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Partial realization for singular systems in standard form

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Abstract

The partial realization problem under consideration consists in finding, for a given sequence $\mathbf{s} = (s_k)_0^{N-1}$ of blocks, matrices (A, E, B, C) of appropriate size such that $s_i = CE^{N-1-i}A^iB$ and the identity matrix is a linear combination of A and E. We discuss the question whether there is always a realization of this form for which the state space dimension is equal to the maximal rank of the underlying Hankel matrices. We show that this question has an affirmative answer if the block size is less than or equal to 2 and some other cases but not in general. The paper strengthens results obtained by Manthey et al. [cf. W. Manthey, U. Helmke, D. Hinrichsen, in: U. Helmke et al. (Eds.), Operators, Systems, and Linear Algebra, Teubner, Stuttgart, 1997, pp. 138–156]. The main tools are the results of the authors obtained in connection with Vandermonde factorization of block Hankel matrices. Finally, an interpretation of the problem in periodic discrete-time systems is given.

Keywords: Linear system; Singular system; Partial realization; Hankel matrix

1. Introduction

The minimal realization problem for linear, time-invariant, discrete-time systems is to find, for a given sequence of blocks of complex numbers[1] $\mathbf{s} = (s_k)_0^\infty$, a system

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[1] All considerations can be extended to matrices over an algebraically closed field.

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$$\begin{aligned} x_{k+1} &= Ax_k + Bu_k \\ y_k &= Cx_k \end{aligned} \quad (k = 0, 1 \ldots) \tag{1.1}$$

with minimal state space dimension, i.e., with minimal size of A, such that $s_k = CA^kB$ ($k = 0, 1, \ldots$). Note that $\mathbf{s}$ is the impulse response of the system. According to Kalman's theorem such a system exists if and only if the rank of the infinite block Hankel matrix $H = [s_{i+j}]$ is finite and, moreover, the minimal state space dimension is equal to the rank of H.

The minimal partial realization problem is to find a system (1.1) with minimal state space dimension from a finite sequence $\mathbf{s} = (s_k)_0^{N-1}$ of blocks, i.e., a triple of matrices (A, B, C) with minimal size such that $s_k = CA^kB$ for $k = 0, \ldots, N-1$. A solution of the problem was presented by Kalman in [13,14] (see also [1,4,6,7,12]). In the solution the block Hankel matrices

$$H_k(\mathbf{s}) = \begin{bmatrix} s_0 & \cdots & s_{k-1} \\ \vdots & & \vdots \\ s_{l-1} & \cdots & s_{N-1} \end{bmatrix} \tag{1.2}$$

$(k + l = N + 1)$ play a significant role. They are the products of the observability and controllability matrices $\mathrm{col}[CA^i]_{i=0}^{l-1}$ and $\mathrm{row}\,[A^jB]_{j=0}^{k-1}$. Therefore, the state space dimension n of a minimal partial realization satisfies the estimation $n \geqslant \rho(\mathbf{s})$, where

$$\rho(\mathbf{s}) := \max\{\mathrm{rank}\ H_k(\mathbf{s}):\ k = 1, \ldots, N\}. \tag{1.3}$$

A natural question is whether equality can be achieved, as for the complete partial realization problem. This is equivalent to the question whether $H_k(\mathbf{s})$ has a rank preserving extension to an infinite Hankel matrix. For some cases the answer is "yes" but in general "no". For example, for $\mathbf{s} = (0, 0, 1)$ the minimal state space dimension equals 3 whereas $\rho(\mathbf{s}) = 1$.

It is natural to seek a possibility to fill the gap between $\rho(\mathbf{s})$ and the minimal state space dimension. The first attempt in this direction was made, as far as we know, by Manthey et al. [16]. In this paper it is proposed to consider the partial realization problem for singular systems

$$\begin{aligned} Ex_{k+1} &= Ax_k + Bu_k \\ y_k &= Cx_k \end{aligned} \quad (k = 0, 1, \ldots), \tag{1.4}$$

where (1.4) is assumed to be in *standard form*. The latter means that the identity matrix is a linear combination of A and E. In this case we have in particular $AE = EA$, and $\det(sE - A)$ does not vanish identically. Systems of this form appear in connection with two-point boundary-value descriptor systems and were studied in [17]. Another motivation to consider such systems arises from the matrix generalization of the Waring problem for binary forms (see [10]).

However, let us point out that for singular systems (1.4) the blocks $CE^{N-1-k}A^kB$ cannot be simply interpreted as the impulse response of the system in the usual sense. We show that, nevertheless, a system's theoretic interpretation is possible as impulse response of periodic systems.

In [16], system (1.4) is called *generalized partial realization* of $\mathbf{s} = (s_k)_0^{N-1}$ if $s_k = CE^{N-1-k}A^kB$ for $k = 0, \dots, N-1$. The main result in [16] is that $\mathbf{s}$ has a generalized partial realization with state space dimension $\rho(\mathbf{s})$ if for certain k the condition

$$\operatorname{rank} H_k(\mathbf{s}) < \min\{k, N+1-k\} \tag{1.5}$$

is fulfilled. For the scalar case $p = q = 1$ this result gives an affirmative answer to the question whether there exists a partial realization with dimension $\rho(\mathbf{s})$. [2] However, in the block case condition (1.5) is very restrictive and the question is whether it can be removed. The aim of the present paper is to discuss this problem.

The motivation in writing this paper was actually the observation that results on canonical representation and affine Vandermonde factorization of block Hankel matrices of the authors presented in [5] can be interpreted in the language of generalized partial realization. The state space dimension of a minimal partial realization will be characterized as the sum of all so-called "regular" degrees of $\mathbf{s}$. The regular degrees can be obtained from the kernels of the matrices $H_k(\mathbf{s})$. From this general result we conclude that a partial realization with state space dimension $\rho(\mathbf{s})$ exists if

$$\rho(\mathbf{s}) \leqslant N,$$

which is weaker than (1.5) but still a rather restrictive condition. But at least it comes out that for 2×2 blocks s_k there always exists a generalized partial realization with state space dimension $\rho(\mathbf{s})$. On the other hand, we show that for 2×3 blocks this fails to be true.

The paper is built as follows. In Section 2, we introduce some basic concepts on singular systems. The material is taken mainly from [2,16]. The main observation is that the general partial realization problem can be reduced to that one in the class $\mathscr{S}(\alpha)$ consisting of all systems with $\alpha E - A = \xi I_n$ for some $\xi \in \mathbb{C}$, $\xi \neq 0$. In Section 3, we discuss the Möbius transformations of systems (1.4), which are the main tools to reduce the generalized partial realization problem for system (1.4) to the classical one. It is shown that Möbius transformations of systems (1.4) are related to Frobenius–Fischer transformations of the sequence of Markov parameters $\mathbf{s}$. Some of the results in these sections can be found similar to those in [15,16] (see also [9,15]). In Section 4, we present shortly the approach of [8] for the solution of the classical minimal partial realization problem in an appropriate form for us.

Using the results of Sections 3 and 4 we describe in Section 5 the general solution of the minimal partial realization problem in the class $\mathscr{S}(\alpha)$. The main results of the paper are presented in Section 6. Section 7 is dedicated to the discussion on the relation between partial realizations and Vandermonde factorization of block Hankel matrices, and in Section 8 we give an interpretation of the quantities $CE^{N-1-k}A^kB$ in the framework of periodic systems.

[2] Note that the scalar result follows also from some earlier results on canonical representations of Hankel matrices. A discussion of this is provided in Section 7.

2. Singular systems

Throughout this paper we consider quadruples of matrices $\Sigma = (A, E, B, C)$, where $A, E \in \mathbb{C}^{n\times n}$, $B \in \mathbb{C}^{n\times q}$, $C \in \mathbb{C}^{p\times n}$, $AE = EA$ and $\det(\lambda E - A)$ does not vanish identically. We will identify the quadruple Σ with system (1.4). In case that $E = I_n$ the system Σ will be called *regular*.

We denote by $\sigma(\Sigma)$ the set of all λ for which $\lambda E - A$ is singular, and we include $\lambda = \infty$ to $\sigma(\Sigma)$ if E is singular. $\sigma(\Sigma)$ is called *spectrum* of Σ. The spectrum of Σ contains the set of the poles of the transfer function

$$F_\Sigma(\lambda) = C(\lambda E - A)^{-1}B.$$

For a fixed *N*, the blocks

$$s_k = CE^{N-1-k}A^kB \quad (k = 0, \ldots, N-1)$$

will be called *N-Markov parameters* of (1.4). If system (1.4) has the *N*-Markov parameters s_k, then it is called (generalized) *partial realization* of $\mathbf{s} = (s_k)_0^{N-1}$ and *n* is said to be its *state space dimension*. If *n* is minimal, then the partial realization is said to be *minimal*. The partial realizations by a regular system will be referred to as *regular* or *classical*.

Let us point out that there is an essential difference between regular and general singular systems (1.4). Two similar regular systems have the same Markov parameters. This is not true for singular systems. Recall that two systems (A, E, B, C) and $(\tilde{A}, \tilde{E}, \tilde{B}, \tilde{C})$ are said to be *equivalent* if there exist nonsingular matrices *P* and *Q* such that

$$\tilde{A} = PAQ, \quad \tilde{E} = PEQ, \quad \tilde{B} = PB, \quad \tilde{C} = CQ.$$

Two equivalent systems have the same transfer function and the same input–output behavior.

For singular systems another type of equivalence, which was introduced in [16], is important. Two systems $\Sigma = (A, E, B, C)$ and $\Sigma = (\tilde{A}, \tilde{E}, \tilde{B}, \tilde{C})$ are said to be *equivalent modulo N-scaling* if there exist a nonsingular matrix *S*, numbers α and β such that $\alpha E + \beta A$ is nonsingular and natural numbers *k* and *l* with $k + l = N + 1$ such that

$$\begin{aligned} \tilde{A} &= S(\alpha E + \beta A)^{-1}AS^{-1}, & \tilde{E} &= S(\alpha E + \beta A)^{-1}ES^{-1}, \\ \tilde{B} &= S(\alpha E + \beta A)^{k-1}B, & \tilde{C} &= C(\alpha E + \beta A)^{l-1}S^{-1}. \end{aligned} \tag{2.1}$$

Note that the two systems which are equivalent modulo *N*-scaling are not necessarily equivalent. In fact, if (2.1) holds, then the transfer functions of $\tilde{\Sigma}$ in terms of (A, E, B, C) is given by

$$F_{\tilde{\Sigma}}(\lambda) = C(\alpha E + \beta A)^N(\lambda E - A)^{-1}B,$$

which is different to $F_\Sigma(\lambda)$.

The following can be easily checked (for details see [16]).

Proposition 1. *If two systems are equivalent modulo N-scaling, then they have the same N-Markov parameters.*

For $\alpha \in \mathbb{C}$ a system is said to be in α-*standard form* if $\alpha E - A = \xi I_n$ for some $\xi \in \mathbb{C}$, $\xi \neq 0$. Systems with $E = \xi I_n$ are said to be in ∞-*standard form*. The class of all systems (1.4), α-standard form, is denoted by $\mathscr{S}(\alpha)$ $(\alpha \in \mathbb{C} \cup \{\infty\})$.

Proposition 2. *A system* (1.4) *is equivalent modulo N-scaling to a system in α-standard form for all $\alpha \notin \sigma(\Sigma)$. More precisely, if Σ is given by* (1.4) *and $\alpha \notin \sigma(\Sigma) \cup \{\infty\}$, then the system $(E(\alpha E - A)^{-1}, (A(\alpha E - A)^{-1}, B, C(\alpha E - A)^{N-1})$ is equivalent modulo N-scaling to Σ and is in α-standard form. If $\infty \notin \sigma(\Sigma)$, then $(I_n, AE^{-1}, B, CE^{N-1})$ is equivalent modulo N-scaling to Σ and is in ∞-standard form.*

In view of Proposition 2 we may state the partial realization problem under consideration in the following form.

Problem MPR(α). For given $\mathbf{s}$, find all systems $\Sigma \in \mathscr{S}(\alpha)$ which are minimal partial realizations of $\mathbf{s}$.

We are particularly interested in the minimal state space dimension. Let us denote this dimension by $d(\alpha)$. It follows from Proposition 2 that $d(\alpha)$ is a constant d for all $\alpha \in \mathbb{C} \cup \{\infty\}$ except for a finite set Λ. For $\alpha \in \Lambda$ we have $d(\alpha) > d$.

3. Möbius transformations

Throughout this section, let

$$\phi = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$$

be a nonsingular 2×2 matrix. Then we will use the fact that the group $\mathrm{GL}(\mathbb{C}^2)$ of these matrices is generated by the matrices

$$\begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 \\ b & 1 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}. \tag{3.1}$$

We associate ϕ with linear fractional function (Möbius transformation) on the Riemann sphere,

$$\phi(\lambda) = \frac{a\lambda + b}{c\lambda + d}\ .$$

We have $(\phi\psi)(\lambda) = \psi(\phi(\lambda))$, where $\phi\psi$ is the product of the two matrices.

The matrix ϕ corresponds to the transformation of linear systems $\Sigma = (A, E, B, C)$ defined by

$$\Sigma(\phi) = (aA + bE, cA + dE, B, C). \tag{3.2}$$

We call $\Sigma(\phi)$ the *ϕ-Möbius transformed* of Σ. It is easily checked that for two non-singular matrices ϕ and ψ

$$\Sigma(\phi\psi) = (\Sigma(\phi))(\psi). \tag{3.3}$$

Proposition 3. *The transformation $\Sigma \to \Sigma(\phi)$ maps the class $\mathscr{S}(\alpha)$ onto the class $\mathscr{S}(\phi(\alpha))$.*

Proof. It is easily checked that for $\alpha \in \mathbb{C}$ the relation

$$(a\alpha + b)(cA + dE) - (c\alpha + d)(aA + bE) = \delta(\alpha E - A)$$

holds where $\delta = ad - bc$. Now if $\alpha, \phi(\alpha) \neq \infty$ and $\alpha E - A = \xi I_n$, then we obtain

$$\phi(\alpha)(cA + dE) - (aA + bE) = \frac{\delta\xi}{c\alpha + d} I_n.$$

Hence, $\Sigma(\phi) \in \mathscr{S}(\phi(\alpha))$. In case that $\phi(\alpha) = \infty$, $\alpha \neq \infty$, we have $c\alpha + d = 0$ and

$$cA + dE = \frac{\delta\xi}{a\alpha + b} I_n.$$

Thus, $\Sigma(\phi) \in \mathscr{S}(\infty)$. Now let $\alpha = \infty$, $\phi(\alpha) \neq \infty$. Then $f\phi(\alpha) = a/c$, and from the relation

$$a(cA + dE) - c(aA + bE) = \delta E = \delta\xi I_n$$

we conclude that $\phi(\infty)(cA + dE) - (aA + bE) \in \mathscr{S}(\phi(\infty))$. That means $\Sigma(\phi) \in \mathscr{S}(\phi(\alpha)$. For $\alpha = \phi(\alpha) = \infty$ the assertion is obvious. Clearly, all transformations are “onto”. □

We now show the transfer functions transfer via Möbius transformations.

Proposition 4. *The transfer function of the ϕ-Möbius transformed $\Sigma(\phi)$ of Σ is given by*

$$F_{\Sigma(\phi)}(\lambda) = \frac{1}{a - c\lambda} F_\Sigma\big(\phi^{-1}(\lambda)\big) \tag{3.4}$$

or

$$F_\Sigma(\lambda) = \frac{\delta}{c\lambda + d} F_{\Sigma(\phi)}(\phi(\lambda)), \tag{3.5}$$

where $F_\Sigma(\lambda)$ is the transfer function of Σ.

Proof. We have

$$F_\Sigma(\lambda) = C(\lambda(cA + dE) - (aA + bE))^{-1} B$$

$$
\begin{aligned}
&= C((d\lambda - b)E - A(a - c\lambda))^{-1}B \\
&= \frac{1}{a - c\lambda}\, C\left(\frac{d\lambda - b}{-c\lambda + a}\, E - A\right)^{-1} B.
\end{aligned}
$$

In view of $\phi^{-1}(\lambda) = (d\lambda - b)/(-c\lambda + a)$ we get (3.4). Replacing λ by $\phi(\lambda)$ we obtain (3.5). □

For fixed ϕ, let $M_{q,m}(\phi)$ denote the $m \times m$ block matrices with $q \times q$ block entries such that the jth column equals the coefficient vector of $\phi(\lambda)^j (c\lambda + d)^{m-1} I_q$. The matrix $M_{q,m}(\phi)$ generates the linear transformation in the space of q-vector polynomials with degree $\leqslant m - 1$ defined by

$$
(M_{q,n}(\phi)x)(\lambda) = x(\phi(\lambda))(c\lambda + d)^{m-1},
$$

which will be identified with the matrix. Since q is always fixed we omit the subscript q. If $m = N$, then we omit the subscript m also. The transformations corresponding to $M(\phi)$ will be called *Frobenius–Fischer transformations*.[3] For Frobenius–Fischer transformations the relation

$$
M(\phi\psi) = M(\phi)M(\psi) \tag{3.6}
$$

holds, where $\phi\psi$ is the product of the two matrices. This implies $M(\phi)^{-1} = M(\phi^{-1})$. Thus, the Frobenius–Fischer transformations form a group isometric to $\mathrm{GL}(\mathbb{C}^2)$.

For a given system Σ, let $\mathrm{mark}_N(\Sigma)$ denote the N-Markov parameters written as a block row matrix. Then the following proposition is crucial for the sequel.

Proposition 5. *The Markov parameter of a system Σ of form* (1.4) *and its ϕ-Möbius transformation are related according to*

$$
\mathrm{mark}_N(\Sigma(\phi)) = \mathrm{mark}_N(\Sigma)M(\phi).
$$

Proof. In view of (3.6) and (3.3) it is sufficient to prove the proposition for generators (3.1). For these cases the assertion is immediately checked. □

4. Fundamental systems and classical partial realization

In this section, we describe shortly the solution of the classical realization problem in the form as it was presented in [8] and characterize those $\mathbf{s}$ for which there exists a regular realization with state space dimension $\rho(\mathbf{s})$. Speaking about partial realization in this section we always mean regular partial realizations. We consider

[3] In [5,9], we called them also *Möbius transformations*. To distinguish them from the classical Möbius transformations we prefer here the present notation, following [16] (see also [11]).

the block Hankel matrices $H_k(\mathbf{s})$. Let $\mathscr{H}_k$ denote the set of all q-vector polynomials $u(\lambda)$ of degree less than k for which the coefficient vector belongs to the kernel of $H_k(\mathbf{s})$. Then $\mathscr{H}_k \subseteq \mathscr{H}_{k+1}$ and $\lambda\mathscr{H}_k \subseteq \mathscr{H}_{k+1}$ for all k.

In [8] it is shown that there exists a system of nonnegative integers d_i, $i = 1, \ldots, t$, where $q \leqslant t \leqslant p + q$, and a system of vector polynomials $u_i \in \mathscr{H}_{d_i+1} \backslash \mathscr{H}_{d_i}$ $(i = 1, \ldots, t)$ such that the system

$$u_i, \lambda u_i, \ldots, \lambda^{k-d_i-1} u_i, \tag{4.1}$$

where i run over all indices with $k > d_i$, forms a basis of $\mathscr{H}_k$ $(k = 1, \ldots, N + 1)$. The integers d_i are uniquely determined by $\mathbf{s}$. In case that $t < p + q$ we put $d_i = N + 1$ for $i = t + 1, \ldots, p + q$. With this definition we have

$$\sum_{i=1}^{p+q} d_i = (N + 1)p. \tag{4.2}$$

The integers d_i are called (*right*) *characteristic degrees of* $\mathbf{s}$. A system of vector polynomials u_i for which system (4.1) is a basis of the subspaces $\mathscr{H}_k(\mathbf{s})$ is said to be a (*right*) *fundamental system of* $\mathbf{s}$.

Fundamental systems can be constructed via recursions $N - 1 \to N$. This leads to algorithms of Schur and Levinson type which are similar to the algorithms for Hankel matrix inversion and have $O(N^2)$ complexity or less (see [1,7]).

The following proposition is not explicitly formulated in [8] but it follows immediately from the results of this paper.

Proposition 6. *Let* $\{d_i\}$ *be the characteristic degrees of* $\mathbf{s}$ *in nondecreasing order. Then for all k satisfying* $d_q \leqslant k \leqslant d_{q+1}$

$$\operatorname{rank} H_k(\mathbf{s}) = \rho(\mathbf{s}) = \sum_{i=1}^{q} d_i.$$

Proof. Since a fundamental system generates bases for all $\mathscr{H}_k$ according to (4.1) we have

$$\dim \mathscr{H}_k = \sum_{d_i \leqslant k} (k - d_i).$$

For k satisfying our assumption we have, in particular,

$$\dim \mathscr{H}_k = qk - \sum_{i=1}^{q} d_i,$$

which implies rank $H_k(\mathbf{s}) = \sum_{i=1}^{q} d_i$.

It remains to show that the maximal rank is attained for these k. Suppose that $\alpha_k = \dim \mathscr{H}_k$ and $r_k = \operatorname{rank} H_k(\mathbf{s})$. Since (4.1) is a basis of $\mathscr{H}_k$ we have

$$\alpha_{k+1} - \alpha_k = \#\{j\colon\ d_j \leqslant k\},$$

where $\#M$ means the number of elements in M. Hence,

$$r_{k+1} - r_k = q - \#\{j\colon d_j \leqslant k\}.$$

From this relation we conclude that the maximal rank is achieved for all k satisfying $d_q \leqslant k \leqslant d_{q+1}$. $\square$

For the solution of the minimal partial realization problem two kinds of characteristic degrees have to be distinguished. Let $u \in \mathscr{H}_k(\mathbf{s})$. Then u is said to be *proper* or *regular at* ∞ if the leading coefficient of u is nonzero, otherwise *improper*. In [8] it is shown that there exists a fundamental system consisting of exactly q proper and $\leqslant p$ improper vector polynomials. The characteristic degrees corresponding to the proper polynomials are uniquely determined by $\mathbf{s}$.

A fundamental system satisfying this condition will be called *canonical* (at ∞). The characteristic degrees corresponding to the proper/improper elements are said to be the *proper/improper characteristic degrees*.

In order to describe the relation between the concept of a fundamental system and the partial realization problem we still need the concept of residual, which is defined as follows. We denote $s(\lambda) = \sum_{i=0}^{N-1} s_i \lambda^i$. Suppose that $u(\lambda) = \sum_{i=0}^{d} u_i \lambda^i \in \mathscr{H}_{d+1}$. Then

$$s\big(\lambda^{-1}\big)\lambda^{-1}u(\lambda) = w(\lambda) + \lambda^{-(N+1-d)}\beta\big(\lambda^{-1}\big),$$

where $w(\lambda)$ and $\beta(\lambda)$ are vector polynomials, $w(\lambda) = \sum_{i=0}^{d-1} w_i \lambda^i$. The polynomial $w(\lambda)$ is called *residual* of $u(\lambda)$ (at ∞).

Let a canonical fundamental system of $\mathbf{s}$ be given. From the q proper elements we form a $q \times q$ matrix polynomial $U_0(\lambda)$. Let the characteristic degree of the jth column be d_j. Then we take all improper elements of this system the characteristic degree of which does not exceed the largest proper degree (if there are any) and form the $q \times r$ $(0 \leqslant r \leqslant p)$ matrix polynomial U_1. Let the characteristic degree of the ith column of U_1 be d_i'. The $p \times q$ and $p \times r$ matrix polynomial formed by the corresponding residuals will be denoted by W_0 and W_1, respectively. Then the following is true (cf. [8]).[4]

Theorem 7. *The general form of the transfer function of a minimal partial realization of* $\mathbf{s}$ *by a regular system is given by*

$$F(\lambda) = (W_0 + W'Z)(U_0 + U'Z)^{-1},$$

where $Z = [z_{ij}]_{i=1}^{r}{}_{j=1}^{q}$ *is an arbitrary* $r \times q$ *matrix polynomial the entries* z_{ij} *of which satisfy*

$$\deg z_{ij} \leqslant d_j - d_i'$$

if $d_j \geqslant d_i'$, *and* $z_{ij} = 0$ *otherwise. In particular, the dimension of the minimal partial realization is equal to the sum of the proper characteristic degrees of* $\mathbf{s}$.

[4] Actually in [8] the theorem is formulated in a slightly different form.

Corollary 8. *There exists a regular partial realization of* **s** *with state space dimension* $\rho(\mathbf{s})$ *if and only if the largest proper degree is less than or equal to the smallest improper one.*

The condition in Corollary 8 can also be formulated as a rank condition, as it was done in [6]. For this we set $\mathbf{s}' = (s_k)_0^{N-1}$, i.e., $\mathbf{s}'$ is obtained from **s** by cancelling the last component. Then the largest proper degree is not larger than the smallest improper degree if and only if

$$\operatorname{rank} H_{k-1}(\mathbf{s}') = \operatorname{rank} H_k(\mathbf{s})$$

for some k with $\operatorname{rank} H_k(\mathbf{s}) = \rho(\mathbf{s})$ (see [8]).

It can be checked that different choices of Z in Theorem 7 will provide different transfer functions. This follows from the fact that the matrix

$$\begin{bmatrix} W_0 & W' \\ U_0 & U' \end{bmatrix}$$

can be extended to a $(p+q) \times (p+q)$ unimodular matrix polynomial. Therefore, the following is true.

Corollary 9. *The transfer function of a minimal partial realization by of* **s** *is unique* (*which means the the partial realization is unique up to similarity*) *if and only if the largest proper characteristic degree is smaller than the smallest improper one.*

If **s** is such that the Hankel matrix $H_n(\mathbf{s})$ is nonsingular, then all characteristic degrees of **s** are equal to n. That means there exist infinitely many nonsimilar partial realizations of **s** by regular systems with state space dimension $\rho(\mathbf{s})$, which is equal to nq.

5. Minimal partial realization in $\mathscr{S}(\boldsymbol{\alpha})$

We now combine the results of Sections 3 and 4 to describe the solution of the partial realization problem in the class $\mathscr{S}(\alpha)$.

First we remember that it is well known that the two classical minimal realizations (A_1, B_1, C_1) and (A_2, B_2, C_2) with the same transfer function are similar, which means that there exists a nonsingular matrix **s** such that

$$A_2 = S^{-1}A_1S, \qquad B_2 = S^{-1}B, \qquad C_2 = C_1S.$$

We show that this is also true for realizations in the class $\mathscr{S}(\alpha)$.

Proposition 10. *Let* $\Sigma_i = (A_i, E_i, B_i, C_i)$ $(i = 1, 2)$ *be two minimal partial realizations of* **s** *in the class* $\mathscr{S}(\alpha)$ *with the same transfer functions. Then there exists a nonsingular matrix* S *and* $\xi \in \mathbb{C}$, $\xi \neq 0$, *such that*

$$A_2 = \xi S^{-1}A_1S, \qquad E_2 = \xi S^{-1}E_1S, \qquad B_2 = \xi S^{-1}B, \qquad C_2 = C_1S.$$

Proof. For the case $\alpha = \infty$ the assertion follows immediately from classical realization theory. For $\alpha \neq \infty$ we employ the Möbius transformation corresponding to

$$\phi = \begin{bmatrix} 0 & 1 \\ 1 & -\alpha \end{bmatrix}. \tag{5.1}$$

Then we have $\phi(\alpha) = \infty$ and, according to Proposition 3, $\Sigma_1(\phi), \Sigma_2(\phi) \in \mathscr{S}(\infty)$. Due to Proposition 4 the transfer functions of $\Sigma_1(\phi)$ and $\Sigma_2(\phi)$ coincide. Suppose that $\Sigma_i(\phi) = (\tilde{A}_i, \tilde{E}_i, B_i, C_i)$. Since the assertion is true for $\alpha = \infty$ there is a nonsingular matrix **s** and $\xi \in \mathbb{C}$ such that

$$\tilde{A}_2 = \xi S^{-1}\tilde{A}_1 S, \qquad \tilde{E}_2 = \xi S^{-1}\tilde{E}_1 S, \qquad B_2 = \xi S^{-1} B, \qquad C_2 = C_1 S.$$

Transforming back we obtain the assertion. $\square$

In view of Proposition 10 it remains to describe all transfer functions of solutions of Problem MPR(α). For this we need the following proposition which can be found in [5, Theorem 30].

Proposition 11. *Let $\{u_1, \ldots, u_t\}$ be a fundamental system of $\mathbf{s} = (s_k)_0^{N-1}$ and let d_i $(i = 1, \ldots, t)$ be the corresponding characteristic degrees. Furthermore, let ϕ be a nonsingular 2×2 matrix. Then the vector polynomials*

$$\tilde{u}_i = M_{d_i+1}\big(\phi^{-1}\big)u_i \quad (i = 1, \ldots, t)$$

form a fundamental system of $\mathbf{s}(\phi) = (\tilde{s}_k)_0^{N-1}$, where

$$\operatorname{row}(\tilde{s}_k)_0^{N-1} = \operatorname{row}(s_k)_0^{N-1} M(\phi). \tag{5.2}$$

Furthermore, the characteristic degrees of $\mathbf{s}$ and $\mathbf{s}(\phi)$ coincide.

A fundamental system is said to be *canonical at* $\alpha \in \mathbb{C}$ if it contains exactly q elements u_i for which $u_i(\alpha) \neq 0$. The elements with this property will be called *α-regular part* and the corresponding characteristic degrees *α-regular* (*characteristic*) *degrees*. As it is explained in [8] for the case $\alpha = \infty$, any canonical system can be transformed into another canonical system which is canonical at α. Furthermore, the α-regular part is linearly independent at α. This follows from the fact that the matrix formed by the elements of a fundamental system has full rank for all λ.

In all what follows in this section we choose ϕ according to (5.1), $\alpha \in \mathbb{C}$. Then

$$\mu = \phi(\lambda) = \frac{1}{\lambda - \alpha} \quad \text{and} \quad \phi^{-1}(\mu) = \frac{\alpha\mu + 1}{\mu}.$$

In particular, $\phi(\alpha) = \infty$.

Let **s** be given. We are going to construct all transfer functions for minimal partial realizations of **s** in the class $\mathscr{S}(\alpha)$. For this we define $\mathbf{s}(\phi) = (\tilde{s}_0, \ldots, \tilde{s}_{N-1})$ by (5.2) and $\tilde{s}(\lambda)$ by

$$\tilde{s}(\lambda) = \sum_{k=0}^{N-1} \tilde{s}_k \lambda^k.$$

It follows from Propositions 3 and 5 that Σ is a partial realization of $\mathbf{s}$ in the class $\mathscr{S}(\alpha)$ if and only if $\Sigma(\phi)$ is a realization of $\mathbf{s}(\phi)$ in the class $\mathscr{S}(\infty)$. That means we have to find the transfer functions for the classical realization of $\mathbf{s}$ and transform back with the help of Proposition 3.2.

Let $\{u_1, \ldots, u_t\}$ be a fundamental system for $\mathbf{s}$ which is canonical at α. Then according to Proposition 11 a fundamental system for $\mathbf{s}(\phi)$ is given by

$$\tilde{u}_i(\mu) = \big(M_{d_i+1}(\phi^{-1})u_i\big)(\mu) = u_i\big(\alpha + \mu^{-1}\big)\mu^{d_i}. \tag{5.3}$$

The corresponding residual system $\tilde{w}_i(\mu)$ is given by

$$\tilde{s}\big(\mu^{-1}\big)\mu^{-1}\tilde{u}_i(\mu) = \tilde{w}_i(\mu) + \mu^{-d_i-N-1}\beta_i\big(\mu^{-1}\big),$$

where $\beta_i(\mu)$ are polynomials. In terms of the variable λ and and original fundamental system this means

$$\tilde{s}(\lambda-\alpha)u_i(\lambda) = w_i(\lambda) + (\lambda-\alpha)^N \beta_i(\lambda-\alpha). \tag{5.4}$$

In order to apply Theorem 7 we form matrices $\tilde{U}_0$, $\tilde{U}'$, $\tilde{W}_0$, $\tilde{W}'$ as it is described before in Theorem 7. Now the general form of a transfer function of all minimal partial realizations of $\mathbf{s}(\phi)$ in the class $\mathscr{S}(\infty)$ is given by

$$\tilde{F}(\mu) = \big(\tilde{W}_0 + \tilde{W}'\tilde{Z}\big)\big(\tilde{U}_0 + \tilde{U}'\tilde{Z}\big)^{-1},$$

where the entries of $\tilde{Z}$ satisfy the conditions in Theorem 7.

Now according to Proposition 3.2 the general form of a transfer function of all minimal partial realizations of $\mathbf{s}$ in the class $\mathscr{S}(\alpha)$ is given by

$$F(\lambda) = -\frac{1}{\lambda-\alpha}\tilde{F}\left(\frac{1}{\lambda-\alpha}\right). \tag{5.5}$$

The ith column of $\tilde{U}_0 + \tilde{U}'\tilde{Z}$ is given by

$$\tilde{u}_i(\mu) + \sum_{k=1}^{r} \tilde{z}_{ik}(\mu)\tilde{u}'_k(\mu) = (\lambda-\alpha)^{-d_i}\left(u_i(\lambda) + \sum_{k=1}^{r} z_{ik}(\lambda)u'_k(\lambda)\right),$$

where $z_{ik}(\lambda) = \tilde{z}_{ik}((\lambda-\alpha)^{-1})(\lambda-\alpha)^{d_i-d_k}$. Similarly, the ith column of $\tilde{W}_0 + \tilde{W}'\tilde{Z}$ is equal to

$$(\lambda-\alpha)^{-d_i+1}\left(w_i(\lambda) + \sum_{k=1}^{r} z_{ik}(\lambda)w'_k(\lambda)\right).$$

From this we see that the polynomials z_{ik} satisfy the same degree conditions as $\tilde{z}_{ik}$. Furthermore, the factors $(\lambda-\alpha)^{-d_i}$ in the denominator, $(\lambda-\alpha)^{-d_i+1}$ in the numerator, and the denominator in (5.5) cancel.

Now we are in a position to formulate the main result of this section. In order to shorten its formulation we recall our construction and collect some notations. Let, for $\alpha \in \mathbb{C}$, a fundamental system which is canonical at α be given. From its α-regular part we form the $q \times q$ matrix polynomial $U_0(\lambda)$. Let the characteristic degree of the jth column be d_j. Then we take all elements not belonging to the regular part the degree of which do not exceed the maximal α-regular degree and form from them a $q \times r$ matrix polynomial $U'(\lambda)$ $(0 \leqslant r \leqslant t - q)$. Let the characteristic degree of the ith column of $U_1(\lambda)$ be d'_i. The residual system is defined by (5.4). The $p \times q$ and $p \times r$ matrix polynomials formed by the corresponding residuals will be denoted by $W_0(\lambda)$ and $W'(\lambda)$, respectively.

Theorem 12. *The general form of the transfer function for a minimal partial realization of* **s** *in the class* $\mathscr{S}(\alpha)$ *is given by*

$$F(\lambda) = -(W_0 + W'Z)(U_0 + U'Z)^{-1},$$

where $Z = [z_{ij}]_{i=1}^{r}{}_{j=1}^{q}$ *is an arbitrary matrix polynomial the entries of which satisfy the conditions*

$$\deg z_{ij}(\lambda) \leqslant d_j - d'_i$$

and $z_{ij}(\lambda) = 0$ *if* $d_j < d'_i$.

6. Main results

In this section, we discuss the original problem of the present paper: Does there exist for any given **s** a system of form (1.4) with the dimension $\rho(\mathbf{s})$? We show that the general answer is "no". Furthermore, we show that under certain conditions the answer is "yes". These conditions are in particular fulfilled if $p, q \leqslant 2$.

For this we introduce first the concept of regular characteristic degrees of the sequence **s**. Let $\{u_1, \ldots, u_t\}$ be a fundamental system of **s** and d_i $(i = 1, \ldots, p + q)$ be the corresponding characteristic degrees in nondecreasing order. The *regular characteristic degrees* d_i^{reg} are defined recursively by:

1. $d_1^{\mathrm{reg}} = d_1$.
2. Suppose that $d_1^{\mathrm{reg}}, \ldots, d_k^{\mathrm{reg}}$ are given and $d_i^{\mathrm{reg}} = d_{l_i}$. Let $i_0 > l_k$ be the smallest integer for which the vector $u_{i_0}(\lambda)$ is linearly independent on $u_{l_1}(\lambda), \ldots, u_{l_k}(\lambda)$ for at least one (which is the same as for almost all) $\lambda \in \mathbb{C}$. Then $d_{k+1}^{\mathrm{reg}} = d_{i_0}$.

Since the matrix formed by a fundamental system has full rank q for all λ, there are exactly q regular degrees. Taking the general form of a fundamental system (see [8, Theorem 2.2]) into account it can be easily verified that they are uniquely determined by **s**, i.e., different fundamental systems give one and the same regular degrees.

It follows immediately from the definition that the regular degrees are the α-regular degrees for all α except for a finite number of α. Therefore, the following is true.

Theorem 13. *The minimal dimension of a partial realization* (1.4) *of a sequence* $\mathbf{s}$ *is equal to the sum of all regular degrees of* $\mathbf{s}$.

Corollary 14. *There exists a partial realization of* $\mathbf{s}$ *of dimension* $\rho(\mathbf{s})$ *if and only if the largest regular degrees is smaller than or equal to the smallest nonregular degree.*

Corollary 15. *The minimal partial realization of* $\mathbf{s}$ *is unique up to equivalence modulo N-scaling if and only if the largest regular degree is smaller than the smallest nonregular degree.*

One may conjecture that the regular degrees are always the smallest ones. For $q = 1$ this is true by definition. For $p = 1$ this follows from duality of left and right degrees (see [8]). This is also the case in some special situations, as it is shown in the following, but it fails to be true for larger p and q. We recall an example from [5].

Example. Let $\mathbf{s} = (s_i)_0^4$ be given by $s_0 = s_4 = 0$,

$$s_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad s_2 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad s_3 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

Then the block Hankel matrix $H_3(\mathbf{s})$ has only a trivial kernel whereas $H_4(\mathbf{s})$ has a kernel spanned by the two vectors

$$[0, 0, 0, 0, 0, 0, 1, 0]^{\mathrm{T}} \quad \text{and} \quad [1, 0, -1, 0, 1, 0, 0, 0]^{\mathrm{T}}.$$

Hence, the smallest characteristic degrees are $d_1 = d_2 = 3$ and $d_3 > 3$. The corresponding vector polynomials of the fundamental system are given by

$$u_1(\lambda) = \begin{bmatrix} \lambda^3 \\ 0 \end{bmatrix} \quad \text{and} \quad u_2(\lambda) = \begin{bmatrix} 1 - \lambda + \lambda^2 \\ 0 \end{bmatrix}.$$

The vectors $u_1(\lambda)$ and $u_2(\lambda)$ are linearly dependent for all λ. Thus, the second regular degree is larger than 3. In view of Theorem 13 this means that $\mathbf{s}$ does not possess a partial realization of dimension $\rho(\mathbf{s})$.

An example for $p = q = 3$ was given in [3]. Our main positive result is the following.

Theorem 16. *If* $\rho(\mathbf{s}) \leqslant N$, *then there exists a partial realization of* $\mathbf{s}$ *of dimension* $\rho(\mathbf{s})$.

The proof follows immediately from the following proposition proved in [5, Lemma 39].

Proposition 17. *Let the q smallest characteristic degrees* d_i $(i = 1, \dots, q)$ *of the sequence* $\mathbf{s} = (s_k)_0^{N-1}$ *satisfy the estimation*

$$d_1 + d_2 + \cdots + d_q \leqslant N. \tag{6.1}$$

Then the $q \times q$ *matrix polynomial formed by the corresponding elements of the canonical system has a determinant not identically equal to* 0.

We now consider the special case $p = q = 2$.

Proposition 18. *For* $p = q = 2$ *inequality* (6.1) *is always fulfilled unless all* d_i $(i = 1, 2, 3, 4)$ *are equal.*

Proof. In view of (4.2) we have $d_1 + d_2 + d_3 + d_4 = 2(N + 1)$. If $d_1 + d_2 > N$, then $d_3 + d_4 > N$, and therefore, $\sum_{i=1}^{4} d_i \geqslant 2(N + 1)$. Since actually equality holds we conclude that in this case $d_1 = d_2 = d_3 = d_4$. □

If $d_1 = d_2 = d_3 = d_4 = m$, then $\mathbf{s}$ has a classical partial realization, according to Corollary 8. In the other case we can apply Theorem 16. In this way we obtain the following result.

Theorem 19. *Let* $\mathbf{s}$ *be a sequence of* $p \times q$ *blocks with* $p = q \leqslant 2$ *or* $p = 1$ *or* $q = 1$. *Then there exists a partial realization of dimension* $\rho(\mathbf{s})$.

7. Partial realization and canonical representations

We show in this section that the results presented above are closely related to those in [5] concerning Vandermonde factorization of block Hankel matrices.

To explain this we introduce another "normal" form for systems (1.4). We will say that (1.4) is in *Jordan form* if E and A have the form

$$E = \begin{bmatrix} J_0 & 0 \\ 0 & I_\nu \end{bmatrix}, \qquad A = \begin{bmatrix} I_{\nu_0} & 0 \\ 0 & J \end{bmatrix}, \tag{7.1}$$

where J_0 is nilpotent and J is a Jordan matrix, i.e., a direct sum of Jordan blocks.

Proposition 20. *For any system* (1.4) *there exists a system in Jordan form with the same N-Markov parameters for any N.*

Proof. According to Proposition 2 system (1.4) is equivalent modulo N-scaling to a system in α-standard form for almost all α. We fix such an α. Then $A = \alpha E - I_n$.

Let $\tilde{J}$ be the Jordan normal form of E. Then (1.4) is equivalent modulo N-scaling to a system $\Sigma = (\tilde{J}, \alpha\tilde{J} - I_n, *, *)$. Now $\tilde{J}$ can be represented as direct sum $\tilde{J} = E_0 \oplus E_1$, where E_0 is nilpotent and E_1 is nonsingular. Hence, Σ is the direct sum of systems $\Sigma_0 = (E_0, A_0, *, *)$ and $\Sigma_1 = (E_1, A_1, *, *)$, where $A_k = \alpha E_k - I_{\nu_k}$ $(k = 0, 1)$. Since A_0 and E_1 are nonsingular, Σ_0 and Σ_1 are equivalent modulo N-scaling to $(E_0 A_0^{-1}, I_{\nu_0}, *, *)$ and $(I_\nu, E_1^{-1} A_1, *, *)$, respectively. It remains now to take the Jordan form of the matrices $E_0 A_0^{-1}$ and $E_1^{-1} A_1$ and to use the fact that $E_0 A_0^{-1}$ is nilpotent to get equivalent modulo N-scaling systems of the form $(J_0, I_\nu, *, *)$ and $(I_{n_0}, J, *, *)$, respectively. By Proposition 1 these systems are also N-Markov equivalent. Hence, their direct sum is N-Markov equivalent to the original system (1.4) and the proposition is proved. □

Let (A, E, B, C) be a partial realization in Jordan form (7.1) where $J = \mathrm{diag}(J_i)_1^r$ and J_i are Jordan blocks, $C = \mathrm{row}(C_i)_0^r$, $B = \mathrm{col}(B_i)_0^r$. Then

$$s_k = C E^{N-1-k} A^k B = C_0 J_0^{N-1-k} B_0 + \sum_{i=1}^{r} C_i J_i^k B_i.$$

If given blocks s_k $(k = 0, \ldots, N-1)$ can be represented in this form, then it is called *canonical representation* of the sequence $\mathbf{s} = (s_k)_0^{N-1}$. The state space dimension of the system is just equal to what is called the rank of the canonical representation.

A canonical representation of $\mathbf{s}$ is equivalent to a representation of the block Hankel matrix as a product of a (affine) Vandermonde, a transpose Vandermonde matrix, and a block diagonal matrix in between.

Vice versa, if a canonical representation is given, this gives immediately a partial realization of a possibly singular system in Jordan form. So the problems discussed in [5] and here are actually equivalent. Furthermore, the existence of a canonical representation with rank $\rho(\mathbf{s})$ for the scalar case is well known (see [9, Part I, Chapter 8]). In this sense the scalar case for the partial realization problem considered here was known before.

8. Realization of discrete-time periodic systems

In this section, we assume that the blocks s_k are square and show that in this case the N-Markov parameters can be interpreted in the framework of N-periodic systems. We consider systems given by (1.4) with $AE = EA$ in which the time variable k belongs to the cyclic group $\mathbf{Z}_N$. Introducing the vectors $u = (u_k)_0^{N-1}$, $x = (x_k)_0^{N-1}$, $y = (y_k)_0^{N-1}$, (1.4) can be written in the form

$$\begin{aligned} M(A, E)x &= \mathrm{diag}(B, \ldots, B)u, \\ y &= \mathrm{diag}(C, \ldots, C)x, \end{aligned} \tag{8.1}$$

where

$$M(A,E)=\begin{bmatrix} -A & E & & & 0 \\ & -A & E & & \\ & & \ddots & \ddots & \\ & & & -A & E \\ E & & & & -A \end{bmatrix}.$$

Note that $M(A,E)$ is a block circulant matrix. The input defines the output uniquely if and only if the matrix $M(A,E)$ is nonsingular. This is equivalent to the nonsingularity of $E^N - A^N$ or, what is the same, to $\det(\omega E - A) \neq 0$ on the group $\mathbb{T}_N$ of all Nth unit roots. We assume that this is the case.

The inverse of $M(A,E)$ must be again a block circulant matrix. It is easily checked that actually

$$M(A,E)^{-1} = D\begin{bmatrix} A^{N-1} & A^{N-2}E & \cdots & E^{N-1} \\ E^{N-1} & A^{N-1} & \cdots & AE^{N-2} \\ \vdots & \ddots & \ddots & \\ A^{N-2}E & A^{N-3}E^2 & \cdots & A^{N-1} \end{bmatrix},$$

where

$$D = \operatorname{diag}\left(\left(E^N - A^N\right)^{-1}, \ldots, \left(E^N - A^N\right)^{-1}\right).$$

Thus, the input–output relation can be described by

$$y = Su,$$

where S is the block circulant matrix

$$S = \begin{bmatrix} s_{N-1} & s_{N-2} & \cdots & s_0 \\ s_0 & s_{N-1} & \cdots & s_1 \\ \vdots & \ddots & \ddots & \\ s_{N-2} & s_{N-3} & \cdots & s_{N-1} \end{bmatrix}$$

with

$$s_k = \tilde{C}A^k E^{N-1-k}B, \quad \tilde{C} = C\left(E^N - A^N\right)^{-1}.$$

In particular, the output for the input $(0,\ldots,0,I_q)$ is just a sequence $(\tilde{C}E^N B, \tilde{C} E^{N-1}AB, \ldots, \tilde{C}A^N B)$, that means the sequence of N-Markov parameters.

The following problem is now natural.

Minimal realization problem for periodic systems. Let a sequence of $q \times q$ blocks $(s_0,\ldots,s_{N-1})$ be given such that the matrix S is nonsingular. Find an N-periodic system (1.4) with $AE = EA$ and minimal state space dimension such that

$$s_k = C\left(E^N - A^N\right)^{-1} A^k E^{N-k-1} B \quad (k = 0,\ldots,N-1).$$

Note that the nonsingularity of S is equivalent to the nonsingularity of $s_0 + s_1\omega + \cdots + s_{N-1}\omega^{N-1}$ for $\omega \in \mathbb{T}_N$ and also to the nonsingularity of $E^N - A^N$.

Let us also recall that we may assume that the system is in standard form. In fact, let any realization be given. We define

$$A_0 = (\alpha E + \beta A)^{-1} A, \quad E_0 = (\alpha E + \beta A)^{-1} E,$$

where α and β are such that $\alpha E + \beta A$ is nonsingular. Then

$$s_k = C(\alpha E + \beta A)^{-1}(E_0^N - A_0^N) A_0^k E_0^{N-1-k} B.$$

Thus, replacing C by $C(\alpha E + \beta A)^{-1}$ we obtain a realization in standard form. That means the results of the previous sections can be applied.

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