JOURNAL OF DIFFERENTIAL EQUATIONS **105**, 167–179 (1993)

Hypergraphs, the Qualitative Solvability of $\kappa \cdot \lambda = 0$, and Volterra Multipliers for Nonlinear Dynamical Systems

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Received April 22, 1991

Certain sign equivalence classes of n-dimensional nonlinear dynamical systems correspond to n-vertex hypergraphs. The global stability of some such dynamical systems can be guaranteed if the associated hypergraphs have a simplicity of structure and meet certain quantitative path product conditions. A purely algebraic version of the same problem can be described as follows. Suppose we are given a rectangular matrix pattern of signs; each entry in the matrix is +, −, or 0. For every real matrix κ of the same sign pattern, is there a real vector λ, each component of which is positive, such that $\kappa \cdot \lambda = 0$? This paper presents graph theoretic sufficient conditions on a hypergraph generated from the sign pattern of κ which guarantee the existence of λ. For κ with more highly connected hypergraphs, this paper also presents sufficient qualitative conditions on the sign pattern of κ and certain quantitative conditions on sums of hypergraph path products which together guarantee the existence of λ. © 1993 Academic Press, Inc.

1. INTRODUCTION

Relations between two mathematical objects are studied in this paper, the first of which is an n-dimensional ($n \geqslant 2$) real dynamical system. Let $\{\phi_l\}$ be a nonempty collection of M of the multiproducts x_1x_2, x_1x_3, ..., $x_{n-1}x_n$, $x_1x_2x_3$, ..., $x_1x_2 \cdots x_n$ (so $1 \leqslant m \leqslant 2^n - n - 1$). The dynamical system we have in mind is

$$dx_i/dt = -x_i + \sum_{l=1}^{M} \kappa_{il} \frac{\partial \phi_l}{\partial x_i}, \tag{1}$$

* Research partly supported by Natural Sciences and Engineering Research Council of Canada Grant A-8965.

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where $\{\kappa_{\iota i}\}$ is an $M \times n$ matrix of real constants, positive, negative, or zero. Clearly any two such n-dimensional systems with the same $\{\phi_\iota\}$ and with coefficients $\{\kappa_{\iota i}\}$ and $\{\underline{\kappa}_{\iota i}\}$ which are pairwise of the same sign can be defined to be members of an equivalence class of dynamical systems; we call all such systems so related to a given system (1) the *sign equivalence class* of the system.

An example of (1) with $n=5$ and $M=3$ is $\phi_1 = x_1 x_2 x_3$, $\phi_2 = x_1 x_2 x_4 x_5$, $\phi_3 = x_2 x_4$ and

$$\begin{aligned} dx_1/dt &= -x_1 + x_2 x_3 \\ dx_2/dt &= -x_2 - x_1 x_3 - x_1 x_4 x_5 - x_4 \\ dx_3/dt &= -x_3 - x_1 x_2 \\ dx_4/dt &= -x_4 + x_1 x_2 x_5 + x_2 \\ dx_5/dt &= -x_5. \end{aligned} \tag{2}$$

Here

$$\kappa = \begin{pmatrix} 1 & -1 & -1 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 & 0 \end{pmatrix}.$$

The results in this paper can be used to show that every trajectory for (2) asymptotically approaches the origin **0** of five-space as $t \to +\infty$, that is, **0** is a global attractor trajectory for (2). Furthermore, for any other five-dimensional dynamical system occurring in the same sign equivalence class, **0** is also a global attractor trajectory provided $\kappa_{24} + (-\kappa_{34}/\kappa_{32})\kappa_{22} = 0$. We mention that $\kappa_{24} + (-\kappa_{34}/\kappa_{32})\kappa_{22}$ is an example of a sum of path products, to be explained below.

The second mathematical object is a *hypergraph* **H** with the following types of components. The hypergraph has n vertices labelled $\{v_1, v_2, ..., v_n\}$. The *barycenters* $\{b_1, b_2, ..., b_M\}$ of **H** are subsets of at least two vertices, each thought of as corresponding to an element of $\{\phi_\iota\}$. We allow the trivial possibilities that the hypergraph consists of one vertex only or one barycenter only. Graphically each barycenter (open circle) is joined to its $p \geq 2$ vertices (filled circles) by p edges (line segments); p is the *degree* of the barycenter. This object is associated with (1) as follows. Each ϕ_ι corresponds to a barycenter connected by edges to the vertices having the same indices as the variables in ϕ_ι. For example, if $\phi_1 = x_1 x_2 x_3$, then barycenter 1 is connected by edges to vertices v_1, v_2, v_3. Given $\kappa_{\iota i}$, the edge between the barycenter ι and vertex i is given the same sign and is called *signed*. A barycenter with at least one "+" edge and at least one "−" edge is said to have *mixed sign*.

As an example, the hypergraph we wish to associate with (2) is shown in Fig. 1.

Our goal is to produce sufficient qualitative conditions on the sign pattern of κ and quantitative conditions on products of certain entries in κ which imply the stability of the origin **0** (making **0** a global attractor trajectory). The present work, in treating systems with hypergraphs with as many signed edges as that in Fig. 1, extends results in [J].

Furthermore, there is a curious relationship between the conditions of the present paper which are used to drive every trajectory to the origin and the conditions of [JvdD] which are used to drive every trajectory to one of several constant trajectories, each with components ± 1. Here heavy use is made of the mixed sign condition; in [JvdD] barycenters having every edge of the same sign are exploited.

The special case of (1) in which each $\{\phi_i\}$ is the set of all $n(n-1)/2$ products of distinct pairs of components of x is, of course, a linear dynamical system

$$dx_i/dt = -x_i + \sum_{\substack{j=1 \\ j \neq i}}^{n} a_{ij} x_j, \tag{3}$$

where $\{a_{ij}\}$ is an $n \times n$ matrix with zero diagonal entries. The stability of (3) can be established by the existence of n positive constants $\{\lambda_1, \lambda_2, ..., \lambda_n\}$ called *Volterra multipliers*. That is, if $\{\lambda_i\}$ can be chosen so that $\{\lambda_i a_{ij}\}$ is a skew symmetric matrix, then the derivative of the positive definite function

$$A = \tfrac{1}{2} \sum_{i=1}^{n} \lambda_i x_i^2$$

is

$$dA/dt = \sum_{i=1}^{n} -\lambda_i x_i^2 + \sum_{\substack{i,j=1 \\ i \neq j}}^{n} x_i (\lambda_i a_{ij}) x_j = \sum_{i=1}^{n} -\lambda_i x_i^2.$$

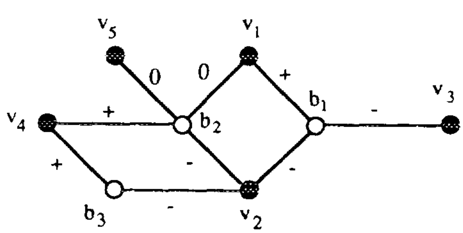

FIG. 1. The hypergraph associated with the dynamical system (2) and other systems of the sign equivalence class. Note that the product $+x_1 x_2 x_5$ occurs in dx_4/dt (hence the $+$ edge between v_4 and b_2) but no $x_1 x_2 x_4$ term occurs in dx_5/dt (hence the 0 edge between v_5 and b_2).

Thus Λ is a natural Lyapunov function for (3) with respect to the constant trajectory $\mathbf{0}$ [H]. More precisely, we say $\mathbf{0}$ is an *attractor trajectory* for a system of the form (1) provided that for any $x(0)$ and any $\varepsilon > 0$ there exists a time interval T such that $t > T$ implies $|\mathbf{x}(t) - \mathbf{0}| < \varepsilon$; T must depend only on ε and the distance from $\mathbf{x}(0)$ to $\mathbf{0}$. The existence of a Lyapunov function such as Λ (positive definite, unbounded along rays from the origin, and differentiable with negative derivative along all trajectories except $\mathbf{0}$) implies $\mathbf{0}$ is an attractor trajectory.

As a clear generalization, $\mathbf{0}$ is an attractor trajectory for a system of the form (1) if positive numbers $\{\lambda_1, \lambda_2, ..., \lambda_n\}$ exist satisfying

$$\sum_{i=1}^{n} \kappa_{ti} \lambda_i = 0 \tag{4}$$

for each $t = 1, ..., M$. Such constants $\{\lambda_i\}$ we also call *Volterra multipliers* for the system (1) or corresponding hypergraph. Thus, given (1), this paper amounts to a specification of *qualitative conditions* on a hypergraph structure derived from $\{\phi_t\}$ and signs of $\{\kappa_{ti}\}$ and *quantitative conditions* on path products derived from certain $\{\kappa_{ti}\}$; these conditions insure the *positive solvability* of $\kappa \cdot \lambda = 0$ and so the stability of $\mathbf{0}$ as a constant trajectory of (1).

So-called neural network models of the form

$$dx_i/dt = k_i(I_i - x_i) + p_i(g(x))$$

appear in [JvdD], where k_i and I_i are positive constants and each p_i is a continuously differentable function of $\{g_i\}$, $i = 1, 2, ..., n$. In turn, each g_i is a function of x_i described as follows. Use is made of some positive $\varepsilon < 0.5$ with respect to which $g_i(x_i) = 0$ if $x_i \leqslant -\varepsilon$; $g_i(x_i) = 1$ if $x_i \geqslant \varepsilon$; and $g_i(x_i) = (x_i + \varepsilon)/(2\varepsilon)$ for $-\varepsilon < x_i < \varepsilon$. Thus each g_i is a ramp function. In order to guarantee that almost all neural network trajectories asymptotically approach one of several "memories" (constant trajectories with components ± 1), all edges connected to each barycenter are required to be of the *same* sign. Use is made of a balanced loop condition so that another type of Volterra multipliers exists.

2. Paths in Hypergraphs

Suppose in a hypergraph $\mathbf{H}$ constructed as above that there exist $q \geqslant 1$ distinct vertices $\{v_1, ..., v_q\}$; q distinct barycenters $\{b_1, ..., b_q\}$ and $2q-1$ distinct *nonzero* edges $\{e_1, e_2, ..., e_{2q-1}\}$ with e_1 connecting v_1 and b_1, e_2 connecting b_1 and v_2, and so on. We call such a triple set a *path* from v_1 to b_q. $\mathbf{H}$ is called *connected* if there is a path from any vertex to any barycenter (so the hypergraph in Fig. 1 is not connected because there is no path from v_5 to b_2). Each maximal connected subset of **H** is called a *component*. If $\{r_i\}$ are in order the real numbers (entries in $\{\kappa_{ti}\}$) associated with the $2q-1$ edges in a path from v_1 to b_q, then we define the *path product* of the path to be $(-r_1/r_2)(-r_3/r_4)\cdots(-r_{2q-3}/r_{2q-2})(r_{2q-1})$.

As an example, the path from v_3 to b_2 in the Fig. 1 hypergraph involves (in order) vertices $\{v_3, v_2, v_4\}$, barycenters $\{b_1, b_3, b_2\}$, and associated edges signed $-$, $-$, $-$, $+$, $+$; the path product is negative.

Suppose between vertex v_i and barycenter b_t are two distinct paths with no common intermediate vertices or barycenters. Then the set of all vertices ($p \geqslant 2$ distinct vertices), barycenters, and edges in the union of the two paths is a *p-cycle*. From a p-cycle as a set of hypergraph elements can arise p^2 pairs of paths. Examples of p-cycles are shown in Fig. 2.

Suppose a hypergraph contains a p-cycle with all barycenters of degree 2. Consider the two associated paths from vertex v_i in the cycle to barycenter b_t in the cycle. If the absolute values of the two path products are equal, then we call the p-cycle *balanced*. Clearly this definition is independent of the choice of the pair of paths. A connected hypergraph without p-cycles is called a *tree*.

As used by Volterra and many others [JvdD, M, RZ], linear Volterra multipliers can be found for (3) satisfying $\lambda_i a_{ij} = -\lambda_j a_{ji}$ (the special linear case of (4)) provided:

(α) $\{i \neq j$ and $a_{ij} \neq 0\}$ imply $a_{ij}a_{ji} < 0$;

(β) for any $p \geqslant 3$ distinct indices $\{i_1, i_2, ..., i_p\}$

$$|a_{i_1 i_2} a_{i_2 i_3} \cdots a_{i_p i_1}| = |a_{i_1 i_p} \cdots a_{i_3 i_2} a_{i_2 i_1}|.$$

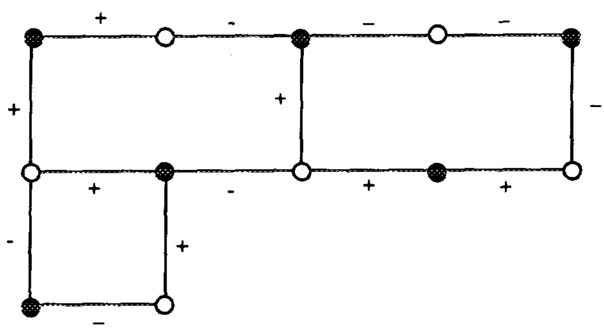

FIG. 2. A hypergraph with one 2-cycle, two 3-cycles, a 4-cycle, a 5-cycle, and a 6-cycle.

In the context of our hypergraph approach, these conditions on (3) amount to:

(α') every barycenter is of mixed sign (and has degree 2);

(β') every p-cycle with $p \geqslant 3$ is balanced.

3. Volterra Multipliers for Special Hypergraphs

We now are in a position to develop conditions sufficient to solve (4). This section contains three little lemmas which set the stage for a fairly general set of conditions.

LEMMA 1 (The Linear Case). *Suppose only edges signed + or − occur in a connected hypergraph* **H** *derived from* (3). *Suppose each barycenter is of mixed sign (and degree* 2). *Then the system admits Volterra multipliers iff all cycles are balanced.*

Proof. Without loss of generality let us choose labels so $a_{12}a_{21} < 0$. Let $\lambda_1 = 1$. Then λ_2 must be $(-a_{12}/a_{21})\lambda_1$, a positive constant. If as well $a_{13}a_{31} < 0$, then λ_3 must be $(-a_{13}/a_{31})\lambda_1$. Suppose $a_{23}a_{32} < 0$. It follows that λ_3 must equal $(-a_{23}/a_{32})\lambda_2$; this additional condition is solvable precisely because $|a_{12}| \cdot |a_{23}| \cdot |a_{31}| = |a_{13}| \cdot |a_{32}| \cdot |a_{21}|$, the balanced cycle or equal path product condition.

Figure 3 contains a general $(p+q)$-cycle in which every barycenter is of degree 2; for the sake of clarity we omit any edges between shown vertices and barycenters not in the cycle.

Let r_i denote the real number in $\{a_{ij}\}$ associated with edge e_i. Note that in Fig. 3 there are two types of paths starting at v_1 and including v_p, namely the paths proceeding clockwise and counterclockwise. Using the

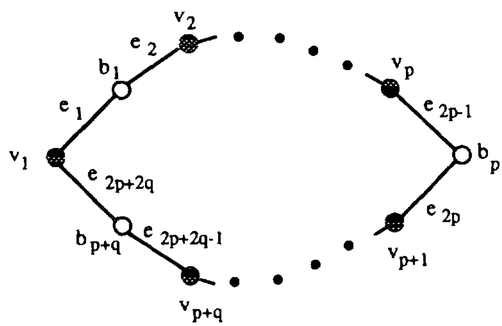

FIG. 3. A typical $(p+q)$-cycle of barycenters of degree 2. Auxiliary edges from shown vertices are omitted.

labels in Fig. 3 and specifying λ_1 to be any positive number, we are led to two values for λ_p corresponding to the two path types. Due to the mixed sign condition, any such λ_p is positive. The balanced path condition is algebraically equivalent to $\lambda_p = (-r_{2p-3}/r_{2p-2}) \cdots (-r_1/r_2)\,\lambda_1$ and $\lambda_p = (-r_{2p+2q}/r_{2p+2q-1}) \cdots (-r_{2p}/r_{2p-1})\,\lambda_1$. Clearly the two specifications of λ_p are equal iff the balanced cycle condition is satisfied, that is, $(-r_1/r_2)(-r_3/r_4) \cdots (-r_{2p+2q-1}/r_{2p+2q}) = 1$. Stating this condition in terms of the sum of the two path products from v_1 to b_p, we have

$$(-r_1/r_2)(-r_3/r_4) \cdots (-r_{2p-3}/r_{2p-2})\, r_{2p-1} + (-r_{2p+2q}/r_{2p+2q-1}) \cdots r_{2p} = 0.$$

Q.E.D.

Now consider a general connected hypergraph. Lemma 1 implies that within a maximal block of barycenters of degree 2 and associated vertices, the balanced loop condition guarantees that specification of any λ value propagates consistently through the block; of course, the same λ values might or might not satisfy other row equations (involving rows of κ with three or more nonzero components) in (4).

LEMMA 2 (The Acyclic Nonlinear Case [J]). *Suppose* **H** *is connected and no cycles occur in* **H**, *that is,* **H** *is a tree. Suppose all edges in* **H** *are signed. Then the system admits Volterra multipliers iff all barycenters are of mixed sign.*

Proof. Clearly Volterra multipliers can exist only if every barycenter is of mixed sign. So suppose every barycenter is of mixed sign. As a typical case, suppose that barycenter b_1 is connected by "+" signed edges to vertices $v_1, ..., v_p$ and "−" signed edges to vertices $v_{p+1}, ..., v_{p+q}$. Let λ_1 be any positive number and set $\lambda_1 = \lambda_2 = \cdots = \lambda_p$. Certainly we may choose positive numbers $\lambda_{p+1}, ..., \lambda_{p+q}$ so that

$$\kappa_{i1}\lambda_1 + \cdots + \kappa_{ip}\lambda_p + \kappa_{ip+1}\lambda_{p+1} + \cdots + \kappa_{ip+q}\lambda_{p+q} = 0.$$

Since all barycenters are of mixed sign, this procedure extends throughout **H** to specify all $\{\lambda_i\}$. Since no cycles occur, redundant specifications do not arise. Q.E.D.

Suppose a hypergraph consists of p vertices, p barycenters of degree 2, and $2p$ edges arranged in a p-cycle, plus a central barycenter b of degree p and its associated p edges. Suppose all barycenters are of mixed sign. We refer to such a hypergraph as a *wagon wheel.* A typical wagon wheel is shown in Fig. 4.

LEMMA 3 (The Wagon Wheel). *Suppose a hypergraph consists of a wagon wheel. Choose a vertex v_1 at random. Select p distinct path products*

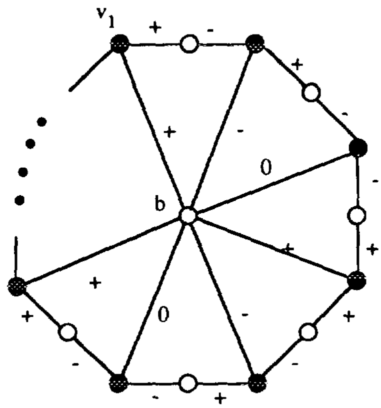

FIG. 4. A typical wagon wheel hypergraph with central barycenter b and vertex v_1.

from v_1 to b, no two having the same last vertex (so all p vertices appear as last vertices). Then the system admits Volterra multipliers iff the rim p-cycle is balanced and the sum of the p distinct path products from v_1 to b is zero.

Proof. Let $\lambda_1 = 1$. Equations in (4) corresponding to barycenters of degree 2 have exactly two nonzero terms, and as in Lemma 1 we can use these equations to determine all of $\{\lambda_2, \lambda_3, ..., \lambda_p\}$. This can be done without contradiction iff the rim p-cycle is balanced. In the equation in (4) corresponding to the central barycenter b we can write each λ_i as a multiple of λ_1 using the path products (due to the balanced cycle condition, it does not matter whether we use clockwise or counterclockwise paths). Thereafter the equation corresponding to b reduces to: zero equals λ_1 times the sum of the path products from v_1 to b. Since λ_1 is positive this equation can be solved iff the sum of the path products is zero. Q.E.D.

Note that Lemma 2 is purely qualitative while Lemmas 1 and 3 are a mixture of qualitative and quantitative conditions on $\{\kappa_{ii}\}$. Roughly speaking, given a hypergraph satisfying the qualitative conditions with arbitrary $\{\kappa_{ii}\}$ values, the quantitative conditions can be met by adjusting the magnitudes of a small fraction of the nonzero entries in $\{\kappa_{ii}\}$. For example, in the wagon wheel at most two such adjustments are needed.

4. CONDENSATION OF HYPERGRAPHS

Standing hypotheses for this section are that every barycenter of $\mathbf{H}$ is of mixed sign and that every cycle containing only barycenters of degree 2 is balanced.

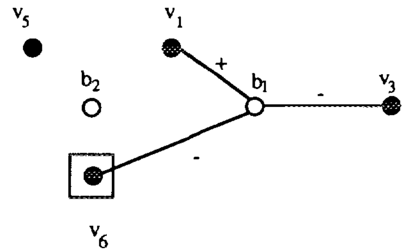

FIG. 5. The condensation of the hypergraph in Fig. 1, assuming that the 2-cycle in Fig. 1 is balanced. Vertex v_6, highlighted by a square, replaces the block in Fig. 1.

Let us define a *block* of vertices in a hypergraph **H** to be a maximal subset of vertices interconnected by paths using only barycenters of degree 2, including such paths which are parts of cycles. (We allow the possibility that the same vertices are otherwise interconnected with other barycenters.) We define the *condensation* $C(\mathbf{H})$ of **H** to be a new hypergraph generated as follows. Edges signed 0 are deleted in $C(\mathbf{H})$. Each block (possibly consisting of a single vertex) becomes a vertex identified with a randomly chosen vertex in the block. All barycenters of degree >2 are retained. If one barycenter of degree >2 is connected by q edges to q vertices in a block, then those q edges are replaced by one new edge; the value of the new edge is the sum of the path products of q paths from the chosen vertex to the barycenter, each having a unique last vertex among the q vertices. Of course, if all q path products have the same sign, then the sign of the resulting edge in the condensed hypergraph is so determined. If the sum of the q path products is 0, then no edge is drawn from the vertex

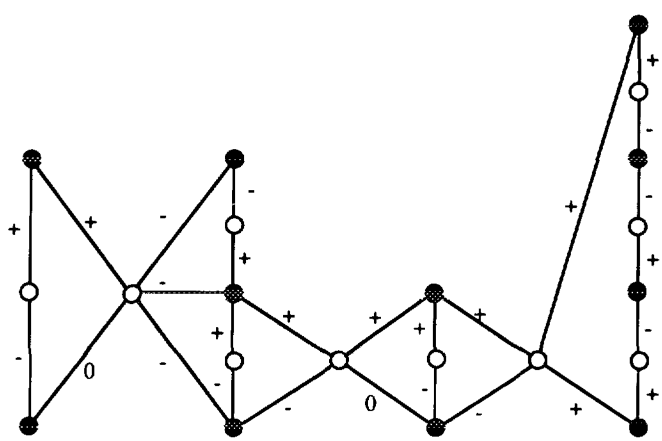

FIG. 6. A hypergraph before condensation.

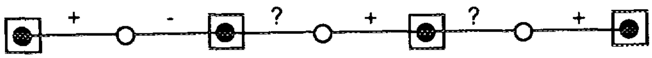

FIG. 7. The condensation of the hypergraph in Fig. 6. The signs of ? edges must be determined by quantitative evaluation of associated sums of path products. If both ? edges are −, then this hypergraph is a tree and all barycenters are of mixed sign.

representing the block to the barycenter. The fact that every barycenter of degree 2 is of mixed sign implies that the sign of the resulting edge in the condensed hypergraph is independent of which vertex serves as chosen vertex in the block. Condensation can be applied nontrivially more than once to some hypergraphs, but after a finite number of condensations, condensation becomes the identity operator.

Let us consider the condensation of the hypergraph in Fig. 1. Assuming the 2-cycle involving vertices v_2, v_4 and barycenters b_2, b_3 is balanced, one condensation produces the hypergraph in Fig. 5. Further condensation is the identity operator.

A more elaborate example is shown in Figs. 6 and 7.

Suppose in the wagon wheel hypergraph the rim p-cycle is balanced and the sum of p path products with distinct last vertices (as in Lemma 3) from any vertex to the central barycenter is 0. The condensation of such a hypergraph is simply one barycenter, one vertex (a block), and no edges.

Using the notions in the above lemmas, we easily deduce the following theorem.

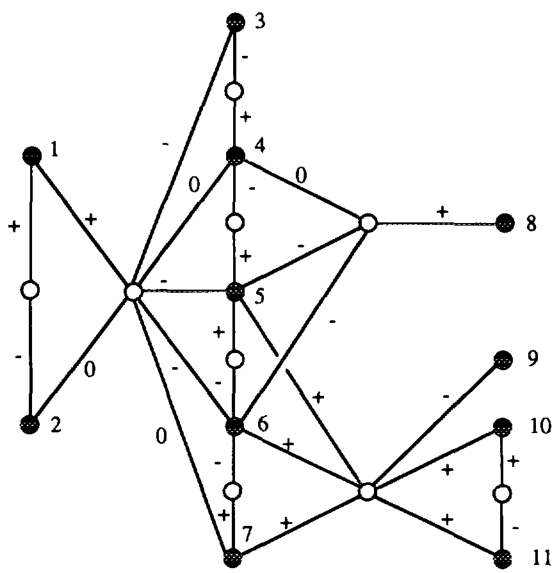

FIG. 8. A hypergraph before condensation.

THEOREM. *Suppose every barycenter of* **H** *is of mixed sign and that every cycle containing only barycenters of degree 2 is balanced. Suppose repeated condensations of a hypergraph (until condensation becomes the identity operator) lead to a hypergraph each component of which is a tree. If every barycenter in the condensed hypergraph is of mixed sign or is attached to no vertices, then the corresponding system* (4) *has Volterra multipliers. The corresponding dynamical system* (1) *has* **0** *as a global attractor trajectory.*

Proof. Volterra multipliers exist for the condensed hypergraph by virtue of Lemma 2. Each value of λ for the chosen vertex in a vertex block can be expanded to generate λ values for all vertices in the block without contradiction by virtue of the balanced cycle condition and the sum of path products condition. Note paths with edges signed 0 do not contribute to such sums. Repetition of this procedure secures all λ values for the original hypergraph. Q.E.D.

A more intricate hypergraph which can be condensed is shown in Fig. 8 (see also Fig. 9).

As an example, a dynamical system corresponding to the hypergraph in Fig. 8 is

$$dx_1/dt = -x_1 + x_2 + x_2x_3x_4x_5x_6x_7$$
$$dx_2/dt = -x_2 - x_1$$
$$dx_3/dt = -x_3 - x_1x_2x_4x_5x_6x_7 - x_4$$
$$dx_4/dt = -x_4 + x_3 - x_5$$
$$dx_5/dt = -x_5 - x_1x_2x_3x_4x_6x_7 + x_4 - x_4x_6x_8 + x_6 + x_6x_7x_9x_{10}x_{11}$$
$$dx_6/dt = -x_6 - x_1x_2x_3x_4x_5x_7 - x_4x_5x_8 - x_5 + x_5x_7x_9x_{10}x_{11} - x_7$$
$$dx_7/dt = -x_7 + x_5x_6x_9x_{10}x_{11} + x_6$$
$$dx_8/dt = -x_8 + x_4x_5x_6$$
$$dx_9/dt = -x_9 - x_5x_6x_7x_{10}x_{11}$$
$$dx_{10}/dt = -x_{10} + x_5x_6x_7x_9x_{11} + x_{11}$$
$$dx_{11}/dt = -x_{11} + x_5x_6x_7x_9x_{10} - x_{10}.$$

The theorem implies that **0** is a global attractor trajectory for this system. One set of Volterra multipliers is $\lambda_1 = \lambda_2 = 1$, $\lambda_3 = \lambda_4 = \lambda_5 = \lambda_6 = \lambda_7 = 1/3$, $\lambda_8 = 2/3$, $\lambda_9 = 3$, $\lambda_{10} = \lambda_{11} = 1$. However, the theorem also implies that any system in the same sign equivalence class likewise has **0** as global attractor (and generally different Volterra multipliers).

It can be said that the above results are weak in the sense that they say

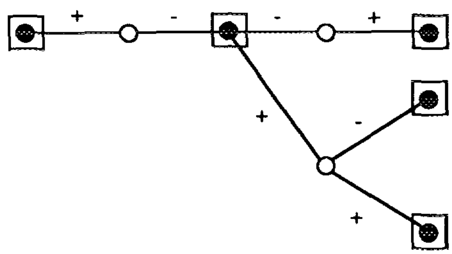

FIG. 9. The result of one condensation of the hypergraph in Fig. 8. Further condensation is the identity operator. The edge signs in the condensed hypergraph are qualitatively determined by the edge signs in the original hypergraph (summands in sums of path products are all the same sign). Also, the condensed hypergraph is a tree with mixed sign barycenters. Thus Lemma 2 implies the condensed hypergraph admits Volterra multipliers. In turn, the theorem then implies the hypergraph itself admits Volterra multipliers. Thus in this case no quantitative conditions on κ are needed.

nothing about hypergraphs with multiple cycles involving only barycenters with degree >2. An example of such a hypergraph is shown in Fig. 10.

On the other hand, it is perhaps worth noting that in the above development we have only used certain ratios of nonzero entries in $\{\kappa_{ii}\}$; it is not necessary that each κ_{ii} be a constant. In fact, all the above pertains just as well to the case that each κ_{ii} is a continuously differentiable function of x and t provided the *ratios* of various elements of $\{\kappa_{ii}\}$ in cycles are constants and admit Volterra multipliers.

Furthermore, it is not necessary that the negative feedback term $-x_i$ appear in each dx_i/dt. Lyapunov stability theory [H] permits replacing

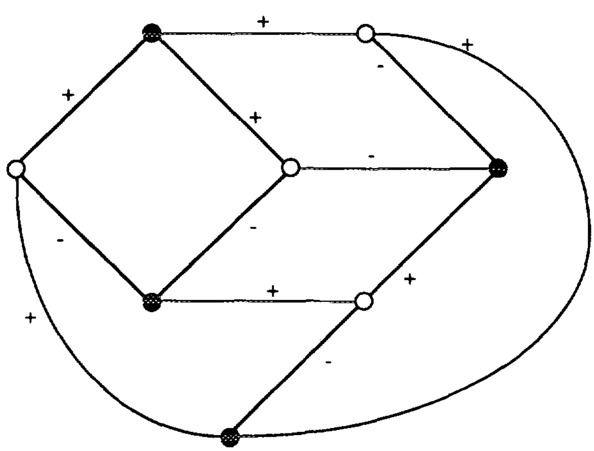

FIG. 10. A hypergraph for which condensation is the identity operator and which cannot be treated by the theorem.

$-x_i$ with any differentiable function $-f_i(x_i, t)$ having the following property: for any $\varepsilon > 0$ there exists $\delta > 0$ such that $\|x\| > \varepsilon$ implies

$$\sum_{i=1}^{n} -f_i(x_i, t)\, x_i < -\delta.$$

This δ value can be used to give a maximum for the time required to go from any initial state to an ε neighborhood of $\mathbf{0}$.

References

[H] W. Hahn, "Stability of Motion," Springer, New York, 1967.

[J] C. Jeffries, Qualitative stability of certain nonlinear systems, *Linear Algebra Appl.* **75** (1986), 133–144.

[JvdD] C. Jeffries and P. van den Driessche, Hypergraph analysis of neural networks, *Physica D* **39** (1989), 315–321.

[M] J. Maybee, Combinatorially symmetric matrices, *Linear Algebra Appl.* **8** (1974), 529–537.

[RZ] R. Redheffer and Z. Zhou, A class of matrices connected with Volterra prey-predator equation, *SIAM J. Algebraic Discrete Methods* **3** (1982), 122–134.