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The Yukawa$_2$ Quantum Field Theory Without Cutoffs

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We construct the Heisenberg picture dynamics for the Yukawa quantum field theory in two-dimensional space-time. All cutoffs are removed in the field operators, and the fields are formally Lorentz covariant. The fields are local, and commute or anticommute at space-like separated points. The field equations are verified.

1. INTRODUCTION

1.1. *Discussion*

Rigorous studies of quantum field theory have among their goals a proof of the mathematical consistency of field theory. Because of the difficulty in solving the equations of quantum field theory, we have worked mainly in two-dimensional space-time. A major step in our program is the construction of solutions to the renormalized field equations. There are difficulties in defining the nonlinear terms in these field equations. The proper definition of the nonlinear terms is called *renormalization* and is motivated by the physical interpretation of the equations.

In this paper we study quantum fields with the nonlinear scalar

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Yukawa interaction $\bar{\psi}\psi\varphi$ in two space-time dimensions. Our main result is the construction of the Heisenberg picture dynamics with all cutoffs removed. The scalar boson field φ and the fermion field ψ satisfy the coupled nonlinear equations

$$\left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} + m_{\mathrm{b}}^2\right) \varphi(x, t) + j(x, t) = 0 \tag{1.1.1}$$

and

$$\left(\gamma_0 \frac{\partial}{\partial t} + \gamma_1 \frac{\partial}{\partial x} + m_{\mathrm{f}}\right) \psi(x, t) + J(x, t) = 0. \tag{1.1.2}$$

The terms j and J are called the renormalized currents. Formally they are

$$j = \lambda\bar{\psi}\psi - \delta m_{\mathrm{b}}^2\varphi, \qquad J = \lambda\varphi\psi - \delta m_{\mathrm{f}}\psi, \tag{1.1.3}$$

where

$$\gamma_0 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \qquad \gamma_1 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \qquad \bar{\psi} = -i\psi^*\gamma_0, \tag{1.1.4}$$

δm_{b}^2 is the divergent boson mass renormalization constant, and δm_{f} is the finite fermion mass shift. The definition of the currents j and J in (1.1.1–1.1.3) will be made precise in Section 3. The constant m_{b} is the boson rest mass and m_{f} is the fermion rest mass.

Quantum fields such as φ and ψ provide a description of particles, and this motivates our studying (1.1.1–1.1.4). In fact, equations of this type in four dimensions are thought to describe the interactions of nucleons and π mesons. These equations also contain complete information about the bound states formed from two or more such particles, and they determine the scattering of these particles.

We now review the qualitative phenomena predicted by the physics of the problem. At large $|t|$, one observes single particles or stable clusters of particles (i.e., bound states), that have separated from one another and do not interact. The stable clusters of mesons and nucleons include the familiar nuclei of atomic physics. The clusters occur because the forces between the particles are attractive. The numbers and types of particles that one observes at large positive and negative times do not necessarily coincide. (In this case the scattering is called inelastic.) A typical inelastic process, which occurs at sufficiently high energy, is the scattering of two incoming protons to produce several π mesons in addition to two outgoing protons. In our Yukawa theory, each particle or stable cluster of particles is uniquely determined by its rest mass and by its nucleon number

$$N_{\mathrm{f}} = \text{Number of nucleons} - \text{Number of antinucleons.}$$

(To describe neutrons, protons, and π mesons, an additional quantum number is needed: electromagnetic charge or isotopic spin.)

We now express this physical picture in mathematical language. The fields $\varphi(x, t)$ and $\psi(x, t)$ are densely defined bilinear forms on some Hilbert space $\mathscr{H}$. The nucleon number operator N_f is a self-adjoint operator on $\mathscr{H}$ with pure discrete spectrum, equal to the integers. In addition we must find the Hamiltonian or energy operator H and the momentum operator P. The operators H, P, and N_f are commuting, self-adjoint operators. The joint spectrum of H and P should lie in the forward light cone. That is, $0 \leqslant H$ and $0 \leqslant H^2 - P^2$. The operator $M = (H^2 - P^2)^{1/2}$ labels the rest mass of our particles. Let $\mathscr{H}(n_f)$ be the eigenspace of N_f for the eigenvalue n_f. In $\mathscr{H}(n_f)$, the operator M should have an absolutely continuous spectrum in an interval

$$[\nu(n_f), \infty), \qquad 0 < \nu(n_f).$$

In the interval $(0, \nu(n_f))$, the operator M should have only a finite number of discrete eigenvalues $m_1(n_f)$, $m_2(n_f)$,... . We expect that

$$\nu(n_f) = \inf\{m_1(n') + m_1(n'') : n' + n'' = n_f\}.$$

The common null space for H, P, and M is called the space V of vacuum vectors; presumably $\dim V = 1$ and $V \subset \mathscr{H}(n_f = 0)$. The remaining discrete eigenvalues of M, the m_i, are the masses of the elementary particles and the stable clusters of the theory. The corresponding values of n_f label the fermion number (also called nucleonic charge). These joint eigenspaces of M and N_f are infinite dimensional; they contain the states in which there is a single particle or cluster of given rest mass $m_i(n_f)$ and fermion number n_f, but undetermined momentum.

The particles explicitly put into the field equations, the meson, the fermion, and the antifermion, are the smallest mass particles in the sectors $\mathscr{H}(n_f)$, $n_f = 0$, $n_f = \pm 1$. That is, $m_b = m_1(0)$, $m_f = m_1(\pm 1)$. The remaining m_i correspond to particle clusters formed from these three elementary particles. The continuous spectrum of M corresponds to states with more than one particle or cluster, as would be observed in the limit $| t | \to \infty$. For more details on the manner in which the multiple particle states are built up out of single particle states, we refer the reader to the Haag–Ruelle formulation of scattering theory [13]. In contrast to most of the above discussion, the Haag–Ruelle theory is a theorem.

We now return to the equations (1.1.1–1.1.4). In spite of the divergences in (1.1.3), our solutions φ and ψ have a mathematical meaning as local quantum fields. A quantum field is an operator-valued distribution. In our case, for each time t and each real $f \in C_0^\infty(R^1)$ we construct self-adjoint operators $\varphi(f, t)$ and bounded operators $\psi(f, t)$ denoted by the formal integrals

$$\int \varphi(x, t) f(x)\, dx, \qquad \int \psi^\alpha(x, t) f(x)\, dx, \qquad \alpha = 1, 2.$$

The fields φ and ψ are also local. This means that for space-like separated points the fields commute or anticommute. More precisely, $\varphi(f, t)$ commutes with $\varphi(g, s)$ if the distance from the support of f to the support of g is greater than $| t - s |$, and so forth for the other fields.

We use the Hamiltonian method, and our starting point is the spatially cutoff, self-adjoint Hamiltonian $H(g)$ of Ref. [10]. We define the fields

$$\varphi(f, t) = e^{itH(g)} \varphi(f, 0)\, e^{-itH(g)}, \tag{1.1.5}$$

$$\psi(f, t) = e^{itH(g)} \psi(f, 0)\, e^{-itH(g)}. \tag{1.1.6}$$

The time-zero values

$$\varphi(f, 0) = \int \varphi(x) f(x)\, dx \qquad \text{and} \qquad \psi(f, 0) = \int \psi(x) f(x)\, dx$$

coincide with time-zero free fields (see Section 1.2). Since $\varphi(f, 0)$ is self-adjoint and $\psi(f, 0)$ is bounded, $\varphi(f, t)$ is also self-adjoint and $\psi(f, t)$ is also bounded. We show in Section 3 that φ and ψ are local, that they satisfy the nonlinear equations of motion (1.1.1–1.1.4) and the equal-time canonical commutation relations

$$\left[\varphi(x, t), \frac{\partial}{\partial t} \varphi(y, t)\right] = i\delta(x - y) \tag{1.1.7}$$

$$\{\psi^\alpha(x, t), \psi^\beta(y, t)^*\} = \delta_{\alpha\beta}\, \delta(x - y). \tag{1.1.8}$$

The equations (1.1.1–1.1.4) are hyperbolic, and so influence propagates no faster than the speed of light. We demonstrate the finite propagation speed in Theorem 2.1.1. This result is central to our discussion of the dynamics. The Hamiltonian $H(g)$ is formally independent of the space cutoff g on the set where $g = 1$. In (1.1.5–1.1.6) we require that $g(y) = 1$, if $\operatorname{dist}(y, \operatorname{supp.} f) \leqslant | t |$. As a consequence of the finite propagation speed, the fields (1.1.5–1.1.6) are independent of g.

Because of the singular nature of the Hamiltonian $H(g)$ in (1.1.5–1.1.6), we defined $H(g)$ in Ref. [10] as the limit (as $\kappa \to \infty$) of approximate, self-adjoint Hamiltonians $H(g, \kappa)$, see (1.2.1). The parameter κ refers to a momentum cutoff. Corresponding to $H(g, \kappa)$ there are approximate, but well-defined, field equations. In these equations the currents j and J of (1.1.1–1.1.2) are replaced by nonlinear, nonlocal integral operators acting on the fields φ and ψ (see Section 3). Since the integral operators are nonlocal, the modified field equations have an infinite propagation speed. We define the true field equations as the limit ($\kappa \to \infty$) of the approximate nonlocal equations. In this way we give a meaning to (1.1.1–1.1.2). While it is not difficult to establish the existence of this limit, the proof of the finite propagation speed for the limiting equations requires the separate argument of Section 2.

In Section 2 we obtain the dynamics for bounded functions of the fields. Let A be a bounded local observable associated with a bounded region B of space at time zero (see Section 2.1). Let B_t be the set of points within distance $| t |$ of B. If $g = 1$ on B_t, then

$$A(t) = e^{itH(g)} A \, e^{-itH(g)} \tag{1.1.9}$$

is independent of g and $A(t)$ is an observable associated with the expanded region B_t. In this fashion we construct the Heisenberg picture dynamics

$$A \to A(t) = \sigma_t(A) \tag{1.1.10}$$

as an automorphism σ_t of the algebra of bounded local observables.

In this paper we have verified that many results of Refs. [6, 7] for the $(\varphi^4)_2$ interaction, and of Ref. [15] for $P(\varphi)_2$ interactions also hold for the Yukawa$_2$ interaction. We anticipate that the major results of Refs. [1, 8] also hold for the Yukawa$_2$ model, as well as for the more general $(P(\varphi) + Q(\varphi)\, \bar{\psi}\psi)_2$ interaction studied in Refs. [11, 12]. In Ref. [10] we showed that $H(g)$ has a vacuum vector Ω_g. We expect that a physical vacuum Ω and a physical Hilbert space $\mathscr{F}_{\text{ren}}$ can be constructed by taking limits of the states $\omega_g(\cdot) = (\Omega_g, \cdot\Omega_g)$ as $g \to 1$. We also expect that for $\tau < 1$,

$$(\Omega_g, N_\tau \Omega_g) \leqslant MV, \tag{1.1.11}$$

where V is the diameter of the support of g. As a consequence of (1.1.11), we believe that Ω and F_{ren} are locally Fock [8]. We expect that the theory is Lorentz covariant [1].

1.2. *Notation*

In general, we follow the notation of Ref. [10]. The Hilbert space $\mathscr{F} = \mathscr{F}_{\mathrm{b}} \otimes \mathscr{F}_{\mathrm{f}}$ is the tensor product of the standard Fock space $\mathscr{F}_{\mathrm{b}}$ for bosons with the Fock space $\mathscr{F}_{\mathrm{f}}$ for fermions. In terms of the densely defined boson annihilation operators $a(k)$, the fermion annihilation operators $b(p)$, and the antifermion annihilation operators $b'(p)$, the time zero fields are given on $\mathscr{F}$ as the bilinear forms

$$\varphi(x) = \varphi(x, 0) = (4\pi)^{-1/2} \int e^{-ikx}\{a(k)^* + a(-k)\}\, \mu(k)^{-1/2}\, dk,$$

$$\pi(x) = \pi(x, 0) = i(4\pi)^{-1/2} \int e^{-ikx}\{a(k)^* - a(-k)\}\, \mu(k)^{1/2}\, dk,$$

$$\psi^{(1)}(x) = \psi^{(1)}(x, 0) = (4\pi)^{-1/2} \int e^{-ipx}\{\nu(-p)\, b'(p)^* + \nu(p)\, b(-p)\}\, \omega(p)^{-1/2}\, dp,$$

$$\psi^{(2)}(x) = \psi^{(2)}(x, 0) = (4\pi)^{-1/2} \int e^{-ipx}\{\nu(p)\, b'(p)^* - \nu(-p)\, b(-p)\}\, \omega(p)^{-1/2}\, dp.$$

Here $\mu(k) = (k^2 + m_{\mathrm{b}}^2)^{1/2}$, $\omega(p) = (p^2 + m_{\mathrm{f}}^2)^{1/2}$, and $\nu(p) = (\omega(p) + p)^{1/2}$. We use the nonrelativistic normalization for the commutation relations, $\{b(p_1), b(p_2)^*\} = \delta(p_1 - p_2)$, etc. The fractional energy operator N_τ is defined by

$$N_\tau = \int a(k)^*\, a(k)\, \mu(k)^\tau\, dk + \int \{b(p)^*\, b(p) + b'(p)^*\, b'(p)\}\, \omega(p)^\tau\, dp,$$

and $N_1 = H_0$, the free field Hamiltonian.

In deriving detailed estimates on bilinear forms or operators in Fock space, it is convenient to use the operator N_τ for the purpose of comparison. We give such inequalities the name N_τ *estimates*. We now state the N_τ estimates needed in this paper. The proof of these, as well as more general N_τ estimates, can be found in Refs. [1, 2, 10].

Let $\alpha(k)$ be either $a(k)$, $b(k)$, or $b'(k)$, and let w be a Schwartz distribution in $\mathscr{S}(R^{l+m})'$. Let $\mathscr{D}$ be the set of vectors in $\mathscr{F}$ with a finite number of particles and wave functions in the Schwartz space $\mathscr{S}$. We consider a *Wick ordered monomial* W defined by

$$W = \int \alpha(k_1)^* \cdots \alpha(k_l)^*\, w(k_1, ..., k_{l+m})\, \alpha(-k_{l+1}) \cdots \alpha(-k_{l+m})\, dk_1 \cdots dk_{l+m}.$$

Then W is a bilinear form on $\mathscr{D} \times \mathscr{D}$. We refer to $\{k_1, ..., k_l\}$ as the creation variables and $\{k_{l+1}, ..., k_{l+m}\}$ as the annihilation variables.

Let $C \subset \{1,..., l\}$, $A \subset \{l + 1,..., l + m\}$, and let

$$A \cup B \cup C = \{1,..., l + m\}$$

be a disjoint union, where $|A|$, $|B|$, and $|C|$ denote the number of elements of A, B, and C.

THEOREM 1.2.1. (a) *There is a constant depending only on* $|B|$ *such that*

$$\|(N_\tau + I)^{-|C|/2} W(N_\tau + I)^{-|A|/2}(N + I)^{-|B|/2}\|$$
$$\leqslant \text{const} \cdot \Big\| \prod_{i \in A \cup C} \mu(k_i)^{-\tau/2} w \Big\|_2 .$$

(b) *Suppose that B contains at least one fermion variable. Then* $|B|$ *may be replaced by* $|B| - 1$ *in the above inequality.*

We now give the basic facts about the Yukawa$_2$ Hamiltonian. The self-adjoint, positive, renormalized Hamiltonian $H(g)$ of Ref. [10] is approximated by the momentum cutoff Hamiltonian

$$H(g, \kappa) = H_0 + H_I(g, \kappa) - \tfrac{1}{2}\delta m^2(\kappa) \int : \varphi(x)^2 : g(x)^2 \, dx - E(g, \kappa). \quad (1.2.1)$$

The cutoff interaction Hamiltonian could be given by a general cutoff of Ref. [10, Section 1.3]. Of particular interest in this paper, however, is a cutoff that is sharp in position space. In this case

$$H_I(g, \kappa) = \lambda \int : \bar{\psi}_\kappa(x) \psi_\kappa(x) : \varphi_\kappa(x) g(x) \, dx, \quad (1.2.2)$$

where

$$\psi_\kappa(x) = \int \delta_\kappa(y) \psi(x - y) \, dy, \quad (1.2.3)$$

$$\varphi_\kappa(x) = \int \delta_\kappa(y) \varphi(x - y) \, dy. \quad (1.2.4)$$

The function $\delta_\kappa(y) = (2\pi)^{-1/2} \kappa\tilde{\chi}(\kappa y) \in C_0^\infty$ is a real approximation to $\delta(y)$ with total integral 1 and positive Fourier transform $(2\pi)^{-1/2} \chi(k/\kappa)$. This sharp cutoff in position space corresponds to inserting the momentum cutoff factor

$$\chi_\kappa(k, p_1, p_2) = \chi(k/\kappa)\, \chi(p_1/\kappa)\, \chi(p_2/\kappa) \quad (1.2.5)$$

in the momentum space expansion of each Wick ordered monomial in (1.2.2).

The mass renormalization constant $\delta m^2(\kappa)$ in (1.2.1) is given by the formula

$$\delta m^2(\kappa) = -\frac{\lambda^2}{2\pi} \int \left| \chi_\kappa \left(0, \frac{\xi}{2}, \frac{-\xi}{2}\right) \right|^2 \omega(\xi)^{-1}\, d\xi + \text{const}, \qquad (1.2.6)$$

where the constant in (1.2.6) is independent of g, κ, and the particular cutoff χ_κ. In the Appendix we show that the choice (1.2.6) is consistent with the choice of the mass renormalization constant in Ref. [10]. In fact (1.2.6) is a special case of Ref. [10, (1.3.12)] that eliminates a possible g dependence in the mass renormalization. For simplicity, we have set $\delta m_{\mathrm{f}} = 0$ and $\delta m_{\mathrm{b}}^2(\kappa) = \delta m^2(\kappa)$.

The vacuum energy counterterm $E(g, \kappa)$ in (1.2.1) is defined implicitly by the equation

$$0 = \inf \text{spectrum}\, H(g, \kappa). \qquad (1.2.7)$$

The constant $E(g, \kappa)$ is logarithmically divergent as $\kappa \to \infty$; it has the form

$$E(g, \kappa) = E_2(g, \kappa) + c + o(1).$$

Here $E_2(g, \kappa)$ is the vacuum energy of $H_0 + H_I(g, \kappa)$ given by second-order perturbation theory,

$$E_2(g, \kappa) = -\| H_0^{-1/2} H_I(g, \kappa)\, \Omega_0 \|^2,$$

where Ω_0 is the no-particle vector. The constant c depends on g, but is independent of κ and χ_κ. It equals the (finite) vacuum energy of

$$\lim_{\kappa \to \infty} \{H_0 + H_I(g, \kappa) - E_2(g, \kappa)\}.$$

The term $o(1)$ in $E(g, \kappa)$ depends on g, χ_κ, and κ, and tends to zero as $\kappa \to \infty$. This term is required to satisfy (1.2.7) for all $\kappa \leqslant \infty$ (see Ref. [10, Section 5]).

The major results concerning $H(g, \kappa)$ of (1.2.1) and the resolvents $R(\kappa, \zeta) = (H(g, \kappa) - \zeta)^{-1}$ are given in the following theorems.

THEOREM 1.2.2. *For any $\tau < 1$ there is a constant M independent of κ such that* [3]

$$N_\tau \leqslant M(H(g, \kappa) + I) \qquad (1.2.8)$$

and [10, 18]

$$(N_{\tau/2})^2 \leqslant M(H(g, \kappa) + I)^2. \tag{1.2.9}$$

THEOREM 1.2.3 [10]. *As $\kappa \to \infty$, the resolvents $R(\kappa, \zeta)$ converge in norm to the resolvent of a positive, self-adjoint operator $H(g)$. The limit $H(g)$ is independent of the particular momentum cutoff function χ_κ.*

2. FINITE PROPAGATION SPEED

2.1 *Introduction*

In this section we study the time translation

$$U(t) =\cdot e^{-itH(g)} = \lim_{\kappa\to\infty} e^{-it(g,\kappa)}, \tag{2.1.1}$$

and we prove that with a proper choice of g, $U(t)$ yields the Heisenberg picture dynamics (1.1.9–1.1.10). Let B be a bounded open region of space at time zero. We define the algebra $\mathfrak{A}(B)$ of local observables as the von Neumann algebra of bounded operators generated by functions of the time-zero boson fields $\varphi(h)$ and $\pi(h)$ and by the fermion currents $\psi^\alpha(h)^* \psi^\beta(h_1)$. Here h and h_1 range over the C^∞ functions with support in B. The algebra $\mathfrak{A}$ of quasilocal observables is the norm closure of $\bigcup_B \mathfrak{A}(B)$. We also study the field algebra $\mathfrak{A}_f(B) \supset \mathfrak{A}(B)$. $\mathfrak{A}_f(B)$ is the von Neumann algebra generated by the bounded functions of $\varphi(h)$ and $\pi(h)$ and by $\psi^\alpha(h)$. As before h is a C^∞ function with support in B, and we define $\mathfrak{A}_f$ to be the norm closure of $\bigcup_B \mathfrak{A}_f(B)$. For $B \cap C = \varnothing$,

$$[\mathfrak{A}(B), \mathfrak{A}(C)] = 0 = [\mathfrak{A}(B), \mathfrak{A}_f(C)], \tag{2.1.2}$$

because the commutation relations (1.1.7–1.1.8) hold at $t = 0$ by the definition of the time-zero fields.

THEOREM 2.1.1. *Let $A \in \mathfrak{A}_f(B)$ and let $g = 1$ on B_t. Then $A(t) = \sigma_t(A)$ of* (1.1.10) *is independent of $g(\cdot)$, and*

$$\begin{aligned} \sigma_t \mathfrak{A}_f(B) &\subset \mathfrak{A}_f(B_t), \\ \sigma_t \mathfrak{A}(B) &\subset \mathfrak{A}(B_t). \end{aligned} \tag{2.1.3}$$

The time translation σ_t extends to a one-parameter automorphism group of the C^* algebras $\mathfrak{A}$ and $\mathfrak{A}_f$.

This theorem is our main result about the Heisenberg picture dynamics. The inclusion (2.1.3) states that influence propagates no faster than the speed of light.

By the strong convergence (2.1.1), we have the strong limit

$$\begin{aligned} A(t) &= \lim_{\kappa\to\infty} e^{itH(g,\kappa)} A\, e^{-itH(g,\kappa)} \\ &= \lim_{\kappa\to\infty} A_\kappa(t). \end{aligned} \tag{2.1.4}$$

However, the ultraviolet cutoff κ destroys locality.

In Section 2.2 we introduce another approximation $A_{\Lambda,\kappa}(t)$ which is g independent and which is local, i.e., belongs to $\mathfrak{A}_f(B_t)$. In Section 2.3 we prove the convergence

$$A(t) = w \cdot \lim_{\Lambda\to\infty} A_{\Lambda,\kappa(\Lambda)}(t).$$

Since $\mathfrak{A}_f(B_t)$ is closed under weak limits, the theorem follows.

2.2 *The Approximation with Finite Propagation Speed*

We use a sharp position cutoff in the Hamiltonian $H(g,\kappa)$ to minimize the loss of locality. If the radius of the support of $\tilde{\chi}_\kappa$ is κ^{-1}, then $H_{I,\kappa}(x) = :\bar{\psi}_\kappa(x)\,\psi_\kappa(x): \varphi_\kappa(x)$ depends on the fields localized in the interval $[x - \kappa^{-1}, x + \kappa^{-1}]$. We do not use a momentum cutoff in the field φ that enters the mass renormalization counterterm. For convenience of notation, we take $B = [-1, 1]$.

The finite propagation speed (2.1.3) would result if the interaction Hamiltonian $H_I(g, \kappa)$ and counterterms $c(g, \kappa)$ were localized in B. Alternatively, $H_I(g, \kappa) + c(g, \kappa)$ could be the sum of an operator localized in B_s and an operator commuting with $\mathfrak{A}_f(B_s)$ for each $s \in [0, |t|]$. However, because of the convolution (1.2.3–1.2.4) in (1.2.2), such a splitting seems to require $\chi_\kappa = \delta$, i.e., $\kappa = \infty$. For $\kappa = \infty$, the counterterms are infinite and $H_I + c$ does not make sense, either as an operator or as a bilinear form. Thus before attempting to split $H_I + c$ into two parts, we modify the problem. We make a time-dependent change in the spatial cutoff $g(x)$ in order to be able to write $H_I(g, \kappa) + c(g, \kappa)$, $\kappa < \infty$, as a sum of two terms, one localized in B_t and the other in $\sim B_t$, the complement of B_t.

In order to modify g, we use a C^∞ function $h_\kappa(x)$ with the following properties:

$$h_\kappa(x) = h(\kappa x), \tag{2.2.1}$$

$$\begin{cases} 0 \leqslant h_\kappa(x) \leqslant 1, \\ h_\kappa(x) = h_\kappa(-x), \end{cases} \tag{2.2.2}$$

$$\begin{cases} h_\kappa(x) = 1 & \text{if } |x| < 4\kappa^{-1}, \\ h_\kappa(x) = 0 & \text{if } |x| > 5\kappa^{-1}. \end{cases} \tag{2.2.3}$$

We define the change in the spatial cutoff $g(x)$ by

$$\delta g_{\kappa,t}(x) = -g(x)\, h_\kappa(|\, x\,| - |\, t\,| - 1). \tag{2.2.4}$$

In other words, we replace $g(x)$ in $H_I(g, \kappa)$ by the cutoff

$$g_{\kappa,t}(x) = g(x) + \delta g_{\kappa,t}(x). \tag{2.2.5}$$

Corresponding to this change, we have the time-dependent Hamiltonian

$$\begin{aligned} H(g, \kappa, t) &= H(g, \kappa) + \delta H(g, \kappa, t) \\ &= H_0 + H_I(g_{\kappa,t}\,; \kappa) + c(g, \kappa), \end{aligned} \tag{2.2.6}$$

where the counterterms $c(g, \kappa)$ are unchanged.

The change in the Hamiltonian $\delta H(g, \kappa, t)$ is localized in strips near light rays emerging from the boundary of B (see Fig. 1).

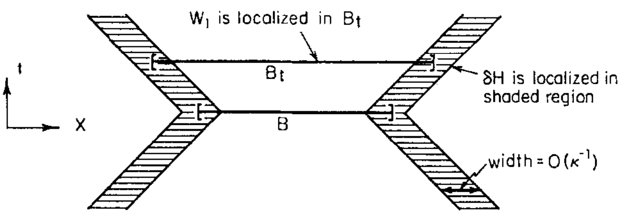


FIG. 1. The localization of δH and of W_1.

We note that the strips have width $10\kappa^{-1} = O(\kappa^{-1})$, and δH is thus small in some sense as $\kappa \to \infty$. On the other hand, the kernels of δH have ultraviolet divergences as $\kappa \to \infty$. We will later study these two effects in detail, in order to show that the total effect of δH is negligible in the limit $\kappa \to \infty$. The operator $H_I(g_{\kappa,t}\,; \kappa)$ is localized outside of two slightly smaller strips. Thus

$$H_I(g_{\kappa,t}; \kappa) = W_1 + W_2$$

is the sum of two terms. W_1 is localized in the bounded region between the strips and hence in B_t, and W_2 is localized in the unbounded region outside the strips, and hence in $\sim B_t$. The term localized in the exterior does not contribute to the Heisenberg picture dynamics. In fact, one could study the Schrödinger equation

$$-i\,\frac{\partial \psi(t)}{\partial t} = H(g, \kappa, t)\, \psi(t) \tag{2.2.7}$$

with the time-dependent Hamiltonian (2.2.6). One expects that there is a unitary propagator $W_\kappa(t)$ such that the solution to (2.2.7) has the form $\psi(t) = W_\kappa(t)^* \psi(0)$, and such that for $A \in \mathfrak{A}(B)$, $W_\kappa(t)^* A W_\kappa(t) \in \mathfrak{A}(B_t)$.

We do not attempt to solve (2.2.7) exactly. We find that it is more convenient to use an approximate propagator $W_\kappa(t)$. Let

$$V_\kappa(t, s) = e^{-itH(g,\kappa,s)}. \tag{2.2.8}$$

The operator $H(g, \kappa, s)$ is self-adjoint, so $V_\kappa(t, s)$ is unitary. We define the approximate propagator $W_\kappa(t)$ by

$$W_\kappa(t) = V_\kappa\left(\frac{t}{n}, 0\right) V_\kappa\left(\frac{t}{n}, \frac{t}{n}\right) V_\kappa\left(\frac{t}{n}, \frac{2t}{n}\right) \cdots V_\kappa\left(\frac{t}{n}, \frac{n-1}{n} t\right), \tag{2.2.9}$$

where n is the largest integer less than $| t | \kappa$.

LEMMA 2.2.1. *If $g(\cdot)$ equals one on B_t, then*

$$W_\kappa(t)^* \mathfrak{A}_f(B)\, W_\kappa(t) \subset \mathfrak{A}_f(B_t), \tag{2.2.10}$$

and the dynamics defined by (2.2.10) *is independent of the cutoff $g(\cdot)$.*

Proof. It is sufficient to prove that if $A \in \mathfrak{A}_f(B_s)$ and $| \tau | \leqslant t/n$, then

$$V_\kappa(\tau, s)^* A V_\kappa(\tau, s) \subset \mathfrak{A}_f(B_{|s|+|\tau|}), \tag{2.2.11}$$

and is independent of $g(\cdot)$. The representation (2.2.9) and the n-fold application of (2.2.11) then yields (2.2.10).

To prove (2.2.11) we substitute the Trotter product formula,

$$V_\kappa(\tau, s) = \operatorname*{st\,lim}_{r\to\infty} \{e^{-i\tau(H_0+c(g,\kappa))/r}\, e^{-i\tau H_I(g,\kappa,s)/r}\}^r \tag{2.2.12}$$

in (2.2.11). The counterterm $c(g, \kappa)$ is local in the following sense: For $a \in [0, | t |]$,

$$c(g, \kappa) = -\frac{1}{2}\, \delta m^2(\kappa) \int_{B_a} : \varphi^2(x) : dx - \frac{1}{2}\, \delta m^2(\kappa) \int_{\sim B_a} : \varphi^2(x) : g^2(x)\, dx - E(g, \kappa). \tag{2.2.13}$$

Each term in (2.2.13) is a self-adjoint operator, and these three operators commute. The bounded functions of the first term belong to $\mathfrak{A}(B_a)$ (see Ref. [6]) and are independent of $g(\cdot)$. Thus by the Trotter formula (see Refs. [16, 6]),

$$e^{iu(H_0+c(g,\kappa))} \mathfrak{A}_f(B_s)\, e^{-iu(H_0+c(g,\kappa))} \subset \mathfrak{A}_f(B_{|s|+|u|}), \tag{2.2.14}$$

and the mapping defined by (2.2.14) is independent of $g(\cdot)$ if $|s| + |u| \leqslant |t|$.

We now study the automorphism induced by the interaction term $H_I(g, \kappa, s)$. Using (2.2.3–2.2.5), we have the decomposition

$$H_I(g, \kappa, s) = \int_{B_{|s|-4\kappa^{-1}}} H_{I,\kappa}(x)\, g_{\kappa,s}(x)\, dx + \int_{\sim B_{|s|+4\kappa^{-1}}} H_{I,\kappa}(x)\, g_{\kappa,s}(x)\, dx$$
$$= W_1 + W_2 . \tag{2.2.15}$$

The first term W_1 in (2.2.15) is localized in the interval $B_{|s|-3\kappa^{-1}} \subset B_s$ (see Fig. 1). This term does not depend on $g(\cdot)$. The second term W_2 is localized outside the interval $B_{|s|+3\kappa^{-1}} \supset B_{|s|+2\kappa^{-1}}$. Each term is a self-adjoint operator with bounded functions in $\mathfrak{A}(B_s)$ and $\mathfrak{A}(B_{|s|+2\kappa^{-1}})'$, respectively. For $|u| \leqslant |t/n| \leqslant 2\kappa^{-1}$, W_2 commutes with $\mathfrak{A}_f(B_{|s|+|u|})$ and for all τ,

$$e^{i\tau H_I(g,\kappa,s)} \mathfrak{A}_f(B_{|s|+|u|})\, e^{-i\tau H_I(g,\kappa,s)}$$
$$= e^{i\tau W_1} \mathfrak{A}_f(B_{|s|+|u|})\, e^{-i\tau W_1} \subset \mathfrak{A}_f(B_{|s|+|u|}). \tag{2.2.16}$$

Since the automorphism (2.2.16) depends only on W_1, it is independent of $g(\cdot)$.

We now apply the inclusions (2.2.14) and (2.2.16) to the Trotter representation (2.2.12) of $V_\kappa(\tau, s)^* A V_\kappa(\tau, s)$ to complete the proof.

Remark. Similarly, $W_\kappa(t)^* \mathfrak{A}(B) W_\kappa(t) \subset \mathfrak{A}(B_t)$. We now prove results for $\mathfrak{A}(B)$ only. The same proofs yield theorems for $\mathfrak{A}_f(B)$.

While the dynamics in (2.2.10) is local, we have not proved that it converges to σ_t as $\kappa \to \infty$. To aid the convergence proof we study a smoothed-out version of (2.2.10). The scaling parameter x is introduced by replacing (2.2.9) by

$$W_\kappa(t; x) = V_\kappa\left(\frac{tx}{n}, 0\right) V_\kappa\left(\frac{tx}{n}, \frac{t}{n}\right) \cdots V_\kappa\left(\frac{tx}{n}, \frac{n-1}{n}\, t\right). \tag{2.2.17}$$

For $x = 1$, $W_\kappa(t; x) = W_\kappa(t)$, and for $0 \leqslant x < 1$, the time steps in W_κ are smaller. Thus if $A \in \mathfrak{A}(B)$ and $|x| \leqslant 1$,

$$W_\kappa(t; x)^* A W_\kappa(t; x) \in \mathfrak{A}(B_t) \tag{2.2.18}$$

and is independent of the spatial cutoff $g(\cdot)$. We average (2.2.18) over $0 \leqslant x \leqslant 1$, preserving the finite propagation speed.

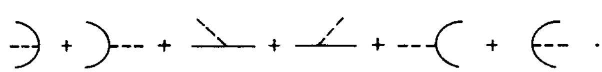
Let $f(x)$ be a positive C^∞ function with support in $[-1, 0]$ and with integral one. The function

$$f_\Lambda(x) = \Lambda f(\Lambda(x-1)) \tag{2.2.19}$$

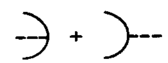
is a positive approximation to the Dirac measure $\delta_1 = \delta(x-1)$ and $\operatorname{supp} f_\Lambda \subset [1 - \Lambda^{-1}, 1]$. Our final approximation to $A(t)$ is

$$A_{\Lambda,\kappa}(t) = \int W_\kappa(t; x)^* \, A W_\kappa(t; x) f_\Lambda(x)\, dx \in \mathfrak{A}(B_t). \tag{2.2.20}$$

For the remainder of this section, we study some elementary properties of the operators $H(g, \kappa, t)$ and $W_\kappa(t, x)$. The interaction Hamiltonian $H_I(g, \kappa)$ of (1.2.2–1.2.5) has the following expansion in terms of diagrams:

$H_I(g, \kappa) =$ 

Each diagram in this expansion stands for a Wick ordered monomial or a sum of similar Wick ordered monomials. The solid lines represent fermions while the dotted lines are bosons. A line pointing to the left represents a creator, while a line pointing to the right stands for an annihilator. The pair creation terms are

$W^C =$ 

$$= \int w^C(k, p_1, p_2)\{a(k)^* + a(-k)\}\, b(p_1)^*\, b'(p_2)^*\, dk\, dp_1\, dp_2,$$

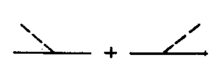
where

$$w^C(k, p_1, p_2) = -\frac{\lambda}{4\pi} \tilde{g}(k + p_1 + p_2)(\mu\omega_1\omega_2)^{-1/2}$$
$$\times (\omega_1\omega_2 - p_1 p_2 - m_f^2)^{1/2} \operatorname{sgn}(p_1 - p_2)\, \chi_\kappa(k, p_1, p_2).$$

The pair annihilation terms

$W^A =$ 

are given by $W^A = (W^C)^*$. The boson emission and absorption terms are

$W =$ 

$$= \int w(k, p_1, p_2)\{a(k)^* + a(-k)\}\{b(p_1)^*\, b(-p_2)$$
$$+ b'(p_1)^*\, b'(-p_2)\}\, dk\, dp_1\, dp_2$$

with

$$w(k, p_1, p_2) = -\frac{\lambda}{4\pi}\tilde{g}(k + p_1 + p_2)(\mu\omega_1\omega_2)^{-1/2}$$

$$\times (\omega_1\omega_2 + p_1 p_2 + m_f^2)^{1/2} \chi_\kappa(k, p_1, p_2).$$

The cutoff χ_κ has the form (1.2.5).

LEMMA 2.2.2. (a) *Let* $0 < \tau, \epsilon$. *The operator* $\delta H(g, \kappa, t)$ *of* (2.2.6) *satisfies*

$$\pm\delta H \leqslant O(\kappa^{-(\tau-\epsilon)/2})(N_\tau + I) \tag{2.2.21}$$

as $\kappa \to \infty$. *Here* $O(\kappa^{(\tau-\epsilon)/2})$ *is independent of* t.

(b) *Let* $0 < \tau < 1$. *Then*

$$N_\tau \leqslant \text{const} \cdot (H(g, \kappa, t) + \text{const}) \tag{2.2.22}$$

for constants independent of κ *and* t.

Proof. It is sufficient to prove (a), since (b) is a consequence of (a) and Theorem 1.2.2. In order to prove part (a), we use the N_τ estimates Theorem 1.2.1 (see Ref. [10]).

The kernels δw^C and δw of δH are the kernels of $H_I(\delta g_{\kappa,t}, \kappa)$. We note that by (2.2.1) and (2.2.4)

$$\|(\delta g_{\kappa,t})^\sim\|_\infty \leqslant \|\delta g_{\kappa,t}\|_1 \leqslant \|h_\kappa(|\cdot| - |t| - 1)\|_1 = 2\kappa^{-1}\|h\|_1 .$$

This yields the bounds

$$|\delta w^C| + |\delta w| \leqslant \text{const} \cdot \kappa^{-1}\mu^{-1/2}\chi_\kappa(k, p_1, p_2), \tag{2.2.23}$$

$$\begin{aligned}\|(\mu + \omega_1 + \omega_2)^{-\tau/2}(|\delta w^C| + |\delta w|)\|_2 &\leqslant O(\kappa^{-2} (\log \kappa)\, \kappa^{2-\tau})^{1/2} \\ &\leqslant O(\kappa^{-(\tau-\epsilon)/2}).\end{aligned} \tag{2.2.24}$$

The bound (2.2.21) now follows by Theorem 1.2.1 or Ref. [10, Corollary 2.1.2], and the proof is complete.

Since $H(g, \kappa, s)$ is bounded from below, uniformly in κ and s for each g, we find it convenient to add constants to these operators (i.e., to renormalize) so that

$$0 = \inf \text{spectrum } H(g, \kappa, s). \tag{2.2.25}$$

COROLLARY 2.2.3. *The unitary operators* $W_\kappa(t; x)$ *are the boundary values of operators* $W_\kappa(t; z)$ *which are analytic in* z *for* $\operatorname{Im} tz < 0$ *and uniformly bounded*: $\| W_\kappa(t; z)\| \leqslant 1$.

Proof. This corollary is a consequence of (2.2.25) and the representation (2.2.8), (2.2.17) for $W_\kappa(t; x)$.

2.3 *Convergence of the Approximation with Finite Propagation Speed*

In this section we prove Theorem 2.1.1, which says that the dynamics $A \to A(t) = \sigma_t(A)$ of (1.1.9–1.1.10) does not depend on any cutoffs and is local. Theorem 2.1.1 is a consequence of the following:

THEOREM 2.3.1. *Let $A_{\Lambda,\kappa}$ be defined by* (2.2.20). *There is a choice $\kappa = \kappa(\Lambda)$ of the cutoff κ such that $\kappa(\Lambda) \to \infty$ as $\Lambda \to \infty$ and*

$$w \cdot \lim_{\Lambda\to\infty} A_{\Lambda,\kappa(\Lambda)}(t) = A(t).$$

The main step in the proof of this theorem is to show that for $\operatorname{Im} tz < 0$ and t real,

$$w \cdot \lim_{\kappa\to\infty} W_\kappa(t; z) - U_\kappa(tz) = 0.$$

LEMMA 2.3.2. *Let $0 \leqslant t$, $H = H(g, \kappa) + I$, $H(s) = H(g, \kappa, s) + I$, $R = H^{-1}$. There is a constant M independent of κ such that*

$$\| H^{1/2} e^{-tH(s)} R^{1/2} \| \leqslant (1 + Mt \log \kappa)^{1/2}. \tag{2.3.1}$$

Proof. Let $F(t) = \| H^{1/2} e^{-tH(s)} R^{1/2} \Omega \|^2$. We have

$$\frac{dF(t)}{dt} = (e^{-tH(s)} R^{1/2}\Omega, -\{HH(s) + H(s)\, H\}\, e^{-tH(s)} R^{1/2}\Omega).$$

Therefore if

$$-\{HH(s) + H(s)\, H\} \leqslant M \log \kappa \tag{2.3.2}$$

then $dF(t)/dt \leqslant M(\log \kappa) \| \Omega \|^2$, and integrating yields (2.3.1). In order to prove (2.3.2), we write

$$-\{HH(s) + H(s)\, H\} = -2H^2 - H\delta H(s) - \delta H(s)\, H.$$

Since $-\{H\, \delta H(s) + \delta H(s) H\} \leqslant H^2 + \delta H(s)^2$,

$$-\{HH(s) + H(s)\, H\} \leqslant -H^2 + \delta H(s)^2. \tag{2.3.3}$$

Our bound on $\delta H(s)^2$ is postponed to Section 2.4. By Lemma 2.4.1,

$$\delta H(s)^2 \leqslant \epsilon H^2 + M \log \kappa \tag{2.3.4}$$

for any positive ϵ and for $\kappa \geqslant \kappa(\epsilon)$ sufficiently large. The inequalities (2.3.3–2.3.4) yield (2.3.2) and complete the proof.

LEMMA 2.3.3. *Let* $0 \leqslant t$. *For* $\tau < 1/2$ *there exists a constant* M_1 *such that*

$$\| R^{1/2}\{e^{-tH(s)} - e^{-tH}\} R^{1/2} \| \leqslant M_1(t + t^2)\, \kappa^{-\tau}. \tag{2.3.5}$$

Proof. We use the identity

$$e^{-tH(s)} - e^{-tH} = -\int_0^t du\, e^{-uH(s)}\, \delta H(s)\, e^{-(t-u)H}, \tag{2.3.6}$$

which is proved by differentiating $e^{-uH(s)}e^{-(t-u)H}$ and integrating the result from 0 to t. By Lemma 2.2.2a,

$$\|(N_{\tau_1} + I)^{-1/2}\, \delta H(s)(N_{\tau_1} + I)^{-1/2} \| \leqslant O(\kappa^{-\tau}), \tag{2.3.7}$$

for any τ_1 such that $2\tau < \tau_1$. If $\tau_1 < 1$, then by (1.2.10), as $\kappa \to \infty$,

$$\|(N_{\tau_1} + I)^{1/2}\, R^{1/2} \| \leqslant \text{const.} \tag{2.3.8}$$

By (2.3.7–2.3.8), for $\tau < \frac{1}{2}$, $\| R^{1/2}\, \delta H R^{1/2} \| \leqslant O(\kappa^{-\tau})$. Therefore, since R commutes with H, we have by (2.3.6)

$$\begin{aligned}
&\| R^{1/2}\{e^{-tH(s)} - e^{-tH}\} R^{1/2} \| \\
&\quad \leqslant \int_0^t du\, \| R^{1/2} e^{-uH(s)} R^{-1/2} \|\, \| R^{1/2}\, \delta H R^{1/2} \|\, \| e^{-(t-u)H} \| \\
&\quad \leqslant \int_0^t du\, \| R^{1/2} e^{-uH(s)} R^{-1/2} \|\, O(\kappa^{-\tau}),
\end{aligned}$$

since H is positive. By Lemma 2.3.2,

$$\int_0^t du\, \| R^{1/2} e^{-uH(s)} R^{-1/2} \| \leqslant \int_0^t (1 + Mt \log \kappa)^{1/2}\, du \leqslant (t + Mt^2) \log \kappa,$$

so that with a new τ,

$$\| R^{1/2}\{e^{-tH(s)} - e^{-tH}\} R^{1/2} \| \leqslant O((t + t^2)\, \kappa^{-\tau})$$

which completes the proof.

THEOREM 2.3.4. *Let* $\operatorname{Im} tz < 0$. *Then*

$$w \cdot \lim_{\kappa \to \infty} \{W_\kappa(t; z) - U_\kappa(tz)\} = 0. \tag{2.3.9}$$

Proof. The functions

$$F_\kappa(z) = (\Omega, \{W_\kappa(t; z) - U_\kappa(tz)\}\,\Omega) \tag{2.3.10}$$

are analytic in z for $\operatorname{Im} tz < 0$. Furthermore, the F_κ are uniformly bounded for all $\kappa < \infty$ by Theorem 1.2.2 and Corollary 2.2.3. Therefore the family (2.3.10), indexed by κ, is a normal family of analytic functions. We prove convergence

$$F_\kappa(z) \to F(z) \tag{2.3.11}$$

for $z = -iy$, $0 < ty < ty_0$. By Vitale's theorem, Ref. [17, p. 168] we have convergence in (2.3.11) in the entire half-plane $\operatorname{Im} tz < 0$, and the convergence is uniform on compact subsets of this half-plane. We now prove the convergence (2.3.11).

Since

$$U_\kappa(tz) = U_\kappa\left(\frac{t}{n}\,z\right)^n,$$

we have by (2.2.17),

$$\begin{aligned} W_\kappa(t, z) - U_\kappa(t, z) = {} & \left\{V_\kappa\left(\frac{tz}{n}\,; 0\right) - U_\kappa\left(\frac{tz}{n}\right)\right\} U_\kappa\left(\frac{tz}{n}\right)^{n-1} \\ & + \sum_{r=1}^{n-1}\left(\prod_{j=0}^{r-1} V_\kappa\left(\frac{tz}{n}\,; \frac{jt}{n}\right)\right) \\ & \times \left\{V_\kappa\left(\frac{tz}{n}\,; \frac{rt}{n}\right) - U_\kappa\left(\frac{tz}{n}\right)\right\} U_\kappa\left(\frac{tz}{n}\right)^{n-1-r}. \end{aligned}$$

Thus for $R_\kappa = (H(g, \kappa) + 1)^{-1}$, $z = iy$, and $\tau < 1/2$,

$$\begin{aligned} & \| R_\kappa^{1/2}\{W_\kappa(t; z) - U_\kappa(tz)\}\, R_\kappa^{1/2} \| \\ & \leqslant \left\| R_\kappa^{1/2}\left\{V_\kappa\left(\frac{tz}{n}\,; 0\right) - U_\kappa\left(\frac{tz}{n}\right)\right\} R_\kappa^{1/2}\right\| \\ & \quad + \sum_{r=1}^{n-1}\prod_{j=0}^{r-1}\left\| R_\kappa^{1/2} V_\kappa\left(\frac{tz}{n}\,; \frac{jt}{n}\right) R_\kappa^{-1/2}\right\| \\ & \quad \times \left\| R_\kappa^{1/2}\left\{V_\kappa\left(\frac{tz}{n}\,; \frac{rt}{n}\right) - U_\kappa\left(\frac{tz}{n}\right)\right\} R_\kappa^{1/2}\right\| \left\| U_\kappa\left(\frac{tz}{n}\right)^{n-1-r}\right\| \\ & \leqslant \sum_{r=0}^{n-1}\left(1 + \frac{Mty}{n}(\log\kappa)\right)^{r/2} M_1\,\frac{ty}{n}\,\kappa^{-\tau}. \end{aligned}$$

We have used Lemma 2.3.2 to bound the $R^{1/2}VR^{-1/2}$ factors and

Lemma 2.2.3 to bound the difference factor $R^{1/2}(V - U)\, R^{1/2}$. We also used $\| U_\kappa(tz/n)\| \leqslant 1$, valid by (1.2.7). Thus

$$\begin{aligned} \| R_\kappa^{1/2}\{W_\kappa(t; z) - U_\kappa(tz)\} R_\kappa^{1/2} \| &\leqslant \left(1 + \frac{Mty}{n} \log \kappa\right)^{n/2} M_1 ty\kappa^{-\tau} \\ &\leqslant M_1 ty\kappa^{-\tau} e^{(Mty \log \kappa)/2} \\ &\leqslant M_1 ty\kappa^{(Mty/2)-\tau} \\ &\leqslant o(1) \text{ for } 0 \leqslant ty < 2\tau/M. \qquad (2.3.12) \end{aligned}$$

We now estimate $F_\kappa(-iy)$ of (2.3.10). Since

$$| F_\kappa(-iy)| \leqslant 2 \| \Omega \|^2, \qquad (2.3.13)$$

it is sufficient to prove convergence for Ω in a dense set, namely $\mathscr{D}(H(g))$. Because

$$H(g) = \operatorname{graph} \lim_{\kappa \to \infty} H(g, \kappa),$$

any $\Omega \in \mathscr{D}(H(g))$ can be approximated by a sequence Ω_κ in $\mathscr{D}(H(g, \kappa))$ such that

$$\| \Omega - \Omega_\kappa \| \leqslant o(1), \qquad \|(H(g, \kappa) + I)^{1/2} \Omega_\kappa \| \leqslant O(1). \qquad (2.3.14)$$

We write the Ω dependence in $F_\kappa(-iy, \Omega)$ explicitly. Then by (2.3.12–2.3.14),

$$\begin{aligned} | F_\kappa(-iy, \Omega)| &= | F_\kappa(-iy, \Omega_\kappa)| + o(1) \\ &= |((H(g, \kappa) + I)^{1/2}\, \Omega_\kappa\, ,\, R^{1/2}\{W_\kappa(t; -iy) - U_\kappa(-iyt)\} \\ &\quad \times R_\kappa^{1/2}(H(g, \kappa) + I)^{1/2}\, \Omega_\kappa)| + o(1) \\ &\leqslant \|(Hg, \kappa) + I)^{1/2}\, \Omega_\kappa \|^2 \\ &\quad \times \| R^{1/2}\{W_\kappa(t; -iy) - U_\kappa(-iyt)\} R_\kappa^{1/2} \| + o(1) \\ &= o(1). \end{aligned}$$

LEMMA 2.3.5. *Let f_Λ be the function* (2.2.19). *There is a choice $\kappa = \kappa(\Lambda)$ of the cutoff κ such that $\kappa(\Lambda) \to \infty$ and*

$$w \cdot \lim_{\Lambda \to \infty} \int \{W_{\kappa(\Lambda)}(t; x) - U(t)\} f_\Lambda(x)\, dx = 0. \qquad (2.3.15)$$

Proof. Since $U_\kappa(s)$ converges strongly to $U(s)$ and since the strong convergence is uniform for s in a bounded interval,

$$\lim_{\Lambda\to\infty} \sup_{x\in[1-\Lambda^{-1},1]} \| [U_{\kappa(\Lambda)}(tx) - U(t)]\,\Omega \| = 0$$

for any $\kappa(\Lambda) \to \infty$. Thus it is sufficient to prove (2.3.15) with $U(t)$ replaced by $U_{\kappa(\Lambda)}(tx)$. We approximate each f_Λ by a function h_Λ, analytic in a complex neighborhood of $[0, 1]$, and such that $\| f_\Lambda - h_\Lambda \|_1 \to 0$. Since as $\Lambda \to \infty$,

$$\left| \left(\Omega, \int_0^1 \{W_\kappa(t; x) - U_\kappa(tx)\}\{f_\Lambda(x) - h_\Lambda(x)\}\, dx\, \Omega\right) \right|$$

$$\leqslant 2\,\| \Omega \|^2 \int_0^1 | f_\Lambda(x) - h_\Lambda(x)|\, dx \leqslant o(1)\, \| \Omega \|^2,$$

it is sufficient to prove that for each Λ,

$$w \cdot \lim_{\kappa\to\infty} \int_0^1 \{W_\kappa(t; x) - U_\kappa(tx)\}\, h_\Lambda(x)\, dx = 0.$$

Let Ω and t be fixed, and

$$G_\kappa(x) = (\Omega, \{W_\kappa(t; x) - U_\kappa(tx)\}\, \Omega).$$

$G_\kappa(x)$ is the boundary value of a function $G_\kappa(z)$ analytic in z in the half-plane $\operatorname{Im} tz < 0$. Let C_Λ be a continuous path of finite length in $\operatorname{Im} tz < 0$, running from 0 to 1 and lying in the domain of analyticity of $h_\Lambda(z)$. By the Cauchy formula,

$$\int_0^1 G_\kappa(x)\, h_\Lambda(x)\, dx = \int_{C_\Lambda} G_\kappa(z)\, h_\Lambda(z)\, dz. \tag{2.3.16}$$

Since $| G_\kappa(z)|$ is uniformly bounded on C_Λ, and since by Theorem 2.3.4

$$\lim_{\kappa\to\infty} G_\kappa(z) = 0, \qquad z \in C_\Lambda,$$

the right side of (2.3.16) converges to zero as $\kappa \to \infty$. Thus for fixed Λ, as $\kappa \to \infty$

$$\left| \int_0^1 G_\kappa(x)\, h_\Lambda\, dx \right| \leqslant o(1),$$

and the proof is complete.

Proof of Theorem 2.3.1. We have

$$\begin{aligned}
&|(\Omega, \{A_{\Lambda,\kappa}(t) - A(t)\}\, \Omega)| \\
&\quad = \left| \left(\Omega, \int \{W_\kappa(t; x)^*\, A W_\kappa(t; x) - U(t)^*\, A U(t)\} f_\Lambda(x)\, dx\, \Omega\right) \right| \\
&\quad \leqslant 2 \,\|A\|\, \|\Omega\| \int \|\{W_\kappa(t; x) - U(t)\}\, \Omega\,\| f_\Lambda(x)\, dx.
\end{aligned}$$

By the Schwarz inequality and the positivity of f_Λ,

$$\begin{aligned}
&\left(\int \| W_\kappa(t; x) - U(t) \| f_\Lambda(x)\, dx\right)^2 \\
&\qquad \leqslant \int \|\{W_\kappa(t; x) - U(t)\}\, \Omega\,\|^2 f_\Lambda(x)\, dx \\
&\qquad = 2 \operatorname{Re} \int (U(t)^* \{U(t) - W_\kappa(t\,; x)\}\, \Omega, \Omega)\, f_\Lambda(x)\, dx.
\end{aligned}$$

We used the fact that $\int f_\Lambda(x)\, dx = 1$. The theorem follows from Lemma 2.3.5.

2.4. *An Estimate for Convergence*

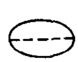
In this section we prove the estimate (2.3.4).

LEMMA 2.4.1. *Let* $\delta H(s) = \delta H(g, \kappa, s)$ *be defined by* (2.2.6). *Let* $\epsilon > 0$. *There are constants* M, κ_ϵ *such that for* $\kappa > \kappa_\epsilon$,

$$\delta H(s)^2 \leqslant \epsilon H(g, \kappa)^2 + M(\log \kappa) \tag{2.4.1}$$

as bilinear forms on $\mathscr{D}(H_0) \times \mathscr{D}(H_0)$.

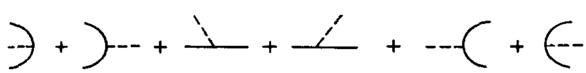
Remark. The $\log \kappa$ in (2.4.1) cannot be eliminated because it comes from the divergent contribution



(2.4.2)

to $\delta H(s)^2$.

Proof. We write $\delta H(s)$ diagramatically as

$\delta H(s) =$ 

$= W_1 + W_2 + W_3 + W_4 + W_5 + W_6$

(2.4.3)

Then

$$\delta H(s)^2 = \sum_{i,j=1}^{6} W_i W_j . \tag{2.4.4}$$

The kernel w_i of each W_i satisfies the bound

$$| w_i(k, p_1, p_2)| \leqslant \text{const} \cdot \kappa^{-1}\mu(k)^{-1/2}\chi(k/\kappa)\,\chi(p_1/\kappa)\,\chi(p_2/\kappa) \tag{2.4.5}$$

by (2.2.23). From (2.4.5) we infer that

$$\| w_i \|_2^2 \leqslant \text{const} \cdot (\log \kappa), \tag{2.4.6}$$

$$\| \omega(p_j)^{-\tau/2} w_i \|_2^2 \leqslant \text{const} \cdot \kappa^{-\tau} \log \kappa, \qquad \tau < 1. \tag{2.4.7}$$

Thus by Theorem 1.2.1 (see also Ref. [10, Theorem 2.1.1b]), for $\tau < 1/2$,

$$\| W_i(N_\tau + I)^{-1} \| \leqslant O(\kappa^{-\tau} \log \kappa), \qquad i = 3, 4, 5, 6, \tag{2.4.8}$$

$$\| W_i(N_\tau + I)^{-1} \| \leqslant O(\log \kappa), \qquad i = 1, 2, \tag{2.4.9}$$

$$\|(N_\tau + I)^{-1} W_i \| \leqslant O(\kappa^{-\tau} \log \kappa), \qquad i = 1, 2, 3, 4, \tag{2.4.10}$$

$$\|(N_\tau + I)^{-1} W_i \| \leqslant O(\log \kappa), \qquad i = 5, 6. \tag{2.4.11}$$

Using (2.4.8–2.4.11) we deal with the terms $W_i W_j$ in

$$W = \sum_{i,j=1}^{6} W_i W_j - \{W_5 W_1 + W_5 W_2 + W_6 W_1 + W_6 W_2\}. \tag{2.4.12}$$

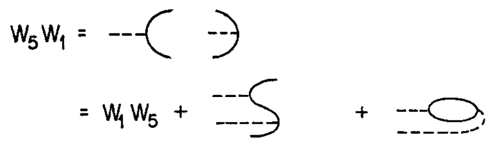
Each term $W_i W_j$ contributing to W satisfies the bound

$$\begin{aligned} \|(N_\tau + I)^{-1} W_i W_j (N_\tau + I)^{-1} \| &\leqslant \|(N_\tau + I)^{-1} W_i \| \, \| W_j (N_\tau + I)^{-1} \| \\ &\leqslant O(\kappa^{-\tau}(\log \kappa)^2) \\ &\leqslant O(\kappa^{-\tau/2}), \end{aligned} \tag{2.4.13}$$

since at least one factor W_i or W_j satisfies either (2.4.8) or (2.4.10). We now bound the remaining terms,

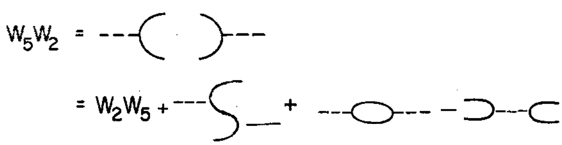
$$W_5 W_1 + W_5 W_2 + W_6 W_1 + W_6 W_2 , \tag{2.4.14}$$

where both factors satisfy (2.4.9) or (2.4.11). We inspect each of the four terms in (2.4.14) individually. By straightforward computation,

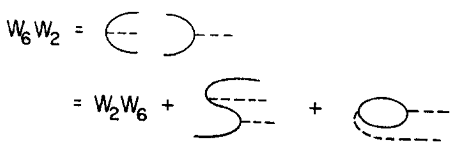


$$= W_1 W_5 + Z_1 + Z_2 . \tag{2.4.15}$$

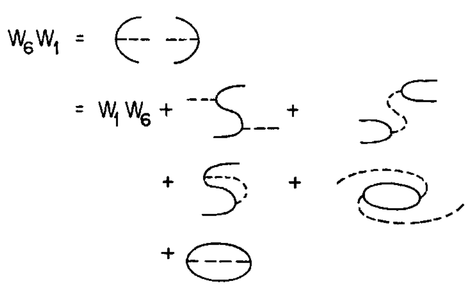
Equation (2.4.15) is the expansion of $W_5 W_1$ into a sum of Wick ordered monomials. See Refs. [2, 12], for instance, for an explanation of the diagramatic notation. Similarly we expand



$$= W_2 W_5 + Z_3 + Z_4 - Z_6 \tag{2.4.16}$$



$$= W_2 W_6 + Z_{10} + Z_{11} \tag{2.4.17}$$



(2.4.18)

$$= W_1 W_6 + Z_5 + Z_6 + Z_7 + Z_8 + Z_9$$

We group the terms $Z_1 ,..., Z_{11}$ into three sets, according to whether they have one, two, or three contractions. Let

$$Z' = Z_1 + Z_3 + Z_5 + Z_6 + Z_{10} , \tag{2.4.19}$$

$$Z'' = Z_2 + Z_4 + Z_7 + Z_8 + Z_{11} . \tag{2.4.20}$$

The once-contracted terms in Z' require second-order estimates but first-order estimates suffice for the twice-contracted terms in Z''. By the above expansions,

$$\delta H(s)^2 = W + W_1 W_5 + W_2 W_5 + W_1 W_6 + W_2 W_6 + Z' + Z'' + Z_9 . \tag{2.4.21}$$

The estimate

$$| Z_9 | \leqslant M \log \kappa \tag{2.4.22}$$

follows from (2.4.6). We later prove

$$\|(N_\tau + I)^{-1} Z'(N_\tau + I)^{-1}\| \leqslant O(\kappa^{-\tau/2}), \tag{2.4.23}$$

$$\|(N_\tau + I)^{-1/2} Z''(N_\tau + I)^{-1/2}\| \leqslant O(1). \tag{2.4.24}$$

Therefore by (2.4.12–2.4.13) and (2.4.21–2.4.24),

$$\delta H(s)^2 \leqslant O(\kappa^{-\tau/2})(N_\tau + I)^2 + O(1)(N_\tau + I) + M \log \kappa. \tag{2.4.25}$$

We choose $0 < \tau < \frac{1}{2}$ and $0 < \epsilon < \tau/2$. We substitute first-order and second-order estimates from Theorem 1.2.1. We use a weaker form of the second order estimate proved in Ref. [10].

$$\begin{aligned} (N_\tau + I)^2 &\leqslant M\kappa^\epsilon(H(g, \kappa) + I)^2 \\ &\leqslant 2M\kappa^\epsilon(H(g, \kappa)^2 + I), \end{aligned} \tag{2.4.26}$$

$$(N_\tau + I) \leqslant \text{const} \cdot (H(g, \kappa) + I) \leqslant \epsilon H(g, \kappa)^2 + O(\epsilon^{-1}) I. \tag{2.4.27}$$

We obtain (2.4.1) from (2.4.25) for κ sufficiently large. Hence the following lemma completes the proof of Lemma 2.4.1.

LEMMA 2.4.2. *The inequalities* (2.4.23–2.4.24) *are valid for* $0 < \tau$.

Proof. We apply N_τ estimates of Theorem 1.2.1 (cf. Ref. [10, Section 2]) to prove an estimate of the type (2.4.23) or (2.4.24) for each of the operators Z_i. The operators Z_2, Z_4, Z_8, and Z_{11} have kernels that are bounded by

$$\text{const} \cdot \mu(k_1)^{-1/2} \mu(k_2)^{-1/2} \kappa^{-2} \left(\int | x(k/\kappa)| \, dk\right)^2 \leqslant \text{const} \cdot (\mu(k_1)^{-1/2} \mu(k_2)^{-1/2}.$$

We use Theorem 1.2.1 or Ref. [10, Corollary 2.1.2, Cases 4–5]. Thus for $i = 2, 4, 8, 11$,

$$\|(N_\tau + I)^{-1/2} Z_i (N_\tau + I)^{-1/2}\|^2 \leqslant \text{const} \cdot \int \mu_1^{-1}\mu_2^{-1}(\mu_1 + \mu_2)^{-\tau}\, dk_1 dk_2$$
$$\leqslant \text{const}. \tag{2.4.28}$$

The kernel of Z_7 is bounded similarly by

$$\text{const} \cdot \kappa^{-1} \log \kappa \mid \chi(p_1/\kappa)\, \chi(p_2/\kappa)|.$$

Thus Theorem 1.2.1a yields

$$\|(N_\tau + I)^{-1/2} Z_7 (N_\tau + I)^{-1/2}\|^2$$
$$\leqslant \text{const} \cdot \kappa^{-2}(\log \kappa)^2 \int |\chi(p_1/\kappa)|^2\, |\chi(p_2/\kappa)|^2\, \omega_1^{-\tau}\omega_2^{-\tau}\, dp_1\, dp_2$$
$$\leqslant \text{const} \cdot \kappa^{-2\tau}(\log \kappa)^2 \leqslant \text{const}. \tag{2.4.29}$$

Thus by (2.4.28–2.4.29) we conclude that (2.4.24) is valid.

The kernel of Z_1, Z_3, Z_5, or Z_{10} is bounded by

$$\text{const} \cdot \mu(k_1)^{-1/2}\, \mu(k_2)^{-1/2}\, \kappa^{-1} \mid \chi(p_1/\kappa)\, \chi(p_2/\kappa)|.$$

Thus we can apply Theorem 1.2.1b to yield

$$\|(N_\tau + I)^{-1} Z_i (N_\tau + I)^{-1}\|^2 \leqslant \text{const} \cdot \kappa^{-2}\kappa^{1+1-\tau} = \text{const} \cdot \kappa^{-\tau}, \tag{2.4.30}$$

for $i = 1, 3, 5, 10$. We need only study Z_6, which has a kernel bounded by

$$\text{const} \left\{ \prod_{i=1}^{4} |\chi(p_i/\kappa)| \right\} \kappa^{-2} \log \kappa.$$

Thus by Theorem 1.2.1a we have

$$\|(N_\tau + I)^{-1} Z_6 (N_\tau + I)^{-1}\|^2 \leqslant \text{const} \cdot \kappa^{-4}(\log \kappa)^2 \left\{ \int |\chi(p/\kappa)|^2\, \omega^{-\tau}\, dp \right\}^4$$
$$\leqslant \text{const} \cdot \kappa^{-4}(\log \kappa)^2 (\kappa^{1-\tau})^4 \tag{2.4.31}$$
$$= \text{const} \cdot \kappa^{-4\tau}(\log \kappa)^2.$$

The inequalities (2.4.30–2.4.31) yield (2.4.32), and hence complete the proof of the lemma.

3. The Field Operators

In this section we give a preliminary discussion of the field operators and the field equations. We deal mainly with the sharp time fields, averaged over space. We use the main theorem of Section 2 and the estimates of Ref. [10]. We expect that stronger results about the fields will be proved, most of which appear to require new estimates.

We first establish basic properties of the space-averaged fields: they are differentiable in time and they are approximated by fields defined with a cutoff Hamiltonian $H(g, \kappa)$. This approximation property proves useful for establishing the field equations. We prove that the sharp time fields have self-adjoint closures. Both the fermion and boson fields are local and are independent of the space cutoff g that occurs in the Hamiltonian $H(g)$. We verify that cutoff field equations are satisfied by the cutoff sharp time fields. Since the sharp time fields and their derivatives converge as $\kappa \to \infty$, the nonlinear terms in the field equations also converge and serve to define the field equations without cutoff (see Section 3.3).

3.1. *Differentiability of the Field Operators*

We consider the sharp time fields

$$\varphi_g(f, t) = e^{itH(g)} \int f(x)\, \varphi(x)\, dx\, e^{-itH(g)}, \tag{3.1.1}$$

$$\pi_g(f, t) = e^{itH(g)} \int f(x)\, \pi(x)\, dx\, e^{-itH(g)}, \tag{3.1.2}$$

$$\psi_g(f, t) = e^{itH(g)} \int f(x)\, \psi(x)\, dx\, e^{-itH(g)}, \tag{3.1.3}$$

and their derivatives as operators or bilinear forms on the domain

$$\mathscr{D}_n(g) = \mathscr{D}(H(g)^n) \qquad \text{or} \qquad \mathscr{D}_\infty(g) = \bigcap_n \mathscr{D}(H(g)^n). \tag{3.1.4}$$

By standard free field calculations, the time-zero operators are well-defined operators on $\mathscr{F}$,

$$\varphi(f, 0) = \int f(x)\, \varphi(x)\, dx, \qquad (-\Delta + 1)^{-1/4} f \in L_2, \tag{3.1.5}$$

$$\pi(f, 0) = \int f(x)\, \pi(x)\, dx, \qquad (-\Delta + 1)^{1/4} f \in L_2, \tag{3.1.6}$$

$$\psi(f, 0) = \int f(x)\, \psi(x)\, dx, \qquad f \in L_2. \tag{3.1.7}$$

The fermion field $\psi(f, 0)$ is a bounded operator, and for real f, $\varphi(f, 0)$ and $\pi(f, 0)$ are self-adjoint. The operators (3.1.1–3.1.2) are unitary transforms of self-adjoint operators. Hence they are self-adjoint and have the domains $\exp(itH(g))\,\mathscr{D}(\varphi(f, 0))$ and $\exp(itH(g))\,\mathscr{D}(\pi(f, 0))$. Similarly, $\psi_g(f, t)$ is a bounded operator defined on all of $\mathscr{F}$.

THEOREM 3.1.1. *Let* $(-\Delta + 1)^{-1/4} f \in L_2$. *Then* $\mathscr{D}_\infty(g)$ *is a core for* $\varphi_g(f, t)$. *If* $(-\Delta + 1)^{1/4} f \in L_2$, $\mathscr{D}_\infty(g)$ *is a core for* $\pi_g(f, t)$.

Proof. We prove the result for φ; the proof for π is the same. Since $\exp(-iH(g)t)\,\mathscr{D}_\infty(g) \subset \mathscr{D}_\infty(g)$, it is sufficient to show that $\mathscr{D}_\infty(g)$ is a core for $\varphi(f, 0)$. Since $\varphi(f, 0)(N_\tau + I)^{-1/2}$ is bounded for $0 \leqslant \tau < 1$,

$$\begin{aligned}\| \varphi(f, 0)(H(g) + I)^{-1/2} \| &\leqslant \| \varphi(f, 0)(N_\tau + I)^{-1/2} \| \, \|(N_\tau + I)^{1/2}(H(g) + I)^{-1/2}\| \\ &\leqslant \text{const.}\end{aligned} \tag{3.1.8}$$

It is sufficient to prove that $\mathscr{D}_{1/2}(g)$ is a core for $\varphi(f, 0)$.

Let $\mathscr{D}$ be the domain of vectors with a finite number of particles and C_0^∞ wave functions in momentum space. We note that by standard free field calculations each vector in $\mathscr{D}$ is an entire vector for $\varphi(f, 0)$, so $\mathscr{D}$ is a core for $\varphi(f, 0)$.

We now make use of the dressing transformations T_ρ [2, 12], which exhibit in closed form a dense domain

$$\mathscr{D}_0 = \bigcup_\rho T_\rho \mathscr{D}$$

for $H(g)$. For $\theta \in \mathscr{D}$,

$$T_\rho \theta \in \mathscr{D}_0 \subset \mathscr{D}_1(g) \subset \mathscr{D}((N + I)^{1/2}) \subset \mathscr{D}(\varphi(f, 0)),$$

and we have the strong limits

$$\lim_{\rho \to \infty} T_\rho \theta = \theta, \qquad \lim_{\rho \to \infty} (N + I)^{1/2}\, T_\rho \theta = (N + I)^{1/2} \theta. \tag{3.1.9}$$

While the second limit in (3.1.9) is not established in Refs. [2, 12], its proof is immediate from the explicit form of T_ρ. Hence

$$\lim_{\rho \to \infty} \varphi(f, 0)\, T_\rho \theta = \varphi(f, 0)\theta,$$

and so

$$\varphi(f, 0) \supset (\varphi(f, 0) \restriction \mathscr{D}_{1/2}(g))^- \supset (\varphi(f, 0) \restriction \mathscr{D}_0)^- \supset (\varphi(f, 0) \restriction \mathscr{D})^-. \tag{3.1.10}$$

Since $\mathscr{D}$ is a core for $\varphi(f, 0)$, the inclusions in (3.1.10) are all equalities and the theorem is proved.

For simplicity, we no longer consider π.

PROPOSITION 3.1.2. *Let $f \in L_2$. Then $\psi_g(f, t)$ is strongly continuous in t and on $D_{1/2}(g)$, $\varphi_g(f, t)$ is strongly continuous in t.*

Proof. For $\psi_g(f, t)$ there is nothing to prove, since $\exp(itH(g))$ is strongly continuous in t. We rewrite (3.1.1) as

$$\varphi_g(f, t) = e^{itH(g)}[\varphi(f, 0)(H(g) + I)^{-1/2}]\, e^{-itH(g)}(H(g) + I)^{1/2},$$

so by (3.1.8), $\varphi_g(f, t)$ is strongly continuous in t on $\mathscr{D}_{1/2}(g)$.

PROPOSITION 3.1.3. *Let $f \in L_2$. The operators $\varphi_g(f, t)$ and $\psi_g(f, t)$, as bilinear forms on the domain $\mathscr{D}_{j+1/2}(g) \times \mathscr{D}_{j+1/2}(g)$ have continuous time derivatives of order up to j, and*

$$\left(\frac{d}{dt}\right)^n (\theta, \varphi_g(f, t)\, \theta) = i^n(\theta, (\operatorname{ad} H(g))^n\, \varphi_g(f, t)\, \theta),$$

$$\left(\frac{d}{dt}\right)^n (\theta, \psi_g(f, t)\, \theta) = i^n(\theta, (\operatorname{ad} H(g))^n\, \psi_g(f, t)\, \theta),$$

for $\theta \in \mathscr{D}_{j+1/2}(g)$, $0 \leqslant n \leqslant j$.

Proof. We write $\theta(t) = \exp(-itH(g))\theta$, and

$$(\theta, \varphi(f, t)\, \theta) = (\theta(t), [\varphi(f, 0)(H(g) + I)^{-1/2}](H(g) + I)^{1/2}\, \theta(t)).$$

Since $\theta(t)$ and $(H(g) + I)^{1/2}\, \theta(t)$ have continuous strong derivatives of order up to j, and since $\varphi(f, 0)(H(g) + I)^{-1/2}$ is bounded, the result follows for $\varphi_g(f, t)$. The proof for $\psi_g(f, t)$ is similar.

In order to study the field equations, we will want to consider the approximate fields

$$\varphi_{g,\kappa}(f, t) = e^{itH(g,\kappa)}\varphi(f, 0)\, e^{-itH(g,\kappa)}, \tag{3.1.11}$$

$$\psi_{g,\kappa}(f, t) = e^{itH(g,\kappa)}\psi(f, 0)\, e^{-itH(g,\kappa)}. \tag{3.1.12}$$

Let R_κ be the resolvent of $H(g, \kappa)$ and let R be the resolvent of $H(g)$.

LEMMA 3.1.4. *Let $f \in L_2$. The following limits exist:*

$$\varphi_g(g, t)\, R^{1/2} = \operatorname*{weak\,lim}_{\kappa\to\infty} \varphi_{g,\kappa}(f, t)\, R_\kappa^{1/2}, \tag{3.1.13}$$

$$\psi_g(f, t) = \operatorname*{strong\,lim}_{\kappa\to\infty} \psi_{g,\kappa}(f, t). \tag{3.1.14}$$

Proof. Since the unitary groups $\exp(itH(g, \kappa))$ converge strongly, it is sufficient to consider the case $t = 0$. For ψ there is nothing to prove. For φ, it is sufficient to prove weak convergence on a dense domain, since the operators $\varphi(f, 0)\, R_\kappa^{1/2}$ are bound uniformly in κ. Let $\theta \in \mathscr{D}(\varphi(f, 0))$,

$$(\theta, \varphi(f, 0)\, R_\kappa^{1/2}\Omega) = (\varphi(f, 0)\, \theta, R_\kappa^{1/2}\Omega)$$
$$\to (\varphi(f, 0)\, \theta, R^{1/2}\Omega) = (\theta, \varphi(f, 0)\, R^{1/2}\Omega),$$

since $\mathscr{D}_{1/2}(g) \subset \mathscr{D}(\varphi(f, 0))$.

We prove the following proposition by the same method.

PROPOSITION 3.1.5. *Let $f \in L_2$. The following limits exist:*

$$\left(\frac{d}{dt}\right)^n R^j \varphi_g(f, t)\, R^{j+1/2} = \operatorname*{weak\,lim}_{\kappa\to\infty} \left(\frac{d}{dt}\right)^n R_\kappa{}^j \varphi_{g,\kappa}(f, t)\, R_\kappa^{j+1/2},$$

$$\left(\frac{d}{dt}\right)^n R^j \psi_g(f, t)\, R^j = \operatorname*{strong\,lim}_{\kappa\to\infty} \left(\frac{d}{dt}\right)^n R_\kappa{}^j \psi_{g,\kappa}(f, t)\, R_\kappa{}^j,$$

for $0 \leqslant n \leqslant j$.

COROLLARY 3.1.6. *Let $(-\Delta + 1)^{1/4} f \in L_2$. Then*

$$\frac{d}{dt}\varphi_g(f, t) = \pi_g(f, t)$$

on the domain $\mathscr{D}_{3/2}(g) \times \mathscr{D}_{3/2}(g)$.

Proof. The corresponding equation holds for the fields $\varphi_{g,\kappa}$ and $\pi_{g,\kappa}$ by an elementary calculation. Thus using Proposition 3.1.5, we can take the limit $\kappa \to \infty$ and obtain the corollary.

3.2. *Locality and Cutoff Independence*

THEOREM 3.2.1. *Let f be in L_2 and let f have compact support. If $g = 1$ on a sufficiently large set, then $\varphi_g(f, t)$ and $\psi_g(f, t)$ are independent of g.*

Proof. Any spectral projection E of one of these operators belongs to a local field algebra $\mathfrak{A}_f(B)$, and in Section 2 we proved that

$$e^{itH(g)}E\, e^{-itH(g)} = \sigma_t(E)$$

is independent of g provided $g = 1$ on a sufficiently large set. Thus the sharp time field operators are also independent of g.

Remark. We now drop the g from the field operators of Theorem 3.2.1. We also drop the g in $\varphi(x, t)$ and $\psi(x, t)$ when there is an expressed or implied integration over x. We note that $(d/dt)\,\varphi(f, t) = \pi(f, t)$ is also independent of g. However, we have not established the g independence of $(d/dt)^2\,\varphi(f, t)$ since the second derivative is given on the domain $\mathscr{D}_{5/2}(g) \times \mathscr{D}_{5/2}(g)$ and we have no g-independent domain for this bilinear form.

THEOREM 3.2.2. *At space-like separation, the boson fields $\varphi(f_1, t_1)$ and $\varphi(f_2, t_2)$ commute and the fermion fields $\psi(f_1, t_1)$ and $\psi(f_2, t_2)$ anticommute. Also the boson fields commute with the fermion fields.*

Proof. We consider the commuting fields first. It is sufficient to show that the corresponding spectral projections commute. The spectral projections of φ belong to a local algebra of observables, while ψ and $\bar{\psi}$ belong to a local field algebra. Since in the case of space-like separation a local algebra of observables commutes with a local field algebra and also with another local algebra of observables, the commutativity in the theorem follows (see Section 2).

To prove that the fermion fields anticommute, we may take one of the field operators $\psi(f, 0)$ at time zero, and we approximate the other by $A_{\Lambda,\kappa}(t)$, defined by (2.2.20), where A in that formula is the appropriate time-zero fermion field operator, $\psi(f_2, 0)$. We use the Trotter product formula to approximate $W_\kappa(t; x)^*$ and $W_\kappa(t; x)$ in (2.2.20). The anticommutativity then follows by an elementary calculation.

THEOREM 3.2.3. *At equal times, the cannonical commutation and anticommutation relations hold.*

Proof. They hold at time zero because the time zero operators are free field operators. They hold at any other time t because the time t operators are unitarily equivalent to time zero operators.

Remark. In the case of the unbounded boson fields φ and π, the commutators are defined as bilinear forms on $\mathscr{D}_{1/2}(g) \times \mathscr{D}_{1/2}(g)$, where $g = 1$ on a sufficiently large set.

We also remark that the fields are covariant under space-time automorphisms (see Ref. [7]).

3.3. *The Field Equations*

We observe that the field equations (1.1.1–1.1.3) are satisfied. We take space derivatives in the sense of distributions and we regard the sharp time field operators $\varphi(f, t)$ and $\psi(f, t)$ as bilinear forms on $\mathscr{D}_3(g) \times \mathscr{D}_3(g)$, where $g = 1$ on a sufficiently large set. The time derivatives are then weak derivatives of bilinear forms, and the time derivatives are continuous in t by Proposition 3.1.3.

The major problem comes from the singular current j in (1.1.1). We can avoid dealing directly with j as follows: We first verify field equations for $\varphi_{g,\kappa}$ and $\psi_{g,\kappa}$,

$$\left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} + m_b{}^2\right) \varphi_{g,\kappa} + j_{g,\kappa} = 0, \tag{3.3.1}$$

$$\left(\gamma_0 \frac{\partial}{\partial t} + \gamma_1 \frac{\partial}{\partial x} + m_f\right) \psi_{g,\kappa} + J_{g,\kappa} = 0, \tag{3.3.2}$$

where $j_{g,\kappa}$ and $J_{g,\kappa}$ are the nonlinear terms. Using Proposition 3.1.5 we get convergence of the linear terms in (3.3.1–3.3.2) as $\kappa \to \infty$. Hence, using the equations of motion we get convergence of the nonlinear terms also. In particular, j is defined as the limit of the cutoff currents, and the space-averaged current $j(f, t)$ is a bilinear form on $\mathscr{D}_3(g) \times \mathscr{D}_3(g)$, continuous in t. On this domain, the current is formally independent of g, but we have no g-independent domain for $j(f, t)$ or $\int j(f, t)\, \alpha(t)\, dt$. It would be desirable to investigate the meaning of the equations (1.1.1–1.1.4) more thoroughly.

The unrenormalized current density is given at time zero by

$$\begin{aligned} j_{g,\kappa,\mathrm{un}}(x) &= -i[H_I(g, \kappa), \pi(x)] \\ &= \lambda \int g(y)\, \delta_\kappa(x - y) : \bar{\psi}_\kappa(y)\, \psi_\kappa(y) : dy \\ &= \int K(x, y_1, y_2) : \bar{\psi}(y_1)\, \psi(y_2) : dy_1\, dy_2\,, \end{aligned} \tag{3.3.3}$$

where

$$K(x, y_1, y_2) = \lambda \int g(y)\, \delta_\kappa(x - y)\, \delta_\kappa(y - y_1)\, \delta_\kappa(y - y_2)\, dy. \tag{3.3.4}$$

The renormalized current density is given at time zero by

$$\begin{aligned} j_{g,\kappa}(x) &= -[H_I(g, \kappa) + c(g, \kappa), \pi(x)] \\ &= j_{g,\kappa,\mathrm{un}}(x) - \delta m^2(\kappa)\, g(x)^2\, \varphi(x), \end{aligned} \tag{3.3.5}$$

and

$$j_{g,\kappa}(x, t) = e^{iH(g,\kappa)t} j_{g,\kappa}(x)\, e^{-iH(g,\kappa)t}. \tag{3.3.6}$$

The expression (3.3.6) is the nonlinear term in (3.3.1).

The nonlinear term in the Dirac equation at time zero is given by

$$J_{g,\kappa}(x) = i\gamma_0[H_I(g,\kappa), \psi(x)] = \lambda \int g(y)\, \delta_\kappa(x-y)\, \varphi_\kappa(y)\, \psi_\kappa(y)\, dy$$
$$= \int K(x, y_1, y_2)\, \varphi(y_1)\, \psi(y_2)\, dy_1\, dy_2 ,$$

where

$$K(x, y_1, y_2) = \lambda \int g(y)\, \delta_\kappa(x-y)\, \delta_\kappa(y-y_1)\, \delta_\kappa(y-y_2)\, dy.$$

The nonlinear term at time t is

$$J_{g,\kappa}(x,t) = e^{iH(g,\kappa)t} J_{g,\kappa}(x)\, e^{-iH(g,\kappa)t}.$$

By Proposition 3.1.5 and (3.3.1–3.3.2) we have

PROPOSITION 3.3.1. *The currents* $j_{g,\kappa}(f,t)$ *and* $J_{g,\kappa}(f,t)$ *converge as* $\kappa \to \infty$ *to bilinear forms on* $\mathscr{D}_3(g) \times \mathscr{D}_3(g)$. *The limiting currents* $j(f,t)$ *and* $J(f,t)$ *uniquely determine symmetric operators*

$$\int j(f,t)\, \alpha(t)\, dt \qquad \textit{and} \qquad \int J(f,t)\, \alpha(t)\, dt$$

on the domain $\mathscr{D}_3(g)$.

The fact that the currents are operators, not merely bilinear forms, follows from the field equations (3.3.1–3.3.2) which express j and J as derivatives of φ and ψ. We already know that the space derivatives of φ and ψ are operators by Theorem 3.1.1. The time derivatives of the fields also determine operators after taking time averages, since the time derivatives can be placed on the test functions.

It is clear that the individual terms in (3.3.5) do not converge as $\kappa \to \infty$, since $\delta m^2(\kappa)$ is divergent and $\varphi(f)$ is an operator on $\mathscr{D}_3(g)$. The cancellations that occur have been exhibited explicitly in Ref. [18].

APPENDIX

We prove that the mass renormalization constant $\delta m^2(\kappa)$ of (1.2.6) is a special case of the mass renormalization constant $\delta m^2(g,\kappa)$ of Ref. [10, (1.3.12)]. We assume that the stability condition $m_b < 2m_f$ is satisfied; otherwise an additional estimate involving condition (c_3) of Ref. [10] is necessary. By Ref. [10, Section 1.3],

$$\delta m^2(g, \kappa) = -\frac{\lambda^2}{4\pi}\int |g^\sim(p_1+p_2)|^2 \left\{\frac{\omega_1\omega_2 - p_1p_2 - m_f^2}{\omega_1\omega_2}\right\} |\chi_\kappa(0, p_1, p_2)|^2$$
$$\times \left\{\frac{1}{\omega_1+\omega_2-m_b} + \frac{1}{m_1+m_2+m_b}\right\} dp_1\, dp_2$$
$$+ M(g) + o(1). \tag{A1}$$

The terms $M(g) + o(1)$ in (A1) are an arbitrary finite renormalization. The constant $M(g)$ depends on g but is independent of κ and the particular cutoff χ_κ. The term $o(1)$ in (A1) may depend on g, κ, and χ_κ, and it converges to zero as $\kappa \to \infty$.

We use the variables $\eta = p_1 + p_2$ and $\xi = p_1 - p_2$. Let

$$\beta_1 = \frac{1}{2}\left\{\frac{1}{\omega_1+\omega_2-m_b} + \frac{1}{\omega_1+\omega_2+m_b}\right\}, \qquad \bar\beta_1 = \omega(\xi)^{-1},$$

$$\beta_2 = \frac{1}{2}\left(\frac{\omega_1\omega_2 - p_1p_2 - m_f^2}{\omega_1\omega_2}\right), \qquad \bar\beta_2 = 1,$$

$$\beta_3 = |\chi_\kappa(0, p_1, p_2)|^2, \qquad \bar\beta_3 = \left|\chi_\kappa\left(0, \frac{\xi}{2}, \frac{-\xi}{2}\right)\right|^2.$$

Thus

$$\delta m^2(g, \kappa) = -\frac{\lambda_2}{2\pi}(\|g\|_2)^{-2}\int |g^\sim(\eta)|^2 \beta_1\beta_2\beta_3\, d\xi\, d\eta + M(g) + o(1).$$

On the other hand, $\delta m^2(\kappa)$ of (1.2.6) is given by

$$\delta m^2(\kappa) = -\frac{\lambda^2}{2\pi}\int \bar\beta_1\bar\beta_2\bar\beta_3\, d\xi + M,$$

where M is a constant independent of g, κ, and χ_κ. Since the $\bar\beta_i$ are independent of η,

$$\delta m^2(\kappa) - \delta m^2(g, \kappa) = -\frac{\lambda^2}{2\pi}(\|g\|_2)^{-2}\int |g^\sim(\eta)|^2 (\bar\beta_1\bar\beta_2\bar\beta_3 - \beta_1\beta_2\beta_3)\, d\xi\, d\eta$$
$$+ M - M(g) + o(1). \tag{A2}$$

We now show that for the proper choice of $o(1)$ in (A2),

$$\delta m^2(\kappa) - \delta m^2(g, \kappa) = -\frac{\lambda^2}{2\pi}(\|g\|_2)^{-2}\int |g^\sim(\eta)|^2(\bar\beta_1 - \beta_1\beta_2)\, d\xi\, d\eta + M - M(g). \tag{A3}$$

The integral in (A3) is finite and independent of κ and χ_κ, so we choose

$$M(g) = M - \frac{\lambda^2}{2\pi} (\| g \|_2)^{-2} \int | \tilde{g}(\eta)|^2 (\bar{\beta}_1 - \beta_1\beta_2)\, d\xi\, d\eta. \qquad \text{(A4)}$$

Then $\delta m^2(\kappa) = \delta m^2(g, \kappa)$ as required.

Since

$$\bar{\beta}_1 - \beta_1\beta_2 = \bar{\beta}_1(1 - \beta_2) + (\bar{\beta}_1 - \beta_1)\, \beta_2\,,$$

elementary inequalities yield

$$| \bar{\beta}_1 - \beta_1\beta_2 | \leqslant \text{const} \cdot \omega(\xi)^{-2}\, \omega(\eta)^2, \qquad \text{(A5)}$$

so that (A4) is a finite, g-dependent constant.

We expand

$$\bar{\beta}_1\bar{\beta}_2\bar{\beta}_3 - \beta_1\beta_2\beta_3 = \beta_1\beta_2(\bar{\beta}_3 - \beta_3) + (\bar{\beta}_1 - \beta_1\beta_2)(\bar{\beta}_3 - 1) + (\bar{\beta}_1 - \beta_1\beta_2) \quad \text{(A6)}$$

and analyze each contribution of (A6) to (A2). By condition (c_2) of Ref. [10, Section 1.3],

$$\int \left| \chi_\kappa(0, p_1, p_2) - \chi_\kappa \left(0, \frac{\xi}{2}, \frac{-\xi}{2}\right) \right| \omega(\xi)^{-1}\, d\xi \leqslant O(\kappa^{-\epsilon})\, \omega(\eta)^{2\epsilon},$$

so that from $| \beta_1\beta_2 | \leqslant \text{const} \cdot \omega(\xi)^{-1}$, we conclude

$$\left| \int | g^\sim(\eta)|^2\, \beta_1\beta_2(\bar{\beta}_3 - \beta_3)\, d\xi\, d\eta \right| \leqslant O(\kappa^{-\epsilon}) \int | g^\sim(\eta)|^2\, \omega(\eta)^{2\epsilon}\, d\eta \leqslant O(\kappa^{-\epsilon})$$

and the first term in (A6) contributes $O(\kappa^{-\epsilon})$ to (A2). For the second term of (A6), we divide the ξ integration into two parts according to whether or not $| \xi | \leqslant \kappa^{1-\beta}$. If $| \xi | \leqslant \kappa^{1-\beta}$, by conditions (a) and (c_1) of Ref. [10, Section 1.3], $| \bar{\beta}_3 - 1 | \leqslant O(\kappa^{-\epsilon})$, so that by (A5) the integral is convergent and $O(\kappa^{-\epsilon})$. If $| \xi | \geqslant \kappa^{1-\beta}$, we bound $| \bar{\beta}_3 - 1 |$ by a constant and use (A5) to establish convergence of the ξ integration and hence to bound the integral by $O(\kappa^{-\epsilon})$. By choosing $o(1)$ in (A2) to be the negative of the $O(\kappa^{-\epsilon})$ terms, we have established (A3).

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