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The dilation property of modulation spaces and their inclusion relation with Besov spaces

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Abstract

We consider the dilation property of the modulation spaces $M^{p,q}$. Let $D_\lambda : f(t) \mapsto f(\lambda t)$ be the dilation operator, and we consider the behavior of the operator norm $\|D_\lambda\|_{M^{p,q} \to M^{p,q}}$ with respect to λ. Our result determines the best order for it, and as an application, we establish the optimality of the inclusion relation between the modulation spaces and Besov spaces, which was proved by Toft [J. Toft, Continuity properties for modulation spaces, with applications to pseudo-differential calculus, I, J. Funct. Anal. 207 (2004) 399–429].

Keywords: Modulation spaces; Besov spaces; Dilation; Inclusion

1. Introduction

The modulation spaces $M^{p,q}$ were first introduced by Feichtinger [3,4] and generalized by Feichtinger and Gröchenig [6]. The exact definition will be given in the next section, but the main idea is to consider the decaying property of a function with respect to the space variable and the variable of its Fourier transform simultaneously. That is exactly the heart of the matter of the time–frequency analysis which is originated in signal analysis or quantum mechanics.

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Based on a similar idea, Sjöstrand [15] independently introduced a symbol class which assures the L^2-boundedness of corresponding pseudo-differential operators. In the last decade, the theory of the modulation spaces has been developed, and its usefulness for the theory of pseudo-differential operators is getting realized gradually. Nowadays Sjöstrand's symbol class is recognized as a special case of the modulation spaces, and many authors used these spaces, as a powerful tool, to discuss the boundedness or compactness properties of pseudo-differential operators. See, for example, Boulkhemair [1], Gröchenig [11], Gröchenig and Heil [12,13], and Toft [18,19]. Consult Feichtinger [5], Gröchenig [10], and Teofanov [17] for further and detailed history of this research fields. Some arguments in these works have their origin in the field of phase space analysis. See also Dimassi and Sjöstrand [2] and Folland [7] for this direction.

Now we are in a situation to start showing fundamental properties of the modulation spaces, in order to apply them to many other problems. Actually in Toft's recent work [18], he investigated the mapping property of convolutions, and showed Young-type results for the modulation spaces. As an application, he showed an inclusion relation between the modulation spaces and Besov spaces. We also mention that some extensions to weighted modulation spaces of the inclusion can be found in Toft [19,20]. We remark that Besov spaces are used in various problems of partial differential equations, and his result will help us to understand how they are translated into the terminology of modulation spaces.

Among many other important properties to be shown, we focus on the dilation property of the modulation spaces in this article. Since $M^{2,2}(\mathbb{R}^n) = L^2(\mathbb{R}^n)$, we have easily $\|f_\lambda\|_{M^{2,2}} = \lambda^{-n/2}\|f\|_{M^{2,2}}$ by the change of variables $t \mapsto \lambda^{-1}t$, where $f_\lambda(t) = f(\lambda t)$ and $t \in \mathbb{R}^n$. But it is not clear how $\|f_\lambda\|_{M^{p,q}}$ behaves like with respect to λ except for the case $(p,q) = (2,2)$. Our objective is to draw the complete picture of the best order of λ for every pair of (p,q) (Theorem 1.1).

We can expect various kinds of applications of this consideration. In fact, this kind of dilation property is frequently used in the "scaling argument," which is a popular tool to know the best possible order of the conditions in problems of partial differential equations. Actually, in this article, we also show the best possibility of Toft's inclusion relation mentioned above, as a side product of the main argument (Theorem 1.2).

In order to state our main results, we introduce several indices. For $1 \leqslant p \leqslant \infty$, we denote by p' the conjugate exponent of p (that is, $1/p + 1/p' = 1$). We define subsets of $(1/p, 1/q) \in [0,1] \times [0,1]$ in the following way:

$$
\begin{aligned}
&I_1\colon \max(1/p, 1/p') \leqslant 1/q, && I_1^*\colon \min(1/p, 1/p') \geqslant 1/q,\\
&I_2\colon \max(1/q, 1/2) \leqslant 1/p', && I_2^*\colon \min(1/q, 1/2) \geqslant 1/p',\\
&I_3\colon \max(1/q, 1/2) \leqslant 1/p, && I_3^*\colon \min(1/q, 1/2) \geqslant 1/p.
\end{aligned}
$$

Let us consider Fig. 1. In [18], Toft introduced the indices

$$
\begin{aligned}
\nu_1(p,q) &= \max\bigl\{0, 1/q - \min(1/p, 1/p')\bigr\},\\
\nu_2(p,q) &= \min\bigl\{0, 1/q - \max(1/p, 1/p')\bigr\}.
\end{aligned}
$$

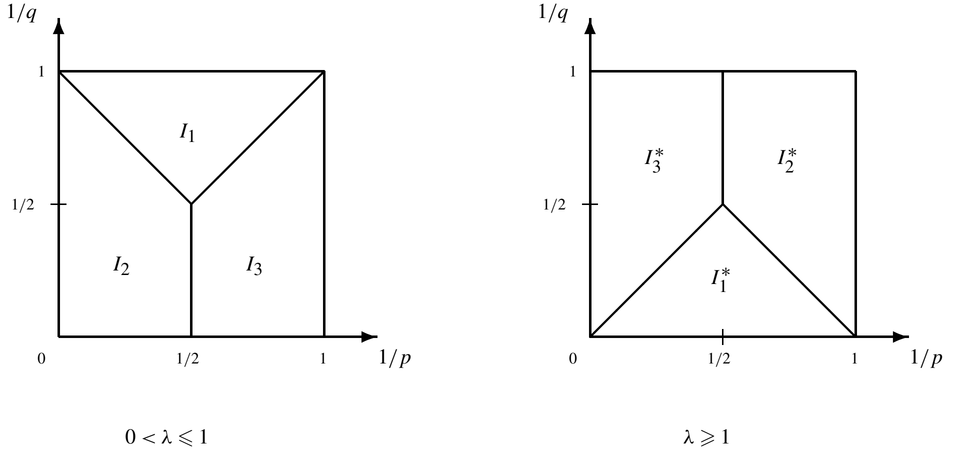

Fig. 1.

Note that

$$\nu_1(p,q) = \begin{cases} 0 & \text{if } (1/p, 1/q) \in I_1^*, \\ 1/p + 1/q - 1 & \text{if } (1/p, 1/q) \in I_2^*, \\ -1/p + 1/q & \text{if } (1/p, 1/q) \in I_3^*, \end{cases}$$

and

$$\nu_2(p,q) = \begin{cases} 0 & \text{if } (1/p, 1/q) \in I_1, \\ 1/p + 1/q - 1 & \text{if } (1/p, 1/q) \in I_2, \\ -1/p + 1/q & \text{if } (1/p, 1/q) \in I_3. \end{cases}$$

We also introduce the indices

$$\mu_1(p,q) = \nu_1(p,q) - 1/p, \qquad \mu_2(p,q) = \nu_2(p,q) - 1/p.$$

Then we have

$$\mu_1(p,q) = \begin{cases} -1/p & \text{if } (1/p, 1/q) \in I_1^*, \\ 1/q - 1 & \text{if } (1/p, 1/q) \in I_2^*, \\ -2/p + 1/q & \text{if } (1/p, 1/q) \in I_3^*, \end{cases}$$

and

$$\mu_2(p,q) = \begin{cases} -1/p & \text{if } (1/p, 1/q) \in I_1, \\ 1/q - 1 & \text{if } (1/p, 1/q) \in I_2, \\ -2/p + 1/q & \text{if } (1/p, 1/q) \in I_3. \end{cases}$$

Our first main result is on the dilation property of the modulation spaces. For a function (or tempered distribution) f on $\mathbb{R}^n$ and $\lambda > 0$, we use the notation f_λ which is defined by $f_\lambda(t) = f(\lambda t)$, $t \in \mathbb{R}^n$.

Theorem 1.1. *Let $1 \leqslant p, q \leqslant \infty$. Then the following are true*:

(1) *There exists a constant $C > 0$ such that*

$$C^{-1}\lambda^{n\mu_2(p,q)}\|f\|_{M^{p,q}} \leqslant \|f_\lambda\|_{M^{p,q}} \leqslant C\lambda^{n\mu_1(p,q)}\|f\|_{M^{p,q}} \tag{1.1}$$

for all $f \in M^{p,q}(\mathbb{R}^n)$ and $\lambda \geqslant 1$. Conversely, if there exist constants $C > 0$ and $\alpha, \beta \in \mathbb{R}$ such that

$$C^{-1}\lambda^{\beta}\|f\|_{M^{p,q}} \leqslant \|f_\lambda\|_{M^{p,q}} \leqslant C\lambda^{\alpha}\|f\|_{M^{p,q}}$$

for all $f \in M^{p,q}(\mathbb{R}^n)$ and $\lambda \geqslant 1$, then $\alpha \geqslant n\mu_1(p,q)$ and $\beta \leqslant n\mu_2(p,q)$.

(2) *There exists a constant $C > 0$ such that*

$$C^{-1}\lambda^{n\mu_1(p,q)}\|f\|_{M^{p,q}} \leqslant \|f_\lambda\|_{M^{p,q}} \leqslant C\lambda^{n\mu_2(p,q)}\|f\|_{M^{p,q}} \tag{1.2}$$

for all $f \in M^{p,q}(\mathbb{R}^n)$ and $0 < \lambda \leqslant 1$. Conversely, if there exist constants $C > 0$ and $\alpha, \beta \in \mathbb{R}$ such that

$$C^{-1}\lambda^{\alpha}\|f\|_{M^{p,q}} \leqslant \|f_\lambda\|_{M^{p,q}} \leqslant C\lambda^{\beta}\|f\|_{M^{p,q}}$$

for all $f \in M^{p,q}(\mathbb{R}^n)$ and $0 < \lambda \leqslant 1$, then $\alpha \geqslant n\mu_1(p,q)$ and $\beta \leqslant n\mu_2(p,q)$.

Since the Gauss function $\varphi(t) = e^{-|t|^2}$ does not change its form under the Fourier transformation, the modulation norm of it can have a "good" property. In this sense, it is reasonable to believe that the Gauss function $f = \varphi$ attains the critical order of $\|f_\lambda\|_{M^{p,q}}$ with respect to λ. But it is not true because $\|\varphi_\lambda\|_{M^{p,q}} \sim \lambda^{n(1/q-1)}$ in the case $\lambda \geqslant 1$ and $\|\varphi_\lambda\|_{M^{p,q}} \sim \lambda^{-n/p}$ in the case $0 < \lambda \leqslant 1$ (see Lemma 2.1). Theorem 1.1 says that they are not critical orders for every pair of (p, q).

It should be pointed out here that the behavior of $\|f_\lambda\|_{M^{p,q}}$ with respect to λ might depend on the choice of $f \in M^{p,q}(\mathbb{R}^n)$. In fact, $f(t) = \sum_{k\in\mathbb{Z}^n} e^{ik\cdot t}\psi(t-k)$, where ψ is an appropriate Schwartz function, has the property $\|f_\lambda\|_{M^{p,\infty}} \sim \lambda^{-2n/p}$ $(0 < \lambda \leqslant 1)$ in the case $1 \leqslant p \leqslant 2$ (Lemma 3.10), while the Gauss function has the different behavior $\|\varphi_\lambda\|_{M^{p,\infty}} \sim \lambda^{-n/p}$ $(0 < \lambda \leqslant 1)$ as mentioned above. On the other hand, the L^p-norm never has such a property since $\|f_\lambda\|_{L^p} = \lambda^{-n/p}\|f\|_{L^p}$ for all $f \in L^p(\mathbb{R}^n)$. That is one of great differences between the modulation spaces and L^p-spaces.

Our second main result is on the optimality of the inclusion relation between the modulation spaces and Besov spaces. In [18, Theorem 3.1], Toft proved the inclusions

$$B^{p,q}_{n\nu_1(p,q)}(\mathbb{R}^n) \hookrightarrow M^{p,q}(\mathbb{R}^n) \hookrightarrow B^{p,q}_{n\nu_2(p,q)}(\mathbb{R}^n)$$

for $1 \leqslant p, q \leqslant \infty$. See also [19, Theorem 2.10] for the case of weighted modulation spaces, and some related results can be seen in Gröbner [9] and Okoudjou [14]. Toft also remarked that the left inclusion is optimal in the case $1 \leqslant p = q \leqslant 2$, that is, if $B^{p,p}_{s_1}(\mathbb{R}^n) \hookrightarrow M^{p,p}(\mathbb{R}^n)$ then $s_1 \geqslant n\nu_1(p,p)$. The same is true for the right inclusion in the case $2 \leqslant p = q \leqslant \infty$, that is, if $M^{p,p}(\mathbb{R}^n) \hookrightarrow B^{p,p}_{s_2}(\mathbb{R}^n)$ then $s_2 \leqslant n\nu_2(p,p)$ [18, Remark 3.11]. The next theorem says that Toft's inclusion result is optimal in the above meaning for every pair of (p, q).

Theorem 1.2. *Let $1 \leqslant p, q \leqslant \infty$ and $s \in \mathbb{R}$. Then the following are true*:

(1) *If $B_s^{p,q}(\mathbb{R}^n) \hookrightarrow M^{p,q}(\mathbb{R}^n)$, then $s \geqslant n\nu_1(p,q)$.*
(2) *If $M^{p,q}(\mathbb{R}^n) \hookrightarrow B_s^{p,q}(\mathbb{R}^n)$ and $1 \leqslant p, q < \infty$, then $s \leqslant n\nu_2(p,q)$.*

During the evaluation process of this paper, a preprint of the independent work by Wang and Huang [22] was sent to the authors, where we can find a related result of Theorem 1.2.

We end this introduction by explaining the plan of this article. In Section 2, we give the precise definition and basic properties of the modulation spaces and Besov spaces. In Sections 3 and 4, we prove Theorems 1.1 and 1.2, respectively.

2. Preliminaries

We introduce the modulation spaces based on Gröchenig [10]. Let $\mathcal{S}(\mathbb{R}^n)$ and $\mathcal{S}'(\mathbb{R}^n)$ be the Schwartz spaces of rapidly decreasing smooth functions and tempered distributions, respectively. We define the Fourier transform $\widehat{f}$ and the inverse Fourier transform $\mathcal{F}^{-1}f$ of $f \in \mathcal{S}(\mathbb{R}^n)$ by

$$\widehat{f}(\xi) = \int_{\mathbb{R}^n} e^{-i\xi\cdot x} f(x)\,dx \quad \text{and} \quad \mathcal{F}^{-1}f(x) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{ix\cdot\xi} f(\xi)\,d\xi.$$

Fix a function $\varphi \in \mathcal{S}(\mathbb{R}^n) \setminus \{0\}$ (called the *window function*). Then the short-time Fourier transform $V_\varphi f$ of $f \in \mathcal{S}'(\mathbb{R}^n)$ with respect to φ is defined by

$$V_\varphi f(x,\xi) = \langle f, M_\xi T_x \varphi\rangle \quad \text{for } x, \xi \in \mathbb{R}^n,$$

where $M_\xi\varphi(t) = e^{i\xi\cdot t}\varphi(t)$, $T_x\varphi(t) = \varphi(t-x)$, and $\langle\cdot,\cdot\rangle$ is the inner product on $L^2(\mathbb{R}^n)$. We can express it in a form of the integral

$$V_\varphi f(x,\xi) = \int_{\mathbb{R}^n} f(t)\,\overline{\varphi(t-x)}e^{-i\xi\cdot t}\,dt,$$

which has actually the meaning for an appropriate function f on $\mathbb{R}^n$. We note that, for $f \in \mathcal{S}'(\mathbb{R}^n)$, $V_\varphi f$ is continuous on $\mathbb{R}^{2n}$ and $|V_\varphi f(x,\xi)| \leqslant C(1+|x|+|\xi|)^N$ for some constants $C, N \geqslant 0$ [10, Theorem 11.2.3]. Let $1 \leqslant p, q \leqslant \infty$. Then the modulation space $M^{p,q}(\mathbb{R}^n)$ consists of all $f \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$\|f\|_{M^{p,q}} = \|V_\varphi f\|_{L^{p,q}} = \left\{\int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n}\bigl|V_\varphi f(x,\xi)\bigr|^p\,dx\right)^{q/p} d\xi\right\}^{1/q} < \infty,$$

(with usual modification when $p = \infty$ or $q = \infty$). We note that $M^{2,2}(\mathbb{R}^n) = L^2(\mathbb{R}^n)$ [10, Proposition 11.3.1] and $M^{p,q}(\mathbb{R}^n)$ is a Banach space [10, Proposition 11.3.5]. The definition of $M^{p,q}(\mathbb{R}^n)$ is independent of the choice of the window function $\varphi \in \mathcal{S}(\mathbb{R}^n) \setminus \{0\}$, that is, different window functions yield equivalent norms [10, Proposition 11.3.2].

We also introduce Besov spaces. Let $1 \leqslant p, q \leqslant \infty$ and $s \in \mathbb{R}$. Suppose that $\varphi_0, \varphi \in \mathcal{S}(\mathbb{R}^n)$ satisfy $\operatorname{supp}\varphi_0 \subset \{\xi\colon |\xi| \leqslant 2\}$, $\operatorname{supp}\varphi \subset \{\xi\colon 1/2 \leqslant |\xi| \leqslant 2\}$ and $\varphi_0(\xi) + \sum_{j=1}^{\infty}\varphi(\xi/2^j) = 1$ for all $\xi \in \mathbb{R}^n$. Set $\varphi_j = \varphi(\cdot/2^j)$ if $j \geqslant 1$. Then the Besov space $B_s^{p,q}(\mathbb{R}^n)$ consists of all $f \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$\|f\|_{B_s^{p,q}} = \left(\sum_{j=0}^{\infty} 2^{jsq} \big\|\mathcal{F}^{-1}[\varphi_j \widehat{f}]\big\|_{L^p}^q\right)^{1/q} = \left(\sum_{j=0}^{\infty} 2^{jsq} \|\Phi_j * f\|_{L^p}^q\right)^{1/q} < \infty,$$

where $\Phi_j = \mathcal{F}^{-1}\varphi_j$ (with usual modification again when $q = \infty$). We remark that $B_s^{p,q}(\mathbb{R}^n)^* = B_{-s}^{p',q'}(\mathbb{R}^n)$ for $1 \leqslant p, q < \infty$.

Finally, we list below the lemmas which will be used in the subsequent section. In this article, we frequently use the Gauss function $\varphi(t) = e^{-|t|^2}$.

Lemma 2.1. *(See [18, Lemma 1.8].) Let φ be the Gauss function. Then*

$$\big\|V_\varphi(\varphi_\lambda)\big\|_{L^{p,q}} = \pi^{n(1/p+1/q+1)/2} p^{-n/2p} q^{-n/2q} 2^{n/q} \lambda^{-n/p} \big(1+\lambda^2\big)^{n(1/p+1/q-1)/2}.$$

Lemma 2.1 says that $\|\varphi_\lambda\|_{M^{p,q}} \sim \lambda^{n(1/q-1)}$ in the case $\lambda \geqslant 1$ and $\|\varphi_\lambda\|_{M^{p,q}} \sim \lambda^{-n/p}$ in the case $0 < \lambda \leqslant 1$.

Lemma 2.2. *(See [10, Corollary 11.2.7].) Let $f \in \mathcal{S}'(\mathbb{R}^n)$ and $\varphi, \psi, \gamma \in \mathcal{S}(\mathbb{R}^n)$. Then*

$$\langle f, \varphi\rangle = \frac{1}{\langle \gamma, \psi\rangle} \int_{\mathbb{R}^{2n}} V_\psi f(x,\xi) \overline{V_\gamma \varphi(x,\xi)}\, dx\, d\xi \quad \textit{for all } \varphi \in \mathcal{S}\big(\mathbb{R}^n\big).$$

We remark that Lemma 2.2 is also found in Folland [7, Proposition 1.92].

Lemma 2.3. *(See [10, Lemma 11.3.3].) Let $f \in \mathcal{S}'(\mathbb{R}^n)$ and $\varphi, \psi, \gamma \in \mathcal{S}(\mathbb{R}^n)$. Then*

$$\big|V_\varphi f(x,\xi)\big| \leqslant \frac{1}{|\langle \gamma, \psi\rangle|} \big(|V_\psi f| * |V_\varphi \gamma|\big)(x,\xi) \quad \textit{for all } x, \xi \in \mathbb{R}^n.$$

Lemma 2.4. *(See [10, Proposition 11.3.4, Theorem 11.3.6].) Let $1 \leqslant p, q < \infty$. Then $\mathcal{S}(\mathbb{R}^n)$ is dense in $M^{p,q}(\mathbb{R}^n)$ and $M^{p,q}(\mathbb{R}^n)^* = M^{p',q'}(\mathbb{R}^n)$ under the duality*

$$\langle f, g\rangle_M = \frac{1}{\|\varphi\|_{L^2}^2} \int_{\mathbb{R}^{2n}} V_\varphi f(x,\xi) \overline{V_\varphi g(x,\xi)}\, dx\, d\xi$$

for $f \in M^{p,q}(\mathbb{R}^n)$ and $g \in M^{p',q'}(\mathbb{R}^n)$.

By Lemmas 2.2 and 2.4, if $1 < p, q \leqslant \infty$ and $f \in M^{p,q}(\mathbb{R}^n)$ then

$$\|f\|_{M^{p,q}} = \sup\big|\langle f, g\rangle_M\big| = \sup\big|\langle f, g\rangle\big|, \tag{2.1}$$

where the supremum is taken over all $g \in \mathcal{S}(\mathbb{R}^n)$ such that $\|g\|_{M^{p',q'}} = 1$.

Lemma 2.5. *(See [3, Corollary 2.3].) Let* $1 \leqslant p_1, p_2, q_1, q_2 \leqslant \infty$ *and* $p_2, q_2 < \infty$. *If* T *is a linear operator such that*

$$\|Tf\|_{M^{p_1,q_1}} \leqslant A_1 \|f\|_{M^{p_1,q_1}} \quad \text{for all } f \in M^{p_1,q_1}(\mathbb{R}^n)$$

and

$$\|Tf\|_{M^{p_2,q_2}} \leqslant A_2 \|f\|_{M^{p_2,q_2}} \quad \text{for all } f \in M^{p_2,q_2}(\mathbb{R}^n),$$

then

$$\|Tf\|_{M^{p,q}} \leqslant C A_1^{1-\theta} A_2^{\theta} \|f\|_{M^{p,q}} \quad \text{for all } f \in M^{p,q}(\mathbb{R}^n),$$

where $1/p = (1-\theta)/p_1 + \theta/p_2$, $1/q = (1-\theta)/q_1 + \theta/q_2$, $0 \leqslant \theta \leqslant 1$ *and* C *is independent of* T.

Remark 2.6. Lemma 2.5 with the cases $p_2 = \infty$ or $q_2 = \infty$ is treated in [18, Remark 3.2], which says that it is true under a modification.

3. The dilation property of modulation spaces

In this section, we prove Theorem 1.1 which appeared in Section 1. We remark that the left-hand sides of inequalities in Theorem 1.1 are obtained from the right-hand sides of them.

Theorem 3.1. *Let* $1 \leqslant p, q \leqslant \infty$. *Then the following are true*:

(1) *There exists a constant* $C > 0$ *such that*

$$\|f_\lambda\|_{M^{p,q}} \leqslant C \lambda^{n\mu_1(p,q)} \|f\|_{M^{p,q}} \quad \text{for all } f \in M^{p,q}(\mathbb{R}^n) \text{ and } \lambda \geqslant 1.$$

Conversely, if there exist constants $C > 0$ *and* $\alpha \in \mathbb{R}$ *such that*

$$\|f_\lambda\|_{M^{p,q}} \leqslant C \lambda^{\alpha} \|f\|_{M^{p,q}} \quad \text{for all } f \in M^{p,q}(\mathbb{R}^n) \text{ and } \lambda \geqslant 1,$$

then $\alpha \geqslant n\mu_1(p,q)$.

(2) *There exists a constant* $C > 0$ *such that*

$$\|f_\lambda\|_{M^{p,q}} \leqslant C \lambda^{n\mu_2(p,q)} \|f\|_{M^{p,q}} \quad \text{for all } f \in M^{p,q}(\mathbb{R}^n) \text{ and } 0 < \lambda \leqslant 1.$$

Conversely, if there exist constants $C > 0$ *and* $\beta \in \mathbb{R}$ *such that*

$$\|f_\lambda\|_{M^{p,q}} \leqslant C \lambda^{\beta} \|f\|_{M^{p,q}} \quad \text{for all } f \in M^{p,q}(\mathbb{R}^n) \text{ and } 0 < \lambda \leqslant 1,$$

then $\beta \leqslant n\mu_2(p,q)$.

Indeed, if $0 < \lambda \leqslant 1$, then the first part of Theorem 3.1(1) gives

$$\|f\|_{M^{p,q}} = \|(f_\lambda)_{1/\lambda}\|_{M^{p,q}} \leqslant C\lambda^{-n\mu_1(p,q)}\|f_\lambda\|_{M^{p,q}}$$

which proves the left-hand side of (1.2) in Theorem 1.1. The others in Theorem 1.1 are given by Theorem 3.1 in a similar way. We also remark that Boulkhemair [1, Proposition 3.2] proved the first part of Theorem 3.1(2) with $(p,q) = (\infty, 1)$.

Now we prove Theorem 3.1. We begin with the following preparing lemma which might be well known.

Lemma 3.2. *Let $1 \leqslant p, q \leqslant \infty$. Then there exists a constant $C > 0$ which only depends on the window functions in the modulation space norms such that*

$$\|f_\lambda\|_{M^{p,q}} \leqslant C\lambda^{-n(1/p-1/q+1)}\left(1+\lambda^2\right)^{n/2}\|f\|_{M^{p,q}}$$

for all $f \in M^{p,q}(\mathbb{R}^n)$ and $\lambda > 0$.

Although the proof of Lemma 3.2 might be found in some literature, we provide it for reader's convenience. Here (and also in other situations) we may assume that the window function is given by the Gauss function $\varphi(t) = e^{-|t|^2}$.

Proof. Let φ be the Gauss function, that is, $\varphi(t) = e^{-|t|^2}$. By a change of variable, we have

$$\|f_\lambda\|_{M^{p,q}} = \|V_\varphi(f_\lambda)\|_{L^{p,q}} = \lambda^{-n(1/p-1/q+1)}\|V_{\varphi_{1/\lambda}} f\|_{L^{p,q}}.$$

From Lemma 2.3 it follows that

$$\left|V_{\varphi_{1/\lambda}} f(x,\xi)\right| \leqslant \|\varphi\|_{L^2}^{-2}\left(|V_\varphi f| * |V_{\varphi_{1/\lambda}}\varphi|\right)(x,\xi).$$

Hence, by Young's inequality and Lemma 2.1, we get

$$\begin{aligned}
\|f_\lambda\|_{M^{p,q}} &\leqslant \lambda^{-n(1/p-1/q+1)}\|\varphi\|_{L^2}^{-2}\|V_{\varphi_{1/\lambda}}\varphi\|_{L^{1,1}}\|V_\varphi f\|_{L^{p,q}} \\
&= \lambda^{-n(1/p-1/q+1)}\|\varphi\|_{L^2}^{-2}\|V_\varphi(\varphi_{1/\lambda})\|_{L^{1,1}}\|V_\varphi f\|_{L^{p,q}} \\
&= \lambda^{-n(1/p-1/q+1)}\|\varphi\|_{L^2}^{-2}\left(\pi^{3n/2}2^n\left(\lambda^{-1}\right)^{-n}\left(1+\lambda^{-2}\right)^{n/2}\right)\|f\|_{M^{p,q}} \\
&= C_{n,\varphi}\lambda^{-n(1/p-1/q+1)}\left(1+\lambda^2\right)^{n/2}\|f\|_{M^{p,q}}.
\end{aligned}$$

The proof is complete. □

We are now ready to prove Theorem 3.1(1) with $(1/p, 1/q) \in I_1^*$ and Theorem 3.1(2) with $(1/p, 1/q) \in I_1$.

Proof of Theorem 3.1(2) with $(1/p, 1/q) \in I_1$. Let $(1/p, 1/q) \in I_1$. Then $\mu_2(p,q) = -1/p$. By Lemma 3.2, we have

$$\|f_\lambda\|_{M^{r,1}} \leqslant C\lambda^{-n/r}\|f\|_{M^{r,1}} \quad \text{for all } f \in M^{r,1}\left(\mathbb{R}^n\right) \text{ and } 0 < \lambda \leqslant 1, \tag{3.1}$$

where $1 \leqslant r \leqslant \infty$. On the other hand, since $M^{2,2}(\mathbb{R}^n) = L^2(\mathbb{R}^n)$, we have

$$\|f_\lambda\|_{M^{2,2}} \leqslant C\lambda^{-n/2}\|f\|_{M^{2,2}} \quad \text{for all } f \in M^{2,2}(\mathbb{R}^n) \text{ and } 0 < \lambda \leqslant 1. \tag{3.2}$$

Take $1 \leqslant r \leqslant \infty$ and $0 \leqslant \theta \leqslant 1$ such that $1/p = (1-\theta)/r + \theta/2$ and $1/q = (1-\theta)/1 + \theta/2$. Then, by interpolation (Lemma 2.5), (3.1) and (3.2) give

$$\|f_\lambda\|_{M^{p,q}} \leqslant C(\lambda^{-n/r})^{1-\theta}(\lambda^{-n/2})^{\theta}\|f\|_{M^{p,q}}$$

for all $f \in M^{p,q}(\mathbb{R}^n)$ and $0 < \lambda \leqslant 1$. Since $(1-\theta)/r = 1/p + 1/q - 1$ and $\theta/2 = -1/q + 1$, we get

$$\|f_\lambda\|_{M^{p,q}} \leqslant C\lambda^{-n/p}\|f\|_{M^{p,q}} \quad \text{for all } f \in M^{p,q}(\mathbb{R}^n) \text{ and } 0 < \lambda \leqslant 1. \tag{3.3}$$

This is the first part of Theorem 3.1(2) with $(1/p, 1/q) \in I_1$.

We next prove the second part of Theorem 3.1(2) with $(1/p, 1/q) \in I_1$. Let $(1/p, 1/q) \in I_1$. Assume that there exist constants $C > 0$ and $\beta \in \mathbb{R}$ such that

$$\|f_\lambda\|_{M^{p,q}} \leqslant C\lambda^{\beta}\|f\|_{M^{p,q}} \quad \text{for all } f \in M^{p,q}(\mathbb{R}^n) \text{ and } 0 < \lambda \leqslant 1.$$

Let φ be the Gauss function. We note that the Gauss function belongs to $M^{p,q}(\mathbb{R}^n)$. Then, by Lemma 2.1 and our assumption, we have

$$\begin{aligned} C_{p,q}\lambda^{-n/p} &\leqslant C_{p,q}\lambda^{-n/p}(1+\lambda^2)^{n(1/p+1/q-1)/2} \\ &= \|V_\varphi(\varphi_\lambda)\|_{L^{p,q}} = \|\varphi_\lambda\|_{M^{p,q}} \leqslant C\lambda^{\beta}\|\varphi\|_{M^{p,q}} \end{aligned}$$

for all $0 < \lambda \leqslant 1$. This is possible only if $\beta \leqslant -n/p$. The proof is complete. □

Proof of Theorem 3.1(1) with $(1/p, 1/q) \in I_1^*$. We recall that $\mu_1(p,q) = -1/p$ if $(1/p, 1/q) \in I_1^*$. Let $(1/p, 1/q) \in I_1^*$. Then $(1/p', 1/q') \in I_1$. We first consider the case $p \neq 1$. Since $1 < p, q \leqslant \infty$, by duality (2.1) and Theorem 3.1(2) with $(1/p', 1/q') \in I_1$, we have

$$\begin{aligned} \|f_\lambda\|_{M^{p,q}} &= \sup|\langle f_\lambda, g\rangle| = \lambda^{-n}\sup|\langle f, g_{1/\lambda}\rangle| \\ &\leqslant \lambda^{-n}\sup\|f\|_{M^{p,q}}\|g_{1/\lambda}\|_{M^{p',q'}} \\ &\leqslant \lambda^{-n}\sup\|f\|_{M^{p,q}}\left(C(\lambda^{-1})^{-n/p'}\|g\|_{M^{p',q'}}\right) = C\lambda^{-n/p}\|f\|_{M^{p,q}} \end{aligned}$$

for all $f \in M^{p,q}(\mathbb{R}^n)$ and $\lambda \geqslant 1$, where the supremum is taken over all $g \in \mathcal{S}(\mathbb{R}^n)$ such that $\|g\|_{M^{p',q'}} = 1$. In the case $p = 1$, by Lemma 3.2, we see that

$$\|f_\lambda\|_{M^{1,\infty}} \leqslant C\lambda^{-n}\|f\|_{M^{1,\infty}} \quad \text{for all } f \in M^{1,\infty}(\mathbb{R}^n) \text{ and } \lambda \geqslant 1.$$

Hence, we obtain the first part of Theorem 3.1(1) with $(1/p, 1/q) \in I_1^*$.

We consider the second part of Theorem 3.1(1) with $(1/p, 1/q) \in I_1^*$. Let $(1/p, 1/q) \in I_1^*$ and $q < \infty$. Note that $1 < p, q < \infty$. Assume that there exist constants $C > 0$ and $\alpha \in \mathbb{R}$ such that

$$\|g_\lambda\|_{M^{p,q}} \leqslant C\lambda^\alpha \|g\|_{M^{p,q}} \quad \text{for all } g \in M^{p,q}(\mathbb{R}^n) \text{ and } \lambda \geqslant 1.$$

Then, by duality and our assumption, we have

$$\begin{aligned}
\|f_\lambda\|_{M^{p',q'}} &= \sup\left|\langle f_\lambda, g\rangle\right| = \lambda^{-n} \sup\left|\langle f, g_{1/\lambda}\rangle\right| \\
&\leqslant \lambda^{-n} \sup \|f\|_{M^{p',q'}} \|g_{1/\lambda}\|_{M^{p,q}} \\
&\leqslant \lambda^{-n} \sup \|f\|_{M^{p',q'}} \left(C\left(\lambda^{-1}\right)^\alpha \|g\|_{M^{p,q}}\right) = C\lambda^{-n-\alpha} \|f\|_{M^{p',q'}}
\end{aligned}$$

for all $f \in M^{p',q'}(\mathbb{R}^n)$ and $0 < \lambda \leqslant 1$, where the supremum is taken over all $g \in \mathcal{S}(\mathbb{R}^n)$ such that $\|g\|_{M^{p,q}} = 1$. Since $(1/p', 1/q') \in I_1$, by Theorem 3.1(2) with $(1/p', 1/q') \in I_1$, we get $-n - \alpha \leqslant -n/p'$. This implies $\alpha \geqslant -n/p$.

We next consider the case $q = \infty$. Let $1 \leqslant r \leqslant \infty$. Assume that there exist constants $C > 0$ and $\alpha \in \mathbb{R}$ such that

$$\|f_\lambda\|_{M^{r,\infty}} \leqslant C\lambda^\alpha \|f\|_{M^{r,\infty}} \quad \text{for all } f \in M^{r,\infty}(\mathbb{R}^n) \text{ and } \lambda \geqslant 1, \tag{3.4}$$

where $\alpha < -n/r$. Since $M^{2,2}(\mathbb{R}^n) = L^2(\mathbb{R}^n)$, we have

$$\|f_\lambda\|_{M^{2,2}} \leqslant C\lambda^{-n/2} \|f\|_{M^{2,2}} \quad \text{for all } f \in M^{2,2}(\mathbb{R}^n) \text{ and } \lambda \geqslant 1. \tag{3.5}$$

Then, by interpolation, (3.4) and (3.5) give

$$\|f_\lambda\|_{M^{p,q}} \leqslant \begin{cases} C\lambda^{(\alpha r+n)(1/p-1/q)-n/p} \|f\|_{M^{p,q}}, & \text{if } 1 \leqslant r < \infty, \\ C\lambda^{\alpha(1-2/q)-n/p} \|f\|_{M^{p,q}}, & \text{if } r = \infty \end{cases}$$

for all $f \in M^{p,q}(\mathbb{R}^n)$ and $\lambda \geqslant 1$, where $1/p = (1-\theta)/r + \theta/2$, $1/q = (1-\theta)/\infty + \theta/2$ and $0 < \theta < 1$. Note that $(1/p, 1/q) \in I_1^*$ and $2 < q < \infty$. However, since $(\alpha r + n)(1/p - 1/q) < 0$ if $1 \leqslant r < \infty$ and $\alpha(1-2/q) < 0$ if $r = \infty$, this contradicts Theorem 3.1(1) with $(1/p, 1/q) \in I_1^*$ and $2 < q < \infty$. Therefore, α must satisfy $\alpha \geqslant -n/r$. The proof is complete. □

Our next goal is to prove Theorem 3.1(1) with $(1/p, 1/q) \in I_2^*$ and Theorem 3.1(2) with $(1/p, 1/q) \in I_2$.

Lemma 3.3. *Let $1 \leqslant p, q \leqslant \infty$ be such that $(1/p, 1/q) \in I_2^*$ and $1/p \geqslant 1/q$. Then there exists a constant $C > 0$ such that*

$$\|f_\lambda\|_{M^{p,q}} \leqslant C\lambda^{-n(2/p-1/q)}\left(1+\lambda^2\right)^{n(1/p-1/2)} \|f\|_{M^{p,q}}$$

for all $f \in M^{p,q}(\mathbb{R}^n)$ and $\lambda > 0$.

Proof. Let $1 \leqslant r \leqslant \infty$. By Lemma 3.2, we have

$$\|f_\lambda\|_{M^{1,r}} \leqslant C\lambda^{n(1/r-2)}\big(1+\lambda^2\big)^{n/2}\|f\|_{M^{1,r}} \tag{3.6}$$

for all $f \in M^{1,r}(\mathbb{R}^n)$ and $\lambda > 0$. Then, by interpolation, (3.2), (3.5) and (3.6) give Lemma 3.3. □

The proof of the following lemma is based on that of [21, Theorem 3].

Lemma 3.4. *Suppose that $\varphi \in \mathcal{S}(\mathbb{R}^n)$ is a real-valued function satisfying $\varphi \geqslant C$ on $[-1/2, 1/2]^n$ for some constant $C > 0$, $\operatorname{supp}\varphi \subset [-1, 1]^n$, $\varphi(t) = \varphi(-t)$ and $\sum_{k\in\mathbb{Z}^n}\varphi(t-k) = 1$ for all $t \in \mathbb{R}^n$. Then*

$$\sup_{k\in\mathbb{Z}^n}\big\|(M_k\Phi)*f\big\|_{L^2} \leqslant \|V_\Phi f\|_{L^{2,\infty}} \leqslant 5^n\|\Phi\|_{L^1}\sup_{k\in\mathbb{Z}^n}\big\|(M_k\Phi)*f\big\|_{L^2}$$

for all $f \in M^{2,\infty}(\mathbb{R}^n)$, where $\Phi = \mathcal{F}^{-1}\varphi$ and $M_k\Phi(t) = e^{ik\cdot t}\Phi(t)$.

Proof. Let $f \in M^{2,\infty}(\mathbb{R}^n)$. Since Φ is a real-valued function and $\Phi(t) = \Phi(-t)$ for all t, we have

$$\begin{aligned}\big|V_\Phi f(x,\xi)\big| &= \left|\int_{\mathbb{R}^n} f(t)\overline{\Phi(t-x)}e^{-i\xi\cdot t}\,dt\right| \\ &= \left|\int_{\mathbb{R}^n} f(t)\Phi(x-t)e^{i\xi\cdot(x-t)}\,dt\right| = \big|(M_\xi\Phi)*f(x)\big|.\end{aligned} \tag{3.7}$$

We first prove

$$\operatorname*{ess\,sup}_{\xi\in\mathbb{R}^n}\left(\int_{\mathbb{R}^n}\big|V_\Phi f(x,\xi)\big|^2\,dx\right)^{1/2} = \sup_{\xi\in\mathbb{R}^n}\left(\int_{\mathbb{R}^n}\big|V_\Phi f(x,\xi)\big|^2\,dx\right)^{1/2}. \tag{3.8}$$

To prove (3.8), it is enough to show that $(\int_{\mathbb{R}^n}|V_\Phi f(x,\xi)|^2\,dx)^{1/2}$ is continuous with respect to ξ. Since $\operatorname{ess\,sup}_{\xi\in\mathbb{R}^n}(\int_{\mathbb{R}^n}|V_\Phi f(x,\xi)|^2\,dx)^{1/2} < \infty$, for each $k \in \mathbb{Z}^n$ there exists $\xi_k \in k/2 + [-1/4, 1/4]^n$ such that $(\int_{\mathbb{R}^n}|V_\Phi f(x,\xi_k)|^2\,dx)^{1/2} < \infty$. Then, by (3.7), we have

$$\frac{1}{(2\pi)^{n/2}}\big\|\varphi(\cdot-\xi_k)\widehat{f}\big\|_{L^2} = \big\|(M_{\xi_k}\Phi)*f\big\|_{L^2} = \left(\int_{\mathbb{R}^n}\big|V_\Phi f(x,\xi_k)\big|^2\,dx\right)^{1/2} < \infty.$$

Since $k/2 + [-1/4, 1/4]^n \subset \xi_k + [-1/2, 1/2]^n$ and $\varphi(\cdot-\xi_k) \geqslant C > 0$ on $\xi_k + [-1/2, 1/2]^n$, we see that $|\widehat{f}|^2$ is integrable on $k/2 + [-1/4, 1/4]^n$. The arbitrariness of $k \in \mathbb{Z}^n$ gives $\widehat{f} \in L^2_{\mathrm{loc}}(\mathbb{R}^n)$. By the Lebesgue dominated convergence theorem, we see that $\|\varphi(\cdot-\xi)\widehat{f}\|_{L^2}$ is continuous with respect to ξ. Hence, $(\int_{\mathbb{R}^n}|V_\Phi f(x,\xi)|^2\,dx)^{1/2}$ is continuous with respect to ξ. We obtain (3.8). Then, from (3.7) and (3.8) it follows that

$$\sup_{k\in\mathbb{Z}^n}\big\|(M_k\Phi)*f\big\|_{L^2}\leqslant\sup_{\xi\in\mathbb{R}^n}\big\|(M_\xi\Phi)*f\big\|_{L^2}=\sup_{\xi\in\mathbb{R}^n}\bigg(\int_{\mathbb{R}^n}\big|V_\Phi f(x,\xi)\big|^2\,dx\bigg)^{1/2}$$
$$=\operatorname*{ess\,sup}_{\xi\in\mathbb{R}^n}\bigg(\int_{\mathbb{R}^n}\big|V_\Phi f(x,\xi)\big|^2\,dx\bigg)^{1/2}=\|V_\Phi f\|_{L^{2,\infty}}.$$

We next prove $\|V_\Phi f\|_{L^{2,\infty}}\leqslant(5^n\|\Phi\|_{L^1})\sup_{k\in\mathbb{Z}^n}\|(M_k\Phi)*f\|_{L^2}$. Let $\xi=(\xi_1,\ldots,\xi_n)\in\mathbb{R}^n$. Since

$$M_\xi\Phi=\mathcal{F}^{-1}\big[\varphi(\cdot-\xi)\big]=\mathcal{F}^{-1}\bigg[\varphi(\cdot-\xi)\bigg(\sum_{k\in\mathbb{Z}^n}\varphi(\cdot-k)\bigg)\bigg]$$
$$=\sum_{\substack{|k_i-\xi_i|\leqslant 2,\\ i=1,\ldots,n}}\mathcal{F}^{-1}\big[\varphi(\cdot-\xi)\varphi(\cdot-k)\big]=\sum_{\substack{|k_i-\xi_i|\leqslant 2,\\ i=1,\ldots,n}}(M_\xi\Phi)*(M_k\Phi),$$

by (3.7), we have

$$\big|V_\Phi f(x,\xi)\big|=\big|(M_\xi\Phi)*f(x)\big|\leqslant\sum_{\substack{|k_i-\xi_i|\leqslant 2,\\ i=1,\ldots,n}}\big|(M_\xi\Phi)*(M_k\Phi)*f(x)\big|.$$

Hence, by (3.8), we get

$$\|V_\Phi f\|_{L^{2,\infty}}=\sup_{\xi\in\mathbb{R}^n}\bigg(\int_{\mathbb{R}^n}\big|V_\Phi f(x,\xi)\big|^2\,dx\bigg)^{1/2}$$
$$\leqslant\sup_{\xi\in\mathbb{R}^n}\sum_{\substack{|k_i-\xi_i|\leqslant 2,\\ i=1,\ldots,n}}\big\|(M_\xi\Phi)*(M_k\Phi)*f\big\|_{L^2}$$
$$\leqslant\sup_{\xi\in\mathbb{R}^n}\sum_{\substack{|k_i-\xi_i|\leqslant 2,\\ i=1,\ldots,n}}\|M_\xi\Phi\|_{L^1}\big\|(M_k\Phi)*f\big\|_{L^2}$$
$$\leqslant\|\Phi\|_{L^1}\Big(\sup_{\ell\in\mathbb{Z}^n}\big\|(M_\ell\Phi)*f\big\|_{L^2}\Big)\bigg(\sup_{\xi\in\mathbb{R}^n}\sum_{\substack{|k_i-\xi_i|\leqslant 2,\\ i=1,\ldots,n}}1\bigg)$$
$$\leqslant 5^n\|\Phi\|_{L^1}\sup_{\ell\in\mathbb{Z}^n}\big\|(M_\ell\Phi)*f\big\|_{L^2}.$$

The proof is complete. □

We remark that Lemma 3.2 implies

$$\|f_\lambda\|_{M^{2,\infty}}\leqslant C\lambda^{-3n/2}\|f\|_{M^{2,\infty}}\quad\text{for all } f\in M^{2,\infty}(\mathbb{R}^n)\text{ and }0<\lambda\leqslant 1.$$

This is not our desired order of λ in the case $(p,q)=(2,\infty)$. But we have

Lemma 3.5. *There exists a constant $C > 0$ such that*

$$\|f_\lambda\|_{M^{2,\infty}} \leqslant C\lambda^{-n}\|f\|_{M^{2,\infty}} \quad \textit{for all } f \in M^{2,\infty}(\mathbb{R}^n) \textit{ and } 0 < \lambda \leqslant 1.$$

Proof. Let $\Phi = \mathcal{F}^{-1}\varphi$, where φ is as in Lemma 3.4. Suppose that $f \in M^{2,\infty}(\mathbb{R}^n)$. We note that $\widehat{f} \in L^2_{\mathrm{loc}}(\mathbb{R}^n)$ (see the proof of Lemma 3.4). Then, by Lemma 3.4, we see that

$$\begin{aligned}
\big\|V_\Phi(f_\lambda)\big\|_{M^{2,\infty}} &\leqslant 5^n\|\Phi\|_{L^1}\sup_{k\in\mathbb{Z}^n}\big\|(M_k\Phi)*f_\lambda\big\|_{L^2} \\
&= 5^n\|\Phi\|_{L^1}\sup_{k\in\mathbb{Z}^n}(2\pi)^{-n/2}\big\|\varphi(\cdot-k)\widehat{f_\lambda}\big\|_{L^2} \\
&= C_n\lambda^{-n/2}\sup_{k\in\mathbb{Z}^n}\big\|\varphi(\lambda\cdot-k)\widehat{f}\big\|_{L^2} \\
&= C_n\lambda^{-n/2}\sup_{k\in\mathbb{Z}^n}\bigg\|\varphi(\lambda\cdot-k)\bigg(\sum_{\ell\in\mathbb{Z}^n}\varphi(\cdot-\ell)\bigg)\widehat{f}\bigg\|_{L^2}.
\end{aligned}$$

Since

$$\begin{aligned}
\bigg|\varphi(\lambda t-k)\bigg(\sum_{\ell\in\mathbb{Z}^n}\varphi(t-\ell)\bigg)\widehat{f}(t)\bigg|^2 &\leqslant 4^n\sum_{\ell\in\mathbb{Z}^n}\big|\varphi(\lambda t-k)\varphi(t-\ell)\widehat{f}(t)\big|^2 \\
&= 4^n\sum_{\substack{|\ell_i-k_i/\lambda|\leqslant 2/\lambda,\\ i=1,\dots,n}}\big|\varphi(\lambda t-k)\varphi(t-\ell)\widehat{f}(t)\big|^2,
\end{aligned}$$

we have

$$\begin{aligned}
&\bigg\|\varphi(\lambda\cdot-k)\bigg(\sum_{\ell\in\mathbb{Z}^n}\varphi(\cdot-\ell)\bigg)\widehat{f}\bigg\|_{L^2} \\
&\quad\leqslant\bigg(4^n\sum_{\substack{|\ell_i-k_i/\lambda|\leqslant 2/\lambda,\\ i=1,\dots,n}}\int_{\mathbb{R}^n}\big|\varphi(\lambda t-k)\varphi(t-\ell)\widehat{f}(t)\big|^2\,dt\bigg)^{1/2} \\
&\quad\leqslant\bigg(4^n(2\pi)^n\|\varphi\|_{L^\infty}^2\sum_{\substack{|\ell_i-k_i/\lambda|\leqslant 2/\lambda,\\ i=1,\dots,n}}\big\|(M_\ell\Phi)*f\big\|_{L^2}^2\bigg)^{1/2} \\
&\quad\leqslant\bigg(4^n(2\pi)^n\|\varphi\|_{L^\infty}^2\Big(\sup_{m\in\mathbb{Z}^n}\big\|(M_m\Phi)*f\big\|_{L^2}\Big)^2\sum_{\substack{|\ell_i-k_i/\lambda|\leqslant 2/\lambda,\\ i=1,\dots,n}}1\bigg)^{1/2} \\
&\quad\leqslant\Big(C_n\|\varphi\|_{L^\infty}^2\lambda^{-n}\Big(\sup_{m\in\mathbb{Z}^n}\big\|(M_m\Phi)*f\big\|_{L^2}\Big)^2\Big)^{1/2} \\
&\quad= C_n\|\varphi\|_{L^\infty}\lambda^{-n/2}\sup_{m\in\mathbb{Z}^n}\big\|(M_m\Phi)*f\big\|_{L^2}.
\end{aligned}$$

Hence, by Lemma 3.4, we get

$$\|f_\lambda\|_{M^{2,\infty}} \leqslant C_n\lambda^{-n}\sup_{m\in\mathbb{Z}^n}\big\|(M_m\Phi)*f\big\|_{L^2} \leqslant C_n\lambda^{-n}\|f\|_{M^{2,\infty}}.$$

The proof is complete. □

Lemma 3.6. *Let* $1 \leqslant p \leqslant \infty$. *Then the following are true*:

(1) *If* $p \leqslant 2$, *then there exists a constant* $C > 0$ *such that*

$$\|f_\lambda\|_{M^{p,1}} \leqslant C\|f\|_{M^{p,1}} \quad \text{for all } f \in M^{p,1}(\mathbb{R}^n) \text{ and } \lambda \geqslant 1.$$

(2) *If* $p \geqslant 2$, *then there exists a constant* $C > 0$ *such that*

$$\|f_\lambda\|_{M^{p,1}} \leqslant C\lambda^{-n(2/p-1)}\|f\|_{M^{p,1}} \quad \text{for all } f \in M^{p,1}(\mathbb{R}^n) \text{ and } \lambda \geqslant 1.$$

Proof. We first consider the case $p \leqslant 2$. By Lemmas 2.2, 2.4 and 3.5, we have

$$\begin{aligned}\|f_\lambda\|_{M^{2,1}} &= \sup\big|\langle f_\lambda, g\rangle_M\big| = \sup\big|\langle f_\lambda, g\rangle\big| \\ &= \lambda^{-n}\sup\big|\langle f, g_{1/\lambda}\rangle\big| \leqslant \lambda^{-n}\sup\|f\|_{M^{2,1}}\|g_{1/\lambda}\|_{M^{2,\infty}} \\ &\leqslant \lambda^{-n}\sup\|f\|_{M^{2,1}}\big(C(1/\lambda)^{-n}\|g\|_{M^{2,\infty}}\big) = C\|f\|_{M^{2,1}}\end{aligned}$$

for all $f \in \mathcal{S}(\mathbb{R}^n)$ and $\lambda \geqslant 1$, where the supremum is taken over all $g \in M^{2,\infty}(\mathbb{R}^n)$ such that $\|g\|_{M^{2,\infty}} = 1$. Since $\mathcal{S}(\mathbb{R}^n)$ is dense in $M^{2,1}(\mathbb{R}^n)$, this gives

$$\|f_\lambda\|_{M^{2,1}} \leqslant C\|f\|_{M^{2,1}} \quad \text{for all } f \in M^{2,1}(\mathbb{R}^n) \text{ and } \lambda \geqslant 1. \tag{3.9}$$

On the other hand, by Lemma 3.2, we see that

$$\|f_\lambda\|_{M^{1,1}} \leqslant C\|f\|_{M^{1,1}} \quad \text{for all } f \in M^{1,1}(\mathbb{R}^n) \text{ and } \lambda \geqslant 1. \tag{3.10}$$

Hence, by interpolation, (3.9) and (3.10) give Lemma 3.6(1).

We next consider the case $p \geqslant 2$. By Lemma 3.2, we have

$$\|f_\lambda\|_{M^{\infty,1}} \leqslant C\lambda^{n}\|f\|_{M^{\infty,1}} \quad \text{for all } f \in M^{\infty,1}(\mathbb{R}^n) \text{ and } \lambda \geqslant 1. \tag{3.11}$$

Therefore, by interpolation, (3.9) and (3.11) give Lemma 3.6(2). □

Lemma 3.7. *Let* $1 \leqslant p \leqslant \infty$. *Then the following are true*:

(1) *If* $p \leqslant 2$, *then there exists a constant* $C > 0$ *such that*

$$\|f_\lambda\|_{M^{p,\infty}} \leqslant C\lambda^{-2n/p}\|f\|_{M^{p,\infty}} \quad \text{for all } f \in M^{p,\infty}(\mathbb{R}^n) \text{ and } 0 < \lambda \leqslant 1.$$

(2) *If* $p \geqslant 2$, *then there exists a constant* $C > 0$ *such that*

$$\|f_\lambda\|_{M^{p,\infty}} \leqslant C\lambda^{-n}\|f\|_{M^{p,\infty}} \quad \text{for all } f \in M^{p,\infty}(\mathbb{R}^n) \text{ and } 0 < \lambda \leqslant 1.$$

Proof. Let $1 < p \leqslant 2$. By duality and Lemma 3.6(2), we have

$$\|f_\lambda\|_{M^{p,\infty}} = \sup\big|\langle f_\lambda, g\rangle\big| = \lambda^{-n} \sup\big|\langle f, g_{1/\lambda}\rangle\big| \\ \leqslant \lambda^{-n} \sup \|f\|_{M^{p,\infty}}\big(C(1/\lambda)^{-n(2/p'-1)}\|g\|_{M^{p',1}}\big) = C\lambda^{-2n/p}\|f\|_{M^{p,\infty}}$$

for all $f \in M^{p,\infty}(\mathbb{R}^n)$ and $0 < \lambda \leqslant 1$, where the supremum is taken over all $g \in \mathcal{S}(\mathbb{R}^n)$ such that $\|g\|_{M^{p',1}} = 1$. In the case $p = 1$, by Lemma 3.2, we have

$$\|f_\lambda\|_{M^{1,\infty}} \leqslant C\lambda^{-2n}\|f\|_{M^{1,\infty}} \quad \text{for all } f \in M^{1,\infty}\big(\mathbb{R}^n\big) \text{ and } 0 < \lambda \leqslant 1.$$

Hence, we obtain Lemma 3.7(1). In the same way, we can prove Lemma 3.7(2). □

Lemma 3.8. *Let $1 \leqslant p, q \leqslant \infty$, $(p, q) \neq (1, \infty), (\infty, 1)$ and $\epsilon > 0$. Set*

$$f(t) = \sum_{k\neq 0} |k|^{-n/q-\epsilon} e^{ik\cdot t} \varphi(t) \quad \text{in } \mathcal{S}'\big(\mathbb{R}^n\big),$$

where φ is the Gauss function. Then $f \in M^{p,q}(\mathbb{R}^n)$ and there exists a constant $C > 0$ such that $\|f_\lambda\|_{M^{p,q}} \geqslant C\lambda^{n(1/q-1)+\epsilon}$ for all $0 < \lambda \leqslant 1$.

Proof. We first prove $f \in M^{p,q}(\mathbb{R}^n)$. Although this fact is an immediate consequence of the discretization properties of the modulation spaces (see Feichtinger and Gröchenig [6], or Gröchenig [10, Theorem 12.2.4]), here we give the proof for reader's convenience. Since

$$\bigg|\int_{\mathbb{R}^n} e^{ik\cdot t}\varphi(t)\varphi(t-x)e^{-i\xi\cdot t}\,dt\bigg| \\ = \bigg|\int_{\mathbb{R}^n} \varphi(t)\varphi(x-t)\big\{\big(1+|\xi-k|^2\big)^{-n}(I-\Delta_t)^n e^{-i(\xi-k)\cdot t}\big\}\,dt\bigg| \\ = \big(1+|\xi-k|^2\big)^{-n}\bigg|\sum_{|\beta_1+\beta_2|\leqslant 2n} C_{\beta_1,\beta_2}\int_{\mathbb{R}^n} \big(\partial^{\beta_1}\varphi\big)(t)\big(\partial^{\beta_2}\varphi\big)(x-t)e^{-i(\xi-k)\cdot t}\,dt\bigg| \\ \leqslant C\big(1+|\xi-k|^2\big)^{-n}\sum_{|\beta_1+\beta_2|\leqslant 2n} \big|\partial^{\beta_1}\varphi\big| * \big|\partial^{\beta_2}\varphi\big|(x),$$

we see that

$$\|f\|_{M^{p,q}} = \|V_\varphi f\|_{L^{p,q}} \\ = \bigg\{\int_{\mathbb{R}^n}\bigg(\int_{\mathbb{R}^n}\bigg|\sum_{k\neq 0}|k|^{-n/q-\epsilon}\int_{\mathbb{R}^n} e^{ik\cdot t}\varphi(t)\varphi(t-x)e^{-i\xi\cdot t}\,dt\bigg|\,dx\bigg)^{q/p} d\xi\bigg\}^{1/q} \\ \leqslant C\bigg\{\int_{\mathbb{R}^n}\bigg(\sum_{k\neq 0}|k|^{-n/q-\epsilon}\big(1+|\xi-k|^2\big)^{-n}\bigg)^q d\xi\bigg\}^{1/q}$$

$$= C\left\{ \sum_{\ell\in\mathbb{Z}^n} \int_{\ell+[-1/2,1/2]^n} \left(\sum_{k\neq 0} |k|^{-n/q-\epsilon}\big(1+|\xi-k|^2\big)^{-n} \right)^q d\xi \right\}^{1/q}$$

$$\leqslant C\left\{ \sum_{\ell\in\mathbb{Z}^n} \left(\sum_{k\neq 0} |k|^{-n/q-\epsilon}\big(1+|\ell-k|^2\big)^{-n} \right)^q \right\}^{1/q}.$$

Since $\{|k|^{-n/q-\epsilon}\}_{k\neq 0} \in \ell^q(\mathbb{Z}^n)$, by Young's inequality, we have $f \in M^{p,q}(\mathbb{R}^n)$.

We next consider the second part. Since $\varphi \in M^{p',q'}(\mathbb{R}^n)$, by duality, we see that

$$\begin{aligned}
\|f_\lambda\|_{M^{p,q}} &= \sup_{\|g\|_{M^{p',q'}}=1} \big|\langle f_\lambda, g\rangle_M\big| \geqslant \big|\langle f_\lambda, \varphi\rangle\big| \\
&= \left| \sum_{k\neq 0} |k|^{-n/q-\epsilon} \int_{\mathbb{R}^n} e^{i(\lambda k)\cdot t} \varphi(\lambda t)\varphi(t)\, dt \right| \\
&= C\big(1+\lambda^2\big)^{-n/2} \sum_{k\neq 0} |k|^{-n/q-\epsilon} e^{-\frac{\lambda^2|k|^2}{4(1+\lambda^2)}} \\
&\geqslant C \sum_{\substack{0<|k_i|\leqslant 1/\lambda,\\ i=1,\dots,n}} |k|^{-n/q-\epsilon} e^{-\frac{\lambda^2|k|^2}{4(1+\lambda^2)}} \\
&\geqslant C\lambda^{n/q+\epsilon} \sum_{\substack{0<|k_i|\leqslant 1/\lambda,\\ i=1,\dots,n}} 1 \geqslant C\lambda^{n(1/q-1)+\epsilon}
\end{aligned}$$

for all $0 < \lambda \leqslant 1$. The proof is complete. □

We are now ready to prove Theorem 3.1(1) with $(1/p, 1/q) \in I_2^*$ and Theorem 3.1(2) with $(1/p, 1/q) \in I_2$.

Proof of Theorem 3.1(2) with $(1/p, 1/q) \in I_2$. We recall that $\mu_2(p,q) = 1/q - 1$ if $(1/p, 1/q) \in I_2$. Let $(1/p, 1/q) \in I_2$ and $1/p \leqslant 1/q$. If $q = 1$ then $p = \infty$, and we have already proved this case in Theorem 3.1(2) with $(1/p, 1/q) \in I_1$. Hence, we may assume $1 < q \leqslant \infty$. Note that $1 \leqslant p', q' < \infty$. Since $(1/p', 1/q') \in I_2^*$ and $1/p' \geqslant 1/q'$, by duality and Lemma 3.3, we have

$$\begin{aligned}
\|f_\lambda\|_{M^{p,q}} &= \sup\big|\langle f_\lambda, g\rangle\big| = \lambda^{-n}\sup\big|\langle f, g_{1/\lambda}\rangle\big| \\
&\leqslant \lambda^{-n} \sup \|f\|_{M^{p,q}} \|g_{1/\lambda}\|_{M^{p',q'}} \\
&\leqslant \lambda^{-n} \sup \|f\|_{M^{p,q}} \Big(C\big(\lambda^{-1}\big)^{-n(2/p'-1/q')}\big(1+\lambda^{-2}\big)^{n(1/p'-1/2)} \|g\|_{M^{p',q'}} \Big) \\
&\leqslant C\lambda^{n(1/q-1)} \|f\|_{M^{p,q}}
\end{aligned} \tag{3.12}$$

for all $f \in M^{p,q}(\mathbb{R}^n)$ and $0 < \lambda \leqslant 1$, where the supremum is taken over all $g \in \mathcal{S}(\mathbb{R}^n)$ such that $\|g\|_{M^{p',q'}} = 1$. This is the first part of Theorem 3.1(2) with $(1/p, 1/q) \in I_2$ and $1/p \leqslant 1/q$. We next consider the case $(1/p, 1/q) \in I_2$, $1/p \geqslant 1/q$ and $q < \infty$. From (3.12) it follows that

$$\|f_\lambda\|_{M^{r,r}} \leqslant C\lambda^{n(1/r-1)}\|f\|_{M^{r,r}} \quad \text{for all } f \in M^{r,r}\big(\mathbb{R}^n\big) \text{ and } 0 < \lambda \leqslant 1, \tag{3.13}$$

where $2 \leqslant r \leqslant \infty$. Then, by interpolation, Lemma 3.5 and (3.13) give

$$\|f_\lambda\|_{M^{p,q}} \leqslant C\lambda^{n(1/q-1)}\|f\|_{M^{p,q}} \quad \text{for all } f \in M^{p,q}\big(\mathbb{R}^n\big) \text{ and } 0 < \lambda \leqslant 1,$$

where $(1/p, 1/q) \in I_2$, $1/p \geqslant 1/q$ and $q < \infty$. In the case $q = \infty$, by Lemma 3.7(2), we have nothing to prove. Hence, we get the first part of Theorem 3.1(2) with $(1/p, 1/q) \in I_2$ and $1/p \geqslant 1/q$.

We next consider the second part of Theorem 3.1(2) with $(1/p, 1/q) \in I_2$. Let $(1/p, 1/q) \in I_2$. Since $(1/\infty, 1/1) \in I_1$, we may assume $(p, q) \neq (\infty, 1)$. Assume that there exist constants $C > 0$ and $\beta \in \mathbb{R}$ such that

$$\|f_\lambda\|_{M^{p,q}} \leqslant C\lambda^{\beta}\|f\|_{M^{p,q}} \quad \text{for all } f \in M^{p,q}\big(\mathbb{R}^n\big) \text{ and } 0 < \lambda \leqslant 1,$$

where $\beta > n(1/q - 1)$. Then we can take $\epsilon > 0$ such that $n(1/q - 1) + \epsilon < \beta$. For this ϵ, we set

$$f(t) = \sum_{k \neq 0} |k|^{-n/q-\epsilon} e^{ik\cdot t}\varphi(t),$$

where φ is the Gauss function. Then, by Lemma 3.8, we see that $f \in M^{p,q}(\mathbb{R}^n)$ and there exists a constant $C' > 0$ such that $\|f_\lambda\|_{M^{p,q}} \geqslant C'\lambda^{n(1/q-1)+\epsilon}$ for all $0 < \lambda \leqslant 1$. Hence,

$$C'\lambda^{n(1/q-1)+\epsilon} \leqslant \|f_\lambda\|_{M^{p,q}} \leqslant C\lambda^{\beta}\|f\|_{M^{p,q}}$$

for all $0 < \lambda \leqslant 1$. However, since $n(1/q - 1) + \epsilon < \beta$, this is a contradiction. Therefore, β must satisfy $\beta \leqslant n(1/q - 1)$. The proof is complete. □

Proof of Theorem 3.1(1) with $(1/p, 1/q) \in I_2^*$. We recall that $\mu_1(p, q) = 1/q - 1$ if $(1/p, 1/q) \in I_2^*$. In every case except for $(p, q) \neq (1, \infty)$, by duality, Theorem 3.1(2) with $(1/p', 1/q') \in I_2$ and the same argument as in the proof of Theorem 3.1(1) with $(1/p, 1/q) \in I_1^*$, we can prove Theorem 3.1(1) with $(1/p, 1/q) \in I_2^*$. For the case $(p, q) = (1, \infty)$, we have already proved in Theorem 3.1(1) with $(1/p, 1/q) \in I_1^*$. □

Our last goal of this section is to prove Theorem 3.1(1) with $(1/p, 1/q) \in I_3^*$ and Theorem 3.1(2) with $(1/p, 1/q) \in I_3$. In the following lemma, we use the fact that there exists $\varphi \in \mathcal{S}(\mathbb{R}^n)$ such that $\operatorname{supp}\varphi \subset [-1/8, 1/8]^n$ and $|\hat{\varphi}| \geqslant 1$ on $[-2, 2]^n$ (see, for example, the proof of [8, Theorem 2.6]).

Lemma 3.9. *Let $1 \leqslant p \leqslant \infty$, $1 \leqslant q < \infty$ and $\epsilon > 0$. Suppose that $\varphi, \psi \in \mathcal{S}(\mathbb{R}^n)$ satisfy $\operatorname{supp}\varphi \subset [-1/8, 1/8]^n$, $\operatorname{supp}\psi \subset [-1/2, 1/2]^n$, $|\hat{\varphi}| \geqslant 1$ on $[-2, 2]^n$ and $\psi = 1$ on $[-1/4, 1/4]^n$. Set*

$$f(t) = \sum_{k \neq 0} |k|^{-n/q-\epsilon} e^{ik\cdot t}\psi(t - k) \quad \text{in } \mathcal{S}'\big(\mathbb{R}^n\big).$$

Then $f \in M^{p,q}(\mathbb{R}^n)$ *and there exists a constant* $C > 0$ *such that*

$$\left\| V_\varphi(f_\lambda) \right\|_{L^{p,q}} \geqslant C\lambda^{-n(2/p-1/q)+\epsilon} \quad \text{for all } 0 < \lambda \leqslant 1.$$

Proof. In the same way as in the proof of Lemma 3.8, we can prove $f \in M^{p,q}(\mathbb{R}^n)$ (see also [6] or [10]). We consider the second part. It is enough to show that $\|V_{\varphi_{1/\lambda}} f\|_{L^{p,q}} \geqslant C\lambda^{-n/p+n+\epsilon}$ for all $0 < \lambda \leqslant 1$, since $\|V_\varphi(f_\lambda)\|_{L^{p,q}} = \lambda^{-n(1/p-1/q+1)}\|V_{\varphi_{1/\lambda}} f\|_{L^{p,q}}$. We note that $\operatorname{supp}\varphi((\cdot - x)/\lambda) \subset \ell + [-1/4, 1/4]^n$ for all $0 < \lambda \leqslant 1$, $\ell \in \mathbb{Z}^n$ and $x \in \ell + [-1/8, 1, 8]^n$. Since $\operatorname{supp}\psi(\cdot - k) \subset k + [-1/2, 1/2]^n$ and $\psi(t-k) = 1$ if $t \in k + [-1/4, 1/4]^n$, it follows that

$$\begin{aligned}
&\left(\int_{\mathbb{R}^n} \left| V_{\varphi_{1/\lambda}} f(x,\xi) \right|^p dx \right)^{1/p} \\
&= \left(\int_{\mathbb{R}^n} \left| \sum_{k\neq 0} |k|^{-n/q-\epsilon} \int_{\mathbb{R}^n} e^{ik\cdot t}\psi(t-k)\overline{\varphi\left(\frac{t-x}{\lambda}\right)} e^{-i\xi\cdot t}\, dt \right|^p dx \right)^{1/p} \\
&\geqslant \left(\sum_{\ell\neq 0} \int_{\ell+[-1/8,1/8]^n} \left| \sum_{k\neq 0} |k|^{-n/q-\epsilon} \int_{\mathbb{R}^n} e^{-i(\xi-k)\cdot t}\psi(t-k)\overline{\varphi\left(\frac{t-x}{\lambda}\right)}\, dt \right|^p dx \right)^{1/p} \\
&= \left(\sum_{\ell\neq 0} \int_{\ell+[-1/8,1/8]^n} \left| |\ell|^{-n/q-\epsilon} \int_{\mathbb{R}^n} e^{-i(\xi-\ell)\cdot t}\overline{\varphi\left(\frac{t-x}{\lambda}\right)}\, dt \right|^p dx \right)^{1/p} \\
&= 4^{-n/p} \left(\sum_{\ell\neq 0} \left| |\ell|^{-n/q-\epsilon}\lambda^n \hat{\varphi}\big(-\lambda(\xi-\ell)\big) \right|^p \right)^{1/p}.
\end{aligned}$$

Hence, using $|\hat{\varphi}| \geqslant 1$ on $[-2,2]^n$, we get

$$\begin{aligned}
\|V_{\varphi_{1/\lambda}} f\|_{L^{p,q}} &\geqslant 4^{-n/p}\lambda^n \left\{ \int_{\mathbb{R}^n} \left(\sum_{\ell\neq 0} \left| |\ell|^{-n/q-\epsilon} \hat{\varphi}\big(-\lambda(\xi-\ell)\big) \right|^p \right)^{q/p} d\xi \right\}^{1/q} \\
&= 4^{-n/p}\lambda^{n-n/q} \left\{ \int_{\mathbb{R}^n} \left(\sum_{\ell\neq 0} \left| |\ell|^{-n/q-\epsilon} \hat{\varphi}(\xi+\lambda\ell) \right|^p \right)^{q/p} d\xi \right\}^{1/q} \\
&\geqslant 4^{-n/p}\lambda^{n-n/q} \left\{ \int_{[-1,1]^n} \left(\sum_{\substack{0<|\ell_i|\leqslant 1/\lambda,\\ i=1,\dots,n}} \left| |\ell|^{-n/q-\epsilon} \hat{\varphi}(\xi+\lambda\ell) \right|^p \right)^{q/p} d\xi \right\}^{1/q} \\
&\geqslant 4^{-n/p}2^{n/q}\lambda^{n-n/q} \left(\sum_{\substack{0<|\ell_i|\leqslant 1/\lambda,\\ i=1,\dots,n}} |\ell|^{-(n/q+\epsilon)p} \right)^{1/p} \\
&\geqslant C_n\lambda^{n-n/q}\lambda^{n/q+\epsilon} \left(\sum_{\substack{0<|\ell_i|\leqslant 1/\lambda,\\ i=1,\dots,n}} 1 \right)^{1/p} \geqslant C_n\lambda^{-n/p+n+\epsilon}
\end{aligned}$$

for all $0 < \lambda \leqslant 1$. The proof is complete. □

For Lemma 3.9, we do not need $\epsilon > 0$ in the case $q = \infty$.

Lemma 3.10. *Let $1 \leqslant p \leqslant \infty$. Suppose that $\varphi, \psi \in \mathcal{S}(\mathbb{R}^n)$ are as in Lemma 3.9. Set*

$$f(t) = \sum_{k \in \mathbb{Z}^n} e^{ik\cdot t} \psi(t-k) \quad \text{in } \mathcal{S}'(\mathbb{R}^n).$$

Then $f \in M^{p,\infty}(\mathbb{R}^n)$ and there exists a constant $C > 0$ such that

$$\| V_\varphi(f_\lambda) \|_{L^{p,\infty}} \geqslant C\lambda^{-2n/p} \quad \text{for all } 0 < \lambda \leqslant 1.$$

In particular, if $1 \leqslant p \leqslant 2$ then there exist constants $C, C' > 0$ such that

$$C\lambda^{-2n/p} \leqslant \|f_\lambda\|_{M^{p,\infty}} \leqslant C'\lambda^{-2n/p} \quad \text{for all } 0 < \lambda \leqslant 1.$$

Proof. In the same way as in the proof of Lemma 3.8, we can prove

$$\left| \int_{\mathbb{R}^n} e^{ik\cdot t} \psi(t-k) \overline{\varphi(t-x)} e^{-i\xi\cdot t}\, dt \right| \leqslant C\big(1 + |x-k|^2\big)^{-n} \big(1 + |\xi - k|^2\big)^{-n}.$$

Hence,

$$\begin{aligned} \big|V_\varphi f(x,\xi)\big| &= \left| \sum_{k\in\mathbb{Z}^n} \int_{\mathbb{R}^n} e^{ik\cdot t} \psi(t-k) \overline{\varphi(t-x)} e^{-i\xi\cdot t}\, dt \right| \\ &\leqslant C \sum_{k\in\mathbb{Z}^n} \big(1 + |x-k|^2\big)^{-n} \big(1 + |\xi-k|^2\big)^{-n} \leqslant C\big(1 + |x-\xi|^2\big)^{-n} \end{aligned}$$

for all $x, \xi \in \mathbb{R}^n$. This implies $f \in M^{p,\infty}(\mathbb{R}^n)$ (which is also a consequence of [6] or [10]).

We next consider the second part. Since $\|V_{\varphi_{1/\lambda}} f(\cdot,\xi)\|_{L^p}$ is continuous with respect to $\xi \in \mathbb{R}^n$, we see that $\|V_{\varphi_{1/\lambda}} f\|_{L^{p,\infty}} = \sup_{\xi\in\mathbb{R}^n} \|V_{\varphi_{1/\lambda}} f(\cdot,\xi)\|_{L^p}$ for each $0 < \lambda \leqslant 1$. Hence, by the same argument as in the proof of Lemma 3.9, we have

$$\begin{aligned} \| V_\varphi(f_\lambda) \|_{L^{p,\infty}} &= \lambda^{-n(1/p+1)} \|V_{\varphi_{1/\lambda}} f\|_{L^{p,\infty}} \geqslant \lambda^{-n(1/p+1)} \big\|V_{\varphi_{1/\lambda}} f(\cdot,0)\big\|_{L^p} \\ &\geqslant C\lambda^{-n(1/p+1)} \bigg(\sum_{\ell\in\mathbb{Z}^n} \big|\lambda^n \hat{\varphi}(\lambda\ell)\big|^p \bigg)^{1/p} \\ &\geqslant C\lambda^{-n/p} \bigg(\sum_{\substack{|\ell_i| \leqslant 1/\lambda, \\ i=1,\dots,n}} \big|\hat{\varphi}(\lambda\ell)\big|^p \bigg)^{1/p} \geqslant C\lambda^{-2n/p} \end{aligned}$$

for all $0 < \lambda \leqslant 1$.

By Lemma 3.7(1), if $1 \leqslant p \leqslant 2$, then $\|f\|_{M^{p,\infty}} \sim \lambda^{-2n/p}$ in the case $0 < \lambda \leqslant 1$. The proof is complete. □

We are now ready to prove Theorem 3.1(1) with $(1/p, 1/q) \in I_3^*$ and (2) with $(1/p, 1/q) \in I_3$.

Proof of Theorem 3.1(2) with $(1/p, 1/q) \in I_3$. We recall that $\mu_2(p,q) = -2/p + 1/q$ if $(1/p, 1/q) \in I_3$. Let $(1/p, 1/q) \in I_3$ and $1/p + 1/q \geqslant 1$. We note that, if $(1/p, 1/q) \in I_3$ and $1/p + 1/q \geqslant 1$, then $(1/p, 1/q) \in I_2^*$ and $1/p \geqslant 1/q$. Then, by Lemma 3.3, there exists a constant $C > 0$ such that

$$\|f_\lambda\|_{M^{p,q}} \leqslant C\lambda^{-n(2/p-1/q)}\|f\|_{M^{p,q}} \quad \text{for all } f \in M^{p,q}(\mathbb{R}^n) \text{ and } 0 < \lambda \leqslant 1. \tag{3.14}$$

This is the first part of Theorem 3.1(2) with $(1/p, 1/q) \in I_3$ and $1/p+1/q \geqslant 1$. We next consider the case $(1/p, 1/q) \in I_3$, $1/p + 1/q \leqslant 1$ and $q < \infty$. (3.14) implies

$$\|f_\lambda\|_{M^{r,r'}} \leqslant C\lambda^{-n(2/r-1/r')}\|f\|_{M^{r,r'}} = C\lambda^{-n(3/r-1)}\|f\|_{M^{r,r'}} \tag{3.15}$$

for all $f \in M^{r,r'}(\mathbb{R}^n)$ and $0 < \lambda \leqslant 1$, where $1 \leqslant r \leqslant 2$. Then, by interpolation, Lemma 3.5 and (3.15) give

$$\|f_\lambda\|_{M^{p,q}} \leqslant C\lambda^{-n(2/p-1/q)}\|f\|_{M^{p,q}} \quad \text{for all } f \in M^{p,q}(\mathbb{R}^n) \text{ and } 0 < \lambda \leqslant 1,$$

where $(1/p, 1/q) \in I_3$, $1/p + 1/q \leqslant 1$ and $q < \infty$. In the case $q = \infty$, by Lemma 3.7(1), we have nothing to prove. Hence, we obtain the first part of Theorem 3.1(2) with $(1/p, 1/q) \in I_3$ and $1/p + 1/q \leqslant 1$.

By using Lemma 3.9 (or 3.10), we can prove the second part of Theorem 3.1(2) with $(1/p, 1/q) \in I_3$ in the same way as in the proof of the second part of Theorem 3.1(2) with $(1/p, 1/q) \in I_2$. We omit the proof. □

Proof of Theorem 3.1(1) with $(1/p, 1/q) \in I_3^*$. We recall that $\mu_1(p,q) = -2/p + 1/q$ if $(1/p, 1/q) \in I_3^*$. In every case except for $(p,q) \neq (\infty, 1)$, by duality, Theorem 3.1(2) with $(1/p', 1/q') \in I_3$ and the same argument as in the proof of Theorem 3.1(1) with $(1/p, 1/q) \in I_1^*$, we can prove Theorem 3.1(1) with $(1/p, 1/q) \in I_3^*$.

For the first part of Theorem 3.1(1) with $(p,q) = (\infty, 1)$, by (3.11), we have nothing to prove. By using interpolation, we can prove the second part in the same way as in the proof of Theorem 3.1(1) with $(1/p, 1/q) \in I_1^*$ and $q = \infty$. □

4. The inclusion between Besov spaces and modulation spaces

In this section, we prove Theorem 1.2 which appeared in Section 1. It is sufficient to prove the first statement only because the first one implies the second one by the duality argument and the elementary relation

$$\nu_2(p,q) = -\nu_1(p',q').$$

See also Section 2 for the dual spaces of the modulation spaces (Lemma 2.4) and Besov spaces.

For the preparation to prove Theorem 1.2(1) with $(1/p, 1/q) \in I_1^*$, we show three lemmas in the below. We denote by B the tensor product of B-spline of degree 2, that is

$$B(t) = \prod_{i=1}^{n} \chi_{[-1/2,1/2]} * \chi_{[-1/2,1/2]}(t_i),$$

where $t = (t_1, \dots, t_n) \in \mathbb{R}^n$. We note that $\operatorname{supp} B \subset [-1,1]^n$ and $\mathcal{F}^{-1}B \in M^{p,q}(\mathbb{R}^n)$ for all $1 \leqslant p, q \leqslant \infty$.

Lemma 4.1. *Let $1 \leqslant p, q \leqslant \infty$, $(p,q) \neq (1,\infty), (\infty,1)$ and $\epsilon > 0$. Suppose that $\psi \in \mathcal{S}(\mathbb{R}^n)$ satisfies $\psi = 1$ on $\{\xi\colon |\xi| \leqslant 1/2\}$ and $\operatorname{supp}\psi \subset \{\xi\colon |\xi| \leqslant 1\}$. Set*

$$f(t) = \sum_{\ell \neq 0} |\ell|^{-n/p-\epsilon} \Psi(t-\ell) \quad \text{in } \mathcal{S}'(\mathbb{R}^n),$$

where $\Psi = \mathcal{F}^{-1}\psi$. Then $f \in M^{p,q}(\mathbb{R}^n)$ and there exists a constant $C > 0$ such that $\|f_\lambda\|_{M^{p,q}} \geqslant C\lambda^{-n/p-\epsilon}$ for all $\lambda \geqslant 2\sqrt{n}$.

Proof. In the same way as in the proof of Lemma 3.8, we can prove $f \in M^{p,q}(\mathbb{R}^n)$ (see also [6] or [10]). We consider the second part. Let $\lambda \geqslant 2\sqrt{n}$. Since $\psi(\cdot/\lambda) = 1$ on $[-1,1]^n$, we have

$$\begin{aligned}
\int_{\mathbb{R}^n} \Psi(\lambda t - \ell)\big(\mathcal{F}^{-1}B\big)(t)\,dt &= (2\pi)^{-n}\lambda^{-n} \int_{\mathbb{R}^n} e^{-i(\ell/\lambda)\cdot t}\psi(t/\lambda)\,B(t)\,dt \\
&= (2\pi)^{-n}\lambda^{-n} \int_{\mathbb{R}^n} e^{-i(\ell/\lambda)\cdot t}\,B(t)\,dt \\
&= (2\pi)^{-n}\lambda^{-n} \prod_{i=1}^{n} \left(\frac{\sin \ell_i/2\lambda}{\ell_i/2\lambda}\right)^2.
\end{aligned}$$

We note that $\prod_{i=1}^n \{(\sin\xi_i)/\xi_i\}^2 \geqslant C$ on $[-\pi/2, \pi/2]^n$ for some constant $C > 0$. Since $\mathcal{F}^{-1}B \in M^{p',q'}(\mathbb{R}^n)$, by Lemmas 2.2 and 2.4, we get

$$\begin{aligned}
\|f_\lambda\|_{M^{p,q}} &= \sup_{\|g\|_{M^{p',q'}}=1} \big|\langle f_\lambda, g\rangle_M\big| \geqslant \big\|\mathcal{F}^{-1}B\big\|_{M^{p',q'}}^{-1} \big|\big\langle f_\lambda, \mathcal{F}^{-1}B\big\rangle_M\big| \\
&= C\left|\sum_{\ell\neq 0} |\ell|^{-n/p-\epsilon} \frac{1}{\|\Phi\|_{L^2}^2} \int_{\mathbb{R}^{2n}} V_\Phi\big[\Psi(\lambda\cdot{} - \ell)\big](x,\xi)\,\overline{V_\Phi\big[\mathcal{F}^{-1}B\big](x,\xi)}\,dx\,d\xi\right| \\
&= C\left|\sum_{\ell\neq 0} |\ell|^{-n/p-\epsilon} \int_{\mathbb{R}^n} \Psi(\lambda t - \ell)\big(\mathcal{F}^{-1}B\big)(t)\,dt\right| \\
&= C\lambda^{-n}\left|\sum_{\ell\neq 0} |\ell|^{-n/p-\epsilon} \prod_{i=1}^{n}\left(\frac{\sin\ell_i/2\lambda}{\ell_i/2\lambda}\right)^2\right| \\
&\geqslant C\lambda^{-n} \sum_{\substack{0<|\ell_i|\leqslant\lambda\pi,\\ i=1,\dots,n}} |\ell|^{-n/p-\epsilon} \prod_{i=1}^{n}\left(\frac{\sin\ell_i/2\lambda}{\ell_i/2\lambda}\right)^2 \\
&\geqslant C\lambda^{-n}\lambda^{-n/p-\epsilon} \sum_{\substack{0<|\ell_i|\leqslant\lambda\pi,\\ i=1,\dots,n}} 1 \geqslant C\lambda^{-n/p-\epsilon}
\end{aligned}$$

for all $\lambda \geqslant 2\sqrt{n}$. The proof is complete. $\square$

Lemma 4.2. *Suppose that* $1 \leqslant p, q \leqslant \infty$, $(p,q) \neq (1,\infty), (\infty,1)$ *and* $\epsilon > 0$. *Let* $\psi \in \mathcal{S}(\mathbb{R}^n)$ *be as in Lemma* 4.1. *Set*

$$f(t) = e^{8it_1} \sum_{\ell \neq 0} |\ell|^{-n/p-\epsilon} \Psi(t-\ell) \quad \text{in } \mathcal{S}'(\mathbb{R}^n), \tag{4.1}$$

where $t = (t_1, \dots, t_n) \in \mathbb{R}^n$ *and* $\Psi = \mathcal{F}^{-1}\psi$. *Then* $f \in M^{p,q}(\mathbb{R}^n)$ *and there exists a constant* $C > 0$ *such that* $\|f_\lambda\|_{M^{p,q}} \geqslant C\lambda^{-n/p-\epsilon}$ *for all* $\lambda \geqslant 2\sqrt{n}$.

Proof. Let $g(t) = \sum_{\ell \neq 0} |\ell|^{-n/p-\epsilon} \Psi(t-\ell)$. Since $f = M_{8e_1} g$ and $f_\lambda = M_{8\lambda e_1} g_\lambda$, we have $V_\Phi(f_\lambda)(x,\xi) = V_\Phi(g_\lambda)(x, \xi - 8\lambda e_1)$, where $e_1 = (1, 0, \dots, 0)$. This gives $\|f_\lambda\|_{M^{p,q}} = \|g_\lambda\|_{M^{p,q}}$. Hence, by Lemma 4.1, we obtain Lemma 4.2. □

Lemma 4.3. *Suppose that* $1 \leqslant p, q \leqslant \infty$, $s \in \mathbb{R}$ *and* $\epsilon > 0$. *Let* f *be defined by* (4.1). *Then there exists a constant* $C > 0$ *such that* $\|f_{2^k}\|_{B_s^{p,q}} \leqslant C 2^{k(s-n/p)}$ *for all* $k \in \mathbb{Z}_+$.

Proof. Let $k \in \mathbb{Z}_+$. Since $\operatorname{supp} \varphi_0 \subset \{\xi \colon |\xi| \leqslant 2\}$, $\operatorname{supp} \varphi_j \subset \{\xi \colon 2^{j-1} \leqslant |\xi| \leqslant 2^{j+1}\}$ if $j \geqslant 1$ (see Section 2), and $\operatorname{supp} \psi(\cdot/2^k - 8e_1) \subset \{\xi \colon |\xi - 2^{k+3}e_1| \leqslant 2^k\}$, we see that

$$\begin{aligned}
&\int_{\mathbb{R}^n} \Phi_j(x-t)\big(e^{8i(2^k t_1)} \Psi\big(2^k t - \ell\big)\big)\, dt \\
&\quad = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix\cdot t} \varphi_j(t)\big(2^{-kn} e^{-i\ell\cdot(t/2^k - 8e_1)} \psi\big(t/2^k - 8e_1\big)\big)\, dt \\
&\quad = \begin{cases} (2\pi)^{-n} e^{8i\ell_1} \int_{\mathbb{R}^n} e^{i(2^k x - \ell)\cdot t} \varphi_j(2^k t) \psi(t - 8e_1)\, dt, & \text{if } k+2 \leqslant j \leqslant k+4, \\ 0, & \text{otherwise.} \end{cases}
\end{aligned}$$

Hence,

$$\begin{aligned}
&\big|\Phi_j * (f_{2^k})(x)\big| \\
&\quad = \left| \sum_{\ell \neq 0} |\ell|^{-n/p-\epsilon} \int_{\mathbb{R}^n} \Phi_j(x-t)\big(e^{8i(2^k t_1)} \Psi\big(2^k t - \ell\big)\big)\, dt \right| \\
&\quad \leqslant C \sum_{\ell \neq 0} |\ell|^{-n/p-\epsilon} \left| \int_{\mathbb{R}^n} \big\{\big(1 + \big|2^k x - \ell\big|^2\big)^{-n} (I - \Delta_t)^n e^{i(2^k x - \ell)\cdot t}\big\} \varphi_j\big(2^k t\big) \psi(t - 8e_1)\, dt \right| \\
&\quad \leqslant C \sum_{\ell \neq 0} |\ell|^{-n/p-\epsilon} \big(1 + \big|2^k x - \ell\big|^2\big)^{-n},
\end{aligned}$$

where $k+2 \leqslant j \leqslant k+4$. On the other hand, $\Phi_j * (f_{2^k}) = 0$ if $j < k+2$ or $j > k+4$. Thus, $\|\Phi_j * (f_{2^k})\|_{L^p} \leqslant C 2^{-kn/p}$ if $k+2 \leqslant j \leqslant k+4$, and $\|\Phi_j * (f_{2^k})\|_{L^p} = 0$ if $j < k+2$ or $j > k+4$. Therefore,

$$\|f_{2^k}\|_{B_s^{p,q}} = \left(\sum_{j=k+2}^{k+4} 2^{jsq} \|\Phi_j * (f_{2^k})\|_{L^p}^q\right)^{1/q}$$
$$\leqslant C2^{-kn/p}\left(\sum_{j=k+2}^{k+4} 2^{jsq}\right)^{1/q} \leqslant C2^{k(s-n/p)}.$$

The proof is complete. □

We are now ready to prove Theorem 1.2(1) with $(1/p, 1/q) \in I_1^*$.

Proof of Theorem 1.2(1) with $(1/p, 1/q) \in I_1^*$. Let $(1/p, 1/q) \in I_1^*$ and $(p,q) \neq (1,\infty)$. Then $\nu_1(p,q) = 0$. We assume that $B_s^{p,q}(\mathbb{R}^n) \hookrightarrow M^{p,q}(\mathbb{R}^n)$, where $s < 0$. Set $s = -\epsilon$, where $\epsilon > 0$. For this ϵ, we define f by

$$f(t) = e^{8it_1} \sum_{\ell \neq 0} |\ell|^{-n/p-\epsilon/2} \Psi(t-\ell),$$

where $t = (t_1, \dots, t_n) \in \mathbb{R}^n$, $\Psi = \mathcal{F}^{-1}\psi$ and ψ is as in Lemma 4.1. Then, by Lemmas 4.2 and 4.3, we have

$$C_1 2^{-k(n/p+\epsilon/2)} \leqslant \|f_{2^k}\|_{M^{p,q}} \leqslant C_2 \|f_{2^k}\|_{B_s^{p,q}} \leqslant C_3 2^{k(s-n/p)} = C_3 2^{-k(n/p+\epsilon)}$$

for any large integer k. However, this is a contradiction. Hence, s must satisfy $s \geqslant 0$.

We next consider the case $(p,q) = (1,\infty)$. Assume $B_s^{1,\infty}(\mathbb{R}^n) \hookrightarrow M^{1,\infty}(\mathbb{R}^n)$. Let $\psi \in \mathcal{S}(\mathbb{R}^n) \setminus \{0\}$ be such that $\operatorname{supp}\psi \subset \{\xi\colon\ 1/2 \leqslant |\xi| \leqslant 2\}$. Since $M^{1,\infty}(\mathbb{R}^n) \hookrightarrow \mathcal{F}L^\infty(\mathbb{R}^n)$ [18, Proposition 1.7], we see that

$$2^{-kn}\|\psi\|_{L^\infty} = \left\|\mathcal{F}[\Psi_{2^k}]\right\|_{L^\infty} \leqslant C\|\Psi_{2^k}\|_{M^{1,\infty}} \quad \text{for all } k \in \mathbb{Z}_+,$$

where $\Psi = \mathcal{F}^{-1}\psi$. On the other hand, it is easy to show that

$$\|\Psi_{2^k}\|_{B_s^{1,\infty}} \leqslant C2^{k(s-n)} \quad \text{for all } k \in \mathbb{Z}_+.$$

Hence, by our assumption, we get

$$2^{-kn}\|\psi\|_{L^\infty} \leqslant C_1\|\Psi_{2^k}\|_{M^{1,\infty}} \leqslant C_2\|\Psi_{2^k}\|_{B_s^{1,\infty}} \leqslant C_3 2^{k(s-n)}$$

for all $k \in \mathbb{Z}_+$. This implies $s \geqslant 0$. The proof is complete. □

Our next goal is to prove Theorem 1.2(1) with $(1/p, 1/q) \in I_2^*$. We remark the following fact, and give the proof for reader's convenience.

Lemma 4.4. *(See [16, Proposition 1.1].) Let $1 \leqslant p, q \leqslant \infty$ and $s > 0$. Then there exists a constant $C > 0$ such that*

$$\|f_\lambda\|_{B_s^{p,q}} \leqslant C\lambda^{s-n/p}\|f\|_{B_s^{p,q}} \quad \textit{for all } f \in B_s^{p,q}(\mathbb{R}^n) \textit{ and } \lambda \geqslant 1.$$

Proof. Let $j_0 \in \mathbb{Z}_+$ be such that $2^{j_0} \leqslant \lambda < 2^{j_0+1}$. Since $\sum_{j=0}^{\infty} \varphi_j(\xi) = 1$ for all $\xi \in \mathbb{R}^n$, we see that

$$\varphi_j(\lambda\xi) = \sum_{\ell=-2}^{1} \varphi_j(\lambda\xi)\, \varphi_{j+\ell}\big(2^{j_0}\xi\big) \quad \text{for all } \xi \in \mathbb{R}^n \text{ and } j \in \mathbb{Z}_+,$$

where $\varphi_{j+\ell} = 0$ if $j + \ell < 0$. Hence, by Young's inequality, we have

$$\begin{aligned}
\|f_\lambda\|_{B_s^{p,q}} &= \left(\sum_{j=0}^{\infty} 2^{jsq} \big\|\mathcal{F}^{-1}[\varphi_j \widehat{f_\lambda}]\big\|_{L^p}^q\right)^{1/q} \\
&= \lambda^{-n/p} \left(\sum_{j=0}^{\infty} 2^{jsq} \big\|\mathcal{F}^{-1}\big[\varphi_j(\lambda\cdot)\widehat{f}\big]\big\|_{L^p}^q\right)^{1/q} \\
&\leqslant \lambda^{-n/p} \sum_{\ell=-2}^{1} \left(\sum_{j=0}^{\infty} 2^{jsq} \big\|\mathcal{F}^{-1}\big[\varphi_j(\lambda\cdot)\varphi_{j+\ell}\big(2^{j_0}\cdot\big)\widehat{f}\big]\big\|_{L^p}^q\right)^{1/q} \\
&\leqslant \lambda^{-n/p} \sum_{\ell=-2}^{1} \left\{\sum_{j=0}^{\infty} 2^{jsq} \Big(\big\|\mathcal{F}^{-1}\big[\varphi_j(\lambda\cdot)\big]\big\|_{L^1} \big\|\mathcal{F}^{-1}\big[\varphi_{j+\ell}\big(2^{j_0}\cdot\big)\hat{f}\big]\big\|_{L^p}\Big)^q\right\}^{1/q} \\
&\leqslant C\lambda^{-n/p} \left(\sum_{j=0}^{\infty} 2^{jsq} \big\|\mathcal{F}^{-1}\big[\varphi_j\big(2^{j_0}\cdot\big)\widehat{f}\big]\big\|_{L^p}^q\right)^{1/q} \\
&= C\lambda^{-n/p} \left\{\left(\sum_{j=0}^{j_0} + \sum_{j=j_0+1}^{\infty}\right) 2^{jsq} \big\|\mathcal{F}^{-1}\big[\varphi_j\big(2^{j_0}\cdot\big)\widehat{f}\big]\big\|_{L^p}^q\right\}^{1/q}.
\end{aligned}$$

For the first term, we see that

$$\begin{aligned}
\sum_{j=0}^{j_0} 2^{jsq} \big\|\mathcal{F}^{-1}\big[\varphi_j\big(2^{j_0}\cdot\big)\widehat{f}\big]\big\|_{L^p}^q &= \sum_{j=0}^{j_0} 2^{jsq} \big\|\mathcal{F}^{-1}\big[\varphi_j\big(2^{j_0}\cdot\big)(\varphi_0 + \varphi_1)\widehat{f}\big]\big\|_{L^p}^q \\
&\leqslant C \sum_{j=0}^{j_0} 2^{jsq} \big\|\mathcal{F}^{-1}\big[(\varphi_0 + \varphi_1)\widehat{f}\big]\big\|_{L^p}^q \\
&\leqslant C\big(2^{j_0 s} \|f\|_{B_s^{p,q}}\big)^q \leqslant C\big(\lambda^s \|f\|_{B_s^{p,q}}\big)^q.
\end{aligned}$$

For the second term, we have

$$\sum_{j=j_0+1}^{\infty} 2^{jsq} \big\|\mathcal{F}^{-1}\big[\varphi_j\big(2^{j_0}\cdot\big)\widehat{f}\big]\big\|_{L^p}^q = \sum_{j=j_0+1}^{\infty} 2^{jsq} \big\|\mathcal{F}^{-1}[\varphi_{j-j_0}\widehat{f}]\big\|_{L^p}^q \leqslant \big(\lambda^s \|f\|_{B_s^{p,q}}\big)^q.$$

Combining these estimates, we obtain the desired result. □

We are now ready to prove Theorem 1.2(1) with $(1/p, 1/q) \in I_2^*$.

Proof of Theorem 1.2(1) with $(1/p, 1/q) \in I_2^*$. Let $(1/p, 1/q) \in I_2^*$. Then $\nu_1(p,q) = 1/p + 1/q - 1$. If $(1/p, 1/q) \in I_2^*$ and $1/p + 1/q = 1$ then $(1/p, 1/q) \in I_1^*$, and we have already proved this case in Theorem 1.2(1) with $(1/p, 1/q) \in I_1^*$. Hence, we may assume $1/p + 1/q > 1$. Suppose that $B_s^{p,q}(\mathbb{R}^n) \hookrightarrow M^{p,q}(\mathbb{R}^n)$, where $s < n(1/p + 1/q - 1)$. Then, since $n(1/p + 1/q - 1) > 0$, we can take $s_0 > 0$ such that $s \leqslant s_0 < n(1/p + 1/q - 1)$. Let φ be the Gauss function. By Lemma 2.1, we see that $\|\varphi_\lambda\|_{M^{p,q}} \geqslant C\lambda^{n(1/q-1)}$ for all $\lambda \geqslant 1$. On the other hand, by Lemma 4.4, we have

$$\|\varphi_\lambda\|_{B_{s_0}^{p,q}} \leqslant C\lambda^{s_0 - n/p}\|\varphi\|_{B_{s_0}^{p,q}} \quad \text{for all } \lambda \geqslant 1.$$

Hence, using $B_{s_0}^{p,q}(\mathbb{R}^n) \hookrightarrow B_s^{p,q}(\mathbb{R}^n) \hookrightarrow M^{p,q}(\mathbb{R}^n)$, we get

$$C_1\lambda^{n(1/q-1)} \leqslant \|\varphi_\lambda\|_{M^{p,q}} \leqslant C_2\|\varphi_\lambda\|_{B_{s_0}^{p,q}} \leqslant C_3\lambda^{s_0-n/p}\|\varphi\|_{B_{s_0}^{p,q}}$$

for all $\lambda \geqslant 1$. However, since $s_0 - n/p < n(1/q - 1)$, this is a contradiction. Therefore, s must satisfy $s \geqslant n(1/p + 1/q - 1)$. The proof is complete. □

Our last goal is to prove Theorem 1.2(1) with $(1/p, 1/q) \in I_3^*$.

Lemma 4.5. *Let $1 \leqslant p \leqslant \infty$, $1 \leqslant q < \infty$ and $\epsilon > 0$. Suppose that $\varphi, \psi \in \mathcal{S}(\mathbb{R}^n) \setminus \{0\}$ satisfy* $\operatorname{supp}\varphi \subset [-1/8, 1/8]^n$, $\operatorname{supp}\psi \subset [-1/2, 1/2]^n$ *and* $\psi = 1$ *on* $[-1/4, 1/4]^n$. *For $j \in \mathbb{Z}_+$, set*

$$f^j(t) = 2^{-jn/p} \sum_{\substack{0<|k_i|\leqslant 2^j,\\ i=1,\dots,n}} |k|^{-n/p-\epsilon} e^{ik\cdot t/2^j}\Psi\bigl(t/2^j - k\bigr), \tag{4.2}$$

where $\Psi = \mathcal{F}^{-1}\psi$. Then $f^j \in M^{p,q}(\mathbb{R}^n)$ and there exists a constant $C > 0$ such that

$$\bigl\|V_\Phi\bigl[\bigl(f^j\bigr)_{2^j}\bigr]\bigr\|_{L^{p,q}} \geqslant C2^{-jn(2/p-1/q)-j\epsilon} \quad \textit{for all } j \in \mathbb{Z}_+,$$

where $\Phi = \mathcal{F}^{-1}\varphi$.

Proof. Since $f^j \in \mathcal{S}(\mathbb{R}^n)$, we have $f^j \in M^{p,q}(\mathbb{R}^n)$. We consider the second part. Note that $\operatorname{supp}\varphi(\cdot - \xi) \subset \ell + [-1/4, 1/4]^n$ for all $\ell \in \mathbb{Z}^n$ and $\xi \in \ell + [-1/8, 1, 8]^n$. Since $\operatorname{supp}\psi(\cdot - k) \subset k + [-1/2, 1/2]^n$ and $\psi(t - k) = 1$ if $t \in k + [-1/4, 1/4]^n$, it follows that

$$\begin{aligned}
&\bigl\|V_\Phi\bigl[\bigl(f^j\bigr)_{2^j}\bigr]\bigr\|_{L^{p,q}} \\
&\quad\geqslant \Biggl\{\sum_{\ell\in\mathbb{Z}^n}\int_{\ell+[-1/8,1/8]^n}\Biggl(\int_{\mathbb{R}^n}\Biggl|2^{-jn/p}\sum_{\substack{0<|k_i|\leqslant 2^j,\\ i=1,\dots,n}}|k|^{-n/p-\epsilon} \\
&\qquad\times\int_{\mathbb{R}^n} e^{ik\cdot t}\Psi(t-k)\overline{\Phi(t-x)}e^{-i\xi\cdot t}\,dt\Biggr|^p dx\Biggr)^{q/p} d\xi\Biggr\}^{1/q}
\end{aligned}$$

$$\geqslant (2\pi)^{-n} 2^{-jn/p} \Biggl\{ \sum_{\substack{0<|\ell_i|\leqslant 2^j,\\ i=1,\dots,n}} \int_{\ell+[-1/8,1/8]^n} \biggl(\int_{\mathbb{R}^n} \biggl| \sum_{\substack{0<|k_i|\leqslant 2^j,\\ i=1,\dots,n}} |k|^{-n/p-\epsilon} e^{i|k|^2}$$

$$\times \int_{\mathbb{R}^n} e^{-ik\cdot t} \psi(t-k) \overline{\varphi(t-\xi)} e^{ix\cdot t}\, dt \biggr|^p dx \biggr)^{q/p} d\xi \Biggr\}^{1/q}$$

$$= (2\pi)^{-n} 2^{-jn/p}$$

$$\times \Biggl\{ \sum_{\substack{0<|\ell_i|\leqslant 2^j,\\ i=1,\dots,n}} \int_{\ell+[-1/8,1/8]^n} \biggl(\int_{\mathbb{R}^n} \biggl| |\ell|^{-n/p-\epsilon} \int_{\mathbb{R}^n} e^{i(x-\ell)\cdot t} \overline{\varphi(t-\xi)}\, dt \biggr|^p dx \biggr)^{q/p} d\xi \Biggr\}^{1/q}$$

$$= 2^{-jn/p} \Biggl\{ \sum_{\substack{0<|\ell_i|\leqslant 2^j,\\ i=1,\dots,n}} |\ell|^{-(n/p+\epsilon)q} \int_{\ell+[-1/8,1/8]^n} \bigl\| \Phi(-\cdot+\ell) \bigr\|_{L^p}^q \, d\xi \Biggr\}^{1/q}$$

$$= 4^{-n/q} \|\Phi\|_{L^p} 2^{-jn/p} \Biggl\{ \sum_{\substack{0<|\ell_i|\leqslant 2^j,\\ i=1,\dots,n}} |\ell|^{-(n/p+\epsilon)q} \Biggr\}^{1/q}$$

$$\geqslant C_n 2^{-jn/p} 2^{-j(n/p+\epsilon)} \Biggl\{ \sum_{\substack{0<|\ell_i|\leqslant 2^j,\\ i=1,\dots,n}} 1 \Biggr\}^{1/q} \geqslant C_n 2^{-jn(2/p-1/q)-j\epsilon}$$

for all $j \in \mathbb{Z}_+$. The proof is complete. □

Lemma 4.6. *Suppose that $1 \leqslant p, q \leqslant \infty$ and $s > 0$. Let f^j be defined by* (4.2). *Then there exists a constant $C > 0$ such that $\|(f^j)_{2^j}\|_{B_s^{p,q}} \leqslant C 2^{j(s-n/p)}$ for all $j \in \mathbb{Z}_+$.*

Proof. By Lemma 4.4, we have $\|(f^j)_{2^j}\|_{B_s^{p,q}} \leqslant C 2^{j(s-n/p)} \|f^j\|_{B_s^{p,q}}$ for all $j \in \mathbb{Z}_+$. Hence, it is enough to prove that $\sup_{j\in\mathbb{Z}_+} \|f^j\|_{B_s^{p,q}} < \infty$. Since

$$\widehat{f^j}(\xi) = 2^{jn(1-1/p)} \sum_{\substack{0<|k_i|\leqslant 2^j,\\ i=1,\dots,n}} |k|^{-n/p-\epsilon}\, e^{-ik\cdot(2^j\xi-k)} \psi\bigl(2^j\xi - k\bigr)$$

and $\operatorname{supp} \psi(2^j \cdot - k) \subset k/2^j + [-2^{-(j+1)}, 2^{-(j+1)}]^n$, we see that $\operatorname{supp} \widehat{f^j} \subset \{\xi\colon |\xi| \leqslant 2\sqrt{n}\}$. Let ℓ_0 be such that $2^{\ell_0 - 1} \geqslant 2\sqrt{n}$. Then,

$$\bigl\| f^j \bigr\|_{B_s^{p,q}} = \Biggl(\sum_{\ell=0}^{\ell_0-1} 2^{\ell s q} \bigl\| \Phi_\ell * f^j \bigr\|_{L^p}^q \Biggr)^{1/q}$$

$$\leqslant \Biggl(\sum_{\ell=0}^{\ell_0-1} 2^{\ell s q} \bigl(\|\Phi_\ell\|_{L^1} \bigl\| f^j \bigr\|_{L^p} \bigr)^q \Biggr)^{1/q} = C_n \bigl\| f^j \bigr\|_{L^p}.$$

Therefore, it is enough to show that $\sup_{j\in\mathbb{Z}_+}\|f^j\|_{L^p}<\infty$. By a change of variable, we have

$$\begin{aligned}\|f^j\|_{L^p} &= \Bigg(\int_{\mathbb{R}^n}\Bigg|\sum_{\substack{0<|k_i|\leqslant 2^j,\\ i=1,\dots,n}}|k|^{-n/p-\epsilon}e^{ik\cdot t}\Psi(t-k)\Bigg|^p dt\Bigg)^{1/p}\\ &\leqslant \Bigg\{\sum_{m\in\mathbb{Z}^n}\int_{m+[-1/2,1/2]^n}\Bigg(\sum_{k\neq 0}|k|^{-n/p-\epsilon}\big|\Psi(t-k)\big|\Bigg)^p dt\Bigg\}^{1/p}\\ &\leqslant C\Bigg\{\sum_{m\in\mathbb{Z}^n}\Bigg(\sum_{k\neq 0}|k|^{-n/p-\epsilon}\big(1+|m-k|\big)^{-n-1}\Bigg)^p\Bigg\}^{1/p}<\infty\end{aligned}$$

for all $j\in\mathbb{Z}_+$. The proof is complete. □

We are now ready to prove Theorem 1.2(1) with $(1/p,1/q)\in I_3^*$.

Proof of Theorem 1.2(1) with $(1/p,1/q)\in I_3^*$. Let $(1/p,1/q)\in I_3^*$. Then $\nu_1(p,q)=-1/p+1/q$. If $(1/p,1/q)\in I_3^*$ and $p=q$ then $(1/p,1/q)\in I_1^*$, and we have already proved this case in Theorem 1.2(1) with $(1/p,1/q)\in I_1^*$. Hence, we may assume $1/q>1/p$. Note that $q\neq\infty$. Suppose that $B_s^{p,q}(\mathbb{R}^n)\hookrightarrow M^{p,q}(\mathbb{R}^n)$, where $s<-n(1/p-1/q)$. Then, since $-n(1/p-1/q)>0$, we can take $s_0>0$ such that $s\leqslant s_0<-n(1/p-1/q)$. Set $s_0=-n(1/p-1/q)-\epsilon$, where $\epsilon>0$. For this ϵ, we define f^j by

$$f^j(t)=2^{-jn/p}\sum_{\substack{0<|k_i|\leqslant 2^j,\\ i=1,\dots,n}}|k|^{-n/p-\epsilon/2}e^{ik\cdot t/2^j}\Psi\big(t/2^j-k\big),$$

where $j\in\mathbb{Z}_+$, $\Psi=\mathcal{F}^{-1}\psi$ and ψ is as in Lemma 4.5. Then, since $B_{s_0}^{p,q}(\mathbb{R}^n)\hookrightarrow B_s^{p,q}(\mathbb{R}^n)\hookrightarrow M^{p,q}(\mathbb{R}^n)$, by Lemmas 4.5 and 4.6, we get

$$\begin{aligned}C_1 2^{-jn(2/p-1/q)-j\epsilon/2} &\leqslant \big\|V_\Phi\big[\big(f^j\big)_{2^j}\big]\big\|_{L^{p,q}} \leqslant C_2\big\|\big(f^j\big)_{2^j}\big\|_{M^{p,q}}\\ &\leqslant C_3\big\|\big(f^j\big)_{2^j}\big\|_{B_{s_0}^{p,q}} \leqslant C_4 2^{j(s_0-n/p)} = C_4 2^{-jn(2/p-1/q)-j\epsilon}\end{aligned}$$

for all $j\in\mathbb{Z}_+$, where $\Phi=\mathcal{F}^{-1}\varphi$ and φ is as in Lemma 4.5. However, this is a contradiction. Therefore, s must satisfy $s\geqslant -n(1/p-1/q)$. The proof is complete. □

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References

[1] A. Boulkhemair, Remarks on a Wiener type pseudodifferential algebra and Fourier integral operators, Math. Res. Lett. 4 (1997) 53–67.
[2] M. Dimassi, J. Sjöstrand, Spectral Asymptotics in the Semi-Classical Limit, London Math. Soc. Lecture Note Ser., vol. 268, Cambridge Univ. Press, Cambridge, 1999.
[3] H.G. Feichtinger, Banach spaces of distributions of Wiener's type and interpolation, in: P. Butzer, B.Sz. Nagy, E. Görlich (Eds.), Proc. Conf. Oberwolfach, Functional Analysis and Approximation, August 1980, in: Int. Ser. Num. Math., vol. 60, Birkhäuser, Basel, 1981, pp. 153–165.
[4] H.G. Feichtinger, Modulation spaces on locally compact abelian groups, in: M. Krishna, R. Radha, S. Thangavelu (Eds.), Wavelets and Their Applications, Chennai, India, Allied Publ., New Delhi, 2003, pp. 99–140, updated version of a technical report, University of Vienna, 1983.
[5] H.G. Feichtinger, Modulation spaces: Looking back and ahead, Sampl. Theory Signal Image Process. 5 (2006) 109–140.
[6] H.G. Feichtinger, K. Gröchenig, Banach spaces related to integrable group representations and their atomic decompositions, I, J. Funct. Anal. 86 (1989) 307–340.
[7] G.B. Folland, Harmonic Analysis in Phase Space, Princeton Univ. Press, Princeton, NJ, 1989.
[8] M. Frazier, B. Jawerth, Decomposition of Besov spaces, Indiana Univ. Math. J. 34 (1985) 777–799.
[9] P. Gröbner, Banachräume Glatter Funktionen und Zerlegungsmethoden, Thesis, University of Vienna, Vienna, 1983.
[10] K. Gröchenig, Foundations of Time–Frequency Analysis, Birkhäuser Boston, Boston, MA, 2001.
[11] K. Gröchenig, Time–frequency analysis of Sjöstrand's class, Rev. Mat. Iberoamericana 22 (2006) 703–724.
[12] K. Gröchenig, C. Heil, Modulation spaces and pseudodifferential operators, Integral Equations Operator Theory 34 (1999) 439–457.
[13] K. Gröchenig, C. Heil, Modulation spaces as symbol classes for pseudodifferential operators, in: M. Krishna, R. Radha, S. Thangavelu (Eds.), Wavelets and Their Applications, Chennai, India, Allied Publ., New Delhi, 2003, pp. 151–170.
[14] K. Okoudjou, Embedding of some classical Banach spaces into modulation spaces, Proc. Amer. Math. Soc. 132 (2004) 1639–1647.
[15] J. Sjöstrand, An algebra of pseudodifferential operators, Math. Res. Lett. 1 (1994) 185–192.
[16] M. Sugimoto, Pseudo-differential operators on Besov spaces, Tsukuba J. Math. 12 (1988) 43–63.
[17] N. Teofanov, Ultramodulation Spaces and Pseudodifferential Operators, Endowment Andrejević, Beograd, 2003.
[18] J. Toft, Continuity properties for modulation spaces, with applications to pseudo-differential calculus, I, J. Funct. Anal. 207 (2004) 399–429.
[19] J. Toft, Continuity properties for modulation spaces, with applications to pseudo-differential calculus, II, Ann. Global Anal. Geom. 26 (2004) 73–106.
[20] J. Toft, Convolution and embeddings for weighted modulation spaces, in: R. Ashino, P. Boggiatto, M.W. Wong (Eds.), Advances in Pseudo-Differential Operators, in: Oper. Theory Adv. Appl., vol. 155, Birkhäuser, Basel, 2004, pp. 165–186.
[21] H. Triebel, Modulation spaces on the Euclidean n-spaces, Z. Anal. Anwendungen 2 (1983) 443–457.
[22] B. Wang, C. Huang, Frequency-uniform decomposition method for the generalized BO, KdV and NLS equations, preprint.