$ elliptic quadric containing $\{X,\infty\}$ which are inequivalent to $\mathcal{O}$, and by the earlier counts there are exactly $q(q-2)(q+1)-q(q-1)=q(q^2-2q-1)$ pairs $(Z,\mathcal{O}')$ where $\mathcal{O}'$ is of this type. Now in this case $|\mathcal{O}\cap\mathcal{O}'|=2$ or $q+2$ and so each $\mathcal{O}'$ gives rise to 0 or q pairs $(Z,\mathcal{O}')$, respectively. From this it follows that there must be q^2-2q-1 elliptic quadrics intersecting $\mathcal{O}$ in $q+2$ points and $(q^2-2q)-(q^2-2q-1)=1$ intersecting $\mathcal{O}$ in the two points X,∞.

Hence there is a unique elliptic quadric of Θ meeting $\mathcal{O}$ in exactly $\{X,\infty\}$ and containing the point $Y\in \mathrm{PG}(3,q)\setminus(\pi_\infty\cup\pi_X\cup\langle\infty,X\rangle\cup\mathcal{O})$.

In Lemma 12 we saw that there are $q-2$ elliptic quadrics of Θ meeting $\mathcal{O}$ in exactly $\{X,\infty\}$. By the above, these $q-2$ elliptic quadrics plus $\mathcal{O}$ cover the $q^3-q^2-q+1=(q^2-1)(q-1)$ points of $\mathrm{PG}(3,q)\setminus(\pi_\infty\cup\pi_X\cup\langle\infty,X\rangle)$. It follows that these elliptic quadrics partition the pointset into $q-2$ sets of size q^2-1. Hence the elliptic quadrics meet pairwise in exactly $\{X,\infty\}$ and have π_X as tangent plane at X; thus forming a transversal of elliptic quadrics. □

Lemma 14. *Let (X,π) be an incident point-plane pair of $\mathrm{PG}(3,q)$ such that $X\notin\pi_\infty$ and $\infty\notin\pi$. Then there are exactly $q-1$ elliptic quadrics of Θ containing X and with tangent plane π at X. Further, these $q-1$ elliptic quadrics form a transversal with one elliptic quadric from each equivalence class of Θ.*

Proof. Since each rosette of elliptic quadrics contained in Θ is generated by the action on one elliptic quadric of the elations of $\mathrm{PG}(3,q)$ with centre ∞ and axis π_∞, it follows that each plane not on ∞ is tangent to exactly one elliptic quadric of a given rosette.

There are $(q-1)q^2$ rosettes of elliptic quadrics in Θ and so $(q-1)q^2$ elliptic quadrics of Θ with π as a tangent plane at one of the points $\pi\setminus\pi_\infty$. By Lemma 13 if π is tangent to one elliptic quadric of Θ at a point, then it is tangent to the $q-1$ elliptic quadrics of a transversal of Θ at that point. Since two elliptic quadrics in the same equivalence class have intersection size 1 or $q+1$, it follows that the elliptic quadrics of the transversal are one from each equivalence class of Θ.

Suppose that for $X\in\pi\setminus\pi_\infty$ there are two transversals $\mathcal{O}_1,\dots,\mathcal{O}_{q-1}$ and $\mathcal{O}'_1,\dots,\mathcal{O}'_{q-1}$ containing X and with tangent plane π. We investigate how $\mathcal{O}'_1$ intersects $\mathcal{O}_1,\dots,\mathcal{O}_{q-1}$. In fact $\mathcal{O}_1,\dots,\mathcal{O}_{q-1}$ partition the q^2-1 points of $\mathcal{O}'_1\setminus\{X,\infty\}$ into $q-1$ sets of size q or $q-1$, which is a contradiction.

Hence there can be only one transversal of elliptic quadrics of Θ with X as a base point and π as the corresponding base plane. Since there are $q^2(q-1)$ elliptic quadrics with tangent plane π, there are q^2 transversals of Θ with base plane π and a unique such transversal with base point X for each $X\in\pi\setminus\pi_\infty$. □

Lemma 15. *Let $\mathcal{T}=\{\mathcal{O}_1,\dots,\mathcal{O}_{q-1}\}$ be a transversal of elliptic quadrics of Θ with base point X and base plane π, $X\notin\pi_\infty$ and $\infty\notin\pi$. Then every plane π' such that*

$X, \infty, \pi_\infty \cap \pi \not\subset \pi'$ is tangent to a unique element of $\mathcal{T}$. Further, any two elements of $\mathcal{T}$ have only π and π_∞ as common tangent planes.

Proof. Let π' be a plane such that $X, \infty, \pi_\infty \cap \pi \not\subset \pi'$. The elements of $\mathcal{T}$ partition the $q^2 - q - 1$ points of $\pi' \setminus (\pi_\infty \cup \pi \cup \langle \infty, X \rangle)$, which can only be into $q - 2$ conics and one single point. That is, π' is tangent to a unique element of $\mathcal{T}$.

There are $(q^2 - 1)(q - 1)$ such planes, which is the same as the number of pairs $(\mathcal{O}_i, \pi'')$ where π'' is tangent to $\mathcal{O}_i \in \mathcal{T}$ at a point not ∞ nor X. Hence no two elements of $\mathcal{T}$ have a common tangent not at ∞ nor X.

Since a plane of $\mathrm{PG}(3, q)$ (distinct from π_∞ and π) on $\pi_\infty \cap \pi$ or on ∞ cannot be a tangent plane to an elliptic quadric of $\mathcal{T}$, the lemma is proved. □

We now work towards proving that the dual of a tetradic set of elliptic quadrics is also a tetradic set. Let $\mathrm{PG}(3, q)^*$ denote the dual space of $\mathrm{PG}(3, q)$ and let $\Theta^* = \{\mathcal{O}^* : \mathcal{O} \in \Theta\}$, a set of $q^3(q - 1)$ ovoids of $\mathrm{PG}(3, q)^*$ each containing the point π_∞^* and with common tangent plane ∞^*.

Lemma 16. *The group G of collineations of $\mathrm{PG}(3, q)$ of order q^5 generated by elations with centre ∞ and elations with axis π_∞ fixes Θ.*

Proof. It is straightforward to check that an elation with axis π_∞ fixes Θ and hence so does G. □

Lemma 17. *Let π be a plane of $\mathrm{PG}(3, q)^*$ distinct from ∞^* and $\mathcal{C}$ a conic in π containing π_∞^* and with tangent line $\pi \cap \infty^*$. If $\mathcal{C}$ is contained in one element of Θ^*, then $\mathcal{C}$ is contained in exactly q elements of Θ^*.*

Proof. Suppose $\mathcal{C} \subset \mathcal{O} \in \Theta^*$ then by Lemma 16 the q images $\mathcal{O}_1 = \mathcal{O}, \mathcal{O}_2, \ldots, \mathcal{O}_q$ of $\mathcal{O}$ under elations with centre π_∞^* are elements of Θ^* containing $\mathcal{C}$. Suppose that there is an elliptic quadric $\mathcal{O}' \in \Theta^*$, not one of the $\mathcal{O}_i$, such that $\mathcal{C} \subset \mathcal{O}'$. Let $X \in \mathcal{C} \setminus \{\pi_\infty^*\}$ and ℓ_X the tangent to $\mathcal{C}$ at X. Each of $\mathcal{O}_1, \ldots, \mathcal{O}_q$ has a distinct tangent plane at X which must contain ℓ_X. Since there are q planes on ℓ_X not containing π_∞^* we can say without loss of generality that $\mathcal{O}$ and $\mathcal{O}'$ have the same tangent plane π_X at X. Dualising this, we have two elements of Θ with common point π_X^* and common tangent plane X^* at π_X^*. Hence by Lemma 14 the two elliptic quadrics are in a common transversal, a contradiction since by Lemma 15 the can only have two common tangent planes, but we know that each element of $\mathcal{C}^*$ is tangent to both elliptic quadrics. □

Theorem 18. *The set Θ^* of elliptic quadrics of $\mathrm{PG}(3, q)^*$ is a tetradic set with respect to (π_∞^*, ∞^*).*

Proof. Each element of Θ^* contains q^2 conics containing π_∞^*, and each such conic is contained in exactly q elements of Θ^* by Lemma 17. Hence there are $q^4(q - 1)$ such conics. Since this is the same as the number of pairs $(\pi, \mathcal{C})$ where $\mathcal{C}$ is a conic in the plane π containing π_∞^* and with tangent $\pi \cap \infty^*$, it follows that all such conics $\mathcal{C}$ are in q elements of Θ^*. Further, these q elliptic quadrics partition the points of $\mathrm{PG}(3, q)^* \setminus (\pi \cup \infty^*)$.

So let $\{X, Y, Z, W\}$ be a tetrad with X, Y, Z, π^*_∞ coplanar in π. By the above there is a unique element of Θ^* containing W and the unique conic containing X, Y, Z, π^*_∞ and with tangent $\pi \cap \infty^*$. Hence Θ^* is a tetradic set of elliptic quadrics with respect to (π^*_∞, ∞^*). □

5. Flocks of the quadratic cone and tetradic sets of elliptic quadrics of PG(3, q)

In this section we will show that a flock of a quadratic cone in PG$(3, q)$ gives rise to a tetradic set of elliptic quadrics and conversely.

Let $\mathcal{K}$ be a quadratic cone in PG$(3, q)$ with vertex V. Let $P \in \mathcal{K} \setminus \{V\}$, $\ell = \langle P, V\rangle$ and let π_ℓ be the plane meeting $\mathcal{K}$ in ℓ. Suppose that π is any plane containing neither V nor P. If we project the points of $\mathcal{K} \setminus \{V\}$ from P onto π, then we have a one-to-one correspondence between the points of $\mathcal{K} \setminus \ell$ and the points of $\pi \setminus (\pi_\ell \cap \pi)$, while the points $\ell \setminus \{P, V\}$ project onto $P' = \ell \cap \pi$. The $q^3 - q^2$ plane sections of $\mathcal{K}$ containing neither P nor V project onto the $q^3 - q^2$ conics of π containing P' and with tangent $\pi \cap \pi_\ell$. If $\mathcal{C}_1$ and $\mathcal{C}_2$ are two planar sections of $\mathcal{K}$ both containing the point $Q \in \ell \setminus \{P, V\}$, then their respective projections $\mathcal{C}'_1, \mathcal{C}'_2$ in π have the property that $\mathcal{C}'_2$ is one of the q^2 images of $\mathcal{C}'_1$ under an elation of π with centre P'. The q^2 planar sections of $\mathcal{K}$ containing P project onto the q^2 lines of π not incident with P'. Hence a flock $\{\mathcal{C}_1, \dots, \mathcal{C}_q\}$ of $\mathcal{K}$ projects to a set $\{\mathcal{C}'_1, \dots, \mathcal{C}'_{q-1}, m\}$ where $\mathcal{C}'_1, \dots, \mathcal{C}'_{q-1}$ are conics of π with common point P', common tangent $\pi \cap \pi_\ell$, m is a line of π not incident with P' and $\mathcal{C}'_1, \dots, \mathcal{C}'_{q-1}, m$ partition the points of $\pi \setminus (\pi \cap \pi_\ell)$. Further, no $\mathcal{C}'_i$ is the image of a $\mathcal{C}'_j$, $i, j \in \{1, \dots, q-1\}$, $i \neq j$, under an elation of π with centre P'. Conversely, any such set $\{\mathcal{C}'_1, \dots, \mathcal{C}'_{q-1}, m\}$ with these properties corresponds to a flock of $\mathcal{K}$.

The following result is straightforward to verify and allows us to provide a correspondence between flocks of quadratic cones and transversals of elliptic quadrics.

Lemma 19. *Let ∞ and R be points of* PG$(3, q)$ *and π, π_∞, π_R be planes of* PG$(3, q)$ *such that $\infty \in \pi_\infty \cap \pi$, $R \in \pi_R \setminus (\pi \cup \pi_\infty)$ and the three planes π, π_∞, π_R meet in a point. If $\mathcal{C}$ is any conic of π such that $\infty \in \mathcal{C}$, $\pi \cap \pi_\infty$ is the tangent to $\mathcal{C}$ at ∞ and $\pi \cap \pi_R$ is external to $\mathcal{C}$, then there exists a unique elliptic quadric $\mathcal{O}$ such that $\mathcal{C} \cup \{R\} \in \mathcal{O}$, π_∞ is the tangent plane to $\mathcal{O}$ at ∞ and π_R is the tangent plane to $\mathcal{O}$ at R.*

Further, suppose that $\mathcal{C}'$ is a second conic of π containing ∞, with tangent line $\pi \cap \pi_\infty$ at ∞, external line $\pi \cap \pi_R$ and that $\mathcal{O}'$ is the unique elliptic quadric containing $\mathcal{C}' \cup \{R\}$ and such that π_∞ is the tangent plane to $\mathcal{O}'$ at ∞ and π_R is the tangent plane to $\mathcal{O}'$ at R. Then $\mathcal{O} \cap \mathcal{O}' = \{\infty, R\}$ if and only if $\mathcal{C} \cap \mathcal{C}' = \{\infty\}$ and *$\mathcal{C}'$ is not the image of $\mathcal{C}$ under an elation of π with centre ∞.*

Theorem 20. *Let $\mathcal{F}$ be a flock of a quadratic cone in* PG$(3, q)$. *Then $\mathcal{F}$ gives rise to a tetradic set of elliptic quadrics of* PG$(3, q)$.

Proof. Let $\mathcal{F}$ be a flock of the quadratic cone $\mathcal{K}$ of PG$(3, q)$. Let V be the vertex of $\mathcal{K}$, P a point of $\mathcal{K} \setminus \{V\}$ and π a plane of PG$(3, q)$ such that $P, V \notin \pi$. If we project the elements of $\mathcal{F}$ from P onto π we obtain a set $\{\mathcal{C}_1, \dots, \mathcal{C}_{q-1}, m\}$ where $\mathcal{C}_1, \dots, \mathcal{C}_{q-1}$ are $q - 1$ conics and m is a line of π. The conics $\mathcal{C}_1, \dots, \mathcal{C}_{q-1}$ meet pairwise in $\infty = \langle V, P\rangle \cap \pi$ and with common tangent ℓ where $\langle \ell, V\rangle$ is the plane meeting $\mathcal{K}$ in the line $\langle V, P\rangle$. The line m is the intersection of π with the plane containing the element of $\mathcal{F}$ containing P. Hence the points of m are disjoint from the $\mathcal{C}_i$.

Now let π_∞ be a plane of $\mathrm{PG}(3, q)$ containing ℓ and distinct from π. Let R be any point of $\mathrm{PG}(3, q) \setminus (\pi \cup \pi_\infty)$ and π_R the plane $\langle m, R\rangle$. Then by Lemma 19 there is a unique set of elliptic quadrics $\mathcal{T} = \{\mathcal{O}_1, \dots, \mathcal{O}_{q-1}\}$ such that $\mathcal{O}_i \cap \pi = \mathcal{C}_i$, $i = 1, \dots, q-1$, R is contained in all of the $\mathcal{O}_i$ and π_∞ and π_R are tangent planes to all of the $\mathcal{O}_i$. Further, $\mathcal{O}_1, \dots, \mathcal{O}_{q-1}$ is a transversal of elliptic quadrics.

Let G be the group of elations of $\mathrm{PG}(3, q)$ with centre ∞ and define $\Theta = \{\mathcal{O}^g : \mathcal{O} \in \mathcal{T},\ g \in G\}$. We show that Θ is a tetradic set of elliptic quadrics. Since no elliptic quadric of $\mathcal{T}$ may be fixed by an element of G, it follows that $|\Theta| = q^3(q-1)$ and it suffices to show that each tetrad of $\mathrm{PG}(3, q)$ with respect to (∞, π_∞) is contained in at least one elliptic quadric of Θ.

Now let π' be a plane containing ∞ distinct from π_∞. Let $\mathcal{O}_i \cap \pi' = \mathcal{C}'_i$, $i = 1, \dots, q-1$ and $m' = \pi_R \cap \pi'$. Any element of G fixes the plane π' and induces an elation with centre ∞ in π'. No such elation in π' fixes a $\mathcal{C}'_i$ and also cannot map $\mathcal{C}'_i$ to $\mathcal{C}'_j$ for $i, j \in \{1, \dots, q-1\}$, $i \neq j$. Hence $\{(\mathcal{C}'_i)^g : i = 1, \dots, q-1,\ g \in G\}$ is a set of $q^2(q-1)$ conics in π' containing ∞ and with $\pi_\infty \cap \pi'$ as tangent. Hence every conic in π' containing ∞ and with $\pi_\infty \cap \pi'$ as tangent is contained in an element of Θ. In fact, taking images under the group of elations with centre ∞ and axis π' we get a set of q elliptic quadrics partitioning the points of $\mathrm{PG}(3, q) \setminus (\pi_\infty \cup \pi')$. It follows from this that every tetrad of $\mathrm{PG}(3, q)$ with respect to (∞, π_∞) is contained in an element of Θ and so Θ is a tetradic set with respect to (∞, π_∞). □

Theorem 21. *Every tetradic set of elliptic quadrics of* $\mathrm{PG}(3, q)$ *arises from a flock of a quadratic cone of* $\mathrm{PG}(3, q)$.

Proof. Suppose that Θ is a tetradic set of elliptic quadrics of $\mathrm{PG}(3, q)$ with respect to (∞, π_∞). Let $\mathcal{T} = \{\mathcal{O}_1, \dots, \mathcal{O}_{q-1}\}$ be a transversal of elliptic quadrics of Θ with base point R and base plane π_R. Let π be any plane containing ∞, but not R and distinct from π_∞. Consider the set $\{\mathcal{C}_1, \dots, \mathcal{C}_{q-1}, m\}$ where $\mathcal{C}_i = \mathcal{O}_i \cap \pi$, $i = 1, \dots, q-1$, and $m = \pi \cap \pi_R$. By Lemma 19 there is no pair $(\mathcal{C}_i, \mathcal{C}_j)$, $i, j \in \{1, \dots, q-1\}$, $i \neq j$, such that $\mathcal{C}_i$ is the image of $\mathcal{C}_j$ under an elation of π with centre ∞. Hence $\{\mathcal{C}_1, \dots, \mathcal{C}_{q-1}, m\}$ is a flock in the planar model of the quadratic cone of $\mathrm{PG}(3, q)$. □

6. Tetradic sets of elliptic quadrics and Property (G)

In this section we use the properties of a tetradic set of elliptic quadrics of $\mathrm{PG}(3, q)$, as established in Section 4, to prove the existence of a GQ of order (q, q^2) which we identify as a dual flock GQ. We characterise a GQ of order (s, s^2), s odd, satisfying Property (G) at a pair of points as a dual flock GQ. We also characterise a GQ of order (s, s^2), s even, satisfying Property (G) at a pair of points and with all associated ovoids elliptic quadrics, as a dual flock GQ.

The construction of the GQ follows Thas [17, Section 5], although there the structure is not proved directly to be a GQ, rather it is shown to be that arising from a dual flock GQ.

Theorem 22. *Let Θ be a tetradic set of elliptic quadrics of* $\mathrm{PG}(3,q)$ *with respect to the incident point-plane pair* (∞, π_∞). *Consider the following incidence structure* $\mathrm{GQ}(\Theta)$:

Points:
- (i) ∞.
- (ii) π_∞.
- (iii) *Equivalence classes of* Θ *under the action of elations with centre* ∞.
- (iv) *Points of* $\mathrm{PG}(3,q) \setminus \pi_\infty$.
- (v) *Planes of* $\mathrm{PG}(3,q)$ *not incident with* ∞.
- (vi) *Elements of* Θ.

Lines:
- (a) $[\infty]$.
- (b) *Lines of* $\mathrm{PG}(3,q)$, *not in* π_∞, *incident with* ∞.
- (c) *Lines of* π_∞ *not incident with* ∞.
- (d) *Rosettes of elliptic quadrics in* Θ.
- (e) *Triples* $(\mathcal{T}, \pi, X)$ *where* $\mathcal{T}$ *is a transversal of elliptic quadrics with distinct base point-base plane pairs* (∞, π_∞) *and* (X, π).

Incidence
- (i): *The point* ∞ *is incident with* $[\infty]$ *and all lines of type* (b).
- (ii): *The point* π_∞ *is incident with* $[\infty]$ *and all lines of type* (c).
- (iii): *An equivalence class E is incident with* $[\infty]$ *and all rosettes contained in E.*
- (iv): *The point* $X \in \mathrm{PG}(3,q) \setminus \pi_\infty$ *is incident with the line* $\langle X, \infty\rangle$ *of* $\mathrm{PG}(3,q)$, *and triples* $(\mathcal{T}, \pi, X)$ *where* $\mathcal{T}$ *is a transversal of ovoids in* Θ *with some base plane* $\pi \neq \pi_\infty$ *and corresponding base point X.*
- (v): *The plane* π, *not incident with* ∞, *is incident with* $\pi \cap \pi_\infty$ *and triples* $(\mathcal{T}, \pi, X)$ *where* $\mathcal{T}$ *is a transversal of elliptic quadrics in* Θ, *and* $(X, \pi) \neq (\infty, \pi_\infty)$ *is a base point-base plane pair of* $\mathcal{T}$.
- (vi): *The elliptic quadric* $\mathcal{O} \in \Theta$ *is incident with the rosette in* Θ *containing it and each triple* $(\mathcal{T}, \pi, X)$ *with* $\mathcal{O} \in \mathcal{T}$.

Then $\mathrm{GQ}(\Theta)$ *is a GQ of order* (q, q^2).

Proof. First we check the order of the incidence structure of $\mathrm{GQ}(\Theta)$. It is straightforward to check that each line of $\mathrm{GQ}(\Theta)$ is incident with $q+1$ points of $\mathrm{GQ}(\Theta)$. The lines are as follows. The line $[\infty]$ contains the points ∞, π_∞ and the $q-1$ equivalence classes. A line ℓ of $\mathrm{PG}(3,q)$, not in π_∞, incident with ∞ contains the point ∞ and the q affine points incident with ℓ. A line m of π_∞ not incident with ∞ contains the points π_∞ and the q planes through m. A line $\mathcal{R}$, a rosette of elliptic quadrics, has points the q ovoids in $\mathcal{R}$, and the equivalence class E containing $\mathcal{R}$. A line $(\mathcal{T}, \pi, X)$ has points X, π and the $q-1$ ovoids in the transversal $\mathcal{T}$.

The points ∞ and π_∞ are both clearly incident with q^2+1 lines. Each equivalence class E of Θ contains q^3 elliptic quadrics partitioned into q^2 rosettes. Together with $[\infty]$ this gives q^2+1 lines incident with E. If $X \in \mathrm{PG}(3,q) \setminus \pi_\infty$, then by Lemma 14 for each plane

π, $\infty \notin \pi$ and $X \in \pi$, there is a unique triple $(\mathcal{T}, \pi, X)$, where $\mathcal{T}$ is a transversal of elliptic quadrics in Θ. Together with the line $\langle \infty, X\rangle$ this gives $q^2 + 1$ incident lines. Similarly to the previous case, we see that a plane π, $\infty \notin \pi$ is incident with $q^2 + 1$ lines. If $\mathcal{O} \in \Theta$, then by Lemma 13 for each $X \in \mathcal{O} \setminus \{\infty\}$ there is a transversal of elliptic quadrics in Θ containing $\mathcal{O}$ and with base point X. Together with the unique rosette of elliptic quadrics in Θ containing $\mathcal{O}$ this gives $q^2 + 1$ incident lines.

We now check those cases of axiom (iii) of a GQ which are not straightforward. Suppose that E is an equivalence class of Θ and $\mathcal{T}$ a transversal of Θ. Then by Lemma 14 there is a unique elliptic quadric of $\mathcal{T}$ in E.

Suppose that X is a point of $\mathrm{PG}(3, q) \setminus \pi_\infty$. If $\mathcal{R}$ is a rosette of Θ not incident with X in $\mathrm{GQ}(\Theta)$, then since the ovoids of $\mathcal{R}$ partition the points of $\mathrm{PG}(3, q) \setminus \pi_\infty$ there is an ovoid $\mathcal{O} \in \mathcal{R}$ such that $X \in \mathcal{O}$. By Lemma 13 there is a unique transversal $\mathcal{T}$ containing $\mathcal{O}$ and with base point X, and if π_X is the tangent plane of $\mathcal{O}$ at X, then $(\mathcal{T}, \pi_X, X)$ is the unique line of $\mathrm{GQ}(\Theta)$ incident with X and meeting $\mathcal{R}$. Next suppose that $(\mathcal{T}, \pi', X')$ is not incident with X in $\mathrm{GQ}(\Theta)$. If $X \in \pi'$, then there is unique transversal with base point X and base plane π'. If $X \in \langle X', \infty\rangle$, then this is the unique line of $\mathrm{GQ}(\Theta)$ incident with X and meeting $(\mathcal{T}, \pi', X')$. Finally, if $X \notin \pi', \langle X', \infty\rangle$ we know that the elements of $\mathcal{T}$ partition the set of such points, so X is contained in a unique element of $\mathcal{T}$ and so is the base point of a unique transversal containing an element of $\mathcal{T}$.

Suppose that π is a plane of $\mathrm{PG}(3, q)$ not incident with ∞. Let $\mathcal{R}$ be a rosette of ovoids of Θ. Then $\mathcal{R}$ is generated by the action of elations with centre ∞ and axis π_∞ on any of the elliptic quadrics of $\mathcal{R}$. It follows that π is tangent to a unique element of $\mathcal{R}$ and so is the base plane of a unique transversal with an elliptic quadric in $\mathcal{R}$. Next suppose that $(\mathcal{T}, \pi', X)$ is not incident in $\mathrm{GQ}(\Theta)$ with π. If $X \in \pi$, then there is a unique transversal with base plane π and base point X. If $\pi \cap \pi' \subset \pi_\infty$, then this is the unique line of $\mathrm{GQ}(\Theta)$ incident with π and meeting $(\mathcal{T}, \pi', X)$. Finally, if $X \notin \pi$ and $\pi \cap \pi' \not\subset \pi_\infty$, then by Lemma 15 π is tangent to a unique elliptic quadric of $\mathcal{T}$ and hence there is a unique transversal with base plane π and containing an elliptic quadric of $\mathcal{T}$.

Now suppose that $\mathcal{O} \in \Theta$. Let $\mathcal{R}$ be a rosette of Θ not containing $\mathcal{O}$. If $\mathcal{O}$ is in the same equivalence class as the elliptic quadrics of $\mathcal{R}$, then the unique rosette containing $\mathcal{O}$ is the unique line of $\mathrm{GQ}(\Theta)$ incident with $\mathcal{O}$ and meeting $\mathcal{R}$. If $\mathcal{O}$ is inequivalent to the elements of $\mathcal{R}$, then by the proof of Corollary 11 there is unique elliptic quadric of $\mathcal{R}$ meeting $\mathcal{O}$ in exactly two points and hence by Lemma 13 contained in a transversal with $\mathcal{O}$. Next suppose $(\mathcal{T}, \pi, X)$ is not incident with $\mathcal{O}$ in $\mathrm{GQ}(\Theta)$. If $X \in \mathcal{O}$, then $\mathcal{O}$ is contained in a unique transversal with base point X and base plane distinct from π. Similarly, if π is a tangent plane to $\mathcal{O}$, then $\mathcal{O}$ is contained in a unique transversal with base plane π and base point distinct from X. Finally, suppose that $X \notin \mathcal{O}$ and that π is not a tangent plane to $\mathcal{O}$. Now $\pi \cap \mathcal{O}$ is a conic $\mathcal{C}$ not containing X and the line $\langle X, \infty\rangle$ of $\mathrm{PG}(3, q)$ contains a unique point Y of $\mathcal{O} \setminus \{\infty\}$ with $Y \neq X$. By Corollary 11 the $q - 1$ elliptic quadrics of $\mathcal{T}$ partition the $q^2 - q - 2$ points of $\mathcal{O} \setminus (\mathcal{C} \cup \{Y, \infty\})$ into sets of size $0, 1, q$ or $q + 1$. There are only two ways in which this may be done. First with one set of size 0 and $q - 2$ sets of size $q + 1$, in other words $\mathcal{O}$ is in a rosette with a unique element of $\mathcal{T}$ and in a transversal with none. Secondly, with one set of size 1, one set of size q and $q - 3$ sets of size $q + 1$, in other words $\mathcal{O}$ is in a transversal with a unique element of $\mathcal{T}$ and in a rosette with none. In either case there is a unique line of $\mathrm{GQ}(\Theta)$ incident with $\mathcal{O}$ and meeting $(\mathcal{T}, \pi, X)$. $\square$

Corollary 23. *Let Θ be a tetradic set of elliptic quadrics of* $\mathrm{PG}(3,q)$ *and* $\mathrm{GQ}(\Theta)$ *the associated GQ of order* (q,q^2). *Then* $\mathrm{GQ}(\Theta)$ *is a dual flock GQ.*

Proof. In [17, Section 7.2] Thas gives the construction of a transversal of elliptic quadrics $\{\mathcal{O}_1,\ldots,\mathcal{O}_{q-1}\}$ from a flock $\mathcal{F}$ that we have employed in the proof of Theorem 20. The elliptic quadrics in the same equivalence class as $\mathcal{O}_i$ are those elliptic quadrics of $\mathrm{PG}(3,q)$ that meet π_∞ in the same two lines as $\mathcal{O}_i$ over $\mathrm{GF}(q^2)$. This is the same as considering the images of $\mathcal{O}_i$ under elations with centre ∞ and hence the set constructed is a tetradic set Θ of elliptic quadrics. Also in [17, Section 7.2] Thas shows that the incidence structure $\mathrm{GQ}(\Theta)$ given in Theorem 22 is the dual flock GQ arising from $\mathcal{F}$. □

Corollary 24. *Let Θ be a tetradic set of ovoids of* $\mathrm{PG}(3,q)$ *with respect to* (∞,π_∞) *and* $GQ(\Theta)^*$ *the corresponding flock GQ. Then* $\mathrm{GQ}(\Theta)$ *is an EGQ with base point* $[\infty]^*$ *and the elation group about* $[\infty]^*$ *is induced by the group of* $\mathrm{PG}(3,q)$ *of order* q^5 *generated by all elations with centre ∞ and all elations with axis* π_∞.

Proof. Any collineation of $\mathrm{PG}(3,q)$ fixing the set Θ induces a collineation of $\mathrm{GQ}(\Theta)$ and by Lemma 16 the group G of collineations of $\mathrm{PG}(3,q)$ of order q^5 generated by all elations with centre ∞ and all elations with axis π_∞ fixes Θ. Calculation shows that this group fixes the equivalence classes of Θ and that for $\mathcal{O}\in\Theta$ the group $G_{\mathcal{O}}$ acts regularly on the points $\mathcal{O}\setminus\{\infty\}$. Hence the induced collineation of $\mathrm{GQ}(\Theta)$ fixes $[\infty]^*$ pointwise and fixes no point not collinear with $[\infty]^*$, and is thus an elation group about $[\infty]^*$. □

Theorem 25. *Let* $\mathcal{S}=(\mathcal{P},\mathcal{B},\mathrm{I})$ *be a GQ of order* (s,s^2) *satisfying Property* (G) *at a pair of collinear points* (X,Y). *If s is odd, then $\mathcal{S}$ is the dual of a flock GQ. If s is even and all ovoids* $\mathcal{O}_Z$ *of* $\mathcal{S}_{XY}$ *for* $Z\in\mathcal{P}\setminus(X^\perp\cup Y^\perp)$ *are elliptic quadrics, then we have the same conclusion.*

Proof. Let $\overline{\mathcal{S}_{XY}}$ be the projective three-space constructed from the pair (X,Y). Hence s is a prime power q. Let Θ be the set of ovoids in $\overline{\mathcal{S}_{XY}}\cong\mathrm{PG}(3,q)$ associated with $\mathcal{S}$. If q is odd, then Θ is a set of elliptic quadrics, while if q is even, then this is also the case by hypothesis. We will show that Θ is a tetradic set of elliptic quadrics in $\mathrm{PG}(3,q)$.

The points of $XY\setminus\{X,Y\}$ divide the elliptic quadrics of Θ into $q-1$ equivalence classes. Two elliptic quadrics in the same equivalence class intersect in either 1 or $q+1$ points, while two elliptic quadrics of Θ in distinct equivalence classes intersect in either 2 or $q+2$ points. We will show that two elliptic quadrics in different equivalence classes cannot have an intersection containing a conic.

Without loss of generality suppose that the elliptic quadrics of Θ have common point $\infty=(0,1,0,0)$ and common tangent plane $X_0=0$. If q is odd let $\mathcal{O}\in\Theta$ with $\mathcal{O}=\{(1,s^2-\eta t^2,s,t)\,:\,s,t\in\mathrm{GF}(q)\}\cup\{(0,1,0,0)\}$, where η is a fixed non-square in $\mathrm{GF}(q)$. Let $\mathcal{C}\subset\mathcal{O}$ be the conic $\mathcal{C}=\{(1,s^2,s,0)\,:\,s\in\mathrm{GF}(q)\}\cup\{(0,1,0,0)\}$. Let $\mathcal{O}'$ be a second elliptic quadric containing $\mathcal{C}$. By the proof of Lemma 7 we may assume that $\mathcal{O}'=\{(1,s^2-\eta t^2+bt,s+ct,dt)\,:\,s,t\in\mathrm{GF}(q)\}\cup\{\infty\}$ for $b,c,d\in\mathrm{GF}(q)$, $d\neq 0$. It follows that $|\mathcal{O}\cap\mathcal{O}'|=k+1$ where k is the number of solution pairs (s,t) to

$$2cst=\eta d^2t^2-c^2t^2-\eta t^2+bt. \tag{1}$$

Any $(s, 0)$ is a solution pair corresponding to the points $\mathcal{C} \setminus \{(0, 1, 0, 0)\}$. If $\mathcal{O}' \in \Theta$ and is inequivalent to $\mathcal{O}$, then it must be that $|\mathcal{O} \cap \mathcal{O}'| = q + 2$ and there is a unique solution to (1) with $t \neq 0$. Under the assumption that $t \neq 0$, (1) becomes $2cs = \eta d^2 t - c^2 t - \eta t + b$. If $c = 0$ we have $\eta d^2 t - \eta t + b = 0$, to which there are either no solutions, or a unique solution for t and q solution pairs (s, t), or all (s, t) are solutions. In each of these cases we do not have a unique solution pair (s, t), so we may suppose that $c \neq 0$. Hence we may write

$$s = \frac{\eta d^2 t - c^2 t - \eta t + b}{2c},$$

which yields a solution in s for each choice of t, that is q solution pairs in total.

If q is even let $\mathcal{O} = \{(1, s^2 + st + \rho t^2, s, t) : s, t \in \mathrm{GF}(q)\} \cup \{\infty\}$ with ρ a fixed element of $\mathrm{GF}(q)$ such that $\mathrm{Tr}(\rho) = 1$. Let $\mathcal{C} \subset \mathcal{O}$ be the conic $\{(1, s^2, s, 0) : s \in \mathrm{GF}(q)\} \cup \{\infty\}$. Let $\mathcal{O}'$ be a second elliptic quadric containing $\mathcal{C}$ which, by the proof of Lemma 7, we may assume is $\mathcal{O}' = \{(1, s^2 + st + \rho t^2 + bt, s + ct, dt) : s, t \in \mathrm{GF}(q)\} \cup \{\infty\}$ for $b, c, d \in \mathrm{GF}(q)$, $d \neq 0$. For $|\mathcal{O} \cap \mathcal{O}'| = q + 2$ we need a unique solution to $(\rho + c^2 + cd + \rho d^2)t + (d + 1)s + b = 0$ with $t \neq 0$. However, the existence of one such solution implies the existence of at least q such solutions.

Now the group of an elliptic quadric is transitive on pairs $(P, \mathcal{C})$ where $\mathcal{C}$ is conic section of the elliptic quadric and $P \in \mathcal{C}$. Hence we may conclude that if two elliptic quadrics of Θ contain a common conic, then they are in the same equivalence class.

Let $\mathcal{C}$ be a conic in the plane π containing ∞ such that $\pi \neq \pi_\infty$ and $\pi \cap \pi_\infty$ is a tangent to $\mathcal{C}$. Then $\mathcal{C}$ is contained in at most q elements of Θ since in $\mathcal{S}$ the set of points $\mathcal{C} \setminus \{X\}$ contains a triad and so has at most $q + 1$ centres, one of which is X. Counting the number of such conics and noting that $|\Theta| = q^3(q - 1)$ we conclude that each such conic is contained in exactly q elliptic quadrics of Θ.

Now let $\{A, B, C, \infty\}$ be a 4-arc in a plane π of $\mathrm{PG}(3, q)$ such that $\pi \neq \pi_\infty$ and $A, B, C \notin \pi_\infty$. The set $\{A, B, C, \infty\}$ uniquely determines a conic $\mathcal{C}$ in π with tangent $\pi \cap \pi_\infty$. The conic $\mathcal{C}$ is contained in q elliptic quadrics $\mathcal{O}_1, \ldots, \mathcal{O}_q$ of Θ all of which must be in the same equivalence class. Hence $\mathcal{O}_1, \ldots, \mathcal{O}_q$ intersect in exactly $\mathcal{C}$ and partition the points of $\mathrm{PG}(3, q) \setminus (\pi_\infty \cup \pi)$. Thus for any 5-cap $\{A, B, C, Z, \infty\}$ of $\mathrm{PG}(3, q)$ with $A, B, C, Z \notin \pi_\infty$, $\infty \in \langle A, B, C\rangle$ and $Z \notin \langle A, B, C\rangle$, there is a unique elliptic quadric on $\{A, B, C, Z\}$. Hence Θ is a tetradic set of elliptic quadrics of $\mathrm{PG}(3, q)$ with respect to (∞, π_∞) and by Corollary 23 $\mathcal{S}$ is the corresponding dual flock GQ. $\square$

References

[1] W. Benz, Vorlesungen über Geometrie der Algebren, Springer, Berlin-New York, 1973.

[2] R.C. Bose, S.S. Shrikhande, Geometric and pseudo-geometric graphs $(q^2 + 1, q + 1, q)$, J. Geom. 2/1 (1973) 74–94.

[3] A. Delandtsheer, Dimensional linear spaces, in: F. Buekenhout (Ed.), Handbook of Incidence Geometry, Elsevier, Amsterdam, 1995(Chapter 6).

[4] H.R. Halder, W. Heise, Einführung in die Kombinatorik, Carl Hanser, Munich-Vienna, 1976.

[5] J.W.P. Hirschfeld, Finite Projective Spaces of Three Dimensions, Oxford University Press, New York, 1985.

[6] J.W.P. Hirschfeld, Projective Geometries Over Finite Fields, Oxford University Press, New York, 1998.

[7] J.W.P. Hirschfeld, J.A. Thas, General Galois Geometries, Oxford University Press, New York, 1991.

[8] N. Johnson, S.E. Payne, Flocks of Laguerre planes and associated geometries, Mostly finite geometries (Iowa City, IA, 1996), Lecture Notes in Pure and Applied Mathematics, vol. 190, Dekker, New York, 1997, pp. 51–122.
[9] W.M. Kantor, Some generalized quadrangles with parameters (q^2, q), Math. Z. 192 (1986) 45–50.
[10] N. Knarr, A geometric construction of generalized quadrangles from polar spaces of rank three, Reultate Math. 21 (1992) 332–334.
[11] S.E. Payne, A new infinite family of generalized quadrangles, Congr. Numer. 49 (1985) 115–128.
[12] S.E. Payne, An essay on skew translation generalized quadrangles, Geom. Dedicata 32 (1989) 93–118.
[13] S.E. Payne, J.A. Thas, Finite Generalized Quadrangles, Pitman, Boston, MA, 1984.
[14] G.F. Steinke, Topological circle geometries, in: F. Buenkenhout (Ed.), Handbook of Incidence Geometry, Elsevier, Amsterdam, 1995(Chapter 24).
[15] J.A. Thas, Generalized quadrangles and flocks of cones, European J. Combin. 8 (1987) 441–452.
[16] J.A. Thas, Generalized quadrangles of order (s, s^2) I, J. Combin. Theory Ser. A 67 (1994) 140–160.
[17] J.A. Thas, Generalized quadrangles of order (s, s^2), III, J. Combin. Theory Ser. A 87 (1999) 247–272.
[18] B.L. van der Waerden, L.J. Smid, Eine Axiomatic der Kreisgeometrie und der Laguerre-Geometrie, Math. Ann. 110 (1907) 379–388.