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The effect on the algebraic connectivity of a tree by grafting or collapsing of edges

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Abstract

Let $G = (V, E)$ be a tree on $n \geqslant 2$ vertices and let $v \in V$. Let $L(G)$ be the Laplacian matrix of G and $\mu(G)$ be its algebraic connectivity. Let $G_{k,l}$, be the graph obtained from G by attaching two new paths $P : vv_1v_2 \ldots v_k$ and $Q : vu_1u_2 \ldots u_l$ of length k and l, respectively, at v. We prove that if $l \geqslant k \geqslant 1$ then $\mu(G_{k-1,l+1}) \leqslant \mu(G_{k,l})$. Let (v_1, v_2) be an edge of G. Let $\widetilde{G}$ be the tree obtained from G by deleting the edge (v_1, v_2) and identifying the vertices v_1 and v_2. Then we prove that $\mu(G) \leqslant \mu(\widetilde{G})$. As a corollary to the above results, we obtain the celebrated theorem on algebraic connectivity which states that among all trees on n vertices, the path has the smallest and the star has the largest algebraic connectivity.

Keywords: Tree; Laplacian matrix; Algebraic connectivity

1. Introduction and preliminaries

Let G be a simple graph with vertex set $V = \{1, 2, \ldots, n\}$ and edge set E. The *adjacency matrix* A of a graph G is defined as $A = [a_{ij}]$, where a_{ij} is equal to 1 if the unordered pair (i, j) is an edge of G and 0 otherwise. Let D be the diagonal matrix of vertex degrees of G. The *Laplacian matrix* of a graph G is defined as $L = D - A$. It is well known (e.g., see [2]) that L is a symmetric, positive semi-definite, M-matrix. The smallest eigenvalue of L is 0 with the vector of all ones as an eigenvector and has multiplicity 1 if and only if G is connected. In other words, the second smallest eigenvalue of L is positive if and only if G is connected. Viewing this eigenvalue as an

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algebraic measure of connectivity, Fiedler termed this eigenvalue as the *algebraic connectivity* (denoted by $\mu(G)$) of G. An eigenvector of L corresponding to the algebraic connectivity $\mu(G)$ is called a *Fiedler vector* of the graph G. We refer the reader to [1–7,9,10] for some interesting facts about the algebraic connectivity and the Fiedler vector. To get a general overview on results related with Laplacians, we refer the reader to [11,12].

Let $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_n = 0$ be the eigenvalues of L with $\mu(G) = \lambda_{n-1}$ and let Y be a Fiedler vector of G. By $Y(v)$, we mean the co-ordinate of Y corresponding to the vertex v. A subvector of Y is called a *Fiedler subvector*. A vertex v of G is called a *characteristic vertex* of G if $Y(v) = 0$ and there exists a vertex w adjacent to v such that $Y(w) \neq 0$. An edge $e = (u, w)$ is called a *characteristic edge* of G if $Y(u)Y(w) < 0$. The *characteristic set* of G is the collection of all characteristic vertices and characteristic edges of G and is denoted by $\mathscr{C}(G, Y)$. For a tree T, it is well known that the cardinality of $\mathscr{C}(T, Y)$ equals 1 (for example, see [1]).

Let G be a connected graph. A vertex v of G is called a *cut-vertex* if $G - v$ (the graph obtained from G by removing v and all its incident edges) is disconnected. Let $C_1, C_2, \ldots, C_k$ be the connected components of $G - v$. We shall refer to these components as the *connected components* of G at v. Note that $k \geqslant 2$ if and only if v is a cut-vertex. Let $\widehat{L}(C_1), \widehat{L}(C_2), \ldots, \widehat{L}(C_k)$ be the principal submatrices of $L(G)$ corresponding to the components $C_1, C_2, \ldots, C_k$, respectively. Since $L(G)$ has nullity 1, it follows that $\widehat{L}(C_i)$ is invertible and since $L(G)$ is an M-matrix, $\widehat{L}(C_i)^{-1}$ is a positive matrix. The matrices $\widehat{L}(C_i)^{-1}$ are called *bottleneck matrices*. By the Perron–Frobenius theorem (see [13]), a bottleneck matrix has a simple dominant eigenvalue called the *Perron value* and the corresponding eigenvector with all entry positive is called the *Perron vector*. A component C_i is called a *Perron component* at v if the Perron value for C_i, denoted by $\rho(\widehat{L}(C_i)^{-1})$, is maximal among the Perron values of $C_1, C_2, \ldots, C_k$. We now state a very useful proposition that gives the description of the bottleneck matrices (see [9]).

Proposition 1.1 [9]. *Let T be a tree on vertices $1, 2, \ldots, n$. Let L_1 be the submatrix of $L(T)$ corresponding to $T - n$. Then $L_1^{-1} = [m_{ij}]$, where m_{ij} is the number of edges in common between the path P_{in} joining i and n and the path P_{jn} joining j and n.*

A connection between Perron components, bottleneck matrices and algebraic connectivity is described in the next two results (see [9]). In particular, they give a relation between Fiedler subvector and Perron vector of bottleneck matrices. In this paper, $\mathbf{e}$ represents the column vector of all ones, $J = \mathbf{e}\mathbf{e}^{\mathrm{t}}$ and $\mathbf{e}_k$ represents the unit column vector having 1 in the kth position. The order of these matrices will be clear from the context.

Theorem 1.2. *Let T be a tree on n vertices. Then T is a tree with a characteristic edge (i, j) if and only if the component C_i at vertex j containing vertex i is the unique Perron component at j, while the component C_j at vertex i containing vertex j is the unique Perron component at i. Moreover, in this case there exists a $\gamma \in (0, 1)$ such that*

$$\frac{1}{\mu(T)} = \rho(\widehat{L}(C_i)^{-1} - \gamma J) = \rho(\widehat{L}(C_j)^{-1} - (1-\gamma)J).$$

Furthermore, any eigenvector of L corresponding to $\mu(T)$ can be permuted so that it has the block form $\left[\frac{Y_1}{-Y_2}\right]$, where Y_1 is a Perron vector for $\rho(\widehat{L}(C_i)^{-1} - \gamma J)$ and Y_2 is a Perron vector for $\rho(\widehat{L}(C_j)^{-1} - (1-\gamma)J)$.

Theorem 1.3. *Let T be a tree on n vertices. Then T is a tree with a characteristic vertex v if and only if there are two or more Perron components of T at v. Moreover, in this case*

$$\mu(T) = \frac{1}{\rho(L_v^{-1})},$$

whenever L_v is a Perron component at v. Furthermore, given two Perron components C_1, C_2 of T at v, an eigenvector Y corresponding to $\mu(T)$ can be chosen so that Y can be permuted and partitioned into the block form $Y^{\mathrm{T}} = [Y_1^{\mathrm{T}} | -Y_2^{\mathrm{T}} | \mathbf{0}^{\mathrm{T}}]$, where Y_1 and Y_2 are Perron vectors for the bottleneck matrices $\widehat{L}(C_1)^{-1}$ and $\widehat{L}(C_2)^{-1}$, respectively, and $\mathbf{0}$ is the zero column vector of an appropriate order.

Identification of the Perron components at a vertex helps to determine the location of the characteristic set. The next Proposition appears in [9].

Proposition 1.4. *Let T be a tree. Then for any vertex v that is neither a characteristic vertex nor an end vertex of the characteristic edge, the unique Perron component at v contains the characteristic set of T.*

For non-negative square matrices A and B (not necessarily of the same order), the notation $A \ll B$ is used to mean that there exists a permutation matrix P such that PAP^{T} is entry wise dominated by a principal submatrix of B, with strict inequality in at least one position in case A and B have the same order. A useful fact from the Perron–Frobenius theory states that if B is irreducible and $A \ll B$, then $\rho(A) < \rho(B)$. The next Theorem is very useful (see [8]).

Theorem 1.5. *Let T be a tree and $\mu(T)$ be its algebraic connectivity. Suppose $C_1, C_2, \ldots, C_k$ are the connected components of $T - v$. Let $C_{i_1}, C_{i_2}, \ldots, C_{i_j}$ be any collection of connected components at v such that the vertex set of $C = \bigcup_{l=1}^{j} C_{i_l}$ does not contain the vertex set of every Perron component at v. Form a new graph $\widetilde{T}$ by replacing C with a single connected component $\widetilde{C}$ at v and let $\widetilde{M}$ be the bottleneck matrix of $\widetilde{C}$. If $\widehat{L}(C)^{-1} \equiv M \ll \widetilde{M}$, then*

$$\mu(\widetilde{T}) \leqslant \mu(T).$$

We now give the outline of the paper. In Sections 2 and 3, we, respectively, recall the definitions of grafting and collapsing of an edge and study their effect on the algebraic connectivity of a tree. As a corollary, we obtain the well known result which states that for a fixed positive integer n, the path has the smallest and the star has the largest algebraic connectivity among all trees on n vertices.

2. Grafting an edge

Let G be a connected graph on n vertices with $n \geqslant 2$. Let v be a vertex of G. For $l \geqslant k \geqslant 1$, let $G_{k,l}$ be the graph obtained from G by attaching two new paths $P : vv_1v_2 \ldots v_k$ and $Q : vu_1u_2 \ldots u_l$ of length k and l, respectively, at v, where $u_1, u_2, \ldots, u_l$ and $v_1, v_2, \ldots, v_k$ are distinct new vertices. For the sake of convention, let $G_{0,l}, l \geqslant 1$ be the graph obtained from G by attaching a new path $Q : vu_1u_2 \ldots u_l$ of length l at v, where $u_1, u_2, \ldots, u_l$ are distinct new vertices. Also, let $\widetilde{G}_{k,l}$ be the graph obtained from $G_{k,l}$ by removing the edge (v_{k-1}, v_k) and

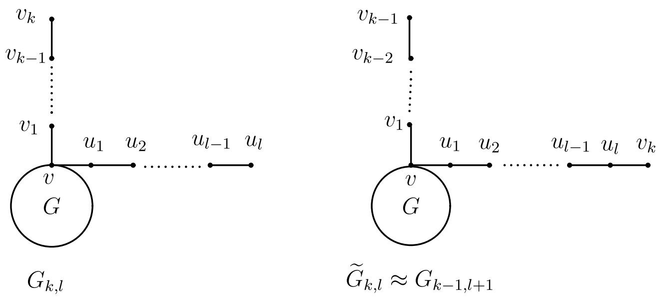

Fig. 1. Grafting an edge.

adding the edge (u_l, v_k) (see Fig. 1). Observe that the graph $\widetilde{G}_{k,l}$ is isomorphic to the graph $G_{k-1,l+1}$. We say that the graph $\widetilde{G}_{k,l}$ is obtained from $G_{k,l}$ by *grafting* an edge.

The next lemma is similar to Theorem 1.5. We will use it in the proof of the next proposition. The proposition is used in the proof of our main result.

Lemma 2.1. *Let (u, v) be the characteristic edge of a tree T such that C is the Perron component of T at v containing u and D is the Perron component of T at u containing v. Also suppose that there exists an $\alpha_0 \in (0, 1)$ such that*

$$\frac{1}{\mu(T)} = \rho(\widehat{L}(C)^{-1} - \alpha_0 J) = \rho(\widehat{L}(D)^{-1} - (1 - \alpha_0)J). \tag{2.1}$$

Form a new tree $\widetilde{T}$ from T by replacing C with a connected component $\widetilde{C}$ at v such that

$$\rho(\widehat{L}(C)^{-1} - \alpha_0 J) < \rho(\widehat{L}(\widetilde{C})^{-1} - \alpha_0 J). \tag{2.2}$$

Then $\mu(\widetilde{T}) < \mu(T)$.

Proof. Let $C = C_1, C_2, \ldots, C_p$ be the connected components of $T - v$. As C is the Perron component of T at v, for $2 \leqslant i \leqslant p$,

$$\rho(\widehat{L}(C_i)^{-1}) < \frac{1}{\mu(T)} = \rho(\widehat{L}(C)^{-1} - \alpha_0 J) < \rho(\widehat{L}(C)^{-1}).$$

Therefore, using (2.2), in the tree $\widetilde{T} - v$, the component $\widetilde{C}$ is the only Perron component as

$$\rho(\widehat{L}(C_i)^{-1}) < \rho(\widehat{L}(C)^{-1} - \alpha_0 J) < \rho(\widehat{L}(\widetilde{C})^{-1} - \alpha_0 J) < \rho(\widehat{L}(\widetilde{C})^{-1}).$$

To complete the proof, we need to consider two cases, depending on the position of the characteristic set of $\widetilde{T}$.

Case 1: Suppose (u, v) is still the characteristic edge of $\widetilde{T}$. In this case, by Theorem 1.2, there exists $\alpha_1 \in (0, 1)$ such that

$$\frac{1}{\mu(\widetilde{T})} = \rho(\widehat{L}(\widetilde{C})^{-1} - \alpha_1 J) = \rho(\widehat{L}(D)^{-1} - (1 - \alpha_1)J). \tag{2.3}$$

We claim that $\alpha_1 > \alpha_0$. On the contrary, assume that $\alpha_1 \leqslant \alpha_0$. Then

$$\begin{aligned}\rho(\widehat{L}(C)^{-1} - \alpha_0 J) &= \rho(\widehat{L}(D)^{-1} - (1 - \alpha_0)J) \quad \text{(from (2.1))}\\ &\geqslant \rho(\widehat{L}(D)^{-1} - (1 - \alpha_1)J) \quad \text{(from } \alpha_1 \leqslant \alpha_0\text{)}\\ &= \rho(\widehat{L}(\widetilde{C})^{-1} - \alpha_1 J) \quad \text{(from (2.3))}\end{aligned}$$

$$\geqslant \rho(\widehat{L}(\widetilde{C})^{-1} - \alpha_0 J) \quad (\text{from } \alpha_1 \leqslant \alpha_0)$$
$$> \rho(\widehat{L}(C)^{-1} - \alpha_0 J) \quad (\text{from (2.2)}).$$

Thus, we have obtained a contradiction. Hence, our claim holds. Therefore, using (2.1) and (2.3), we get $\mu(\widetilde{T}) < \mu(T)$.
Case 2: Suppose (u, v) is not the characteristic edge of $\widetilde{T}$. Then, for the component D of $\widetilde{T} - u$ containing v, we have

$$\frac{1}{\mu(\widetilde{T})} \geqslant \rho(\widehat{L}(D)^{-1}) > \rho(\widehat{L}(D)^{-1} - (1-\alpha_0)J) = \frac{1}{\mu(T)}.$$

That is, in this case as well, we get $\mu(\widetilde{T}) < \mu(T)$. □

Remark 2.2. Lemma 2.1 assumes the existence of $\alpha_0 \in (0, 1)$ satisfying

$$\frac{1}{\mu(T)} = \rho(\widehat{L}(C)^{-1} - \alpha_0 J) = \rho(\widehat{L}(D)^{-1} - (1-\alpha_0)J).$$

Note that the existence of such an α_0 is guaranteed by Theorem 1.2.

We are now ready to prove the proposition which will be used in the proof of the main result.

Proposition 2.3. *Let G be a tree on $n \geqslant 2$ vertices and let v be a vertex of G. Let $G_{k,l}$ be the graph defined earlier. Let Y be a Fiedler vector of $G_{k,l}$ such that along with $Y(v)$, $Y(u_i)$ for $i = 1, \ldots, l$ and $Y(v_j)$ for $j = 1, \ldots, k$ are positive. Then $\mu(G_{k-1,l+1}) < \mu(G_{k,l})$ whenever $1 \leqslant k \leqslant l$.*

Proof. By the given conditions, the components of $G_{k,l} - v$, containing u_1 and v_1 are not Perron components. We need to consider two cases depending on whether $G_{k,l}$ has a characteristic edge or a characteristic vertex.
Case 1: The characteristic set of $G_{k,l}$ contains an edge.

Let (w_1, w_2) be the characteristic edge of $G_{k,l}$ and let the vertex w_2 be farthest from v. Let $C_1, C_2, \ldots, C_p$ be the components of $G_{k,l} - w_2$ with C_1 containing the vertex v. Let $V(C_1)$ denote the set of vertices of C_1 and let B be the bottleneck matrix of C_1. Then by Theorem 1.2, C_1 is the only Perron component and their exists a $\gamma \in (0, 1)$ such that

$$\frac{1}{\mu(G_{k,l})} = \rho(B - \gamma J).$$

Let $\overline{B}$ be the bottleneck matrix for the component obtained from C_1 by deleting the edge (v_{k-1}, v_k) and adding the edge (u_l, v_k). Then, the new graph corresponds to the graph $\widetilde{G}_{k,l}$. As $\widetilde{G}_{k,l}$ is isomorphic to $G_{k-1,l+1}$, $\mu(\widetilde{G}_{k,l}) = \mu(G_{k-1,l+1})$.

Claim. $\rho(\overline{B} - \gamma J) > \rho(B - \gamma J) = \frac{1}{\mu(G_{k,l})}$.

By Proposition 1.1,

$$B = \left[\begin{array}{c|c} 1 + M_{v_{k-1},v_{k-1}} & \mathbf{e}_{v_{k-1}}^{\mathrm{T}} M \\ \hline M\mathbf{e}_{v_{k-1}} & M \end{array}\right] \quad \text{and} \quad \overline{B} = \left[\begin{array}{c|c} 1 + M_{u_l,u_l} & \mathbf{e}_{u_l}^{\mathrm{T}} M \\ \hline M\mathbf{e}_{u_l} & M \end{array}\right],$$

where the first row and column of B and $\overline{B}$ corresponds to the vertex v_k and $M = [M_{ij}]$ is the matrix that corresponds to the vertices $V(C_1) - v_k$. Let Z be the unit positive eigenvector of

$B - \gamma J$ associated with the eigenvalue $\rho(B - \gamma J) \equiv r$, say. That is, $Z^{\mathrm{T}}(B - \gamma J)Z = r$. By Theorem 1.2, Z is a subvector of the Fiedler vector Y. Thus,

$$Z^T(\overline{B} - \gamma J)Z - Z^T(B - \gamma J)Z = Z^T(\overline{B} - B)Z$$
$$= Y(v_k)^2(M_{u_l,u_l} - M_{v_{k-1},v_{k-1}}) + 2Y(v_k)(\mathbf{e}_{u_l}^{\mathrm{T}} M\widehat{Y} - \mathbf{e}_{v_{k-1}}^{\mathrm{T}} M\widehat{Y}) \quad (2.4)$$

where $Z = \left[\frac{Y(v_k)}{\widehat{Y}}\right]$ is partitioned conformally with B. The lower block of the eigenvalue–eigenvector equation $(B - \gamma J)Z = rZ$, gives

$$(M\mathbf{e}_{v_{k-1}} - \gamma\mathbf{e})Y(v_k) + (M - \gamma J)\widehat{Y} = r\widehat{Y}.$$

Hence

$$\mathbf{e}_{u_l}^{\mathrm{T}} M\widehat{Y} = rY(u_l) - Y(v_k)M_{u_l,v_{k-1}} + \gamma Y(v_k) + \gamma\mathbf{e}^{\mathrm{T}}\widehat{Y} \quad (2.5)$$

and

$$\mathbf{e}_{v_{k-1}}^{T} M\widehat{Y} = rY(v_{k-1}) - Y(v_k)M_{v_{k-1},v_{k-1}} + \gamma Y(v_k) + \gamma\mathbf{e}^{\mathrm{T}}\widehat{Y}. \quad (2.6)$$

Now substituting the value of $\mathbf{e}_{u_l}^{\mathrm{T}} M\widehat{Y}$ from (2.5) and the value of $\mathbf{e}_{v_{k-1}}^{\mathrm{T}} M\widehat{Y}$ from (2.6) in (2.4), we have

$$Z^T(\overline{B} - \gamma J)Z - Z^T(B - \gamma J)Z$$
$$= Y(v_k)^2(M_{u_l,u_l} - M_{v_{k-1},v_{k-1}}) + 2rY(v_k)(Y(u_l) - Y(v_{k-1}))$$
$$+ 2Y(v_k)^2(M_{v_{k-1},v_{k-1}} - M_{u_l,v_{k-1}}). \quad (2.7)$$

As $l \geqslant k$, by Proposition 1.1, $M_{u_l,u_l} > M_{v_{k-1},v_{k-1}}$ and $M_{v_{k-1},v_{k-1}} \geqslant M_{u_l,v_{k-1}}$. So, to complete the proof of our claim, we need to show that $Y(u_l) \geqslant Y(v_{k-1})$. We do this by looking at the behaviour of the Fiedler vector Y of the graph $G_{k,l}$. Observe that the linear equation $L(G_{k,l})Y = \mu(G_{k,l})Y$ gives

$$Y(u_{l-1}) = c_1Y(u_l), \quad Y(u_{l-2}) = c_2Y(u_{l-1}), \ldots, Y(v) = c_lY(u_1),$$

and

$$Y(v_{k-1}) = c_1Y(v_k), \quad Y(v_{k-2}) = c_2Y(v_{k-1}), \ldots, Y(v) = c_kY(v_1),$$

where $c_1, c_2, \ldots, c_l$ depend only on the entries of $L(G_{k,l})$ and $\mu(G_{k,l})$. It follows from a theorem of Fiedler (see [4]) that $0 \leqslant c_j \leqslant 1$ for $1 \leqslant j \leqslant l$. As $l \geqslant k$, $Y(u_l) \geqslant Y(v_{k-1})$. Hence using (2.7), we have been able to show that

$$Z^T(\overline{B} - \gamma J)Z - Z^T(B - \gamma J)Z > 0.$$

That is, $Z^T(\overline{B} - \gamma J)Z > \rho(B - \gamma J)$. Therefore, the claim holds true as

$$\rho(\overline{B} - \gamma J) \geqslant Z^T(\overline{B} - \gamma J)Z > \rho(B - \gamma J).$$

Thus, by Lemma 2.1, $\mu(G_{k-1,l+1}) < \mu(G_{k,l})$.
Case 2: The characteristic set of $G_{k,l}$ contains a vertex.

Let w be the characteristic vertex of $G_{k,l}$. Let $C_1, C_2, \ldots, C_p$ be the connected components of $G_{k,l} - w$ and let C_1 be the component containing the vertex v. Form D_1 from C_1 by deleting the edge (v_{k-1}, v_k) and adding the edge (u_l, v_k).

By the given condition $(Y(v) > 0)$ and Theorem 1.3, C_1 is a Perron component and thus $\mu(G_{k,l}) = \frac{1}{\rho(\widehat{L}(C_1)^{-1})}$. An argument similar to the one used in Case 1, helps us to prove that

$\rho(\widehat{L}(D_1)^{-1}) > \rho(\widehat{L}(C_1)^{-1})$. Therefore, $\mu(G_{k-1,l+1}) < \mu(G_{k,l})$. Hence the required result follows. $\square$

We now state and prove our main result. This result compares the algebraic connectivity of the trees $G_{k,l}$ and $G_{k-1,l+1}$ defined earlier.

Theorem 2.4. *Let G be a tree on $n \geqslant 2$ vertices and let v be a vertex of G. Let $G_{k,l}$ be the graph defined earlier. If $l \geqslant k \geqslant 1$, then $\mu(G_{k-1,l+1}) \leqslant \mu(G_{k,l})$.*

Proof. Let $P : vv_1v_2 \ldots v_k$ and $Q =: vu_1u_2 \ldots u_l$ be two paths of length k and $l, l \geqslant k \geqslant 1$, respectively, attached at v. Clearly the new graph, denoted as $G_{k,l}$ is a tree. The characteristic set of $G_{k,l}$ is either a vertex or an edge.

Claim. *In $G_{k,l}$, neither the characteristic set lies in the v_1–v_k path nor (v, v_1) is the characteristic edge.*

In $G_{k,l}$, assume that (v, v_1) is the characteristic edge or the characteristic set lies in the v_1–v_k path. Let $C_1, C_2, \ldots, C_p$ be the components of $G_{k,l} - v$, with C_1 as the u_1–u_l path and C_2 as the v_1–v_k path. By our assumption and Proposition 1.4, C_2 is the only Perron branch. So $\rho(\widehat{L}(C_2)^{-1}) > \rho(\widehat{L}(C_1)^{-1})$. As $l \geqslant k \geqslant 1$, by Proposition 1.1 and the Perron–Frobenius theorem $\rho(\widehat{L}(C_2)^{-1}) \leqslant \rho(\widehat{L}(C_1)^{-1})$. Thus we arrive at a contradiction to our assumption. Hence the claim holds.

To complete the proof, we need to consider two cases depending on whether C_1 is a Perron component of $G_{k,l} - v$ or not.

Case 1: C_1 (containing u_1) is a Perron component in $G_{k,l} - v$.

Let $C \equiv \bigcup_{j=2}^{p} C_j$. Let $\widehat{L}(C)^{-1} \equiv M$. Let D_1 be the component of $G_{k-1,l+1} - u_1$ containing v. Form a new graph $\widetilde{G}$, after replacing C by D_1 in such a way that the vertex v in $G_{k,l}$ is joined to the vertex v of D_1. Observe that the graph $\widetilde{G}$ is isomorphic to $G_{k-1,l+1}$. Suppose the bottleneck matrix of D_1 in $\widetilde{G} - v$ is $\widetilde{M}$. Then by Proposition 1.1, $\widetilde{M} \gg M$ and therefore by Theorem 1.5, $\mu(G_{k-1,l+1}) \leqslant \mu(G_{k,l})$.

Case 2: C_1 is not a Perron component in $G_{k,l} - v$.

In this case, C_2 (containing v_1) is also not a Perron component as $l \geqslant k$. Therefore, the Perron components of $G_{k,l} - v$ belong to the graph G. Now look at a Fiedler vector Y. Then without loss of generality, either $Y(v) = 0$ or $Y(v) > 0$.

If $Y(v) > 0$ then by Proposition 2.3, $\mu(G_{k-1,l+1}) < \mu(G_{k,l})$. So, let us assume that $Y(v) = 0$. As C_1 and C_2 are not the Perron components

$$Y(v_1) = \cdots = Y(v_k) = 0 = Y(u_1) = \cdots = Y(u_l).$$

So, $\mu(G_{k,l} - v_k) = \mu(G_{k,l})$. As $G_{k-1,l+1}$ is obtained by adding a new vertex to the vertex u_l of $G_{k,l} - v_k$ it follows that

$$\mu(G_{k,l}) = \mu(G_{k,l} - v_k) \geqslant \mu(G_{k-1,l+1}).$$

Hence the proof of the theorem is complete. $\square$

To complete the proof of our main theorem we considered two cases. If we carefully break the cases further into subcases depending on the position of the characteristic set, we get the following observations.

Remark 2.5. Consider the graph $G_{k,l}$, $l \geqslant k \geqslant 1$. Then in Theorem 2.4, $\mu(G_{k,l}) = \mu(G_{k-1,l+1})$ if $G_{k,l}$ has a characteristic set consisting of a vertex w with $w \in V(G)$ and if one of the following conditions hold:

1. Suppose $w = v$. In this case, let $C_1, C_2, \ldots, C_p$ be the components of $G_{k,l} - v$ with C_1 as the component containing the vertex u_1 and C_2 as the component containing the vertex v_1. Then the required condition is:
   If one of the C_i, $3 \leqslant i \leqslant t$ is a Perron component for both $G_{k,l} - v$ and $G_{k-1,l+1} - v$.
2. Suppose $w \neq v$. In this case, let C_1 be the component of $G_{k,l} - w$ containing the vertex u_1. Suppose D_1 is the graph obtained from C_1 by deleting the edge (v_{k-1}, v_k) and adding the edge (u_l, v_k). Then the condition is:
   If C_1 is not a Perron component and $\mu(G_{k,l}) \leqslant \frac{1}{\rho(\widehat{L}(D_1)^{-1})}$.

As an immediate corollary to Theorem 2.4, we have the following important result. The proof is omitted as it is an easy consequence of the theorem.

Corollary 2.6. *Fix a positive integer n. Then among all trees on n vertices the path has the smallest algebraic connectivity.*

3. Collapsing an edge

Let $G = (V, E)$ be a graph with an edge $e = (v_1, v_2)$ not lying on a cycle in G. Let $\widetilde{G} = (\widetilde{V}, \widetilde{E})$ be the graph obtained from G by deleting the edge e and identifying v_1 and v_2. We say $\widetilde{G}$ is obtained from G by *collapsing* an edge (see Fig. 2).

Theorem 3.1. *Let (u, v) be an edge of a tree T. Let $\widetilde{T}$ be the tree obtained from T by collapsing the edge (u, v). Then $\mu(\widetilde{T}) \geqslant \mu(T)$.*

Proof. Let $C_1, C_2, \ldots, C_k$ be the components of $T - u$ with C_1 containing the vertex v and let $D_1, D_2, \ldots, D_\ell$ be the components of $T - v$ with D_1 as the component containing the vertex u. In $\widetilde{T}$, let w represent the collapsed vertices u and v.

The proof of the theorem is based on the position of the characteristic set of T. We consider two cases, depending on whether (u, v) is the characteristic edge of T or not.

Case 1: (u, v) is the characteristic edge of T.

By Theorem 1.2, C_1 is the only Perron component of $T - u$ and D_1 is the only Perron component of $T - v$. Thus,

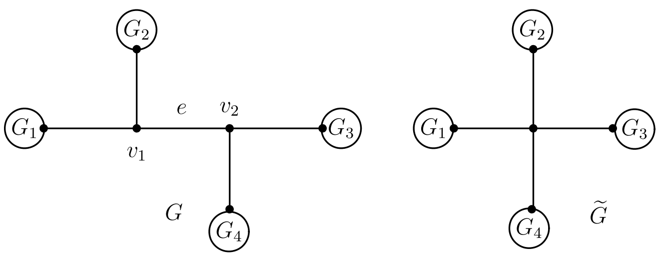

Fig. 2. Collapsing the edge e.

$$\frac{1}{\rho(\widehat{L}(C_1)^{-1})} < \mu(T), \quad \mu(T) < \frac{1}{\rho(\widehat{L}(C_i)^{-1})} \quad \text{for } i = 2, \ldots, k \tag{3.1}$$

and

$$\frac{1}{\rho(\widehat{L}(D_1)^{-1})} < \mu(T), \quad \mu(T) < \frac{1}{\rho(\widehat{L}(D_j)^{-1})} \quad \text{for } j = 2, \ldots, \ell. \tag{3.2}$$

Clearly, the tree $\widetilde{T} - w$ has $k + \ell - 2$ components. Let the components of $\widetilde{T} - w$ be $A_1, A_2, \ldots, A_{k+\ell-2}$. Note that the components $A_1, A_2, \ldots, A_{k+\ell-2}$ of $\widetilde{T} - w$ are a permutation of the components $C_2, C_3, \ldots, C_k$ and $D_2, \ldots, D_\ell$. In $\widetilde{T} - w$, at least one of the A_i's, for $i = 1, 2, \ldots, k + \ell - 2$ is a Perron component, say A_1. Then using (3.1) and (3.2), we have

$$\mu(T) < \min_{i=2,\ldots,k j=2,\ldots,\ell} \left\{ \frac{1}{\rho(\widehat{L}(C_i)^{-1})}, \frac{1}{\rho(\widehat{L}(D_j)^{-1})} \right\} = \frac{1}{\rho(\widehat{L}(A_1)^{-1})} \leqslant \mu(\widetilde{T}).$$

Therefore $\mu(T) < \mu(\widetilde{T})$.

Case 2: (u, v) is not the characteristic edge of T.

Let the characteristic set $\mathscr{C}(T, Y)$ be nearer to u. Construct $\widetilde{T}$ from T by removing C_1 and adding $D_2, D_3, \ldots, D_\ell$ at u. Let M be the bottleneck matrix of C_1 and let $\widetilde{M}$ be the bottleneck matrix of $D \equiv \bigcup_{i=2}^{\ell} D_i$. As we have removed the edge (u, v), by Proposition 1.1, $M \gg \widetilde{M}$. Clearly $\widetilde{T}$ is the tree obtained from T by collapsing the edge (u, v). As $M \gg \widetilde{M}$, by Theorem 1.5,

$$\mu(T) \leqslant \mu(\widetilde{T}).$$

Hence the proof of the theorem is complete. □

Let T be a tree with characteristic vertex v. By Theorem 1.3, $T - v$ has at least two Perron components. Let C_1 and C_2 be any two Perron components of $T - v$. Let $\widehat{T}$ be the tree obtained from T by adding a pendant vertex w to the vertex v. Let D_1 be the component of $\widehat{T} - v$ containing the single vertex w. Then, by Proposition 1.1, $\rho(\widehat{L}(C_1)^{-1}) = \rho(\widehat{L}(C_2)^{-1}) \geqslant 1 = \rho(\widehat{L}(D_1)^{-1})$. Therefore, $\widehat{T} - v$ still has C_1 and C_2 as two Perron components. Hence $\mu(T) = \mu(\widehat{T})$. So, if we add a pendant vertex to a characteristic vertex of a tree T, the algebraic connectivity does not change.

We use this observation and Theorem 3.1 to obtain the following corollary. Hence the proof is omitted.

Corollary 3.2. *Fix a positive integer n. Then among all trees on n vertices the star has the largest algebraic connectivity.*

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References

[1] R.B. Bapat, Sukanta Pati, Algebraic connectivity and the characteristic set of a graph, Linear and Multilinear Algebra 45 (1998) 247–273.

[2] M. Fiedler, Algebraic connectivity of graphs, Czechoslovak Math. J. 23 (98) (1973) 298–305.

[3] M. Fiedler, Eigenvectors of acyclic matrices, Czechoslovak Math. J. 25 (100) (1975) 607–618.

[4] M. Fiedler, A property of eigenvectors of nonnegative symmetric matrices and its application to graph theory, Czechoslovak Math. J. 25 (100) (1975) 619–633.
[5] R. Grone, R. Merris, Algebraic connectivity of trees, Czechoslovak Math. J. 37 (112) (1987) 660–670.
[6] R. Grone, R. Merris, V.S. Sunder, The Laplacian spectrum of a graph, SIAM J. Matrix Anal. Appl. 11 (2) (1990) 218–238.
[7] S. Kirkland, S. Fallat, Perron components and algebraic connectivity for weighted graphs, Linear and Multilinear Algebra 44 (2) (1998) 131–148.
[8] S. Kirkland, M. Neuman, Algebraic connectivity of weighted trees under perturbation, Linear and Multilinear Algebra 42 (1997) 187–203.
[9] S. Kirkland, M. Neumann, B.L. Shader, Characteristic vertices of weighted trees via Perron values, Linear and Multilinear Algebra 40 (1996) 311–325.
[10] S. Kirkland, M. Neumann, B.L. Shader, Distances in weighted trees and group inverses of Laplacian matrices, SIAM J. Matrix Anal. Appl. 18 (1997) 827–841.
[11] R. Merris, Laplacian matrices of graphs: a survey, Linear Algebra Appl. 197/198 (1994) 143–176.
[12] B. Mohar, The Laplacian spectrum of graphs, in: Y. Alavi (Ed.), Graph Theory, Combinatorics, and Applications, John Wiley, New York, 1991, pp. 871–898.
[13] H. Minc, Nonnegative Matrices, Wiley Interscience Publication, 1987.