Variational iteration method for unsteady flow of gas through a porous medium using He's polynomials and Pade approximants

Muhammad Aslam Noor*, Syed Tauseef Mohyud-Din

Department of Mathematics, COMSATS Institute of Information Technology, Islamabad, Pakistan

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ABSTRACT

In this paper, we apply the variational iteration method using He's polynomials for finding the analytical solution of unsteady flow of gas through a porous medium. The proposed method is an elegant combination of He's variational iteration and the homotopy perturbation methods. The suggested algorithm is quite efficient and is practically well suited for use in such problems. The proposed iterative scheme finds the solution without any discretization, linearization or restrictive assumptions. The diagonal Pade approximants are effectively used in the analysis to capture the essential behavior of the solution and to determine the initial slope. A clear advantage of this technique over the decomposition method is that no calculation of Adonian's polynomials is needed.

1. Introduction

In this paper, we outline a reliable strategy for developing an analytic solution to the nonlinear ordinary differential equation [1–6] given by

\[ y''(x) + \frac{2x}{\sqrt{1-\alpha y}} y'(x) = 0, \quad 0 < \alpha < 1, \]  

which appears in the problem of the transient flow of gas [1–6] within one-dimensional semi-infinite porous medium. A useful analysis has been carried out to distinguish the difference between unsteady flow of gas and conduction of heat in solids. The study shows that the unsteady flow of gas through porous medium is closely analogous to the unsteady conduction of heat in solids; see [1–6] and the references therein. Moreover, the unsteady flow of gas in a porous medium is modeled by a nonlinear partial differential equation, whereas the unsteady conduction of heat in solids is characterized by a linear partial differential equation. The unsteady conduction of heat is usually handled by the typical analytic techniques such as Laplace or Fourier transforms. On the other hand, the unsteady flow of gas through a porous medium was investigated by several authors [3,6], where no analytic approaches of many forms have been proposed for the study of (1). The main concern of all works was to obtain numerical solution of (1). The approaches consist of approximating the nonlinear partial differential equation that models this phenomenon by an appropriate finite difference equation, then solving the resulting difference equation by numerical methods. The analytic solution was constructed and the results obtained were of moderate accuracy. However, the complexity of the calculations increases rapidly with increasing order of terms especially beyond the second order term, see [7,1–6] and the references therein. The nonlinear partial differential equation that describes the unsteady flow of gas through a semi-infinite porous medium has been derived [3] in the form

\[ \nabla^2 (p^2) = \frac{2\phi \mu}{k} \frac{\partial p}{\partial t}, \]
where $p$ is the pressure within porous medium, $\phi$ the porosity, $\mu$ the viscosity, $k$ the permeability, and $t$ the time. New variables were introduced by Kidder and Davis [3] to transform the nonlinear partial differential equation (2) to the nonlinear ordinary differential equation

$$y''(x) + \frac{2x}{\sqrt{1 - \alpha y}} y'(x) = 0, \quad 0 < \alpha < 1.$$ 

with typical boundary conditions imposed by the physical properties [1–3,6]

$$y(0) = 1, \quad \lim_{x \to \infty} y(x) = 0.$$ 

A substantial amount of numerical and analytical work has been invested so far [7,1–6] on this model. The main reason of this interest is that the approximation can be used in many diversified engineering problems. As stated before, the problem (1) was handled by Kidder [3] to produce a series of linear differential equations where the following terms were obtained

$$y^{(0)}(x) = 1 - \text{erf}(x),$$

$$y^{(1)}(x) = -\frac{1}{2\pi} \left( y^{(0)} \left[ 1 + \sqrt{\pi} x e^{-x^2} \right] - e^{-x^2} \right),$$

$$y^{(2)}(x) = -\frac{1}{\pi} y^{(1)} + \left( \frac{1}{8\pi^{3/2}} \right) x e^{-3x^2} - \frac{1}{2\pi} y^{(0)} - \left( \frac{1}{16\sqrt{\pi}} \right) x (5 - 2x^2) e^{-x^2} \left( y^{(0)} \right)^2$$

$$+ \left( \frac{1}{4\pi} \right) (2 - x^2) e^{-2x^2} y^{(0)} + \left( \frac{3^{3/2}}{16\pi} \right) \left[ \text{erf}(\sqrt{3}x) - \text{erf}(x) \right],$$

having the error function

$$\text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt.$$ 

Recently, Adomian’s decomposition method [6] was employed in a straight forward manner for finding solution of the unsteady flow of gas problem. Inspired and motivated by the ongoing research in this area, we apply the variational iteration method using He’s polynomials for finding the solution of unsteady flow of gas through a porous medium. It is worth mentioning that our proposed technique is an elegant combination of He’s variational iteration and the homotopy perturbation methods.

He [8–20] developed the variational iteration and the homotopy perturbation methods for solving linear, nonlinear, initial and boundary value problems. It is worth mentioning that the origin of variational iteration method can be traced back to Inokuti, Sekine and Mura [21], but the real potential of this technique was explored by He [8–14]. Moreover, He realized the physical significance of the variational iteration method, its compatibility with the physical problems and applied this promising technique to a wide class of linear and nonlinear, ordinary, partial, deterministic or stochastic differential equation; see [8–14]. The homotopy perturbation method [8,15–20] was also developed by He by merging two techniques, the standard homotopy and the perturbation. The homotopy perturbation method was formulated by taking the full advantage of the standard homotopy and perturbation methods. The variational iteration and homotopy perturbation methods have been applied to a wide class of functional equations; see [22–28,8–20,29–46] and the references therein. In these methods the solution is given in an infinite series usually converging to an accurate solution, see [8–20] and the references therein. In a later work Ghorbani et al. [27,28] split the nonlinear term into a series of polynomials calling them as the He’s polynomials. The basic motivation of this paper is to apply the variational iteration method coupled with He’s polynomials (VIMHP) for finding the solution of unsteady flow of gas through a porous medium. In this algorithm, the correct functional is developed [8–14] and the Lagrange multipliers are calculated optimally via variational theory. The use of Lagrange multipliers reduces the successive application of the integral operator and the cumbersome of huge computational work while still maintaining a very high level of accuracy. Finally, the He’s polynomials are introduced in the correct functional and the comparison of like powers of $p$ gives solutions of various orders. The developed algorithm takes full advantage of He’s variational iteration and the homotopy perturbation methods. It is worth mentioning that the VIMHP is applied without any discretization, restrictive assumption or transformation and is free from round off errors. Unlike the method of separation of variables that require initial and boundary conditions, the VIMHP provides an analytical solution by using the initial conditions only. The proposed method work efficiently and the results so far are very encouraging and reliable. The fact that VIMHP solves nonlinear problems without using Adomian’s polynomials can be considered as a clear advantage of this technique over the decomposition method. To make the work more concise and to get a better understanding of the solution behavior, the series solutions are replaced by the powerful Padé approximants [7,47,48,6,49]. The use of Padé approximants shows real promise in solving boundary value problems in an infinite domain. The proposed VIMHP solves effectively, easily and accurately a large class of linear, nonlinear, partial, deterministic or stochastic differential equations with approximate solutions which converge very rapidly to accurate solutions.
2. Variational iteration method

To illustrate the basic concept of the technique, we consider the following general differential equation

\[ Lu + Nu = g(x), \]

where \( L \) is a linear operator, \( N \) a nonlinear operator and \( g(x) \) is the forcing term. According to the variational iteration method [22–26,8–14,21,29–31,38–44,46], we can construct a correct functional as follows

\[ u_{n+1}(x) = u_n(x) + \int_0^x \lambda \left( Lu_n(s) + N \tilde{u}_n(s) - g(s) \right) ds, \]

where \( \lambda \) is a Lagrange multiplier [8–14], which can be identified optimally via variational iteration method. The subscript \( n \) denotes the \( n \)th approximation, \( \tilde{u}_n \) is considered as a restricted variation, i.e. \( \delta \tilde{u}_n = 0 \); (3) is called as a correct functional. The solution of the linear problems can be solved in a single iteration step due to the exact identification of the Lagrange multiplier. The principles of variational iteration method and its applicability for various kinds of differential equations are given in [8–14]. In this method, it is required first to determine the Lagrange multiplier \( \lambda \). Optimally. The successive approximation \( u_{n+1} \), \( n \geq 0 \) of the solution \( u \) will be readily obtained upon using the determined Lagrange multiplier and any selective function \( u_0 \), consequently, the solution is given by \( u = \lim_{n \to \infty} u_n \).

3. Homotopy perturbation method

To explain the homotopy perturbation method, we consider a general equation of the type,

\[ L(u) = 0, \]

where \( L \) is any integral or differential operator. We define a convex homotopy \( H(u, p) \) by

\[ H(u, p) = (1 - p)F(u) + pL(u), \]

where \( F(u) \) is a functional operator with known solutions \( v_0 \), which can be obtained easily. It is clear that, for \( H(u, p) = 0 \), we have

\[ H(u, 0) = F(u), \quad H(u, 1) = L(u). \]

This shows that \( H(u, p) \) continuously traces an implicitly defined curve from a starting point \( H(v_0, 0) \) to a solution function \( H(f, 1) \). The embedding parameter monotonically increases from zero to unity as the trivial problem \( F(u) = 0 \) continuously deforms the original problem \( L(u) = 0 \). The embedding parameter \( p \in (0, 1] \) can be considered as an expanding parameter \([8,15–20,32–37,45]\). The homotopy perturbation method uses the homotopy parameter \( p \) as an expanding parameter \([8,15–20]\) to obtain

\[ u = \sum_{i=0}^{\infty} p^i u_i = u_0 + p u_1 + p^2 u_2 + p^3 u_3 + \cdots. \]

If \( p \to 1 \), then (7) corresponds to (5) and becomes the approximate solution of the form,

\[ f = \lim_{p \to 1} u = \sum_{i=0}^{\infty} u_i. \]

It is well known that series (8) is convergent for most of the cases and also the rate of convergence is dependent on \( L(u) \); see \([8,15–20]\). We assume that (8) has a unique solution. The comparisons of like powers of \( p \) give solutions of various orders. In sum, according to [28], He’s HPM considers the solution, \( u(x) \), of the homotopy equation in a series of \( p \) as follows:

\[ u(x) = \sum_{i=0}^{\infty} p^i u_i = u_0 + p u_1 + p^2 u_2 + \cdots, \]

and the method considers the nonlinear term \( N(u) \) as

\[ N(u) = \sum_{i=0}^{\infty} p^i H_i = H_0 + p H_1 + p^2 H_2 + \cdots, \]

where \( H_i \)'s are the so-called He’s polynomials \([28]\), which can be calculated by using the formula

\[ H_n(u_0, \ldots, u_n) = \frac{1}{n!} \frac{\partial^n}{\partial p^n} \left( N \left( \sum_{i=0}^{n} p^i u_i \right) \right) \bigg|_{p=0}, \quad n = 0, 1, 2, \ldots. \]
4. Variational Iteration Method Using He's Polynomials (VIMHP)

To illustrate the basic concept of the variational iteration method using He's polynomials, we consider the following general differential equation

\[ Lu + Nu = g(x), \]  

(9)

where \( L \) is a linear operator, \( N \) a nonlinear operator and \( g(x) \) is the forcing term. According to the variational iteration method [22–26,8–14,21,29–31,38–44,46], we can construct a correct functional as follows

\[ u_{n+1}(x) = u_n(x) + \int_0^x \lambda(\xi) \left( L u_n(\xi) + N \tilde{u}_n(\xi) - g(\xi) \right) \, d\xi, \]  

(10)

where \( \lambda \) is a Lagrange multiplier [8–14], which can be identified optimally via variational iteration method. The subscript \( n \) denotes the \( n \)th approximation, \( \tilde{u}_n \) is considered as a restricted variation, i.e., \( \delta u_n = 0 \); (10) is called as a correct functional. It is well known that He’s homotopy perturbation method provides the solution as a series; whereas He’s variational iteration method provides the solution as a sequence. Now, we apply the homotopy perturbation method

\[ \sum_{n=0}^{\infty} p^{(n)} u_n = u_0(x) + p \int_0^x \lambda(\xi) \left( \sum_{n=0}^{\infty} p^{(n)} L u_n + \sum_{n=0}^{\infty} p^{(n)} N \tilde{u}_n \right) \, d\xi - \int_0^x \lambda(\xi) g(\xi) \, d\xi, \]  

(11)

which is the coupling of variational iteration method and He’s polynomials [28] and comparison of like powers of \( p \) gives solutions of various orders.

5. Pade approximants

A Pade approximant is the ratio of two polynomials constructed from the coefficients of the Taylor series expansion of a function \( u(x) \). The \([L/M]\) Pade approximants to a function \( y(x) \) are given by [7,47,48].

\[ \left[ \frac{L}{M} \right] = \frac{P_L(x)}{Q_M(x)}, \]  

(12)

where \( P_L(x) \) is polynomial of degree at most \( L \) and \( Q_M(x) \) is a polynomial of degree at most \( M \). The formal power series

\[ y(x) = \sum_{i=1}^{\infty} a_i x^i, \]  

(13)

\[ y(x) - \frac{P_L(x)}{Q_M(x)} = O(x^{L+M+1}), \]  

(14)

determine the coefficients of \( P_L(x) \) and \( Q_M(x) \) by the equation. Since we can clearly multiply the numerator and denominator by a constant and leave \([L/M]\) unchanged, we imposed the normalization condition

\[ Q_M(0) = 1.0. \]  

(15)

Finally, we require that \( P_L(x) \) and \( Q_M(x) \) have noncommon factors. If we write the coefficient of \( P_L(x) \) and \( Q_M(x) \) as

\[
\begin{align*}
P_L(x) &= p_0 + p_1 x + p_2 x^2 + \cdots + p_L x^L, \\
Q_M(x) &= q_0 + q_1 x + q_2 x^2 + \cdots + q_M x^M.
\end{align*}
\]  

(16)

Then by (15) and (16), we may multiply (12) by \( Q_M(x) \), which linearizes the coefficient equations. We can write out (14) in more details as

\[
\begin{align*}
a_{l+1} + a_0 q_1 + \cdots + a_{l-M} q_M &= 0, \\
a_{l+2} + a_{l+1} q_1 + \cdots + a_{l-M+2} q_M &= 0, \\
&\vdots \\
a_{l+M} + a_{l+M-1} q_1 + \cdots + a_l q_M &= 0,
\end{align*}
\]  

(17)

\[
\begin{align*}
a_0 &= p_0, \\
a_0 + a_0 q_1 + \cdots + a_l q_l &= p_1, \\
&\vdots \\
a_0 + a_0 q_1 + \cdots + a_l q_l &= p_L
\end{align*}
\]  

(18)

To solve these equations, we start with Eq. (17), which is a set of linear equations for all the unknown \( q \)'s. Once the \( q \)'s are known, then Eq. (18) gives an explicit formula for the unknown \( p \)'s, which complete the solution. If Eqs. (17) and (18) are nonsingular, then we can solve them directly and obtain Eq. (19) [48], where Eq. (19) holds, and if the lower index on a sum exceeds the upper, the sum is replaced by zero:
To obtain diagonal Padé approximants of different order such as \([2/2], [4/4]\) or \([6/6]\), we can use the symbolic calculus software Maple.

6. Numerical application

In this section, we apply the variational iteration method using He’s polynomials (VIMHP) for finding the analytical solution of the unsteady flow of gas through a porous medium. Finally, the series solution will be replaced by the powerful Padé approximants in order to get a better understanding of the solution behavior and to determine the initial slope \(y'(0)\).

Consider Eq. (1)

\[
y''(x) + \frac{2x}{\sqrt{1-\alpha y}}y'(x) = 0, \quad 0 < \alpha < 1. \tag{1}
\]

with the following typical boundary conditions imposed by the physical properties \([3,6]\)

\[
y(0) = 1, \quad \lim_{x \to \infty} y(x) = 0.
\]

The correct functional is given as

\[
y_{n+1}(x) = y_n(x) + \int_0^x \lambda(s) \left( y''(s) + \frac{2s}{\sqrt{1-\alpha y(s)}} y'(s) \right) \, ds \quad 0 < \alpha < 1.
\]

Making the correct functional stationary, the Lagrange multipliers can be identified as \(\lambda = s - x\), we get the following iterative formula

\[
y_{n+1}(x) = y_n(x) + \int_0^x (s - x) \left( y''(s) + \frac{2s}{\sqrt{1-\alpha y_n(s)}} y'_n(s) \right) \, ds, \quad 0 < \alpha < 1.
\]

Applying the variational iteration method using He’s polynomials

\[
y_0 + py_1 + p^2y_2 + \cdots = y_0 + \int_0^x (s - x) \left( y''_0 + py''_1 + p^2y''_2 + \cdots \right) \, ds
\]

\[
+ 2 \int_0^x (s - x)(1 - \alpha \left(y'_0 + py_1 + p^2y_2 + \cdots\right)) \frac{1}{2!} \left(y'_{2} + py_1 + p^2y_2 + \cdots\right) \, ds.
\]

where \(A = y'(0)\). Comparing the coefficient of like powers of \(p\), consequently, the following approximants are obtained

\[
p^{(0)} : y_0(x) = 1,
\]

\[
p^{(1)} : y_1(x) = 1 + Ax,
\]

\[
p^{(2)} : y_2(x) = 1 + Ax - \frac{A}{3\sqrt{1-\alpha}}x^3,
\]

\[
p^{(3)} : y_3(x) = 1 + Ax - \frac{A}{3\sqrt{1-\alpha}}x^3 - \frac{\alpha A^2}{12(1-\alpha)^{3/2}}x^4 + \frac{A}{10(1-\alpha)}x^5,
\]

\[
p^{(4)} : y_4(x) = 1 + Ax - \frac{A}{3\sqrt{1-\alpha}}x^3 - \frac{\alpha A^2}{12(1-\alpha)^{3/2}}x^4 + \frac{A}{10(1-\alpha)}x^5 - \frac{3\alpha^2A^3}{80(1-\alpha)^{5/2}}x^5 + \frac{\alpha A^2}{15(1-\alpha)^2}x^6 + O(x^7),
\]

\[
p^{(5)} : y_5(x) = 1 + Ax - \frac{A}{3\sqrt{1-\alpha}}x^3 - \frac{\alpha A^2}{12(1-\alpha)^{3/2}}x^4 + \frac{A}{10(1-\alpha)}x^5 - \frac{3\alpha^2A^3}{80(1-\alpha)^{5/2}}x^5 + \frac{\alpha A^2}{15(1-\alpha)^2}x^6 - \frac{\alpha^3A^4}{48(1-\alpha)^{7/2}}x^7 + O(x^8),
\]

\[
\vdots
\]
Table 6.1
Exhibits the initial slopes $A = y'(0)$ for various values of $\alpha$.

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>$B_{2/2} = y'(0)$</th>
<th>$B_{3/3} = y'(0)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>−3.556558821</td>
<td>−1.957208953</td>
</tr>
<tr>
<td>0.2</td>
<td>−2.441894334</td>
<td>−1.786475516</td>
</tr>
<tr>
<td>0.3</td>
<td>−1.928338405</td>
<td>−1.478270843</td>
</tr>
<tr>
<td>0.4</td>
<td>−1.606856838</td>
<td>−1.231801809</td>
</tr>
<tr>
<td>0.5</td>
<td>−1.373178096</td>
<td>−1.025529704</td>
</tr>
<tr>
<td>0.6</td>
<td>−1.185519607</td>
<td>−0.840046085</td>
</tr>
<tr>
<td>0.7</td>
<td>−1.021413090</td>
<td>−0.6612047893</td>
</tr>
<tr>
<td>0.8</td>
<td>−0.8633400217</td>
<td>−0.4776697286</td>
</tr>
<tr>
<td>0.9</td>
<td>−0.6844600642</td>
<td>−0.2772628386</td>
</tr>
</tbody>
</table>

The series solution is given as

$$y(x) = 1 + Ax - \frac{A}{3\sqrt{1-\alpha}}x^3 - \frac{\alpha A^2}{12(1-\alpha)^{3/2}}x^4 + \left(\frac{A}{10(1-\alpha)} - \frac{3\alpha^2A^3}{80(1-\alpha)^{5/2}}\right)x^5 + \left(\frac{\alpha A^2}{15(1-\alpha)^2} - \frac{\alpha^3A^4}{48(1-\alpha)^{7/2}}\right)x^6 + O(x^7),$$

which is in full agreement with [6]. Now, we investigate the mathematical behavior of the solution $y(x)$ in order to determine the initial slope $y'(0)$. This goal can be achieved by forming Padé approximants [7,47,48,6,49] which have the advantage of manipulating the polynomial approximation into a rational function to gain more information about $y(x)$. It is well known that Padé approximants will converge on the entire real axis [7,47,48,6,49] if $y(x)$ is free of singularities on the real axis. It is of interest to note that Padé approximants give results with no greater error bounds than approximation by polynomials. More importantly, the diagonal approximant is the most accurate approximant; therefore we will construct only the diagonal approximants in the following discussions. Using the boundary condition $y(\infty) = 0$, the diagonal approximant $[M/M]$ vanishes if the coefficient of $x$ with the highest power in the numerator vanishes. The computational work can be performed by using the mathematical software MAPLE. In applying the boundary condition $y(\infty) = 0$ to the diagonal approximant, a polynomial equation for the initial slope $A = y'(0)$ results that gives many roots although the Kidder equation (1) has a unique solution. The physical behavior indicates that $y(x)$ is a decreasing function, hence $y'(0) < 0$. The complex roots and nonphysical positive roots should be excluded, see [7,6]. Based on this, the $[2/2]$ Padé approximant produced the slope $A$ to be

$$A = -\frac{2(1-\alpha)^{1/4}}{\sqrt{3\alpha}},$$  \hspace{1cm} (20)

and using $[3/3]$ Padé approximants we find

$$A = -\frac{\sqrt{(-4674\alpha + 8664)\sqrt{1-\alpha - 144\gamma}}}{57\alpha},$$  \hspace{1cm} (21)

where

$$\gamma = \sqrt{5(1-\alpha)(1309\alpha^2 - 2280\alpha + 1216)}.\hspace{1cm} (22)$$

Using (20)–(22) gives the values of the initial slope $A = y'(0)$ listed in the Table 6.1. The formulas (19) and (20) suggest that the initial slope $A = y'(0)$ depends mainly on the parameter $\alpha$, where $0 < \alpha < 1$. Table 6.1 shows that the initial slope $A = y'(0)$ increases with the increase of $\alpha$. The mathematical structure of $y(x)$ was successfully enhanced by using the Padé approximants. Table 6.2 indicates the values of $y(x)$ in [15] and by using the $[2/2]$ and $[3/3]$ approximants for specific value of $\alpha = 0.5$.

7. Conclusion

In this paper, we applied the He’s variational iteration method coupled with He’s polynomials (VIMHP) by combining the traditional variational iteration and the homotopy perturbation methods for finding the analytical solution of unsteady flow of gas through porous medium. The proposed method is employed without using linearization, discretization or restrictive assumptions. It may be concluded that the variational iteration method using He’s polynomials is very powerful and efficient in finding the analytical solutions for a wide class of boundary value problems. The method gives more realistic series solutions that converge very rapidly in physical problems. To make the work more concise and to get the better understanding of the solution behavior, the series solutions were replaced by the powerful Padé-approximants. The fact that the VIMHP solves nonlinear problems without using the Adomian’s polynomials is a clear advantage of this technique over the decomposition method.
Table 6.2
Exhibits the values of $y(x)$ for $\alpha = 0.5$ for $x = 0.1$ to $1.0$.

<table>
<thead>
<tr>
<th>x</th>
<th>y kider</th>
<th>$y(2/2)$</th>
<th>$y(3/3)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>0.8816588283</td>
<td>0.8633060641</td>
<td>0.8979160728</td>
</tr>
<tr>
<td>0.2</td>
<td>0.7663076781</td>
<td>0.7301262261</td>
<td>0.7985228199</td>
</tr>
<tr>
<td>0.3</td>
<td>0.6565379995</td>
<td>0.6030541400</td>
<td>0.7041257030</td>
</tr>
<tr>
<td>0.4</td>
<td>0.5544024032</td>
<td>0.4848898717</td>
<td>0.6165057901</td>
</tr>
<tr>
<td>0.5</td>
<td>0.4613650295</td>
<td>0.3761603869</td>
<td>0.5370537396</td>
</tr>
<tr>
<td>0.6</td>
<td>0.3783109315</td>
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</tr>
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<td>0.7</td>
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<td>0.1896843371</td>
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<tr>
<td>0.8</td>
<td>0.2431325473</td>
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</tr>
<tr>
<td>0.9</td>
<td>0.1904632681</td>
<td>0.0432637236</td>
<td>0.3179966614</td>
</tr>
<tr>
<td>1.0</td>
<td>0.1587688826</td>
<td>0.0164750847</td>
<td>0.2900259005</td>
</tr>
</tbody>
</table>

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