On Invertibility of Nonsquare Generalized Bezoutians

M. I. Gekhtman* and M. Shmoish
Department of Theoretical Mathematics
The Weizmann Institute of Science
Rehovot 76100, Israel

Dedicated to M. Fiedler and V. Pták.

Submitted by Nicholas J. Young

ABSTRACT

Conditions for inverting $H$-Bezoutians of nonsquare matrix polynomials are studied. Necessary and sufficient conditions for invertibility of the Anderson-Jury Bezoutian are obtained. An application to the inversion of block Hankel matrices is given.

1. INTRODUCTION

It is well known that the Bezoutian matrix $B$ of two scalar polynomials is invertible if and only if they are coprime, and in that case the inverse $B^{-1}$ is a Hankel matrix (see [18], [20], and [8]).

In [3] Anderson and Jury introduced a generalized Bezoutian of four matrix polynomials and among other things studied its connection with block Hankel matrices. We recall their original definition.

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*Recipient of a Dov Biegun Postdoctoral Fellowship.
†E-mail: mtgekht@weizmann.weizmann.ac.il.
Let \( W(z) \) be a real rational \( p \times q \) matrix valued function with an expansion at infinity of the form

\[
W(z) = \sum_{i=1}^{\infty} W_i z^{-i}, \quad (1.1)
\]

and let

\[
A(z), \quad B(z), \quad C(z), \quad \text{and} \quad D(z) \quad (1.2)
\]

be matrix polynomials of sizes \( p \times p, p \times q, p \times q, \) and \( q \times q, \) respectively, such that

\[
A(z) = \sum_{j=0}^{\alpha} A_j z^j, \quad D(z) = \sum_{k=0}^{\delta} D_k z^k \quad (1.3)
\]

are nonsingular (i.e., their determinants are not identically equal to zero), \( A_{\alpha} \neq 0 \) and \( D_{\delta} \neq 0, \) and

\[
W(z) = A^{-1}(z)B(z) = C(z)D^{-1}(z). \quad (1.4)
\]

The \( \alpha p \times \delta q \) matrix

\[
\Gamma = \begin{bmatrix} \Gamma_{ij} \end{bmatrix}, \quad i = 0, 1, \ldots, \alpha - 1, \quad j = 0, 1, \ldots, \delta - 1, \quad (1.5)
\]

whose \( p \times q \) block entries \( \Gamma_{ij} \) can be found from the expansion

\[
\Delta(z, y) = \frac{A(z)C(y) - B(z)D(y)}{z - y} = \sum_{i=0}^{\alpha-1} \sum_{j=0}^{\delta-1} z^i \Gamma_{ij} y^j, \quad (1.6)
\]

is called the (generalized) Bezoutian of the quadruple \((A, B; C, D)\).

It is proved in [3] that \( \Gamma \) is congruent to a block Hankel matrix \( H_{\alpha\delta} \) with \( p \times q \) block entries:

\[
\Gamma = S(\alpha, A) H_{\alpha\delta} S(\delta, D), \quad (1.7)
\]
NONSQUARE GENERALIZED BEZOUTIANS

where

\[
S(\alpha, A) := \begin{bmatrix}
A_1 & \cdots & A_{\alpha-1} & A_{\alpha} \\
A_2 & \cdots & A_{\alpha} & 0 \\
\vdots & \vdots & \vdots & \vdots \\
A_\alpha & \cdots & 0 & 0
\end{bmatrix},
\]

\[
H_{\alpha\delta} := \begin{bmatrix}
W_1 & W_2 & \cdots & W_\delta \\
W_2 & W_3 & \cdots & W_{\delta+1} \\
\vdots & \vdots & \ddots & \vdots \\
W_\alpha & W_{\alpha+1} & \cdots & W_{\alpha+\delta-1}
\end{bmatrix},
\]

and \(S(\delta, D)\) has the same pattern as \(S(\alpha, A)\).

It is known that if the Bezoutian \(\Gamma\) is an invertible matrix, then its inverse itself has a block Hankel structure but with \(q \times p\) matrix entries (see [28]). Conversely, the inverse of an invertible block Hankel matrix can be represented as the Bezoutian of quadruple of certain matrix polynomials (see [22] for the square block case). This result for rectangular \(q \times p\) blocks will be justified in the present paper.

If the matrix polynomials (1.2) satisfy the condition

\[
A(z)C(z) = B(z)D(z), \quad z \in \mathbb{C},
\]

where \(A(z)\) and \(D(z)\) are not restricted to be nonsingular polynomials, then we are still able to define the Bezoutian

\[
\Gamma = \mathfrak{B}(A, B; C, D)
\]

by (1.5) and (1.6), where it is assumed that

\[
\alpha = \deg A \geq \deg B, \quad \delta = \deg D \geq \deg C.
\]

It turns out that four square singular matrix polynomials can produce a nonsingular Bezoutian, as the following simple example shows.
EXAMPLE 1.1. Let
\[ A(z) = \begin{bmatrix} z & z - 1 \\ 0 & 0 \end{bmatrix}, \quad B(z) = \begin{bmatrix} 0 & 0 \\ z & z - 1 \end{bmatrix}, \]
\[ C(z) = \begin{bmatrix} 0 & 1 - z \\ 0 & z \end{bmatrix}, \quad D(z) = \begin{bmatrix} z - 1 & 0 \\ -z & 0 \end{bmatrix} \]
be $2 \times 2$ matrix polynomials. Clearly,
\[ A(z)C(z) = 0 = B(z)D(z), \quad z \in \mathbb{C}. \quad (1.11) \]
Thus, all the matrix polynomials are singular, but their Bezoutian
\[ \mathcal{B}(A, B; C, D) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad (1.12) \]
is an invertible matrix.

We shall establish necessary and sufficient conditions for the invertibility of the square Bezoutian matrix.

**Theorem 1.1.** Let $A(z), B(z), C(z),$ and $D(z)$ be matrix polynomials as in (1.2) which meet the conditions (1.8) and (1.10). Assume in addition
\[ \alpha p = \delta q. \quad (1.13) \]
Then the square $\alpha p \times \delta q$ Bezoutian matrix
\[ \Gamma = \mathcal{B}(A, B; C, D) \]
is invertible if and only if
1. $\text{rank}[A(z) \quad B(z)] = p$ for all $z \in \mathbb{C},$
2. $\text{rank}[C(z) \quad D(z)]^T = q$ for all $z \in \mathbb{C},$
3. $\text{rank}[A_\alpha \quad B_\alpha] = p,$
4. $\text{rank}[C_\delta \quad D_\delta] = q,$
where $B_\alpha$ and $C_\delta$ are assumed to be zero $p \times q$ matrices whenever $\text{deg} B < \alpha$ and $\text{deg} C < \delta$ respectively. Moreover, if $\Gamma$ is invertible, then $\Gamma^{-1}$ has a block Hankel structure with respect to the partition into $q \times p$ blocks.
This result will be obtained as a corollary of more general considerations concerning with the one-sided invertibility of nonsquare Bezoutians. We found it more convenient to work with the Bezoutian of only two matrix polynomials as defined by Gohberg and Shalom [15].

In the sequel the following setup will be fixed. The matrix polynomials

\[ M(\lambda) = \sum_{i=0}^{\mu} M_i \lambda^i, \quad N(\lambda) = \sum_{j=0}^{\nu} N_j \lambda^j, \quad M_i \in \mathbb{C}^{p \times s}, \quad N_j \in \mathbb{C}^{s \times q}, \]

are always assumed to satisfy the condition

\[ M(\lambda) N(\lambda) = 0, \quad \lambda \in \mathbb{C}. \]

The Bezoutian (or the H-Bezoutian in the terminology of [15])

\[ B = \mathfrak{B}(M, N) \]

of the polynomials \( M(\lambda) \) and \( N(\lambda) \) is, by definition, the matrix

\[ B = \begin{bmatrix} b_{ij} \end{bmatrix}, \quad i = 0, 1, \ldots, \mu - 1, \quad j = 0, 1, \ldots, \nu - 1 \]

with entries \( b_{ij} \in \mathbb{C}^{p \times q} \) which are given by

\[ \frac{M(\lambda) N(\omega)}{\lambda - \omega} = \sum_{i=0}^{\mu-1} \sum_{j=0}^{\nu-1} \lambda^i b_{ij} \omega^j. \]

Note that this concept of Bezoutian is equivalent to that discussed just above when

\[ s = p + q. \]

To see this it is enough to write

\[ M(\lambda) = \begin{bmatrix} A(\lambda) & B(\lambda) \end{bmatrix}, \quad N(\lambda) = \begin{bmatrix} C(\lambda) \\ -D(\lambda) \end{bmatrix}. \]
The paper is organized as follows. In Section 2 we establish some preliminary results on nonsquare Bezoutians (1.16), including necessary conditions for right invertibility and an important characterization of a nonsquare Bezoutian. Section 2 concludes with some technical lemmas which are needed for further considerations.

In Section 3 we prove several results on the right invertibility of a nonsquare Bezoutian and pass on to the important case when nonsquare blocks constitute a square Bezoutian matrix. In this case we obtain under the assumption (1.19) the necessary and sufficient conditions for invertibility.

In Section 4 we compare the Bezoutian defined by Wimmer in [28] with the definition we work with and discuss the related invertibility conditions.

In Section 5 we prove that the inverse of a block Hankel matrix with rectangular block entries can be represented as a generalized Bezoutian. This allows us to establish some results on the invertibility of block Hankel matrices. In particular, we obtain the formula for the inverse of Hermitian block Hankel matrix in terms of only two block columns of the inverse (this formula appears implicitly in [5]).

Some words on the notation: The symbols C and \( \mathbb{C}^{p \times q} \) will denote the complex numbers and the complex \( p \times q \) matrices, respectively, whereas \( \mathbb{C}^{p} \) is short for \( \mathbb{C}^{p \times 1} \). The symbol \( I_{n} \) designates the \( n \times n \) identity matrix. If \( A \) is a matrix, then \( A^{*} \) denotes its conjugate transpose, or just its complex conjugate if \( A \) is a \( 1 \times 1 \) matrix, that is, a number.

The block column matrix

\[
\begin{bmatrix}
A_{1} \\
A_{2} \\
\vdots \\
A_{n}
\end{bmatrix}
\]

will be denoted by \([A_{1}, A_{2}, \ldots, A_{n}]^{bT}\) (where the superscript \( bT \) stands for block transpose). We write

\[
\text{diag}(A_{1}, A_{2}, \ldots, A_{n})
\]

for the block-diagonal matrix \( A_{1}, A_{2}, \ldots, A_{n} \) on the main diagonal.

If \( W(\lambda) \) is a rational matrix valued function with expansion at infinity

\[
W(\lambda) = \sum_{i=-n}^{\infty} W_{i} \lambda^{-i}, \quad n \geq 0,
\]
then we denote by

\[ \pi_+ W := \sum_{i = -n}^{0} W_i \lambda^{-i} = \sum_{j = 0}^{n} W_{-j} \lambda^{j} \quad (1.21) \]

its polynomial part and by

\[ \pi_- W := W - \pi_+ W \quad (1.22) \]

its strictly rational part.

If \( M(\lambda) = \sum_{i=0}^{\mu} M_i \lambda^i \) is a matrix polynomial (\( M_{\mu} \) is nonzero), then we define

\[ M^*(\lambda) := M(\lambda^*)^*, \quad (1.23) \]
\[ M(\infty) := M_{\mu}, \quad (1.24) \]

and

\[ T(\nu, M) := \begin{bmatrix} M_0 & 0 \\ M_1 & M_0 \\ \vdots & \vdots \\ M_{\mu} & \cdots & M_0 \\ 0 & \cdots & \cdots & \cdots & M_{\mu} \end{bmatrix}, \quad (1.25) \]

where \( \nu \) indicates the number of block columns of the block Toeplitz matrix \( T(\nu, M) \).

The symbol \( \mathcal{B} \) is used throughout this paper to indicate the Bezoutian of four matrix polynomials, while the script \( \mathcal{B} \) stands for the H-Bezoutian of two nonsquare matrix polynomials.
2. PRELIMINARY RESULTS

To begin with, we establish some necessary conditions for a nonsquare Bezoutian $\mathcal{B}(M, N)$ to be right or left invertible.

**Lemma 2.1.** Let

$$M(\lambda) = \sum_{i=0}^{\mu} M_i \lambda^i \quad \text{and} \quad N(\lambda) = \sum_{j=0}^{\nu} N_j \lambda^j \quad (2.1)$$

be $p \times s$ and $s \times q$ matrix polynomials, respectively, which satisfy the condition

$$M(\lambda)N(\lambda) = 0 \quad \text{for all} \quad \lambda \in \mathbb{C}. \quad (2.2)$$

If the Bezoutian

$$B = \mathcal{B}(M, N)$$

is right [left] invertible, then

$$\text{rank } M_\mu = p \quad [\text{rank } N_\nu = q] \quad (2.3)$$

and

$$\text{rank } M(\lambda) = p \quad [\text{rank } N(\lambda) = q] \quad \text{for all } \lambda \in \mathbb{C}. \quad (2.4)$$

**Proof.** Assume that rank $M_\nu < p$. Then there exists a nonzero vector $\xi \in \mathbb{C}^p$ such that $\xi^*M_\mu = 0$. Therefore, for every fixed $\omega$,

$$\xi^* \frac{M(\lambda)N(\omega)}{\lambda - \omega} = \xi^* \left( \frac{M(\lambda) - M(\omega)}{\lambda - \omega} \right) N(\omega)$$

is a polynomial in $\lambda$ of degree at most $\mu - 2$. Hence, for every $\omega$,

$$\xi^* \left[ I_p \lambda I_p \cdots \lambda^{\mu-1} I_p \right] B \left[ I_q \omega I_q \cdots \omega^{\nu-1} I_q \right]^* \quad (br)$$
is also a polynomial in $\lambda$ of degree at most $\mu - 2$. Consider the natural partition of the Bezoutian $B$ into $p \times q$ blocks

$$B = [b_{ij}], \quad i = 0, \ldots, \mu - 1, \quad j = 0, \ldots, \nu - 1.$$  

Then it is obvious that

$$\xi^* [b_{\mu-1,0} \quad b_{\mu-1,1} \quad \cdots \quad b_{\mu-1,\nu-1}] = 0,$$

which contradicts to the right invertibility of $B$. The contradiction shows that

$$\text{rank } M_{\mu} = p.$$  

To prove (2.4) let us observe that in view of the right invertibility of $B$, for every nonzero vector $\eta \in \mathbb{C}^p$ and for every fixed $\lambda \in \mathbb{C}$ there exists some $\omega_0 \in \mathbb{C}$ such that

$$\eta^* \left[ I_p \quad \lambda I_p \quad \cdots \quad \lambda^{\mu-1} I_p \right] B \left[ I_q \quad \omega_0 I_q \quad \cdots \quad \omega_0^{\nu-1} I_q \right]^r \neq 0.$$  

By the very definition of the Bezoutian $B = \mathcal{B}(M, N)$ this means that

$$\frac{\eta^* M(\lambda) N(\omega_0)}{\lambda - \omega_0} \neq 0, \quad \text{i.e.,} \quad \eta^* M(\lambda) \neq 0 \quad \text{for all } \lambda \in \mathbb{C},$$

which is equivalent to (2.4). Similar arguments work when $B$ is a left invertible matrix.  

**Remark.** The condition in Lemma 2.1

$$\text{rank } M(\lambda) = p \quad \text{for all } \lambda \in \mathbb{C}$$

is equivalent to the existence of the $s \times p$ matrix polynomial $K(\lambda)$ such that

$$M(\lambda) K(\lambda) = I_p \quad (2.5)$$

(see e.g., Theorem 6.1 in [24]).
The following theorem contains an important characterization of the $H$-Bezoutian (cf. [9, Theorem 4.2], where the Anderson-Jury Bezoutian is considered).

**THEOREM 2.1.** Let $R(\lambda)$ be a rational $q \times p$ matrix valued function with expansion at infinity

$$R(\lambda) = \sum_{j=1}^{\infty} r_j \lambda^{-j}, \quad r_j \in \mathbb{C}^{q \times p}, \quad j = 1, 2, \ldots . \quad (2.6)$$

Let $B = \mathcal{B}(M, N)$ be the Bezoutian of the matrix polynomials $M(\lambda), N(\lambda)$. Then

$$M(\lambda) \pi_{+} [N(\lambda) R(\lambda)] = -\left[ I_p \lambda I_p \cdots \lambda^{\mu - 1} I_p \right] B \left[ r_1 r_2 \cdots r_{\mu} \right]^{br}$$

and

$$\pi_{+} [R(\lambda) M(\lambda)] N(\lambda) = \left[ r_1 r_2 \cdots r_{\mu} \right] B \left[ I_q \lambda I_q \cdots \lambda^{\nu - 1} I_q \right]^{br}, \quad (2.8)$$

where $\pi_{+} [\cdot]$ denotes the polynomial part as in (1.21), and $\mu$ and $\nu$ are the degrees of $M(\lambda)$ and $N(\lambda)$ respectively.

**Proof.** It is easy to verify that the polynomial

$$F(\lambda) = \pi_{+} [N(\lambda) R(\lambda)]$$

is of degree at most $\nu - 1$ with its coefficients given by

$$[F_0 \ F_1 \cdots \ F_{\nu - 1}]^{br} = S(\nu, N) [r_1 \ r_2 \cdots \ r_{\nu}]^{br}, \quad (2.10)$$

where the matrix

$$S(\nu, N) = \begin{bmatrix} N_1 & \cdots & N_{\nu - 1} & N_{\nu} \\ N_2 & \cdots & N_{\nu} & 0 \\ \vdots & \vdots & \vdots & \vdots \\ N_{\nu} & \cdots & 0 & 0 \end{bmatrix} \quad (2.11)$$

has a block Hankel structure with $\nu$ block columns and $\nu$ block rows.
Clearly enough, the product $M(\lambda)F(\lambda)$ can be expressed in the following two equivalent forms:

\[
M(\lambda)F(\lambda) = \begin{bmatrix} I_p & \lambda I_p & \cdots & \lambda^k I_p \end{bmatrix} T(\nu, M) \begin{bmatrix} F_0 & F_1 & \cdots & F_{\nu-1} \end{bmatrix}^T \tag{2.12}
\]

\[
= \begin{bmatrix} M_0 & M_1 & \cdots & M_\mu \end{bmatrix} T(\mu, F)^T \begin{bmatrix} I_p & \lambda I_p & \cdots & \lambda^k I_p \end{bmatrix}^T, \tag{2.13}
\]

where $T(\nu, M)$ and $T(\mu, F)$ denote the block Toeplitz matrices of the form (1.25) with $\nu$ and $\mu$ block columns respectively; $k := \mu + \nu - 1$.

Combining the formulas (2.9), (2.10), and (2.12), we get

\[
M(\lambda)\pi_+ \left[ N(\lambda) R(\lambda) \right] = \begin{bmatrix} I_p & \lambda I_p & \cdots & \lambda^k I_p \end{bmatrix} T(\nu, M) S(\nu, N) \begin{bmatrix} r_1 & r_2 & \cdots & r_\nu \end{bmatrix}^T. \tag{2.14}
\]

On the other hand, for all $\lambda, \omega \in \mathbb{C}$ we have

\[
\begin{bmatrix} I_p & \lambda I_p & \cdots & \lambda^{\mu-1} I_p \end{bmatrix} B \begin{bmatrix} I_q & \omega I_q & \cdots & \omega^{r-1} I_q \end{bmatrix}^T
\]

\[
= M(\lambda) \left( -\frac{N(\lambda) - N(\omega)}{\lambda - \omega} \right)
\]

\[
= -M(\lambda) \begin{bmatrix} I_q & \lambda I_q & \cdots & \lambda^{r-1} I_q \end{bmatrix} S(\nu, N) \begin{bmatrix} I_q & \omega I_q & \cdots & \omega^{r-1} I_q \end{bmatrix}^T
\]

\[
= -\begin{bmatrix} I_p & \lambda I_p & \cdots & \lambda^k I_p \end{bmatrix} T(\nu, M) S(\nu, N) \begin{bmatrix} I_q & \omega I_q & \cdots & \omega^{r-1} I_q \end{bmatrix}^T, \tag{2.15}
\]

where $k = \mu + \nu - 1$. Since $\omega$ is arbitrary, it follows immediately that

\[
\begin{bmatrix} I_p & \lambda I_p & \cdots & \lambda^{\mu-1} I_p \end{bmatrix} B = -\begin{bmatrix} I_p & \lambda I_p & \cdots & \lambda^k I_p \end{bmatrix} T(\nu, M) S(\nu, N). \tag{2.16}
\]

Comparing (2.14) and (2.16), we get the desired identity (2.7). The relation (2.8) can be proved in much the same way.
COROLLARY. The Bezoutian $\mathcal{B}(M, N)$ of $M(\lambda), N(\lambda)$ as in (2.1), (2.2) is right invertible if and only if there exist $R_0(\lambda), R_1(\lambda), \ldots, R_{\mu-1}(\lambda)$ of the form (2.6) such that

$$M(\lambda) \pi_s \left[ N(\lambda) R_s(\lambda) \right] = \lambda^s I_p, \quad s = 0, 1, \ldots, \mu - 1. \quad (2.17)$$

REMARK. Equation (2.16) provides another proof of the representation of the $H$-Bezoutian in terms of the coefficients of matrix polynomials $M(\lambda)$ and $N(\lambda)$ (see [15, p. 266]). Different explicit expressions for (classical) Bezoutians were presented in [23].

Our next objective is to find out how the Bezoutian $\mathcal{B}(M, N)$ changes when the matrix polynomial $N(\lambda)$ undergoes the elementary column operations (see e.g., [17, p. 74]).

Recall that the performing of a finite sequence of elementary column operations on $N(\lambda)$ is equivalent to postmultiplication by some square matrix polynomial $P(\lambda)$ which is unimodular:

$$\det P(\lambda) \equiv \text{const} \neq 0. \quad (2.18)$$

We consider even more general transformations of matrix polynomials (cf. Proposition 2.2 in [3]):

**Lemma 2.2.** Let $M(\lambda)$ and $N(\lambda)$ be matrix polynomials as in (2.1), (2.2) such that

$$M(\lambda) N(\lambda) = 0, \quad \lambda \in \mathbb{C}. \quad (2.19)$$

Let $K(\lambda) = \sum_{\ell=0}^{r} K_{\ell} \lambda^\ell$ be any $q \times r$ matrix polynomial. Then for an arbitrary $n_r \times (\kappa + v - n)r$ constant matrix $A$ we have

$$\mathcal{B}(M, NK) = \mathcal{B}(M, N) T(\nu, K)^{br} [I_{nr} - A]^{br}, \quad (2.20)$$

where $\nu := \deg N, n := \deg(NK)$, and $T(\nu, K)$ is of the form (1.25).

Proof. To compute $\mathcal{B}(M, NK)$ we write

$$\left( \frac{M(\lambda) N(\omega)}{\lambda - \omega} \right) K(\omega)$$

$$= \left[ I_p, \lambda I_p, \ldots, \lambda^{\mu-1} I_p \right] \mathcal{B}(M, N) \left[ I_q, \omega I_q, \ldots, \omega^{v-1} I_q \right]^{br} K(\omega)$$

$$= \left[ I_p, \lambda I_p, \ldots, \lambda^{\mu-1} I_p \right] \mathcal{B}(M, N) T(\nu, K)^{br} \left[ I_r, \omega I_r, \ldots, \omega^{n} I_r \right]^{br},$$
where the second equality is justified by (2.12); \( m := \kappa + \nu - 1 \). On the other hand, since \( \text{deg}(NK) = n \), the Bezoutian \( \mathcal{B}(M, NK) \) is a \( \mu p \times nr \) matrix:

\[
\begin{align*}
\frac{M(\lambda)\{N(\omega)K(\omega)\}}{\lambda - \omega} &= \left[ I_p \ \lambda I_p \ \ldots \ \lambda^{\mu-1}I_p \right] \mathcal{B}(M, NK) \\
&\times \left[ I_r \ \omega I_r \ \ldots \ \omega^{n-1}I_r \right]^\text{tr}.
\end{align*}
\] (2.21)

This means that in the matrix product

\[
\mathcal{B}(M, N)T(\nu, K)^\text{tr}
\]

the last \((\kappa + \nu - n)r\) columns are zero, and therefore (2.20) holds true for any constant matrix \( A \) of the appropriate size. ■

**Lemma 2.3.** Let \( N(\lambda) \) be an \( s \times q \) matrix polynomial which has full column rank, i.e.,

\[
\text{rank } N(\lambda_0) = q \quad \text{for some } \lambda_0 \in \mathbb{C}.
\] (2.22)

Then there exists a \( q \times q \) matrix polynomial \( T(\lambda) \) such that the top coefficient of the \( s \times q \) matrix polynomial \( Q(\lambda) = N(\lambda)T(\lambda) \) has rank \( q \), and

\[
\text{deg } Q = \text{deg } N.
\] (2.23)

**Proof.** It is known that any matrix polynomial \( N(\lambda) \) which has full column rank can be transformed to column-reduced form by finite sequence of elementary column operations (see, e.g., [17, p. 387]). Denote the corresponding \( q \times q \) unimodular matrix by \( P(\lambda) \). Then the matrix polynomial \( S(\lambda) = N(\lambda)P(\lambda) \), by the very definition of the column-reduced form [17, p. 386], has the following shape:

\[
S(\lambda) = \left[ S^{(1)}(\lambda) \ S^{(2)}(\lambda) \ \ldots \ S^{(q)}(\lambda) \right],
\] (2.24)

where the top coefficients \( s_j \in \mathbb{C}^s \) of the \( s \times 1 \) vector polynomials \( S^{(j)}(\lambda) \), \( j = 1, 2, \ldots, q \), are linearly independent:

\[
\text{rank} \begin{bmatrix} s_1 & s_2 & \ldots & s_q \end{bmatrix} = q.
\] (2.25)
It turns out that the unimodular matrix $P(\lambda)$ can be chosen in such a way that

$$\deg S \leq \nu := \deg N. \quad (2.26)$$

To complete the proof it is enough to set

$$T(\lambda) = P(\lambda) \text{ diag}\{\lambda^{\nu-d_1}, \lambda^{\nu-d_2}, \ldots, \lambda^{\nu-d_q}\}, \quad (2.27)$$

where $d_j := \deg S^{(j)}, j = 1, 2, \ldots, q.$

3. INVERTIBILITY CONDITIONS FOR GENERALIZED BEZOUTIANS

We have already shown that the right invertibility of the Bezoutian $\mathcal{B}(M,N)$ imposes some restrictions on $M(\lambda)$:

$$\text{rank } M_\mu = p = \text{rank } M(\lambda) \text{ for all } \lambda \in \mathbb{C}. \quad (3.1)$$

Clearly, these necessary conditions are far from being sufficient. A certain additional assumption about $N(\lambda)$ has to be made to insure the right invertibility of $\mathcal{B}(M,N)$.

The following theorem gives sufficient conditions for the right invertibility of a nonsquare Bezoutian provided

$$s = p + q. \quad (3.2)$$

**Theorem 3.1.** Let

$$M(\lambda) = \sum_{i=0}^{\mu} M_i \lambda^i \text{ and } N(\lambda) = \sum_{j=0}^{\nu} N_j \lambda^j, \quad \mu, \nu \geq 1, \quad (3.3)$$

be $p \times (p + q)$ and $(p + q) \times q$ matrix polynomials, respectively, such that

$$M(\lambda) N(\lambda) = 0, \quad \lambda \in \mathbb{C}. \quad (3.4)$$
Assume that the following conditions are met:

\[ \text{rank } M(\lambda) = p \quad \text{for every } \lambda \in \mathbb{C}, \]
\[ \text{rank } M_{\mu} = p \quad \text{(3.6)} \]
\[ \text{rank } N_{\nu} = q \quad \text{(3.7)} \]

Then the \( m \times n \) Bezoutian \( B = B(M, N) \) is right invertible. Moreover, there exists a block Hankel matrix \( H \) with \( q \times p \) blocks such that

\[ BH = I_{mp}. \quad \text{(3.8)} \]

**Proof.** It follows from (3.5) that there exists a \( s \times p \) matrix polynomial \( K(\lambda) \) such that

\[ M(\lambda) K(\lambda) = I_p. \quad \text{(3.9)} \]

We proceed in steps.

**Step 1.** The polynomial \( K(\lambda) \) can be chosen in such a way that

\[ \deg K < \nu := \deg N. \]

**Proof of step 1.** Let \( K(\lambda) = K_0 + K_1 \lambda + \cdots + K_{\kappa} \lambda^\kappa \). Then \( M_{\mu} K_{\kappa} = 0 \) by (3.9), and \( M_{\mu} N_{\nu} = 0 \) by (3.4). Therefore (3.7) insures that

\[ K_{\kappa} = N_{\nu} h \quad \text{(3.10)} \]

for some \( h \in \mathbb{C}^{q \times p} \). If \( \kappa \geq \nu \) then the matrix polynomial

\[ \tilde{K}(\lambda) = K(\lambda) - \lambda^{\kappa-\nu} N(\lambda) h \]

satisfies the conditions

\[ \deg \tilde{K} < \kappa \quad \text{and} \quad M(\lambda) \tilde{K}(\lambda) = I_p. \quad \text{(3.11)} \]

By repeating this process appropriate number of times, one can complete the proof of step 1.

**Step 2.** There exists a matrix valued function of the form

\[ R(\lambda) = \sum_{i=0}^{\mu + \nu - 2} h_i \lambda^{-i}, \quad h_i \in \mathbb{C}^{q \times p}, \quad \text{(3.12)} \]
such that

$$\pi_+ \left[ \lambda^{\mu-1} \{ K(\lambda) - N(\lambda) R(\lambda) \} \right] = 0,$$

(3.13)

where \( K(\lambda) \) is chosen as in step 1.

Proof of step 2. Let \( K(\lambda) = \sum_{j=0}^{r-1} K_j \lambda^j \), where \( K_{r-1} \) can be the zero matrix. It follows from (3.9) that \( M_{\mu} K_{r-1} = 0 \), and therefore, by (3.7), there exists \( h_0 \in \mathbb{C}^{q \times p} \) such that

$$K_{r-1} = N_{r} h_0.$$

Then the rational function

$$Q_0(\lambda) = K(\lambda) - N(\lambda) \lambda^{-1} h_0 = Q^{(0)}_{r-2} \lambda^{r-2} + Q^{(0)}_{r-3} \lambda^{r-3} + \cdots + Q^{(0)}_{r-1} \lambda^{-1}$$

is still subject to the equation of the form (3.9):

$$M(\lambda) Q_0(\lambda) = I_p.$$

Similarly, one can choose successively \( q \times p \) matrices \( h_1, h_2, \ldots, h_{r+\mu-2} \) such that the rational function

$$Q_j(\lambda) = K(\lambda) - N(\lambda) \sum_{i=0}^{j} h_i \lambda^{-i-1}, \quad j = \mu + \nu - 2,$$

(3.14)

has the following expansion:

$$Q_j(\lambda) = Q_{-\mu}^{(j)} \lambda^{-\mu} + Q_{-\mu-1}^{(j)} \lambda^{-\mu-1} + \cdots.$$  

(3.15)

It is easily seen now that the rational function

$$R(\lambda) = \sum_{i=0}^{\mu + \nu - 2} h_i \lambda^{-i-1}$$

(3.16)

meets (3.13).

Step 3 is to establish the formula

$$M(\lambda) \pi_+ \left[ N(\lambda) \tilde{R}_s(\lambda) \right] = \lambda^s I_p, \quad s = 0, 1, \ldots, \mu - 1,$$

(3.17)
Proof of step 3. By (3.15), (3.16) we have

\[ K(\lambda) = Q_j(\lambda) + N(\lambda)R(\lambda). \]

Thus, by step 2 we get

\[ \lambda^sK(\lambda) = \pi_+ [\lambda^sQ_j(\lambda)] + \pi_+ [\lambda^sN(\lambda)R(\lambda)] \]

\[ = \pi_+ [\lambda^sN(\lambda)R(\lambda)], \quad s = 0, 1, \ldots, \mu - 1. \quad (3.19) \]

Then

\[ N(\lambda)\tilde{R}_s(\lambda) = \lambda^sN(\lambda)R(\lambda) - N(\lambda)\pi_+ [\lambda^sR(\lambda)], \]

and it follows from (3.19) that

\[ \pi_+ [N(\lambda)\tilde{R}_s(\lambda)] = \lambda^sK(\lambda) - N(\lambda)\pi_+ [\lambda^sR(\lambda)]. \quad (3.20) \]

Premultiplying both sides of (3.20) by \( M(\lambda) \) and invoking

\[ M(\lambda)K(\lambda) = I_p, \quad M(\lambda)N(\lambda) = 0, \]

we get the desired formula (3.17).

Step 4 is to complete the proof of the theorem. By step 3 we have

\[ M(\lambda)\pi_+ [N(\lambda)\tilde{R}_s(\lambda)] = \lambda^sI_p, \quad s = 0, 1, \ldots, \mu - 1, \]

where

\[ \tilde{R}_s(\lambda) = \pi_- [\lambda^sR(\lambda)] = \sum_{i=0}^{\mu + \nu - 2} h_{i+s} \lambda^{-i-1}. \]
On the other hand, (3.17) and (2.7) imply that

\[
\lambda^t I_p = M(\lambda) \pi_s \left[ N(\lambda) \tilde{R}_s(\lambda) \right] \\
= \left[ I_p \lambda I_p \cdots \lambda^{t-1} I_p \right] B[h_s, h_{s+1}, \ldots, h_{s+\nu-1}]^T, \quad (3.21)
\]

where \( B = \mathcal{B}(M, N) \) is a Bezoutian, \( s = 0, 1, \ldots, \mu - 1 \). Clearly, (3.21) is equivalent to the equality

\[
BH = I_{\mu p},
\]

where the matrix \( H \) is a block Hankel with \( q \times p \) block entries \((-h_i)\):

\[
H = \begin{bmatrix}
    h_0 & h_1 & \cdots & h_{\mu-1} \\
    h_1 & h_2 & \cdots & h_{\mu-1} \\
    \vdots & \vdots & \ddots & \vdots \\
    h_{\nu-1} & h_{\nu} & \cdots & h_{\mu+\nu-2}
\end{bmatrix}.
\]

The condition (3.7) turns out to be rather restrictive, as the following example shows.

**Example 3.1.** Let

\[
M(\lambda) = \begin{bmatrix} 1 & \lambda - 1 & \lambda^2 \end{bmatrix} \quad \text{and} \quad N(\lambda) = \begin{bmatrix} \lambda - 2\lambda^2 & \lambda^2 \\
\lambda & 0 \\
1 & -1 \end{bmatrix}.
\]

Then it is easily checked that the Bezoutian

\[
\mathcal{B}(M, N) = \begin{bmatrix} 0 & 0 & 2 & 1 \\
1 & 1 & 0 & 0 \end{bmatrix}
\]

is a right invertible \( 2 \times 4 \) matrix, even though

\[
\text{rank } N_2 = 1 < q = 2,
\]

where \( N_2 \) is the top coefficient of the matrix polynomial \( N(\lambda) \).
Theorem 3.2. If, in the setting of Theorem 3.1, the condition (3.7) is replaced by the assumption that \( N(\lambda) \) has full column rank:

\[
\text{rank } N(\lambda_0) = q \quad \text{for some } \lambda_0 \in \mathbb{C}, \tag{3.22}
\]

then the Bezoutian \( B(M, N) \) is right invertible and there exists a block Hankel matrix \( H \) with \( q \times p \) blocks such that

\[
BH = I_{\mu_p}. \tag{3.23}
\]

Proof. By Lemma 2.3, there exists a \( q \times q \) matrix polynomial \( K(\lambda) \) of degree \( \kappa \) such that the \( s \times q \) matrix polynomial \( Q(\lambda) = N(\lambda)K(\lambda) \) has the top coefficient of rank \( q \) and \( \deg Q = \deg N = \nu \). Then by Theorem 2.2 the Bezoutian \( \tilde{B} = B(M, Q) \) has a block Hankel right inverse \( \tilde{H} = [\tilde{h}_{i+j}] \), \( i = 0, \ldots, \nu - 1, \; j = 0, \ldots, \mu - 1 \), with \( q \times p \) blocks. Using Lemma 2.2, one has

\[
I_{\mu_p} = \tilde{B}\tilde{H} = BT(\nu, K)^{br} [\tilde{H} \; A]^{br}, \tag{3.24}
\]

where \( A \) is an arbitrary matrix of appropriate size. Since all the block columns of the matrix \( BT(\nu, K)^{br} \) starting from \((\nu + 1)\)th are zero, one can choose a matrix \( A \) in such a way that the matrix \( \tilde{H} = [\tilde{h}_{i+j}] \) is a \((\kappa + \nu)q \times \mu p\) block Hankel matrix with \( q \times p \) blocks \( \tilde{h}_{ij} \). Let us compute the \((i, j)\) block entry of the matrix

\[
H = [h_{ij}] = T(\nu, K)^{br} \tilde{H}, \tag{3.25}
\]

namely,

\[
h_{ij} = \sum_{t=0}^{\kappa} K_{ij} \tilde{h}_{i+j+t}, \tag{3.26}
\]

where \( i = 1, \ldots, \nu - 1, \; j = 0, \ldots, \mu - 1 \).

It follows from (3.24) and (3.25) that \( BH = I_{\mu_p} \), while (3.26) implies that \( H \) is a block Hankel matrix with \( q \times p \) blocks. \( \blacksquare \)

The two previous theorems deal with the case when \( s = p + q \). The following corollary of Theorem 3.2 gives a simple sufficient condition for the right invertibility of the Bezoutian when \( s > p + q \).
COROLLARY. Let $M(A)$ and $N(A)$ be $p \times s$ and $s \times q$ matrix polynomials, which satisfy the conditions (3.4)–(3.6) of Theorem 3.1. Suppose that for some $\lambda_0 \in \mathbb{C}$

$$\dim\{\ker\begin{bmatrix} M_0 & M_1 & \cdots & M_\mu \end{bmatrix} + \text{Range } N(\lambda_0)\} \geq s - p. \quad (3.27)$$

Then the Bezoutian $B = \mathcal{B}(M, N)$ is right invertible.

**Proof.** A vector $\xi \in \mathbb{C}^s$ belongs to $\ker\begin{bmatrix} M_0 & M_1 & \cdots & M_\mu \end{bmatrix}$, i.e., $\begin{bmatrix} M_0 & M_1 & \cdots & M_\mu \end{bmatrix} \xi = 0$, if and only if $M(A)\xi = 0$. The inequality (3.27) implies that there exist vectors $\xi_1, \xi_2, \ldots, \xi_k \in \ker\begin{bmatrix} M_0 & M_1 & \cdots & M_\mu \end{bmatrix}$ such that for some $\lambda_0 \in \mathbb{C}$ we have $\text{rank}\begin{bmatrix} N(\lambda_0) & \xi_1 & \cdots & \xi_k \end{bmatrix} = s - p$. We may assume without loss of generality that $\text{rank } N(\lambda_0) = q$ and $k = s - p - q$, and consider a matrix polynomial $\tilde{N}(\lambda) = \begin{bmatrix} N(\lambda) & \xi_1 & \cdots & \xi_k \end{bmatrix}$, which evidently satisfies the conditions of the Theorem 3.2. Therefore the Bezoutian $\tilde{B} = \mathcal{B}(M, \tilde{N})$ is right invertible. But it follows from the definition of $\tilde{N}(\lambda)$ that $(i, j)$ block entry of $\tilde{B}$ has a form $\tilde{b}_{ij} = [b_{ij} 0]$, where $b_{ij}$ is a $(i, j)$ block entry of $B$ and 0 stands for a $p \times k$ zero matrix. Hence, $\text{rank } \tilde{B} = \text{rank } B$. \hfill \blacksquare

Now we pass on to the important case when the nonsquare blocks $b_{ij}$ fill out a square generalized Bezoutian matrix $B$. First, we give some necessary conditions for $B$ to be invertible.

**THEOREM 3.3.** Let

$$M(\lambda) = \sum_{i=0}^{\mu} M_i \lambda^i \quad \text{and} \quad N(\lambda) = \sum_{j=0}^{\nu} N_j \lambda^j \quad (3.28)$$

be $p \times s$ and $s \times q$ matrix polynomials, respectively, such that $\mu p = \nu q$, $M(\lambda)N(\lambda) = 0$. Assume that their generalized Bezoutian is an invertible matrix.

Then for every point $\lambda \in \mathbb{C} \cup \{\infty\}$ one has

$$\text{rank } M(\lambda) = p \quad \text{and} \quad \text{rank } N(\lambda) = q, \quad (3.29)$$

$$s \geq p + q. \quad (3.30)$$
Proof. The assertion (3.29) follows easily from Lemmas 2.1 and 2.2. To prove (3.30) let us fix any $\lambda_0 \in \mathbb{C}$. Then in view of

$$M(\lambda_0)N(\lambda_0) = 0,$$

by Sylvester's inequality for ranks we have

$$\text{rank } M(\lambda_0) + \text{rank } N(\lambda_0) - s \leq \text{rank } [M(\lambda_0)N(\lambda_0)] = 0,$$

i.e., $\text{rank } M(\lambda_0) + \text{rank } N(\lambda_0) \leq s$, or $s \geq p + q$. 

In general these necessary conditions are not sufficient for the invertibility of the generalized Bezoutian, as is clear from the following

Example 3.2. Let $p = q = 1$, $\mu = \nu = 3$, $s = 4$,

$$M(\lambda) = \begin{bmatrix} -\lambda - 2\lambda^3 & 1 - \lambda + 2\lambda^2 + \lambda^3 & 1 - 2\lambda - \lambda^3 + \lambda^3 & 2 - \lambda^2 \end{bmatrix},$$

$$N(\lambda) = \begin{bmatrix} 1 & \lambda & \lambda^2 & \lambda^3 \end{bmatrix}^t.$$

Since the (scalar) polynomial entries in $M(\lambda)$ have no roots in common, it is easily checked that all the necessary conditions hold, but the Bezoutian

$$\mathcal{B}(M, N) = \begin{bmatrix} -1 & -1 & -2 \\ 0 & 0 & 0 \\ -2 & 1 & 1 \end{bmatrix}$$

is a singular matrix.

It turns out that for the case

$$s = p + q,$$  \hfill (3.31)

i.e., for the Anderson-Jury Bezoutian, the conditions (3.29) are not only necessary but also sufficient for invertibility of $B$. The following theorem is an immediate consequence of Theorems 3.1 and 3.2.

**Theorem 3.4.** Let $M(\lambda)$ and $N(\lambda)$ be a pair of matrix polynomials which satisfy the conditions (3.28) of Theorem 3.3 and the condition (3.31).
Then the Bezoutian $B = \mathcal{B}(M, N)$ is an invertible matrix if and only if either

\[ \text{rank } M(\lambda) = p \quad \text{for every } \lambda \in \mathbb{C} \cup \{\infty\}, \quad (3.32) \]

\[ \text{rank } N(\lambda_0) = q \quad \text{for some } \lambda_0 \in \mathbb{C} \cup \{\infty\} \]

or

\[ \text{rank } M(\lambda_0) = p \quad \text{for some } \lambda_0 \in \mathbb{C} \cup \{\infty\}, \quad (3.33) \]

\[ \text{rank } N(\lambda) = q \quad \text{for every } \lambda \in \mathbb{C} \cup \{\infty\}. \]

Moreover, if $B$ is invertible, then its inverse has block Hankel structure with $q \times p$ blocks.

It is easy to reformulate this result directly in terms of the quadruple of matrix polynomials $(A, B; C, D)$ and the Anderson-Jury Bezoutian. In particular, Theorem 3.4 implies Theorem 1.1.

It is worth mentioning that the generalized Bezoutians come into play naturally in the investigation of the root distribution of matrix polynomials (see, e.g. [21, 6, 27] and references therein) and the study of matrix boundary value problems ([25, 26]). Since the Bezoutian matrices appearing in this context are Hermitian, it is useful to formulate the following criterion of invertibility for such Bezoutians:

**Theorem 3.5.** Let $A(z)$ and $B(z)$ be $p \times p$ and $p \times q$ matrix polynomials of degrees $\alpha$ and $\beta$, respectively. Let

\[ A(z)A^*(z) = B(z)B^*(z), \quad z \in \mathbb{C}, \quad (3.34) \]

and put $\mu := \max(\alpha, \beta)$. Then the $\mu p \times \mu p$ generalized Bezoutian

\[ \Gamma = \mathcal{B}(A, B; A^*, B^*) \]

defined by

\[ \frac{A(z)A^*(y) - B(z)B^*(y)}{z - y} = \sum_{i,j=0}^{\mu-1} z^i \Gamma_{ij} y^j \]

is a Hermitian matrix. It is invertible if and only if

1. $A(z)$ and $B(z)$ are left coprime;
2. $\text{rank } [A_{\mu} B_{\mu}] = p$. 


Moreover, if $\Gamma$ is invertible, then its inverse is a Hermitian block Hankel matrix with $p \times p$ blocks.

Proof. It is enough to take

$$M(\lambda) = \begin{bmatrix} A(\lambda) & B(\lambda) \end{bmatrix}, \quad N(\lambda) = \begin{bmatrix} A^\#(\lambda) & -B^\#(\lambda) \end{bmatrix}^t$$

and invoke Theorem 3.4. ■

Remarks.

(1) It follows from this theorem that under certain conditions the Gram matrix $G$ appearing in Theorem 4.2 in [6] has the block Hankel structure. This observation is important because of the close connection between the inertia of a Hermitian block Hankel matrix and the root location of the associated orthogonal matrix polynomials (see [27] for details).

(2) This theorem provides conditions for a matrix polynomial $X(\lambda)$ to generate a reproducing kernel Pontryagin space $\mathcal{H}(X)$ (see [1] and [2] for a detailed exposition on these spaces).

4. WIMMER'S GENERALIZED BEZOUTIAN

In [28] the following definition of Bezoutian was introduced. Let

$$w_i = \sum_{i=1}^{\infty} \frac{W_i}{z^i}$$

be a $p \times q$ matrix strictly proper rational function, and let

$$F(z) = \sum_{j=0}^{r} F_j z^j, \quad V(z) = \sum_{j=0}^{s} V_j z^j$$

be $p \times p$ and $q \times q$ nonsingular matrix polynomials respectively. Assume that $p \times q$ matrix polynomials $G(z)$ and $U(z)$ are chosen in such a way that

$$\pi_- [F^{-1}(z)G(z)] = \pi_- [U(z)V^{-1}(z)] = W(z).$$

(4.3)
DEFINITION [28]. The Bezoutian $B_w = \mathcal{B}_w(F, G; U, V)$ of the quadruple $(F, G; U, V)$ is the $rp \times sq$ matrix

$$B_w = [b_{ij}], \quad i = 1, \ldots, r, \quad j = 1, \ldots, s, \quad (4.4)$$

where the $p \times q$ block entries $b_{ij}$ are given by

$$\Delta_w(z, y) = F(z) \frac{W(y) - W(z)}{z - y} V(y) = \sum_{i=1}^{r} \sum_{j=1}^{s} z^{i-1} b_{ij} y^{j-1}. \quad (4.5)$$

Wimmer studied the generalized inverse of the Bezoutian matrix $B_w$. In particular, he obtained the following result, which seems to be inconsistent with the theory developed in the preceding sections.

THEOREM 4.1 [28]. The $pr \times pr$ square Bezoutian

$$B_w = \mathcal{B}_w(F, G; U, V) \quad (4.6)$$

is invertible if and only if

1. $F$ and $G$ are left coprimes;
2. $U$ and $V$ are right coprimes;
3. the top coefficients $F_r$ and $V_s$ of the polynomials $F(z)$ and $V(z)$ are both invertible matrices.

The contradiction is only apparent, since Theorem 4.1 and Theorem 1.1 deal with different Bezoutian matrices. To see this let us assume that the quadruple of the matrix polynomials $(F, G; U, V)$, as given at the beginning of this section, satisfies (4.3) and, at the same time, meets the following condition:

$$F(z)U(z) = G(z)V(z). \quad (4.7)$$

Then the Bezoutian form corresponding to (1.8), (1.6) is

$$\Delta(z, y) = \frac{F(z)U(y) - G(z)V(y)}{z - y}$$

$$= F(z) \frac{U(y)V^{-1}(y) - F^{-1}(z)G(z)}{z - y} V(y)$$

$$= F(z) \frac{S(y) - P(z)}{z - y} V(y) + F(z) \frac{W(y) - W(z)}{z - y} V(y)$$

$$= \Delta_p(z, y) + \Delta_w(z, y). \quad (4.8)$$
where $P(z)$ and $S(z)$ are the polynomial parts of $F^{-1}G$ and $UV^{-1}$, respectively, $\Delta_w(z, y)$ is given by (5.4), and $\Delta_p(z, y) = \Delta(z, y) - \Delta_w(z, y)$ is a polynomial in two variables.

It follows from (4.8) that the Bezoutians

$$B = \mathcal{B}(F, G; U, V) \quad \text{and} \quad B_w = \mathcal{B}_w(F, G; U, V)$$

are different unless

$$\Delta_p(z, y) = 0,$$  \hspace{1cm} (4.9)

or, equivalently, unless

$$P(z) = S(z) = \text{const.} \hspace{1cm} (4.10)$$

Note that for the original definition of Bezoutian by Anderson and Jury when $F, G, U, V$ satisfy $W = F^{-1}G = UV^{-1}$ rather than (4.7), the condition (4.10) holds, i.e., $B = B_w$.

Clearly, for every quadruple $(F, G; U, V)$ of the matrix polynomials which meet the conditions (4.2) and (4.3) one can easily pick up another quadruple $(F, G_1; U_1; V)$ in such a way that

$$\mathcal{B}(F, G_1; U_1, V) = \mathcal{B}_w(F, G; U, V). \hspace{1cm} (4.11)$$

For this purpose it is enough to set

$$G_1(z) = G(z) - F(z)P(z),$$

$$U_1(z) = U(z) - S(z)V(z),$$

where $P$ and $S$ are the polynomial parts of $F^{-1}G$ and $UV^{-1}$ respectively.

In fact, one can show Theorems 4.1 and 1.1 are equivalent.

5. BLOCK HANKEL MATRIX INVERSION

The objective of this section is to apply the results on generalized Bezoutians for the inversion of block Hankel matrices. It was first recognized by L. Lerer and M. Tismenetsky [22] that if a block Hankel matrix with
square blocks is invertible, then its inverse can be represented as the generalized Bezoutian of a quadruple of certain matrix polynomials (see also [10] for another proof, which is in a sense closer to our considerations below). On the other hand, starting from the works of Gohberg with coauthors [13, 12, 11] many formulas have been obtained which express the inverses of scalar and block Hankel and Toeplitz matrices in terms of so-called standard equations (see, e.g., [22, 4, 14, 19]). We are going to show that under certain conditions, the inverse of a Hermitian block Hankel matrix can be constructed via its two last columns.

**Theorem 5.1.** Let $H := H_n = [h_{i+j}]$, $i, j = 0, \ldots, n$, be a Hermitian block Hankel matrix with $p \times p$ Hermitian entries $h_0, h_1, \ldots, h_{2n}$. Assume that there exist solutions to the block equations

\[
H[B_0 B_1 \cdots B_n]^{br} = [0 \cdots 0 I_p]^{br},
\]

\[
H[C_0 C_1 \cdots C_n]^{br} = [0 \cdots I_p 0]^{br},
\]

where $C_i$ and $B_i$ are $p \times p$ matrices, $i = 0, 1, \ldots, n$, and

\[
B_n \text{ is an invertible matrix}. \tag{5.2}
\]

Then

1. $H$ is invertible;
2. the inverse $\Gamma = H^{-1} = [\Gamma_{ij}]$ can be found as a generalized Bezoutian of the quadruple of matrix polynomials $(A(\lambda), B(\lambda); B^*(\lambda), A^*(\lambda))$:

\[
\frac{A(\lambda)B^*(\omega) - B(\lambda)A^*(\omega)}{\lambda - \omega} = \sum_{i,j=0}^{n} \lambda^i \Gamma_{ij} \omega^j, \tag{5.3}
\]

where

\[
B(\lambda) = \sum_{i=0}^{n} B_i \lambda^i, \quad A(\lambda) = \lambda B(\lambda)B_n^{-1} + B(\lambda)B_n^{-1}C_n - \sum_{i=0}^{n} C_i \lambda^i.
\]

**Remark.** Under the assumptions that the Hermitian block Hankel matrix $H$ and its block submatrix $H_{n-1}$ are invertible, (5.3) appears in implicit form in [5]: one need only combine the identities (3.5) and (3.27) in that
paper. Note that the "continuous" counterpart of (5.3) was extensively used in [7] for investigation of the root location of "continuous" matrix orthogonal polynomials.

We shall obtain Theorem 5.1 as an immediate consequence of a similar result for non-Hermitian block Hankel matrices. First of all, we shall prove the following.

**Theorem 5.2.** Let \( h_0, h_1, \ldots, h_{m+n} \) be \( q \times p \) matrices which constitute the square block Hankel matrix

\[
H = \begin{bmatrix} h_{i+j} \end{bmatrix}, \quad i = 0, 1, \ldots, m, \quad j = 0, 1, \ldots, n.
\]

If \( H \) is invertible, then there exist a \( p \times p \) matrix polynomial \( P(A) \), a \( q \times q \) matrix polynomial \( P'(A) \), and \( p \times q \) matrix polynomials \( Q(A) \) and \( Q'(A) \) such that the inverse \( F = H^{-1} = \begin{bmatrix} F_{ij} \end{bmatrix}, i = 0, 1, \ldots, n, j = 0, 1, \ldots, m, \) can be found as a generalized Bezoutian of the quadruple \((P(\lambda), Q(\lambda); P'(\lambda), Q'(\lambda))\):

\[
\frac{P(\lambda)Q'(\omega) - Q(\lambda)P'(\omega)}{\lambda - \omega} = \sum \lambda^i \Gamma_{ij} \omega^j.
\]

**Proof.** Since \( H \) is invertible, there exist \( p \times q \) matrices \( Q_0, Q_1, \ldots, Q_n \) and \( Q'_0, Q'_1, \ldots, Q'_m \) such that

\[
H[Q_0 \quad Q_1 \quad \cdots \quad Q_n]^{br} = [0 \quad \cdots \quad 0 \quad I_q]^{br}. \tag{5.4}
\]

\[
[Q'_0 \quad Q'_1 \quad \cdots \quad Q'_m]H = [0 \quad \cdots \quad 0 \quad I_p]. \tag{5.4'}
\]

Furthermore, for any \( q \times p \) matrix \( h_{m+n+1} \), there exist \( p \times p \) matrices \( P_0, P_1, \ldots, P_n \) and \( q \times q \) matrices \( P'_0, P'_1, \ldots, P'_m \) such that

\[
H[P_0 \quad P_1 \quad \cdots \quad P_n]^{br} = -[h_{n+1} \quad h_{n+2} \quad \cdots \quad h_{m+n+1}]^{br}, \tag{5.5}
\]

\[
[P'_0 \quad P'_1 \quad \cdots \quad P'_m]H = -[h_{m+1} \quad h_{m+2} \quad \cdots \quad h_{m+n+1}]. \tag{5.5'}
\]

Define recurrently

\[
h_{m+n+k} = -\sum_{j=0}^{n} h_{m+j+k-1}P_j, \quad k = 1, 2, \ldots, \tag{5.6}
\]
and denote

\[ P(\lambda) = \lambda^{n+1}I_p + \sum_{j=0}^{n} \lambda^j P_j, \quad P'(\lambda) = \lambda^{m+1}I_q + \sum_{j=0}^{m} \lambda^j P'_j, \]

\[ Q(\lambda) = \sum_{j=0}^{n} \lambda^j Q_j, \quad Q'(\lambda) = \sum_{j=0}^{m} \lambda^j Q'_j, \] (5.7)

\[ h(\lambda) = \sum_{j=0}^{\infty} \lambda^{-j-1} h_j. \]

Then (5.4)–(5.6) imply the following identities:

\[ \pi_- [h(\lambda)Q(\lambda)] = \lambda^{-n} \varphi(\lambda), \] (5.8)

\[ \pi_- [Q'(\lambda)h(\lambda)] = \lambda^{-m} \varphi'(\lambda), \] (5.8')

and

\[ \pi_- [h(\lambda)P(\lambda)] = 0, \] (5.9)

where

\[ \varphi(\lambda) = \sum_{j=0}^{\infty} \lambda^{-j-1} \varphi_j, \quad \varphi'(\lambda) = \sum_{j=0}^{\infty} \lambda^{-j-1} \varphi'_j, \quad \varphi_0 = I_q, \quad \varphi'_0 = I_p. \]

Moreover, since \( P(\lambda) \) is monic, it follows from (5.5') and (5.9) that

\[ \pi_- [P'(\lambda)h(\lambda)] = 0. \] (5.9')

Denote

\[ P_+(\lambda) := \pi_+ [h(\lambda)P(\lambda)] = h(\lambda)P(\lambda), \]

\[ P'_+(\lambda) := \pi_+ [P'(\lambda)h(\lambda)] = P'(\lambda)h(\lambda), \]

\[ Q_+(\lambda) := \pi_+ [h(\lambda)Q(\lambda)], \quad Q'_+(\lambda) := \pi_+ [Q'(\lambda)h(\lambda)]. \]
Then

\[ P'(\lambda)Q(\lambda) = \{P'(\lambda)h(\lambda)\}Q(\lambda) = P'(\lambda)\{h(\lambda)Q(\lambda)\} \]

\[(5.8) = P'(\lambda)Q_+(\lambda) + P'(\lambda)\lambda^{-m}\varphi(\lambda). \quad (5.10)\]

\[ Q'_+(\lambda)Q(\lambda) + \lambda^{-n}\varphi'(\lambda)Q(\lambda) \]

\[(5.8') = Q'(\lambda)h(\lambda)Q(\lambda), \]

\[ = Q'(\lambda)Q_+(\lambda) + \lambda^{-m}\varphi(\lambda)Q'(\lambda). \quad (5.11)\]

Applying \( \pi_- \) to (5.10), we get \( \lambda^{-m}P'(\lambda)\varphi(\lambda) - I_q = 0 \), or

\[ \lambda^{-m}\varphi(\lambda) = P'(\lambda)^{-1}. \quad (5.12) \]

Similarly,

\[ \lambda^{-n}\varphi'(\lambda) = P(\lambda)^{-1}. \quad (5.12') \]

Then it follows from (5.11) that

\[ P(\lambda)^{-1}Q(\lambda) = Q'(\lambda)P'(\lambda)^{-1}. \]

Hence,

\[ P(\lambda)Q'(\lambda) = Q(\lambda)P'(\lambda). \quad (5.13) \]

The identity (5.13) ensures the existence of Anderson-Jury generalized Bezoutian

\[ \Gamma = \mathcal{B}(P, Q; P', Q') \]

of the quadruple \((P(\lambda), Q(\lambda); P'(\lambda), Q'(\lambda))\):

\[ \frac{P(\lambda)Q'(\omega) - Q(\lambda)P'(\omega)}{\lambda - \omega} = \sum_{i,j} \lambda^i \Gamma_{ij} \omega^j. \]
Let us observe that in view of Theorem 1.1 the matrix $\Gamma$ is invertible:

1. The polynomials $P(\lambda), Q(\lambda)$ are left coprime. Indeed,

\[
Q(\lambda)P'(\lambda) - P(\lambda)Q'(\lambda) \quad (5.9')
\]

\[
= Q(\lambda)P'(\lambda)h(\lambda) - P(\lambda)Q'(\lambda) \quad (5.13)
\]

\[
= P(\lambda)\{Q'(\lambda)h(\lambda) - Q'(\lambda)\} \quad (5.12')
\]

\[
= P(\lambda)\lambda^{-n}a'(\lambda) \quad (5.14)
\]

(2) Similarly, the polynomials $P'(\lambda), Q'(\lambda)$ are right coprime.

Conditions (3) and (4) of Theorem 1.1 are trivially satisfied, since the polynomials $P(\lambda)$ and $P'(\lambda)$ are monic.

Therefore, the Bezoutian $\Gamma$ is invertible and its inverse is a block Hankel matrix with $q \times p$ blocks. It remains only to show that $\Gamma^{-1} = H$. To do this let us take

\[
M(\lambda) = [P(\lambda), Q(\lambda)], \quad N(\lambda) = \begin{bmatrix} Q'(\lambda) \\ -P'(\lambda) \end{bmatrix}
\]

and compute

\[
T(\lambda) := \pi_+ [h(\lambda)M(\lambda)]N(\lambda) = P_+(\lambda)Q'(\lambda) - Q_+(\lambda)P'(\lambda) = I_q
\]

[cf. (5.14)].

On the other hand, by (2.8) of Theorem 2.1,

\[
T(\lambda) = \begin{bmatrix} h_0 & h_1 & \cdots & h_n \end{bmatrix}B(M, N)\begin{bmatrix} I_1 & \lambda I_q & \cdots & \lambda^m I_q \end{bmatrix}^{br}
\]

\[
= \begin{bmatrix} h_0 & h_1 & \cdots & h_n \end{bmatrix}\Gamma \begin{bmatrix} I_q & \lambda I_q & \cdots & \lambda^m I_q \end{bmatrix}^{br}.
\]

Therefore,

\[
\begin{bmatrix} h_0 & h_1 & \cdots & h_n \end{bmatrix} = \begin{bmatrix} I_q & 0 & \cdots & 0 \end{bmatrix}\Gamma^{-1},
\]

which means that the upper block rows of $H$ and $\Gamma^{-1}$ coincide.

Applying the matrix $H - \Gamma^{-1}$ to $[P_0 \quad P_1 \quad \cdots \quad P_n]^{br}$ and using (5.5), we obtain the needed equality $\Gamma^{-1} = H$. ■
It is known that for any block matrix $A$ with $p \times q$ block entries there exist an integer $s$, a $p \times s$ matrix polynomial $M(\lambda)$, and an $s \times q$ matrix polynomial $N(\lambda)$ such that

$$A = \mathfrak{B}(M, N)$$

is an $H$-Bezoutian (see [22, 15]).

Analogously to the definition of $T$-Bezoutian rank given in [15] (see also [16]), let us define the $H$-Bezoutian rank of the matrix $A$ to be

$$\text{rank}_H A := \min s,$$

where minimum is taken over all pairs $M(\lambda), N(\lambda)$ such that $A = \mathfrak{B}(M, N)$.

Theorem 5.2 shows that the $H$-Bezoutian rank of the inverse of block Hankel matrix does not exceed $p + q$. On the other hand, by Theorem 3.3 it cannot be less than $p + q$. This proves the following

**COROLLARY.** If $H$ is an invertible block Hankel matrix with $q \times p$ blocks, then

$$\text{rank}_H(H^{-1}) = p + q. \quad (5.15)$$

Note that similarly one has $\text{rank}_T(T^{-1}) = p + q$ for every invertible block Toeplitz matrix with $q \times p$ blocks (cf. [22]).

Let us return to the case of block Hankel matrices with square blocks and consider the matrix

$$H = [h_{i+j}], \quad i, j = 0, \ldots, n,$$

with $p \times p$ blocks $h_0, \ldots, h_{2n}$. Suppose there exist solutions to block equations

$$H[Q_0 \quad Q_1 \quad \cdots \quad Q_n]^{b_T} = [0 \quad \cdots \quad 0 \quad I_p]^{b_T}, \quad (5.16)$$

$$H[S_0 \quad S_1 \quad \cdots \quad S_n]^{b_T} = [0 \quad \cdots \quad I_p \quad 0]^{b_T}, \quad (5.17)$$

where

$$Q_n \text{ is an invertible } p \times p \text{ matrix.} \quad (5.18)$$
Denote

\[ Q(\lambda) = \sum_{j=0}^{n} \lambda^{j}Q_{j}, \]

\[ R(\lambda) = \sum_{j=0}^{n} \lambda^{j}S_{j} - Q(\lambda)Q_{n}^{-1}S_{n} = \sum_{j=0}^{n-1} \lambda^{j}R_{j}, \]

\[ P(\lambda) = \left\{ \lambda Q(\lambda) - R(\lambda) \right\} Q_{n}^{-1} = \lambda^{-n+1}I_{p} + \sum_{j=0}^{n} \lambda^{j}P_{j}. \]

Clearly,

\[ H_{n-1} \begin{bmatrix} R_{0} & R_{1} & \cdots & R_{n-1} \end{bmatrix}^{\text{br}} = \begin{bmatrix} 0 & 0 & \cdots & I_{p} \end{bmatrix}^{\text{br}}, \]

where \( H_{n-1} = [h_{i+j}], \ i, j = 0, \ldots, n - 1, \) and hence the block column \( [P_{0} \ P_{1} \ \cdots \ P_{n}]^{\text{br}} \) satisfies (5.5) with \( m = n \) and \( h_{2n+1} = -\sum_{j=0}^{n} h_{n+j}P_{j}. \)

Using (5.6), which is now equivalent to

\[ \sum_{j=0}^{n} h_{n+j+k}Q_{j} = \sum_{j=0}^{n-1} h_{n+j+k-1}R_{j}. \]

we can build a series

\[ h(\lambda) = \sum_{j=0}^{\infty} \lambda^{-j-1}h_{j}, \]

which satisfies (5.9). Note also that in view of (5.16) the equality (5.8) with \( m = n \) holds true. Moreover, (5.20) and (5.21) imply that

\[ \pi_{-}(h(\lambda)R(\lambda)) = \lambda^{-n+1}\varphi(\lambda). \]

Remark. If the block Hankel matrix in the statement of Theorem 5.2 is Hermitian, then the extension (5.21)\textendash(5.22) is Hermitian as well.

We proceed with the following
Lemma 5.1. Let (5.16)–(5.18) be satisfied. Then there exist solutions to the block equations

\[
\begin{bmatrix}
Q_0 & Q_1 & \cdots & Q_n
\end{bmatrix}H = [0 \cdots 0 I_p],
\]
(5.24)
\[
\begin{bmatrix}
S_0' & S_1' & \cdots & S_n'
\end{bmatrix}H = [0 \cdots I_p 0],
\]
(5.25)

and

\[Q'_n = Q_n.\]

Proof. It suffices to prove the existence of matrix polynomials

\[
P'(\lambda) = \lambda^{n+1}I_p + \sum_{j=0}^{n} \lambda^j P_j' \quad \text{and} \quad Q'(\lambda) = \sum_{j=0}^{n} \lambda^j Q_j'
\]
(5.26)

such that \(Q'_n = Q_n\) and (5.8'), (5.9') hold true. Then \(S_0', \ldots, S'_n\) can be found as the coefficients of the polynomial

\[
S'(\lambda) = \lambda Q'(\lambda) - Q_n P'(\lambda) - \varphi_1 Q'(\lambda).
\]
(5.27)

By (5.9) we have

\[
H_{n+1} \begin{bmatrix}
P_0 & \cdots & P_n & I_p
\end{bmatrix}^{br} = 0,
\]
where \(H_{n+1} = [h_{i+j}], \ i, j = 0, \ldots, n + 1\). Hence, there exist at least \(p\) linearly independent solutions \(x_1, x_2, \ldots, x_p\) of the equation

\[
x^*H_{n+1} = 0,
\]

where \(x \in \mathbb{C}^d, \ d = (n + 2)p\). Let us build the \(p \times (n + 2)p\) matrix \(X := [x_1 \ x_2 \ \cdots \ x_p]^*,\) and consider its partition into \(p \times p\) blocks \(\tilde{P}_j, \ j = 0, 1, \ldots, n + 1:\)

\[
X = \begin{bmatrix}
\tilde{P}_0 & \tilde{P}_1 & \cdots & \tilde{P}_{n+1}
\end{bmatrix}.
\]

By applying the arguments from the proof of Theorem 5.2 one can show that polynomial \(\tilde{P}(\lambda) = \sum_{j=0}^{n+1} \lambda^j \tilde{P}_j\) satisfies (5.9').
Let us suppose that $\eta^*\tilde{P}_{n+1} = 0$ for some vector $\eta \in \mathbb{C}^p$. Then $\deg[\eta^*\tilde{P}(\lambda)] < n$, and it follows from (5.8) that

$$0 = \pi_- \left[ \tilde{P}(\lambda)h(\lambda)Q(\lambda) \right] = \lambda^{-n}\eta^*\tilde{P}(\lambda)\varphi(\lambda). \quad (5.28)$$

Bearing in mind that $\varphi_0 = I_p$, one may conclude that $\eta^*[\tilde{P}_0 \tilde{P}_1 \cdots \tilde{P}_{n+1}] = 0$, which contradicts the linear independence of rows of the matrix $X$. Therefore, $\tilde{P}_{n+1}$ is invertible, and we can put $P'_j = \tilde{P}_{n+1}^{-1}P_j$, $j = 0, \ldots, n$.

Analogously, let us consider the equation $y^*G = 0$, where the $(n + 1)p \times np$ matrix

$$G := [h_{i+j}], \quad i = 0, 1, \ldots, n, \quad j = 0, 1, \ldots, n - 1,$

is obtained from $H$ by deleting the last block column. This equation has at least $p$ linearly independent solutions

$$y_1, y_2, \ldots, y_p \quad \text{in} \quad \mathbb{C}^l, \quad l := (n + 1)p,$

which form a $p \times (n + 1)p$ matrix $Y := [y_1 \ y_2 \cdots \ y_p]^*$. Consider the partition of $Y$ into $p \times p$ blocks:

$$Y = [E_0 \ E_1 \cdots \ E_n].$$

Then the corresponding polynomial $E(\lambda) = \sum_{j=0}^n \lambda^jE_j$ satisfies the following condition:

$$\pi_- \left[ E(\lambda)h(\lambda) \right] = \lambda^{-n}\psi(\lambda) \quad (5.29)$$

with some rational function $\psi(\lambda) = \sum_{j=0}^n \psi_j\lambda^{-j-1}$.

Let us show the invertibility of $E_n$ and $\psi_0$. Assume that $E_n$ is singular, i.e., for some vector $\xi \in \mathbb{C}^p$ and integer $k \geq 1$

$$\deg[\xi^*E(\lambda)] \leq n - k,$

and consider the following identity:

$$\{E(\lambda)h(\lambda)\}Q(\lambda) = E(\lambda)\{h(\lambda)Q(\lambda)\}. \quad (5.30)$$
Using (5.8), (5.29), we obtain

\[ \xi^* \psi(\lambda) Q(\lambda) = \xi^* E(\lambda) \varphi(\lambda) \]

and hence \( \deg[\xi^* \psi(\lambda)] \leq -k - 1 \). Similarly, considering \( E(\lambda) h(\lambda) R(\lambda) \), we arrive at \( \lambda \xi^* E(\lambda) \varphi(\lambda) = \xi^* \psi(\lambda) R(\lambda) \). But this means that \( \deg[\xi^* E(\lambda)] \leq n - k - 2 \).

By repeating the arguments we get \( \xi^* E(\lambda) = 0 \), which contradicts the definition of \( E(\lambda) \). Thus, \( E_n \) is an invertible matrix.

By matching the coefficients of \( \lambda^{-1} \) on the right and left hand sides of (5.30) we get

\[ E_n = \psi_0 Q_n \]

Hence \( \psi_0 \) is invertible too. Then the choice \( Q'(\lambda) = \psi_0^{-1} E(\lambda) \) completes the proof of the lemma.

Now we are able to prove

**Theorem 5.3.** Let \( H = [h_{i+j}], i, j = 0, \ldots, n \), be a block Hankel matrix with \( p \times p \) blocks. Assume that the \( p \times p \) matrices \( Q_0, \ldots, Q_n; S_0, \ldots, S_n \) satisfy the conditions (5.16)–(5.18). Then

1. \( H_n \) is invertible;
2. the inverse \( \Gamma = H^{-1} = [\Gamma_{ij}] \) can be expressed as the generalized Bezoutian of the quadruple of matrix polynomials \( (P(\lambda), Q(\lambda); P'(\lambda), Q'(\lambda)) \):

\[ \frac{P(\lambda)Q'(\omega) - Q(\lambda)P'(\omega)}{\lambda - \omega} = \sum_{i,j=n}^{n} \lambda^i \Gamma_{ij} \omega^j, \]

where

\[ P(\lambda) = \{\lambda Q(\lambda) + Q(\lambda) Q_n^{-1} S_n - S(\lambda)\} Q_n^{-1}, \]
\[ P'(\lambda) = Q_n^{-1}\{\lambda Q'(\lambda) + S_n Q_n^{-1} Q'(\lambda) - S'(\lambda)\}, \]

and the polynomials \( Q(\lambda), S(\lambda), Q'(\lambda), S'(\lambda) \) correspond to solutions of (5.16), (5.17), (5.24), and (5.25).
Proof. Once the existence of solutions for (5.24) and (5.25) is established by Lemma 5.1, the theorem can be proved by simply repeating arguments which were used to prove Theorem 5.2.

Remark. It follows easily from our considerations that a block Hankel matrix with square blocks is invertible if and only if Equations (5.16) and (5.5) are solvable. This result was proved by different methods in [22].

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