On the reconstruction problem for factorizable homeomorphism groups and foliated manifolds

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Abstract

For a group $G$ of homeomorphisms of a regular topological space $X$ and a subset $U \subseteq X$, set $G|U| := \{g \in G : g \restriction (X \setminus U) = \text{Id}\}$. We say that $G$ is a factorizable group of homeomorphisms, if for every open cover $\mathcal{U}$ of $X$, $\bigcup_{U \in \mathcal{U}} G|U|$ generates $G$.

Theorem I. Let $G$, $H$ be factorizable groups of homeomorphisms of $X$ and $Y$ respectively, and suppose that $G$, $H$ do not have fixed points. Let $\varphi$ be an isomorphism between $G$ and $H$. Then there is a homeomorphism $\tau$ between $X$ and $Y$ such that $\varphi(g) = \tau \circ g \circ \tau^{-1}$ for every $g \in G$.

Theorem A strengthens known theorems in which the existence of $\tau$ is concluded from the assumption of factorizability and some additional assumptions.

Theorem II. For $\ell = 1, 2$ let $(X_\ell, \Phi_\ell)$ be a countably paracompact foliated (not necessarily smooth) manifold and $G_\ell$ be any group of foliation-preserving homeomorphisms of $(X_\ell, \Phi_\ell)$ which contains the group $H_0(X_\ell, \Phi_\ell)$ of all foliation-preserving homeomorphisms which take every leaf to itself. Let $\varphi$ be an isomorphism between $G_1$ and $G_2$. Then there is a foliation-preserving homeomorphism $\tau$ between $X_1$ and $X_2$ such that $\varphi(g) = \tau \circ g \circ \tau^{-1}$ for every $g \in G_1$.

In both Theorems I and II, $\tau$ is unique.

1. Introduction

For a topological space $X$ let $H(X)$ denote the group of all auto-homeomorphisms of $X$. Let $G$ be a subgroup of $H(X)$. For a subset $U$ of $X$ define

$$G|U| = \{g \in G : g \restriction (X \setminus U) = \text{Id}\}.$$ 

So $G|U|$ is a subgroup of $G$. We say that $G$ is a factorizable group of $X$ if for every open cover $\mathcal{U}$ of $X$, the set $\bigcup_{U \in \mathcal{U}} G|U|$ generates $G$. For $x \in X$ define $G(x) = \{g(x) : g \in G\}$. We say that $G$ is a non-fixing group of $X$ if $\{x\} \subseteq G(x)$ for every $x \in X$.

Suppose that $X$ is a regular space and $G$ is a subgroup of $H(X)$. Then $(X, G)$ is called a space-group pair. If $G$ is a factorizable group of $X$, then $(X, G)$ is called a factorizable space-group pair, and if $G$ is a non-fixing group, then $(X, G)$ is called a non-fixing space-group pair.

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If $G$, $H$ are groups, then the notation $\varphi : G \cong H$ means that $\varphi$ is an isomorphism between $G$ and $H$. If $X$, $Y$ are topological spaces, then the notation $\tau : X \equiv Y$ means that $\tau$ is a homeomorphism between $X$ and $Y$.

The following theorem strengthens a theorem of Wensor Ling [1, Theorem 2.1].

**Theorem A.** Let $(X, G)$, $(Y, H)$ be factorizable non-fixing space-group pairs and $\varphi : G \cong H$. Then there is $\tau : X \equiv Y$ such that $\varphi(g) = \tau \circ g \circ \tau^{-1}$ for every $g \in G$.

Ling discovered that, in essence, given a topological space $X$ and a non-fixing factorizable group $G$ of homeomorphisms of $X$, the space $X$ can be reconstructed from $G$. However, the exact factorizability requirement in his proof was about the commutator group. He required that for every open cover $U$ of $X$, the commutator group $[G, G]$ is generated by $\bigcup_{U \in \mathcal{U}} [G[U], G[U]]$. This is a bit stronger than requiring that $[G, G]$ be factorizable. Ling also needed three additional requirements, one of which is similar to (but weaker than) assumption (B1) below.

In fact, the assumption that $G$ is factorizable in Theorem A can be replaced by a much weaker assumption which is called weak factorizability. This is done in Theorem 2.1. So Theorem A is a corollary of 2.1.

The next theorem strengthens the main result of Tomasz Rybicki in [9]. (That result is named there as THEOREM.) A subset of a topological space is somewhere dense, if its closure contains a nonempty open set.

**Theorem B.** Let $(X, G)$, $(Y, H)$ be space-group pairs. Assume that:

(A1) There are $G_1 \leq G$ and $H_1 \leq H$ such that $G_1$, $H_1$ are factorizable non-fixing groups of $X$ and $Y$ respectively.

(A2) For every $x \in X$, $G(x)$ is somewhere dense, and for every $y \in Y$, $H(y)$ is somewhere dense.

Suppose that $\varphi : G \cong H$. Then there is $\tau : X \equiv Y$ such that $\varphi(g) = \tau \circ g \circ \tau^{-1}$ for every $g \in G$.

Also in Theorem B, factorizability can be replaced by weak factorizability. This strengthening of Theorem B is proved in Corollary 2.6. The THEOREM in Rybicki [9] has the same conclusion as Theorem B, namely, every isomorphism between $G$ and $H$ is induced by a homeomorphism between $X$ and $Y$. But his assumptions on $(X, G)$ and $(Y, H)$ are stronger. He formulates three axioms which $(X, G)$, $(Y, H)$ need to satisfy. His Axiom 1 is a bit stronger than (A1). (He requires that all members of $G_1$ and $H_1$ be compactly isotopic to the identity.) The main differences, though, are the following additional requirements, which appear in Rybicki’s Axioms 2 and 3.

(B1) For every $x \in X$ there is an open $U_0 \ni x$ such that for every open neighborhood of $x$, $U \subseteq U_0$ there is $g \in G_1$ such that

$$\{x \in X \mid g(x) = x\} = (X \setminus U) \cup \{x\}.$$

(B2) $G$ is $n$-transitive for every $n$. (In the case that $\dim(X) = 1$ only $n$-order transitivity is required.)

Theorem B is a trivial corollary of the main result of Section 2. This result is stated in Theorem 2.5, and it says that if $(X, G)$, $(Y, H)$ fulfill only clause (A1) of Theorem B, then there are subsets $\hat{X} \subseteq X$, $\hat{Y} \subseteq Y$, respectively invariant under $G$ and $H$, and $\tau : \hat{X} \equiv \hat{Y}$ such that $X \setminus \hat{X}$ and $Y \setminus \hat{Y}$ are nowhere dense subsets of $X$ and $Y$, and $\tau$ induces $\varphi$.

In fact, [1, Corollary 2.6] has a better statement. It says that under assumptions similar to those of Theorem 2.5, there is $\tau : X \equiv Y$ such that $\tau$ induces $\varphi$. However, the corollary is not proved there, and seems to be incorrect due to a counterexample. See Example 4.2.

Section 3 deals with foliated manifolds. In this paper we do not require that the foliations be smooth. Let $X$ be a foliated manifold. Let $H_0(X)$ denote the group of foliation-preserving homeomorphisms and

$$H_0(X) = \{g \in H_1(X) \mid \text{for every leaf } L \text{ of } X, g(L) = L\}.$$

**Theorem C.** Let $X$ and $Y$ be countably paracompact foliated manifolds, $H_0(X) \subseteq G \subseteq H_1(X)$ and $H_0(Y) \subseteq H \subseteq H_1(Y)$. Suppose that $\varphi : G \cong H$. Then there is a unique $\tau : X \equiv Y$ such that $\varphi(g) = \tau \circ g \circ \tau^{-1}$ for every $g \in G$. Furthermore, $\tau$ is foliation-preserving.

Theorem C is restated as Theorem 3.3. The reconstruction problem for foliated manifolds is more difficult than the analogous question for manifolds equipped with other auxiliary structures. The main property of automorphism groups $G$ of those other structures, which is not present here, is the fact that an orbit of a point $x$ under $G[U]$ is somewhere dense in $U$, for every neighborhood $U$ of $x$. In some cases, every $x \in X$ has this property. So the following theorem from [4] is applicable. (See also [6, Theorem 3.1] or [7, Theorem 2.5].)

A space-group pair $(X, G)$ is locally densely conjugated (LDC) if it has no isolated points and for every $x \in X$ and a neighborhood $U$ of $x$, the closure of $G[U](x)$ has a nonempty interior.

**Theorem D.** ([4, Theorem 3.5]) Let $(X, G)$, $(Y, H)$ be locally compact LDC space-group pairs and $\varphi : G \cong H$. Then there is $\tau : X \equiv Y$ such that $\varphi(g) = \tau \circ g \circ \tau^{-1}$ for every $g \in G$. 

A variant of Theorem D applies to space-group pairs in which the set $\text{LDC}(X, G)$ defined below is dense in $X$:

$$\text{LDC}(X, G) := \{x \in X \mid \text{for every neighborhood } U \text{ of } x, \ G[U \setminus (x) \text{ is somewhere dense}]\}.$$ 

In such cases $\text{LDC}(X, G)$ can be recovered from $G$, and the recovery of all of $X$ from $G$ requires extra assumptions and extra steps.

If $X$ is a foliated manifold and $X \subseteq H_1(X)$, then $\text{LDC}(X, G) = \emptyset$, so Theorem D does not apply to such $(X, G)$’s. Thus we use the less easily verifiable assumption of weak factorizability to deal with this case. The reconstruction problem for smooth foliated manifolds will be dealt with in a subsequent work.

### 2. Factorizable groups

We state the first theorem to be proved in this section. Let $(X, G)$ be a space-group pair and $x \in X$. Set

$$G_{[x]} := \{g \in G \mid \text{for some neighborhood } U \text{ of } x, \ g \mid U = \text{Id}\}.$$ 

We say that $G$ is weakly factorizable and that $(X, G)$ is a weakly factorizable space-group pair, if for every distinct $x, y \in X$, $G_{[x]} \cup G_{[y]}$ generates $G$.

For a function $g$, $\text{Dom}(g)$ and $\text{Rng}(g)$ denote respectively the domain and range of $g$, and $\text{supp}(g)$ denotes the set $\{x \in \text{Dom}(g) \mid g(x) \neq x\}$. Let $(X, G)$ be a space-group pair. If for every nonempty open set $U \subseteq X$, $G[U] \neq \{\text{Id}\}$, then $G$ is called a locally moving group of $X$ and $(X, G)$ is called a local movement system.

It is worthwhile introducing the following notion. A class $K$ of space-group pairs is called a faithful class, if for every $(X, G), (Y, H) \in K$ and $\varphi : G \cong H$, there is $\tau : X \cong Y$ such that $\varphi(g) = \tau \circ g \circ \tau^{-1}$ for every $g \in G$.

**Theorem 2.1.** Let $K_{\text{WF}}$ be the class of all space-group pairs $(X, G)$ such that: (1) $X$ is locally compact; (2) $G$ is locally moving; (3) for every $g \in G$, $\text{cl}(\text{supp}(g))$ is compact; (4) $(X, G)$ is weakly factorizable and non-fixing. Then $K_{\text{WF}}$ is faithful.

Let $K_{\text{LDC}}$ be the class of all factorizable non-fixing space-group pairs. Recall that Theorem A in the Introduction states that $K_{\text{LDC}}$ is faithful. We shall see that $K_{\text{LDC}} \subseteq K_{\text{WF}}$. So Theorem A is a special case of Theorem 2.1.

Let $K_{\text{LDC}}$ be the class whose faithfulness was stated in Theorem D in the Introduction. Namely, $K_{\text{LDC}}$ is the class of locally compact locally densely conjugated space-group pairs. We shall also see (Example 4.3) that $K_{\text{WF}} \setminus (K_{\text{LDC}} \cup K_{\text{WF}}) \neq \emptyset$. Indeed, the Cantor set together with its group of isometries belongs to $K_{\text{WF}} \setminus K_{\text{LDC}}$, and if we add a “foliation” to the Cantor set we obtain a space-group pair which belongs to $K_{\text{WF}} \setminus (K_{\text{LDC}} \cup K_{\text{WF}})$.

For a subgroup $G \subseteq H(X)$ set $\text{Fix}(G) := \{x \in X \mid (\forall g \in G)(g(x) = x)\}$. For $x \in X$ let $\text{Nbr}^X(x)$ denote the set of open neighborhoods of $x$ in $X$.

**Proposition 2.2.** (a) Suppose that $(X, G)$ is a space-group pair and $G$ is factorizable and non-fixing. Then for every open set $U \subseteq X$,

$$\text{Fix}(G[U]) = X \setminus U.$$ 

In particular, $G$ is locally moving.

(b) Suppose that $(X, G)$ is a space-group pair and $G$ is factorizable and non-fixing. Then for every $g \in G$, $\text{cl}(\text{supp}(g))$ is compact, and $X$ is locally compact.

(c) Suppose that $(X, G)$ is a local movement system. Then $\text{Fix}(G)$ is nowhere dense.

**Proof.** (a) Let $U \subseteq X$ be open and $x \in U$. There is $g \in G$ such that $g(x) \neq x$. $[U, X \setminus \{x\}]$ is an open cover of $X$. So $g$ is a composition of members of $G$ supported by $U$ or $X \setminus \{x\}$. One of these members moves $x$, and such a member can’t be supported by $X \setminus \{x\}$. Hence it must be supported by $U$.

(b) Let $g \in G$. Denote $\text{supp}(g)$ and $\text{cl}(\text{supp}(g))$ by $W, F$ respectively. Let $U$ be an open cover of $F$. For every $U \in U$ and $x \in U$ let $V_x, U$ be an open set such that $x \in V_x, U$ and $\text{cl}(V_x, U) \subseteq U$. And let $V = \{X \setminus F\} \cup \bigcup_{x \in U} V_x, U$. Then $V$ is an open cover of $X$. So there are $V_1, \ldots, V_k \subset V$ and $g_1, \ldots, g_k \in G$ such that $\text{supp}(g_i) \subseteq V_i$ and $g = \bigcap_{i \leq k} g_i$. Clearly, $\bigcap_{i \leq k} V_i \supseteq W$. Let $U_i \in U$ be such that for some $x \in U_i$, $V_i = V_x, U_i$. Hence $[U_1, \ldots, U_k]$ covers $F$. So $F$ is compact.

Let $x \in X$. Since $G$ is non-fixing there is $g \in G$ such that $g(x) \neq x$. Hence $\text{supp}(g) \in \text{Nbr}(x)$ and $\text{cl}(\text{supp}(g))$ is compact. So $X$ is locally compact.

(c) Let $U$ be a nonempty open subset of $X$. Choose $g \in G[U \setminus \{\text{Id}\}$ and $x$ such that $g(x) \neq x$. Since $X$ is Hausdorff, there is $V \in \text{Nbr}(x)$ such that $g(V) \cap V = \emptyset$. Clearly $V \subseteq U$ and $V \cap \text{Fix}(G) = \emptyset$. So $\text{Fix}(G)$ is nowhere dense.

**Corollary 2.3.** $K_{\text{LDC}} \subseteq K_{\text{WF}}$.

**Proof.** Let $(X, G) \in K_{\text{LDC}}$. By Proposition 2.2(a), $G$ is locally moving, and by 2.2(b), $X$ is locally compact. By 2.2(b), for every $g \in G$, $\text{cl}(\text{supp}(g))$ is compact. Let $x, y \in X$ be distinct. Since $X$ is regular, $x$ and $y$ have closed disjoint neighborhoods $F$ and $K$. Since $[X \setminus F, X \setminus K]$ is an open cover of $X$, it follows that $G[X \setminus F] \cup G[X \setminus K]$ generates $G$. However, $G[X \setminus F] \subseteq G[x]$ and $G[X \setminus K] \subseteq G[y]$. So $G[x] \cup G[y]$ generates $G$. Hence $(X, G) \in K_{\text{WF}}$. 


We quote a theorem from [6] which is a basic tool in this work. For a subset $A \subseteq X$, $\text{int}^X(A)$ and $\text{cl}^X(A)$ denote respectively the interior and closure of $A$ in $X$, and $A$ is called a regular open set if $A = \text{int}^X(\text{cl}^X(A))$. Let $\text{Ro}(X)$ denote the set of regular open subsets of $X$. If we define the following operations on $\text{Ro}(X)$:

$$
U + V := \text{int}(\text{cl}(U \cup V)), \quad U \cdot V := U \cap V, \quad -U := \text{int}(X \setminus U),
$$

then the structure $(\text{Ro}(X); +, \cdot, -)$ is a Boolean algebra whose 0 and 1 are $\emptyset$ and $X$ and whose partial ordering is $\subseteq$. So we regard $\text{Ro}(X)$ as a Boolean algebra. For Boolean algebras $B$ and $C$ the notation $\psi : B \cong C$ means that $\psi$ is an isomorphism between $B$ and $C$.

Let $g$ be a homeomorphism between $X$ and $Y$. Then $g$ induces an isomorphism $g^{\text{Ro}}$ between $\text{Ro}(X)$ and $\text{Ro}(Y)$. $g^{\text{Ro}}$ is defined by

$$
g^{\text{Ro}}(V) = g[V] := \{g(x) \mid x \in V\}, \quad V \in \text{Ro}(X).
$$

It always holds that $g \mapsto g^{\text{Ro}}$ is a group homomorphism from $H(X)$ into $\text{Aut}(\text{Ro}(X))$. Whenever $X$ is Hausdorff, then $g \mapsto g^{\text{Ro}}$ is also 1–1 and thus an embedding.

**Theorem 2.4.** ([6, Corollary 1.4]) Let $(X, G)$ and $(Y, H)$ be local movement systems, and $\varphi : G \cong H$. Then there is a unique $\psi : \text{Ro}(X) \cong \text{Ro}(Y)$ such that $\psi$ induces $\varphi$ in the following sense: $\varphi(g)^{\text{Ro}} = \psi \circ g^{\text{Ro}} \circ \psi^{-1}$ for every $g \in G$. In other words, for every $U, V \in \text{Ro}(X)$ and $g \in G$,

$$
V = g[U] \iff \psi(V) = \varphi(g)[\psi(U)].
$$

**Remarks.**

(a) The uniqueness of the isomorphism $\psi$ is trivial.

(b) The isomorphism $\psi$ need not come from a homeomorphism between $X$ and $Y$. It may happen that there is no $\tau : X \cong Y$ such that $\psi = \tau^{\text{Ro}}$.

For a space-group pair $(X, G)$, consider the following object:

$$
M(X, G) := (\text{Ro}(X), G; +, \cdot, -, \circ, \text{Ap}),
$$

where $+, \cdot, -$ are the operations on $\text{Ro}(X)$ defined above, $\circ$ is the composition in $G$, and $\text{Ap} : G \times \text{Ro}(X) \rightarrow \text{Ro}(X)$ is the "application function". It is defined by $\text{Ap}(g, U) := g^{\text{Ro}}(U)$. $M(X, G)$ is called the RO-system of $(X, G)$. An isomorphism between the RO-systems $M(X, G)$ and $M(Y, H)$ is defined to be a bijection $\eta : \text{Ro}(X) \cup G \rightarrow \text{Ro}(Y) \cup H$ which takes $\text{Ro}(X)$ to $\text{Ro}(Y)$ and $G$ to $H$, and which preserves the operations $+, \cdot, -, \circ$ and Ap. Using the notion of an RO-system, Theorem 2.4 can be restated as follows.

**Theorem 2.4*.** Let $(X, G)$ and $(Y, H)$ be LM systems and $\varphi : G \cong H$. Then there is $\psi : \text{Ro}(X) \rightarrow \text{Ro}(X)$ such that $\varphi \cup \psi$ is an isomorphism between $M(X, G)$ and $M(Y, H)$.

If $F \subseteq X$ is closed, then $\text{int}(F)$ is regular open. If $S \subseteq X$ is open, then $\text{int}(\text{cl}(S))$ is the smallest regular open set containing $S$. For $g \in H(X)$ set $\text{var}(g) = \text{int}(\text{supp}(g))$. So for $U \in \text{Ro}(X)$, $\text{supp}(g) \subseteq U$ iff $\text{var}(g) \subseteq U$.

**Proof of Theorem 2.1.** Let $(X, G)$, $(Y, H)$ be space-group pairs and $\eta : M(X, G) \cong M(Y, H)$. By definition, $\eta$ preserves $+, \cdot, -, \circ$ and Ap. So every property of $g_1, \ldots, g_k, U_1, \ldots, U_\ell \in G \cup \text{Ro}(X)$, which is expressed in terms $+, \cdot, -, \circ$ and Ap is also preserved by $\eta$. The partial ordering $\subseteq$ of $\text{Ro}(X)$ (which is just set inclusion) is expressed by

$$
U \subseteq V \iff U \cdot V = U.
$$

So $\eta$ preserves $\subseteq$. Since $\emptyset$ and $X$ are the minimum and maximum of $(\text{Ro}(X), \subseteq)$, they are sent by $\eta$ to $\emptyset$ and $Y$. Let $(\ast)$ be the following property of $g \in G$ and $U \in \text{Ro}(X)$:

"For every $V \in \text{Ro}(X)$, if $V \cdot U = 0$, then $\text{Ap}(g, V) = V$".

Then $g, U$ have property $(\ast)$ iff $\eta(g), \eta(U)$ have property $(\ast)$. The following claim is trivial.

**Claim 1.** Let $g \in G$ and $U \in \text{Ro}(X)$. Then $\text{supp}(g) \subseteq U$ iff $g, U$ have property $(\ast)$.

It follows that $\text{supp}(g) \subseteq U$ iff $\text{supp}(\eta(g)) \subseteq \eta(U)$.

**Claim 2.** Let $g \in G$ and $U \in \text{Ro}(X)$. Then $\text{var}(g) = U$ iff $g, U$ have property $(\ast)$ and for every $V \subseteq U$, the pair $g, V$ does not have property $(\ast)$. 

It follows that \( \text{var}(g) = U \) iff \( \text{var}(\eta(g)) = \eta(U) \).

Let \( p \subseteq \text{Ro}(X) \) be an ultrafilter of \( \text{Ro}(X) \). (That is, \( p \) is closed under finite intersections, \( \emptyset \notin p \) and for every \( U \in \text{Ro}(X), U \in p \) iff \(-U \notin p \).) It is trivial that \( |\bigcup_{U \in p} \text{cl}(U)| \leq 1 \). It is also trivial that for every ultrafilter \( p \) and \( x \in X \), \( \bigcap_{U \in p} \text{cl}(U) = \{x\} \) iff \( \text{Nbr}(x) \cap \text{Ro}(X) \subseteq p \).

We say that \( p \) is a good ultrafilter (with respect to \( (X, G) \)), if for some \( g \in G \), \( \text{var}(g) \in p \). Clearly, if \( \eta : M(X, G) \cong M(Y, H), \) then \( \eta \) takes ultrafilters to ultrafilters. By Claim 2, for every ultrafilter \( p \), \( p \) is good iff \( \eta[p] \) is good. Let Good\((X, G)\) denote the set of good ultrafilters of \((X, G)\).

Let \((X, G) \in K_{WF}\). Recall that for every \( g \in G \), \( \text{cl}(\text{supp}(g)) \) is compact. It follows trivially from this fact that for every \( p \in \text{Good}(X, G), \bigcap_{U \in p} \text{cl}(U) \) is a singleton. Denote this singleton by \( x_p \).

For an ultrafilter \( p \) let \( G[p] = \{g \in G | \text{ for some } U \in p, g \mid U = \text{Id} \} \). Then \( G[p] \) is a subgroup of \( G \).

**Claim 3.** Let \((X, G) \in K_{WF} \) and \( p, q \in \text{Good}(X, G) \). Then \( x_p = x_q \) iff \( G[p] \cup G[q] \) does not generate \( G \).

**Proof.** Denote by \( G[p, q] \) the group generated by \( G[p] \cup G[q] \). Clearly, for every \( g \in G[p] \), \( g(x_p) = x_p \). So if \( x_p = x_q \), then \( x_p \in \text{Fix}(G[p, q]) \). Since \( G \) is non-fixing, it follows that \( G[p, q] \neq G \).

Now assume that \( x_p \neq x_q \). By the weak factorizability of \((X, G)\), \( G[x_p] \cup G[x_q] \) generates \( G \). Note that for every open set \( U \) and \( g \in G : g \mid U = \text{Id} \) iff \( g \mid \text{int}(\text{cl}(U)) = \text{Id} \). Hence \( G[x_p] \subseteq G[p] \) and \( G[x_q] \subseteq G[q] \). Hence \( G[p, q] \subseteq G \). This proves Claim 3. \( \square \)

Claim 3 means that the fact that \( x_p = x_q \) is equivalent to a property of \( p \) and \( q \) which is expressed entirely in terms of \( +, \cdot, - , \circ \) and \( A \) and \( p \). So if \((X, G) \), \((Y, H) \in K_{WF}, \eta : M(X, G) \cong M(Y, H), p, q \in \text{Good}(X, G) \), then \( x_p = x_q \) iff \( x_{\eta[p]} = x_{\eta[q]} \).

**Claim 4.** Let \((X, G) \in K_{WF}\), \( p \in \text{Good}(X, G) \) and \( U \in \text{Ro}(X) \). Then \( x_p \in U \) iff for every \( q \in \text{Good}(X, G) \), \( x_q = x_p \), then \( U \in q \).

**Proof.** Suppose first that \( x_p \notin U \), then \( x_p \notin \text{Int}(\text{cl}(U)) \). This means that \( x_p \in \text{cl}(X \setminus \text{cl}(U)) \). It is easy to verify that \( X \setminus \text{cl}(U) = \text{Int}(X \setminus U) \). Hence \( x_p \in \text{cl}(X \setminus U) \). Consider the set \( \{ x_p \cap \text{Ro}(X) \} \cup \{ -U \} \). This set has the finite intersection property, so it can be extended to an ultrafilter \( q \). Clearly, \( x_p = x_q \) and \( U \notin q \). Let \( g \in G \) be such that \( g(x_p) \neq x_p \). Such a \( g \) exists, since \( G \) is non-fixing. Hence \( \text{supp}(g) \in \text{Nbr}(x_p) \). So \( \text{var}(g) \in \text{Nbr}(x_p) \cap \text{Ro}(X) \subseteq q \). It follows that \( q \) is a good ultrafilter. This proves Claim 4. \( \square \)

We have shown that for \( p \in \text{Ro}(X) \) and \( U \in \text{Ro}(X) \), the fact \( x_p \in U \) is equivalent to a property of \( p \) and \( U \) which is expressed in terms of \( +, \cdot, - , \circ \) and \( A \) and \( p \). So if \((X, G) \), \((Y, H) \in K_{WF}, \eta : M(X, G) \cong M(Y, H), p \in \text{Good}(X, G) \) and \( U \in \text{Ro}(X) \), then \( x_p \in U \) iff \( x_{\eta[p]} = \eta(U) \).

We now prove the theorem. Let \((X, G) \), \((Y, H) \in K_{WF}\) and \( \varphi : G \cong H \). By Theorem 2.4*, there is \( \psi : \text{Ro}(X) \to \text{Ro}(Y) \) such that \( \eta : = \varphi \circ \psi \) is an isomorphism between \( \text{MR}(X, G) \) and \( \text{MR}(Y, H) \). We define \( \tau : X \to Y \). (In fact, the definition of \( \tau \) and the proof that \( \tau \) induces \( \varphi \) is a standard argument. For the sake of completeness we give the details of this argument.) Let \( x \in X \). There is an ultrafilter \( p \) such that \( x_p = x \). Since \( G \) is non-fixing, \( p \) is good. So \( \eta[p] \) is good. Define \( \tau(x) = x_{\eta[p]} \). By Claim 3, if \( q \) is another good ultrafilter such that \( x_q = x \), then \( x_{\eta[q]} = x_{\eta[p]} \). So the definition of \( \tau \) is valid.

Claim 3 also implies that \( \tau \) is 1–1. It is trivial that \( \tau \) is onto \( Y \). So \( \tau \) is a bijection between \( X \) and \( Y \). We show that \( \tau \) is a homeomorphism. It suffices to show that for every \( U \in \text{Ro}(X) \), \( \tau[U] = \eta(U) \). Let \((X, p) \in \text{Good}(X, G) \) be such that \( x_p = x \). Using Claim 4 we obtain

\[
x \in U \quad \text{iff} \quad x_p \in U \quad \text{iff} \quad x_{\eta[p]} \in \eta(U) \quad \text{iff} \quad \tau(x) \in \eta(U).
\]

The fact that \( \tau \) is surjective together with (1) implies that \( \tau[U] = \eta(U) \). Hence \( \tau : X \cong Y \).

Finally we show that for every \( g \in G \), \( \varphi(g) = \tau \circ g \circ \tau^{-1} \). Let \( g \in G \) and \( y \in X \). Choose \( p \in \text{Good}(X, G) \) such that \( x_p = \tau^{-1}(y) \). Then

\[\tau \circ g \circ \tau^{-1}(y) = \tau \circ g(x_p) = \tau(g(x_p)).\]

Note that \( g(x_p) = x_{\eta(g)[p]} \). Hence

\[\tau(g(x_p)) = \tau(x_{\eta(g)[p]}) = x_{\eta[\eta(g)[p]]} = x_{\eta[\eta(g)[p]]}.
\]

Now we use the fact that \( \varphi(g)^{\text{Ro}} = \psi \circ g^{\text{Ro}} \psi^{-1} \). This fact implies that \( \psi[\eta(g)^{\text{Ro}}[p]] = \varphi(g)^{\text{Ro}}[\psi[p]] \).

So

\[x_{\eta[\eta(g)[p]]} = x_{\varphi[\eta(g)^{\text{Ro}}][\psi[p]]}.
\]

For any homeomorphism \( h \) and a good ultrafilter \( q \), \( x_\eta[q] = h(x_q) \). In particular, this is true for \( h = \varphi(g) \) and \( q = \psi[p] \).

Hence

\[\varphi(g)^{\text{Ro}}[\psi[p]] = \varphi(g)(\psi[p]) = \varphi(g)(x_{\eta[p]}) = \varphi(g)(\tau(x_p)) = \varphi(g)(y).
\]

It follows from (2)–(5) that \( \tau \circ g \circ \tau^{-1}(y) = \varphi(g)(y) \). \( \square \)
2.1. Groups which contain factorizable subgroups

In the context of Euclidean manifolds, many naturally arising groups of homeomorphisms are not factorizable but do contain non-fixing factorizable subgroups. Suppose that \( \langle X, G \rangle, \langle Y, H \rangle \) are two such pairs. Then an isomorphism \( \varphi \) between \( G \) and \( H \) is not always induced by a homeomorphism between \( X \) and \( Y \). However, there are co-nowhere-dense subsets \( \hat{X} \subseteq X \) and \( \hat{Y} \subseteq Y \) such that \( \varphi \) is induced by a homeomorphism between \( \hat{X} \) and \( \hat{Y} \). In fact, it suffices that \( G \) and \( H \) contain weakly factorizable (rather than factorizable) subgroups.

**Theorem 2.5.** Let \( \langle X, G \rangle \) and \( \langle Y, H \rangle \) be space-group pairs, and assume that there are \( G_0 \leq G \) and \( H_0 \leq H \) such that \( \langle X, G_0 \rangle, \langle Y, H_0 \rangle \in K_{WF} \). Then there are \( \hat{X} \subseteq X \), \( \hat{Y} \subseteq Y \) and \( \tau \) such that \( \hat{X} \) and \( \hat{Y} \) are invariant under \( G \) and \( H \) respectively, \( X \setminus \hat{X} \) and \( Y \setminus \hat{Y} \) are nowhere dense, \( \tau : X \cong \hat{Y} \) and for every \( g \in G \), \( \varphi(g) \mid \hat{Y} = \tau \circ (g \mid \hat{X}) \circ \tau^{-1} \).

**Proof.** Let \( \langle X, G \rangle \) be any space-group pair. If \( \mathcal{G} \) is a family of subgroups of \( G \), and every member of \( \mathcal{G} \) is weakly factorizable, then \( \bigcup \mathcal{G} \) generates a weakly factorizable group. Hence \( G \) contains a maximal weakly factorizable group \( G^\ast \), and every weakly factorizable subgroup of \( G \) is contained in \( G^\ast \). We show that \( G^\ast \) is normal. It is obvious that if a subgroup \( G_1 \) of \( G \) is weakly factorizable and \( g \in G \), then \( g \circ G_1 \circ g^{-1} \) is weakly factorizable. By the maximality of \( G^\ast \), \( g \circ G^\ast \circ g^{-1} \subseteq G^\ast \) for every \( g \in G \). Hence \( G^\ast \) is normal. Suppose now that for some \( G_0 \leq G \), \( \langle X, G_0 \rangle \in K_{WF} \). Then \( \langle X, G^\ast \rangle \in K_{WF} \).

Let \( \langle X, G \rangle, \langle Y, H \rangle \) be as in the theorem, and let \( G^\ast \) and \( H^\ast \) be the maximal weakly factorizable subgroups of \( G \) and \( H \) respectively. Clearly, \( \langle X, G \rangle, \langle Y, H \rangle \) are local movement systems. Hence by Theorem 2.4\(*\), there is \( \psi : \mathrm{Ro}(X) \to \mathrm{Ro}(X) \) such that \( \varphi \cup \psi : M(X, G) \cong M(Y, H) \). For a Boolean algebra \( A \), denote the set of ultrafilters of \( A \) by \( \text{Ult}(A) \). Set \( B = \text{Ro}(X) \) and \( C = \text{Ro}(Y) \). An ultrafilter \( p \) of \( B \) is called a good ultrafilter, if for some \( g \in G^\ast \), \( \varphi(g) \in p \), and for some \( h \in H^\ast \), \( \varphi^{-1}(h) \in p \). Similarly, an ultrafilter \( q \) of \( C \) is good, if the analogous facts hold. That is, for some \( h \in H^\ast \), \( \varphi(h) \in q \) and for some \( g \in G^\ast \), \( \varphi(g) \in q \). Denote by \( \text{Good}(B) \) and \( \text{Good}(C) \) the set of good ultrafilters of \( B \) and \( C \).

Note that \( \varphi^{-1}(H^\ast) \) is a normal subgroup of \( G \). This is so, since \( H^\ast \) is normal in \( G \) and \( \varphi^{-1} : H \cong G \). The facts that \( G^\ast \) and \( \varphi^{-1}(H^\ast) \) are normal in \( G \) imply that \( \text{Good}(B) \) is invariant under \( G \). That is, for every \( g \in G \) and \( p \in \text{Ult}(B) \), \( g \circ \text{Good}(B) \circ g^{-1} \subseteq \text{Good}(B) \).

Recall that by Claim 2 in the proof of Theorem 2.1, \( \varphi(\varphi(g)) = \psi(\varphi(g)) \). This implies that:

**Claim 1.** For any \( p \in \text{Ult}(B) \), \( p \) is good iff \( \psi(p) \) is good.

Let \( Z \) be a regular locally compact space. For \( p \in \text{Ult}(\text{Ro}(Z)) \) set \( M^Z_p = \bigcap_{U \in p} \text{cl}(U) \). Then \( M^Z_p = \emptyset \) or \( M^Z_p \) is a singleton. The following are equivalent:

1. \( M^Z_p \) is a singleton.
2. For some \( z \in Z \), \( \text{Nbr}(z) \cap \text{Ro}(X) \subseteq p \).
3. There is \( U \in p \) such that \( \text{cl}(U) \) is compact.

Denote the set of ultrafilters satisfying the above by \( \text{PU}(Z) \) (point ultrafilters), and for \( p \in \text{PU}(Z) \) let \( m^Z_p \) be such that \( \{ m^Z_p \} = M^Z_p \). For \( p \in \text{PU}(X) \) denote \( m^Z_p \) by \( x_p \), and for \( q \in \text{PU}(Y) \) denote \( m^Z_q \) by \( y_q \).

Recall that for every \( k \in G^\ast \cup H^\ast \), \( \text{cl}(\text{Var}(k)) \) is compact. It thus follows from (3) that:

**Claim 2.** \( \text{Good}(B) \subseteq \text{PU}(X) \) and \( \text{Good}(C) \subseteq \text{PU}(Y) \).

Let \( X_1 = \text{Fix}(\varphi^{-1}(H^\ast)) \) and \( Y_1 = \text{Fix}(\varphi(G^\ast)) \).

**Claim 3.** \( Y_1 \) is nowhere dense and \( X_1 \) is nowhere dense.

**Proof.** \( G^\ast \) is a locally moving group of \( X \). So \( G^\ast \) satisfies:

For every \( U \in \text{Ro}(X) \setminus \{ \emptyset \} \) there is \( g \in G^\ast \setminus \{ \text{Id} \} \) such that \( \text{Var}(g) \subseteq U \).

The above statement is preserved by any isomorphism between \( M(X, G) \) and \( M(Y, H) \). Since \( \varphi \cup \psi \) is such an isomorphism, \( \varphi(G^\ast) \) satisfies the same statement. That is,

For every \( U \in \text{Ro}(Y) \setminus \{ \emptyset \} \) there is \( g \in \varphi(G^\ast) \setminus \{ \text{Id} \} \) such that \( \text{Var}(g) \subseteq U \).

This means that \( \varphi(G^\ast) \) is a locally moving group of \( Y \). By Proposition 2.2(c), \( \text{Fix}(\varphi(G^\ast)) \) is nowhere dense. That is, \( Y_1 \) is nowhere dense. The same argument holds for \( X_1 \). \( \Box \)

**Claim 4.** For every \( x \in X \setminus X_1 \) and \( p \in \text{Ult}(B) \): if \( \text{Nbr}(x) \cap B \subseteq p \), then \( p \in \text{Good}(B) \). The same holds in \( Y \).
Proof. Let \( x \in X \setminus X_1 \) and \( p \in \text{Ult}(B) \) be such that \( \text{Nbr}(x) \cap B \subseteq p \). Since \( G^* \) is non-fixing, there is \( g \in G^* \) such that \( g(x) \neq x \). Hence \( \text{var}(g) \in \text{Nbr}(x) \). So \( \text{var}(g) \in p \). Similarly, since \( x \notin X_1 \), there is \( h \in H^* \) such that \( \varphi^{-1}(h)(x) \neq x \). Hence \( \text{var}(\varphi^{-1}(h)) \in \text{Nbr}(x) \). So \( \text{var}(\varphi^{-1}(h)) \in p \). Hence \( p \in \text{Good}(B) \).

Claim 5. Let \( X_2 = \{ x_p \mid p \in \text{Good}(B) \} \) and \( y_{\varphi[p]} \in Y_1 \). Then \( X_2 \) is nowhere dense.

Proof. Suppose by contradiction that \( X_2 \) is dense in \( U \). Let \( V = \psi(U) \). Choose \( W \subseteq V \) such that \( Y_1 \cap \text{cl}(W) = \emptyset \) and let \( S = \psi^{-1}(W) \). Choose \( x \in X_2 \cap S \). Then there is \( p \in \text{Good}(B) \) such that \( x_p = x \) and \( y_{\varphi[p]} \in Y_1 \). Then \( S \subseteq p \). Hence \( W \subseteq \psi[p] \). So \( y_{\varphi[p]} \in \text{cl}(W) \). Hence \( y_{\varphi[p]} \notin Y_1 \). A contradiction.

Let \( Y_2 = \{ q \mid q \in \text{Good}(C) \} \) and \( x_{\varphi^{-1}[q]} \in X_1 \). Just as for \( X_2 \), it holds that \( Y_2 \) is nowhere dense.

Claim 6. \( X_1 \) and \( X_2 \) are invariant under \( G \) and \( Y_1 \) and \( Y_2 \) are invariant under \( H \).

Proof. For every \( x \in X \) a subgroup \( L \) of \( G \) and \( g \in G \): \( x \in \text{Fix}(L) \) iff \( g(x) \in \text{Fix}(g \circ L \circ g^{-1}) \). Since \( \varphi^{-1}(H^*) \) is a normal in \( G \), it follows that \( x \in \text{Fix}(\varphi^{-1}(H^*)) \) iff \( g(x) \in \text{Fix}(\psi^{-1}(H^*)) \). Hence \( X_1 \) is \( G \)-invariant.

Similarly, \( Y_1 \) is \( H \)-invariant.

We show that \( X_2 \) is \( G \)-invariant. Let \( x \in X_2 \) and \( g \in G \). So for some \( p \in \text{Good}(B) \), \( x_p = x \) and \( y_{\varphi[p]} \in Y_1 \). Since \( \text{Good}(B) \) is \( G \)-invariant, \( g[p] \in \text{Good}(B) \). Since \( Y_1 \) is \( H \)-invariant, \( \psi(g)(y_{\varphi[p]}) \in Y_1 \). Clearly,
\[
g(x_p) = y_{g[p]} \quad \text{and} \quad \psi(g)(y_{\varphi[p]}) = y_{\psi g[p]}.
\]

Since \( \varphi \cup \psi : M(X, G) \cong M(Y, H) \), it follows that \( \psi(g)(y_{\varphi[p]}) = \psi(g[p]) \). Hence \( y_{\psi g[p]} = \varphi(g)(y_{\varphi[p]}) \in Y_1 \). So \( x_{g[p]} \in X_2 \). That is, \( g(x_p) \in X_2 \). Hence \( X_2 \) is invariant under \( G \). Similarly, \( Y_2 \) is \( H \)-invariant.

Claim 7. Let \( p_1, p_2 \in PU(X) \) be such that \( x_{p_1} \neq x_{p_2} \). Then \( G^* \subseteq G_{[p_1, p_2]} \). The analogous claim holds for \( q_1, q_2 \in PU(Y) \).

Proof. Clearly, for \( i = 1, 2 \), \( G_{[p_i]} \supseteq G_{[x_{p_i}]} \supseteq G_{[x_{p_i}]}^* \). So \( G_{[p_1, p_2]} \supseteq G_{[x_{p_1}]}^* \cup G_{[x_{p_2}]}^* \). By the weak factorizability of \( G^* \), \( G_{[p_1, p_2]} \subseteq G^* \).

Claim 8. For every \( p_1, p_2 \in PU(X) \): if \( x_{p_1} \neq x_{p_2} \), \( \psi_{[p_1]} \in PU(Y) \) and \( y_{\varphi[p_1]} = y_{\varphi[p_2]} \), then \( y_{\varphi[p_1]} \in Y_1 \). The same holds for \( q_1, q_2 \in PU(Y) \).

Proof. Set \( q_i = \psi_{[p_i]} \) and \( y = y_{q_i} \). Since \( \varphi \cup \psi \) is an isomorphism between \( M(X, G) \) and \( M(Y, H) \), \( \psi(G_{[p_i]}) = H_{[q_i]} \). Hence \( y \in \text{Fix}(\varphi(G_{[p_i]})) \). It follows that \( y \in \text{Fix}(\psi(G_{[p_i]})) \). By Claim 7, \( G_{[p_1, p_2]} \supseteq G^* \). So \( y_{\varphi[p_1]} \in \text{Fix}(\psi(G^*)) \). By definition, this means that \( y_{\varphi[p_1]} \in Y_1 \).

Let \( \hat{X} = X \setminus (X_1 \cup X_2) \) and \( \hat{Y} = Y \setminus (Y_1 \cup Y_2) \).

Claim 9. Let \( x \in \hat{X} \) and \( p \in PU(X) \) be such that \( x_p = x \). Then \( p \) is good and \( y_{\varphi[p]} \in \hat{Y} \). The analogous claim holds for \( y \in \hat{Y} \).

Proof. By Claim 4, \( p \) is good. In Claim 1 we have seen that \( \psi \) takes \( \text{Good}(B) \) to \( \text{Good}(C) \), so \( q := \psi[p] \) is good. Since \( x \notin X_2 \), it follows that \( y_q \notin Y_1 \). Suppose by contradiction that \( y_q \in Y_2 \). So there is \( z \in X_1 \) and \( r \in \text{Good}(B) \) such that \( z = x_p \) and \( y_{\varphi[r]} = y_q \). Since \( x \notin X_1 \), it follows that \( x \neq z \). Hence by Claim 8, \( y_q \notin Y_1 \). A contradiction. We have proved Claim 9.

We define a relation \( \tau \) as follows: \( \langle x, y \rangle \in \tau \) if \( x \in \hat{X} \) and there is \( p \in PU(X) \) such that \( x = x_p \) and \( y = y_{\varphi[p]} \).

Claim 10. \( \tau \) is a function and \( \tau : \hat{X} \cong \hat{Y} \).

Proof. Suppose by contradiction that \( \langle x, y_1 \rangle, \langle x, y_2 \rangle \in \tau \) and \( y_1 \neq y_2 \). For \( i = 1, 2 \) let \( p_i \in PU(X) \) be such that \( x = x_{p_i} \) and \( y_i = y_{\varphi[p_i]} \). By Claim 9, \( p_1, p_2 \in \text{Good}(B) \). Hence by Claim 1, \( \psi[p_1] \in \text{Good}(C) \). By Claim 2, \( \psi[p_i] \in PU(Y) \). Hence by Claim 8 applied to \( Y, x \in X_1 \). A contradiction. We have shown that \( \tau \) is a function.

By Claim 9, if \( \langle x, y \rangle \in \tau \), then \( y \in \hat{Y} \). This means that \( \text{Rng}(\tau) \subseteq \hat{Y} \).

Let \( x \in \hat{X} \). Choose \( p \in \text{Ult}(B) \) such that \( p \supseteq \text{Nbr}(x) \cap \text{Ro}(X) \). Then \( p \in PU(X) \) and \( x_p = x \). By the definition of \( \tau \), \( \langle x, y_{\varphi[p]} \rangle \in \tau \). This shows that \( \text{Dom}(\tau) = \hat{X} \).

Define a relation \( \rho \) as follows: \( \langle y, x \rangle \in \rho \) if \( y \in \hat{Y} \) and there is \( q \in PU(Y) \) such that \( y = y_q \) and \( x = x_{\varphi^{-1}[q]} \). Note that the definition of \( \rho \) is obtained from the definition of \( \tau \) by reversing the roles of \( X \) and \( Y \). So \( \rho : \hat{Y} \cong \hat{X} \). We show that \( \tau \subseteq \rho^{-1} \).

Let \( \langle y, x \rangle \in \tau \). There is \( p \in PU(X) \) such that \( x = x_p \) and \( y = y_{\varphi[p]} \). Set \( q = \psi[p] \). Then \( y \in \hat{Y} \) and \( q \) is an evidence that \( \langle y, x \rangle \in \rho \). We have seen that \( \tau \subseteq \rho^{-1} \). Reversing the roles of \( X \) and \( Y \) we conclude that \( \rho \subseteq \tau^{-1} \). Hence \( \rho = \tau^{-1} \). This implies that \( \tau \) is a bijection between \( \hat{X} \) and \( \hat{Y} \).
We shall show that for every \( U \subseteq B \), \( \tau(U \cap \tilde{X}) = \psi(U) \cap \tilde{Y} \). We first prove that \( \tau(U \cap \tilde{X}) \subseteq \psi(U) \cap \tilde{Y} \). Suppose by contradiction that \( x \in U \cap \tilde{X} \) and \( y := \tau(x) \notin \psi(U) \). Let \( p \in PU(X) \) be such that \( x_p = x \). Then \( U \subseteq B_n \). Since \( y \notin \psi(U) \), there is \( q \in PU(Y) \) such that \( \psi(U) \subseteq q \) and \( y_q = y \). Since \( y \in \tilde{Y} \), \( q \) is good. Set \( r = \psi^{-1}[q] \). Hence \( r \) is good. Since \( x \in U \subseteq B_n \) it follows that \( x \neq x_r \). So \( x_p \neq x_r \) but \( y_p[x] = y_r[x] \). By Claim 8, \( y \in Y \). A contradiction. So \( \tau(x) \in \psi(U) \).

By applying the same argument to \( \rho \) and recalling that \( \rho = \tau^{-1} \), we conclude that for every \( V \subseteq C \), \( \tau^{-1}[V \cap \tilde{Y}] \subseteq \psi^{-1}(V) \cap \tilde{X} \). By the above to \( V = \psi(U) \). We obtain that \( \psi(U) \cap \tilde{Y} \subseteq \tau(V \cap \tilde{X}) \).

This shows that for every \( U \subseteq B \), \( \tau(U \cap \tilde{X}) = \psi(U) \cap \tilde{Y} \). Note that \( \text{Rng}(\tilde{X}) = (U \cap \tilde{X}) \cup \text{Rng}(\tilde{X}) \), and the same holds for \( \tilde{Y} \). We have thus shown that \( \tau \) takes an open base of \( \tilde{X} \) to an open base of \( \tilde{Y} \). Hence \( \tau \colon \tilde{X} \cong \tilde{Y} \). We have proved Claim 10. □

By Claim 6, \( \tilde{X} \) is \( G \)-invariant and \( \tilde{Y} \) is \( H \)-invariant. By Claims 3 and 5, \( X \setminus \tilde{X} \) and \( Y \setminus \tilde{Y} \) are nowhere dense. It remains to show that \( \langle * \rangle \) for every \( g \in G \), \( \psi(g) \cap \tilde{Y} = \tau \circ (g \downarrow \tilde{X}) \circ \tau^{-1} \).

**Claim 11.** For every \( x \in \tilde{X} \) and \( p \in PU(X) \): if \( x_p = x \), then \( y_{\psi[p]} = \tau(x) \).

**Proof.** This fact follows from the definition of the relation \( \tau \) and the fact that the relation \( \tau \) is a function. □

Using Claim 11, the proof of \( \langle * \rangle \) is identical to the proof of the analogous fact in Theorem 2.1. □

The following corollary strengthens Theorem B from the Introduction.

**Corollary 2.6.** Let \( \langle X, G \rangle, \langle Y, H \rangle \) be space-group pairs. Assume that:

1. \( G_{0} \leq G \) and \( H_{0} \leq H \) such that \( \langle X, G_{0} \rangle, \langle Y, H_{0} \rangle \in K_{WF} \).
2. For every \( x \in X, G(x) \) is somewhere dense, and for every \( y \in Y, H(y) \) is somewhere dense.

Suppose that \( \psi : G \cong H \). Then there is \( \tau : X \cong Y \) such that \( \psi(g) = \tau \circ g \circ \tau^{-1} \) for every \( g \in G \).

**Proof.** (a) Let \( \langle X, G \rangle, \langle Y, H \rangle \) and \( \psi \) be as in the theorem. Let \( \tilde{X}, \tilde{Y} \) and \( \tau \) be as assured by Theorem 2.5. Suppose by contradiction that \( \tilde{X} \neq X \), and let \( x \in X \setminus \tilde{X} \). So \( G(x) \subseteq X \setminus \tilde{X} \) and \( G(x) \) is somewhere dense. A contradiction, so \( \tilde{X} = X \).

Similarly, \( \tilde{Y} = Y \). □

### 3. Foliations

**Definition 3.1.** (a) Let \( n \in \mathbb{N} \) and \( 1 \leq \ell < n \). For \( i = 1, 2 \) let \( (u_i, v_i) \in \mathbb{R}^{\ell} \times \mathbb{R}^{n-\ell} \). Define \( (u_1, v_1) \sim^{n, \ell} (u_2, v_2) \) if \( v_1 = v_2 \).

(b) Denote by \( B_n \) and \( B_n \) the open and closed unit balls of \( \mathbb{R}^n \). Set \( B_n, \ell = B_\ell \times B_{n-\ell} \), \( B_n, \ell = B_\ell \times B_{n-\ell} \), and \( B_n, \ell(u, r) = u + r \cdot B_n, \ell \). Let \( 1 \leq \ell < n \in \mathbb{N} \).

(c) An \((n, \ell)\)-foliated manifold is a pair \( \langle X, \Phi \rangle \), where \( X \) is a topological space and \( \Phi \) is a set of maps with the following properties:

1. For every \( \psi \in \Phi, \psi : B_{n, \ell} \to X, \psi : \text{Dom}(\psi) \supseteq \text{Rng}(\psi) \), \( \text{Rng}(\psi) \) is closed in \( X \) and \( \psi(B_{n, \ell}) \) is open in \( X \).
2. For every \( x \in X \) there is \( \psi \in \Phi \) and \( y \in B_{n, \ell} \) such that \( x = \psi(y) \).
3. For every \( \psi, \psi' \in \Phi \) and \( x, y \in \text{Rng}(\psi) \cap \text{Rng}(\psi') \), \( \psi^{-1}(x) \sim^{n, \ell} \psi'^{-1}(y) \) iff \( \psi^{-1}(x) \sim^{n, \ell} \psi^{-1}(y) \).

(d) Let \( \langle X, \Phi \rangle \) be an \((n, \ell)\)-foliated manifold. For \( \psi \in \Phi \) and \( x, y \in \text{Rng}(\psi) \) we define \( x \sim^\psi y \) if \( \psi^{-1}(x) \sim^{n, \ell} \psi^{-1}(y) \). Let \( \sim^\psi \) be the transitive closure of \( \cup_{\psi \in \Phi} \sim^\psi \). The leaf of \( x \) in \( \langle X, \Phi \rangle \) is defined as \( L^\psi(X) = x / \sim^\psi \). Also define \( L(\Phi) = \{ L^\psi(X) : x \in X \} \).

(e) Let \( \langle X, \Phi \rangle \) and \( \langle Y, \Psi \rangle \) be respectively \((m, k)\) and \((n, \ell)\) foliated manifolds and \( g : X \cong Y \). We say that \( g \) is foliation-preserving, if for every \( x \in X \), \( \psi \in \Phi \) and \( \psi' \in \Psi \), if \( x \in \psi(B_{m, k}) \) and \( g(x) \in \psi'(B_{n, \ell}) \), then there is \( W \in \text{Nbr}(x) \) such that \( W \subseteq \psi'(B_{n, \ell}) \), \( g(W) \subseteq \psi(B_{m, k}) \), and for every \( u, v \in W \), \( u \sim^\psi v \) if \( g(u) \sim^{n, \ell} g(v) \).

(f) Let \( H_1(X, \Phi) \) denote the group of foliation-preserving auto-homeomorphisms of \( \langle X, \Phi \rangle \). A homeomorphism \( g \in H_1(X, \Phi) \) is called leaf-fixing, if \( g(L) = L \) for every \( L \in L(\Phi) \). We let \( H_0(X, \Phi) \) denote the group of leaf-fixing homeomorphisms of \( \langle X, \Phi \rangle \).

Note that if \( \langle X, \Phi \rangle \) is an \((n, \ell)\)-foliated manifold, then \( \Phi \) determines \( n \) and \( \ell \). And \( \Phi \) also determines the set \( X \) and the topology of \( X \). That is,

1. \( n \) and \( \ell \) are determined by \( \Phi \), since \( \text{Dom}(\psi) = B_\ell \times B_{n-\ell} \), for every \( \psi \in \Phi \). And \( B_\ell \times B_{n-\ell} \) is not the same set as \( B_{n, \ell} = B_{n, \ell} \).
2. \( X = \cup_{\psi \in \Phi} \text{Rng}(\psi) \).
3. A set \( U \subseteq X \) is open if for every \( \psi \in \Phi \), \( \psi^{-1}(U \cap \psi(B_{n, \ell})) \) is open in \( \mathbb{R}^\ell \times \mathbb{R}^{n-\ell} \).
We shall avoid mentioning $\Phi$ explicitly. So $X$ stands for $(X, \Phi)$ and $H_1(X, \Phi)$ stand for $H_1(X, \Phi)$. Also, $L_\Phi(x)$ is abbreviated by $L(x)$, and $L(\Phi)$ is also denoted by $L(X)$.

**Remark.** It is tempting to adopt a simpler definition of a foliation-preserving homeomorphism. Namely:

(D) Let $(X, \Phi)$ and $(Y, \Psi)$ be foliated manifolds and $g : X \cong Y$. Then $g$ is foliation-preserving if for every $L \in L(\Phi)$, $g(L) \in L(\Psi)$.

This definition does not give the desired notion. Indeed, in Section 4.6 we construct a $(3, 2)$-foliated manifold $X$ which has only one leaf. If (D) is taken to be the definition of foliation-preservation, then every member of $H(X)$ is foliation-preserving. This is certainly not one has in mind.

The following fact is obvious.

**Proposition 3.2.** Let $(X, \Phi)$ and $(Y, \Psi)$ be respectively $(k, m)$ and an $(\ell, n)$ foliated manifolds and $g : X \cong Y$ be foliation-preserving. Then $m = n, k = \ell$ and

$$\{ g(L) \mid L \in L(\Phi) \} = L(\Psi).$$

The goal of this section is to prove Theorem C from the introduction. It is restated below.

**Theorem 3.3.** Let $X$ and $Y$ be countably paracompact foliated manifolds, $H_0(X) \subseteq G \subseteq H_1(X)$ and $H_0(Y) \subseteq H \subseteq H_1(Y)$. Suppose that $\zeta : G \cong H$. Then there is a unique $\tau : X \cong Y$ such that $\zeta(g) = \tau \circ g \circ \tau^{-1}$ for every $g \in G$. Also, $\tau$ is foliation-preserving.

Part (a) of the following theorem is due to Palis and Smale [3]. Indeed, they did not deal with foliated manifolds. But their proof applies without change to the foliated case. Part (b) is a trivial conclusion from part (a).

**Theorem 3.4.** (Palis and Smale [3, Theorem 3.1]) (a) For $r > 0$ let $H^{n,r}_{\infty}$ be the group of all homeomorphisms $g \in H(B_n, 1)$ such that $g$ is isotopic to the identity through an $r$-smooth foliation-preserving isotopy supported by a compact subset of $B_n$. Then $H^{n,r}_{\infty}$ is factorizable.

(b) Let $X$ be a foliated manifold. Then $H_0(X)$ contains a factorizable non-fixing subgroup.

Let $(X, G)$ be a space-group pair and $X_0$ be a $G$-invariant dense subset of $X$. Define

$$\text{MPO}(X_0, G) = (X_0, \sigma^{X_0}, G; \varepsilon, \circ, \text{Ap}),$$

where $\sigma^{X_0}$ is the relative topology of $X_0$, $\varepsilon$ is the belonging relation on $X_0 \times \sigma^{X_0}$, and $\text{Ap}$ is the application function. That is, $\text{Ap} : G \times (X_0 \cup \sigma^{X_0}) \to X_0 \cup \sigma^{X_0}$, for $x \in X_0$, $\text{Ap}(g, x) = g(x)$ and for $U \in \sigma^{X_0}$, $\text{Ap}(g, U) = g[U] := \{g(x) \mid g \in U\}$.

**Proposition 3.5.** (a) Let $X$ and $Y$ be foliated manifolds, $H_0(X) \subseteq G \subseteq H_1(X)$ and $H_0(Y) \subseteq H \subseteq H_1(Y)$. Suppose that $\zeta : G \cong H$. Then there are $X_0 \subseteq X$, $Y_0 \subseteq Y$ and $\tau_0 : X_0 \cong Y_0$ such that $X \setminus X_0$ and $Y \setminus Y_0$ are nowhere dense subsets of $X$ and $Y$ respectively. $X_0$ is $G$-invariant and $Y_0$ is $H$-invariant, and $\tau_0$ induces $\zeta$.

(b) Let $X, Y, G, H$ and $\zeta$ be as in (a), and $X_0, Y_0$ and $\tau_0$ be as in the conclusion of (a). Then $\zeta \otimes \tau_0 : \text{MPO}(X_0, G) \cong \text{MPO}(Y_0, H)$.

**Proof.** Part (a) follows trivially from Theorems 2.5 and 3.4(b), and part (b) is just a restatement of (a). □

**Remark.** We shall not use the fact that $X \setminus X_0$ and $Y \setminus Y_0$ are nowhere dense. It will suffice to know that $X_0$ and $Y_0$ are dense sets.

In what follows we assume that $(X, \Phi)$ and $(Y, \Psi)$ are respectively $(n, \ell)$ and $(m, k)$ foliated manifolds, $H_0(X) \subseteq G \subseteq H_1(X)$ and $H_0(Y) \subseteq H \subseteq H_1(Y)$, $: G \cong H$. $X_0$ and $Y_0$ are dense subsets of $X$ and $Y$ respectively. $X_0$ is $G$-invariant and $Y_0$ is $H$-invariant, $\tau_0 : X_0 \cong Y_0$ and $\tau_0$ induces $\zeta$.

**Definition 3.6.** Let $Z$ be a topological space. An arc function in $Z$ is a $1$–$1$ continuous function from a proper closed interval of $\mathbb{R}$ to $Z$. So the range of an arc function is an arc. Let $A \subseteq X_0$ be an arc. We say that $A$ is a good arc, if for every $x, y \in A$ and an open subset $U$ of $X_0$ containing the subarc of $A$ whose endpoints are $x$ and $y$, there is $g \in G$ such that $g(x) = y$ and $g \upharpoonright (X_0 \setminus U) = \text{Id}$. We say that $x, y \in X_0$ are well-connected, if there are $k \in \mathbb{N}$, good arcs $A_0, \ldots, A_{k-1}$ and $x_0, \ldots, x_k \in X_0$ such that $x_0 = x, x_k = y$ and for every $i < k$ the endpoints of $A_i$ are $x_i$ and $x_{i+1}$. 


Note that the fact "A is a good arc" is a property that A has as a subset of MPO(X₀, G), so for every isomorphism \( \chi : \text{MPO}(X₀, G) \cong \text{MPO}(Y₀, H) \) and \( A \subseteq X₀ \) is a good arc iff \( \chi(A) \) is a good arc. Similarly, the fact that \( x, y \) are well-connected is expressible in \( \text{MPO}(X₀, G) \). So well-connectedness too is preserved under isomorphisms.

Let \( (Z, \sigma) \) be a topological space and \( Z₀ \) be a dense subset of \( Z \). Denote by \( \sigma₀ \) the relative topology of \( Z₀ \). For \( A \subseteq Z₀ \) define

\[
\text{int}^Z_{Z₀}(A) := \bigcup \{ U \in \sigma \mid U \cap Z₀ \subseteq A \}.
\]

Then \( \text{int}^Z_{Z₀} | \sigma₀ = 1 \to 1 \) and (\( \text{int}^Z_{Z₀} | \text{Ro}(Z₀) : \text{Ro}(Z₀) \cong \text{Ro}(Z) \)). If in addition, \( G \subseteq H(Z) \) and \( Z₀ \) is \( G \)-invariant, then denote \( G \mid Z₀ := \{ g \mid Z₀ \mid g \in G \} \).

**Proposition 3.7.** (a) Let \( \varphi \in \Phi \) and \( v ∈ B_{n−l} \). Suppose that \( A₀ \subseteq B_1 \times \{ v \} \) is an arc and let \( A = \varphi(A₀) \). Suppose that \( A \subseteq X₀ \). Then \( A \) is a good arc.

(b) Let \( A \) be a good arc, \( x, y \in A \) and \( U \) be an open set of \( X₀ \) containing the subarc of \( A \) whose endpoints are \( x \) and \( y \). Set \( \hat{U} = \text{int}^X_{X₀}(U) \). Then there is \( g \in H₀(X₀)\{\hat{U}\} \) such that \( g(x) = y \).

(c) For every \( x, y \in X₀ \),

\[
y \in \text{Lf}(x) \text{ iff } x \text{ and } y \text{ are well connected.}
\]

So the fact that for \( x, y \in X₀ \), \( y \in \text{Lf}(x) \) is a property expressible in \( \text{MPO}(X₀, G) \). Hence it is preserved under isomorphisms of \( \text{MPO}(X₀, G) \).

**Proof.** The proofs of parts (a) and (c) are trivial. We prove (b). Let \( A, x, y, U, \hat{U} \) be as in (b). Let \( A' \) be the subarc of \( A \) whose endpoints are \( x \) and \( y \). There are \( x₀, \ldots, xₖ \in A' \) and \( ϕ₁, \ldots, ϕₖ \in \Phi \) such that \( x₀ = x, xₖ = y \) and \( xₖ = N\big(ϕₖ\big) \) for every \( i = 1, \ldots, k \). Let \( Aᵢ \) be the subarc of \( A' \) whose endpoints are \( xᵢ₋₁ \) and \( xᵢ \). Then \( Aᵢ \subseteq \hat{U} \). There is an open set \( Uᵢ \) of \( X \) such that \( Aᵢ \subseteq Uᵢ \) and \( \overline{cl}^X(Uᵢ) \subseteq U \cap \overline{ϕ(Bᵢ)} \). Set \( Uᵢ = Uᵢ \cap X₀ \). For \( i = 1, \ldots, k \) let \( gᵢ \in G \) be such that \( gᵢ \mid X₀ \in (G \mid X₀)|\{\hat{U}\} \) and \( gᵢ(xᵢ₋₁) = xᵢ \). The existence of \( gᵢ \) follows from the goodness of \( A \). It follows that \( gᵢ \mid G|\{\hat{U}\} \). Let \( i \in \{1, \ldots, k\} \) Then for every \( x \in X, \text{Lf}(x) \setminus \hat{U} \neq \emptyset \). Let \( y \in \text{Lf}(x) \setminus \hat{U} \). Then \( gᵢ(y) = y \). So \( gᵢ(\text{Lf}(x)) = \text{Lf}(x) \). Hence \( gᵢ \in H₀(X₀) \). Let \( g = g₁ \circ \cdots \circ gₖ \). Then \( g(x) = y \) and \( g \in H₀(\hat{U}) \).

For a subset \( A \subseteq X \), \( \text{acc}(A) \) denotes the set of accumulation points of \( A \). We use the notation \( \hat{x} \) to denote the sequence \( \{xₙ\}_{n∈N} \), where \( \sigma \) is an infinite subset of \( N \). Also, we denote \( \sigma \) by \( \sigma₂ \). The notations \( \hat{y} \) and \( \hat{t} \) are used in the same way. Rather than writing \( \text{Rng}(\hat{x}) \subseteq A \), we write \( \hat{x} \subseteq A \). We denote \( \text{acc}(\text{Rng}(\hat{x})) \) by \( \text{acc}(\hat{x}) \). Let \( \tilde{x}, \tilde{y} \) be sequences. We say that \( \tilde{y} \) is a subsequence of \( \tilde{x} \) if \( \tilde{y} = \tilde{x} \mid \sigma₂ \). For \( \tilde{x}, \tilde{y} \subseteq X \) we define \( \tilde{x} ≜ G \tilde{y} \) if \( \sigma₂ = \sigma₂ \) and there is \( g \in G \) such that \( g(xₙ) = yₙ \) for every \( n \in \sigma₂ \). We say that \( \tilde{x} \) is a discrete sequence in \( X \), if \( \tilde{x} \) is \( 1 \to 1 \) and \( \text{acc}^X(\tilde{x}) = \emptyset \).

Let \( A \) be a family of pairwise disjoint subsets of a topological space \( X \). Then the set of accumulation points of \( A \) in \( X \) is defined as

\[
\text{acc}^X(A) = \{ x ∈ X \mid \text{for every } U ∈ \text{Nbr}(x), \{ A ∈ A \mid A ∩ U \neq \emptyset \} \text{ is infinite} \}.
\]

We say that a family \( A \) of pairwise disjoint subsets of a topological space \( X \) is discrete in \( X \) if \( \text{acc}(A) = \emptyset \). We say that \( \lim A = x \) if for every \( U ∈ \text{Nbr}(x) \), \( \{ A ∈ A \mid A ∩ U \neq \emptyset \} \) is finite. Let \( \mathcal{A} \) be a sequence of sets. We say that \( \lim \mathcal{A} = x \) if for every \( U ∈ \text{Nbr}(x) \), \( \{ i ∈ N \mid Aᵢ ∩ U \neq \emptyset \} \) is finite.

The next lemma is a key fact in the proof of Theorem 3.3.

**Lemma 3.8.** Let \( X, Y \) be countably paracompact foliated manifolds, \( H₀(X₀) \subseteq G \subseteq H₁(X₀) \) and \( H₀(Y₀) \subseteq H \subseteq H₁(Y₀) \), \( X₀ \) be a dense subset of \( X \) invariant under \( G \) and \( Y₀ \) be a dense subset of \( Y \) invariant under \( H \). (Note that by Proposition 3.7(c), \( X₀ \) and \( Y₀ \) are the union of leaves.) Suppose that \( \chi : \text{MPO}(X₀, G) \cong \text{MPO}(X₀, H) \). Then:

1. For every sequence \( \tilde{x} \subseteq X₀ \), \( \tilde{x} \) is convergent in \( X \) and \( \lim \tilde{x} ∈ X \setminus X₀ \) iff \( \chi(\tilde{x}) \) is convergent in \( Y \) and \( \lim \chi(\tilde{x}) ∈ Y \setminus Y₀ \).
2. For every \( \tilde{x}, \tilde{x'} \subseteq X₀ \) if \( \tilde{x}, \tilde{x'} \) converge to members of \( X \setminus X₀ \), then

\[
\lim \tilde{x} = \lim \tilde{x'} \quad \text{iff} \quad \lim \chi(\tilde{x}) = \chi(\lim \tilde{x'}).\]

To prove Lemma 3.8, we consider only one structure of the form \( \text{MPO}(X₀, G) \). We show that for a sequence \( \tilde{x} \subseteq X₀ \), the property that it converges to a member of \( X \setminus X₀ \) is equivalent to a certain property that \( \tilde{x} \) has as a sequence of members of \( \text{MPO}(X₀, G) \). Having shown this equivalence, we conclude that the property of converging to a member of \( X \setminus X₀ \) is preserved under isomorphisms between \( \text{MPO}(X₀, G) \) and \( \text{MPO}(Y₀, H) \). The property stated in (2) is treated in the same way.
Remark. The above approach can be rigorously formalized using notions from model theory. We will not, however, present this model-theoretic framework. It is called “the interpretation of one class of structures in another class of structures” and it is fully described in [5, Chapter 1] and in [2, Chapter 6].

Proof of Lemma 3.8. Let $I$ denote the unit interval $[0,1]$. Let $Z$ be a topological space. An arc sequence in $Z$ is a sequence $A(t) = \{ A_i(t) | i \in \sigma \}$ such that:

(A1) For every $i \in \sigma$, $A_i(t)$ is an arc function in $Z$.
(A2) Dom$(A_i(t)) = I$.
(A3) $\text{Rng}(A_i(t)) \mid i \in \sigma \}$ is a pairwise disjoint family.

Denote $\text{Rng}(A_i(t))$ by $A_i$. For $t_0 \in I$ set $\tilde{A}(t_0) = \{ A_i(t_0) \mid i \in \sigma \}$ and for $\tilde{t} = \{ t_i \mid i \in \sigma \} \subseteq I$ set $\tilde{A}(\tilde{t}) = \{ A_i(t_i) \mid i \in \sigma \}$.

Let $\tilde{x} = \{ x_i \mid i \in \sigma \} \subseteq X_0$ be a sequence and $H$ be any subgroup of $G$. An arc sequence $\tilde{A}(t) = \{ A_i(t) \mid i \in \sigma \}$ in $X_0$ is called a tester for $\tilde{x}$ and $H$ if:

(T1) For every $i \in \sigma$, $A_i(0) = x_i$.
(T2) For every $i \in \sigma$, $A_i$ is a good arc.
(T3) For every monotonic sequence $\tilde{t} = \{ t_i \mid i \in \sigma \} \subseteq I$, $\tilde{x} \sim_H \tilde{A}(\tilde{t})$.

Let $E_1(\tilde{x})$ be the following property of a sequence $\tilde{x} \subseteq X_0$:

$$E_1(\tilde{x}) \equiv \text{for every subsequence } \tilde{y} \text{ of } \tilde{x}, \text{ a tester } \tilde{A}(t) \text{ for } \tilde{y} \text{ and } G \text{ and}
$$

a subset $\eta \subseteq \sigma \tilde{y}$, $\tilde{y} \setminus (\sigma \tilde{y} \setminus \eta) \cup (\tilde{A}(1) \setminus \eta) \sim_G \tilde{y}$.

Note that $E_1(\tilde{x})$ is expressed using only the relations and operations of MPO$(X_0, G)$. So for every $\chi : \text{MPO}(X_0, G) \cong \text{MPO}(Y, H)$ and $\tilde{x} \subseteq X_0$; $E_1(\tilde{x})$ holds iff $E_1(\chi(\tilde{x}))$ holds.

Claim 1. For every $1$–$1$ sequence $\tilde{x} \subseteq X_0$ the following are equivalent:

(i) $\tilde{x}$ is discrete in $X$.
(ii) $\tilde{x}$ is discrete in $X_0$ and $\tilde{x}$ fulfills $E_1$.

Proof. (i) $\Rightarrow$ (ii) Let $\tilde{x} \subseteq X_0$ be a $1$–$1$ sequence and suppose that $\tilde{x}$ is discrete in $X$. Let $\tilde{y}$ be a subsequence of $\tilde{x}$ and $\tilde{A}(t)$ be a tester for $\tilde{y}$ and $G$. Denote $\sigma \tilde{y}$ by $\sigma$. Set $A_i = \text{Rng}(A_i(t))$ and $\tilde{A} = \{ A_i \mid i \in \sigma \}$. We show that $\text{acc}(\tilde{A}) = \emptyset$. Suppose by contradiction that $y \in \text{acc}(\tilde{A})$. Then there is an infinite $\eta \subseteq \sigma$ and a monotonic sequence $\tilde{t} = \{ t_i \mid i \in \sigma \}$ such that $y = \lim(\tilde{A}(\eta) \tilde{t} \eta)$. By (T3), there is $g \in G$ such that $g(\tilde{y}) = \tilde{A}(\tilde{t})$. Hence $\lim \tilde{y} \eta = g^{-1}(y)$. That is, $\tilde{y}$ is not discrete. A contradiction.

Let $A = \bigcup_{i \in \sigma} A_i$. Since each $A_i$ is closed in $X$ and $\tilde{x}$ is discrete, $A$ is closed in $X$ and hence $V := X \setminus A$ is open. We define by induction on $m \in \sigma$ a sequence $\{ V_m \mid m \in \sigma \}$ of open sets of $X$, such that

(1) for every $m \in \sigma$, $V_m \supseteq A_m$.
(2) $\text{cl}(V_m) \cap (\bigcup_{i < m} \text{cl}(V_i) \cup \bigcup_{i > m} A_i) = \emptyset$.

To simplify the notation we assume momentarily that $\sigma = \mathbb{N}$. Suppose that $V_i$ has been defined for every $i < m$. For every $x \in A_m$ let $V_x \in \text{Nbr}(x)$ be such that $\text{cl}(V_x) \cap (\bigcup_{i < m} \text{cl}(V_i) \cup \bigcup_{i > m} A_i) = \emptyset$. Let $V_m$ be a finite subset of $\{ V_x \mid x \in A_m \}$ such that $V_m := \bigcup_{V \in V_m} A_m$. Then $V_m$ is as required.

Clearly, $V := \{ V \mid V \subseteq \bigcup \{ V \mid m \in \sigma \} \}$ is an open cover of $X$. By the countable paracompactness of $X$, there is an open cover $U$ of $X$ such that $U$ is a refinement of $V$ and $U$ is locally finite. For every $i \in \sigma$ let $U_i = \bigcup \{ U \subseteq U \mid U \subseteq V_i \}$. Then $A_i \subseteq U_i$. Also, $\{ U_i \mid i \in \sigma \}$ is a pairwise disjoint family and it is discrete.

Let $\eta \subseteq \sigma$. Then for every $i \in \eta$ there is $g_i \in H_0(X)\setminus U_i$ such that $g(\eta) = A_i(1)$. This follows from Proposition 3.7(b). Define $g = g_i (g_i \mid i \in \eta)$. Then since $\{ U_i \mid i \in \sigma \}$ is discrete, $g \in H(X)$. It follows that $g \in H_0(X)$. Clearly, $g(\tilde{y}) = (\tilde{y} \setminus (\sigma \setminus \eta)) \cup (\tilde{A}(1) \setminus \eta)$. This proves (ii).

(ii) $\Rightarrow$ (i) Let $\tilde{x} \subseteq X_0$ be a $1$–$1$ sequence. Assume that $\tilde{x}$ is discrete in $X_0$ and that $\text{acc}^c(\tilde{x}) \neq \emptyset$. We show that $\tilde{x}$ does not fulfill $E_1$. Let $\tilde{y}$ be a subsequence of $\tilde{x}$ and $z \in X$ be such that $\lim \tilde{y} = z$. So $z \notin X_0$. There is $\varphi \in \Phi$ and $(u, v) \in B_{\eta, \ell}$ such that $z = \varphi((u, v))$. By removing finitely many members of $\varphi$, we may assume that $\varphi \subseteq \text{Rng}(B_{\eta, \ell})$. Denote $\sigma \tilde{y}$ by $\sigma$. Let $y^0 = \varphi^{-1}(\tilde{y})$. For every $i \in \sigma$ write $y^0_i = (u_i, v_i)$, where $u_i \in B_{\eta}$ and $v_i \in B_{\eta, \ell}$. Set $\tilde{u} = \{ u_i \mid i \in \sigma \}$ and $\tilde{v} = \{ v_i \mid i \in \sigma \}$. Since $\varphi((u, v)) \notin X_0$ and $X_0$ is a union of leaves, $\varphi(B_{\eta} \times \{ v \}) \cap \emptyset = \emptyset$. So for every $i \in \sigma$, $v_i \neq v$. Since $\varphi((u, v)) \neq v$, we may assume that $\tilde{v}$ is $1$–$1$. Let $w \in B_{\eta} \setminus \{ 0 \}$. Since $\lim \tilde{v} \in B_{\eta}$, it follows that for all but finitely many $i \in \sigma$, $u_i + w \in B_{\eta}$. We may assume that $u_i + w \in B_{\eta}$ for all $i \in \sigma$. The function $\gamma_i$ which is defined by

$$t \mapsto (u_i + tw, v_i), \quad t \in I,$$
is the affine function which takes $t$ to $[(u_i, v_i), (u_i + w, v_i)]$. Let $A_i(t) := \varphi \circ \gamma_i(t)$, and set $\hat{A}(t) := \{A_i(t) \mid i \in \sigma\}$. We check that $\hat{A}(t)$ is a tester for $\tilde{y}$ and $G$. Denote $\text{Rng}(A_i(t))$ by $A_i$. Then by Proposition 3.7(a), each $A_i$ is a good arc. Hence clause (T2) in the definition of a tester holds. It is trivial to verify that $\tilde{y}$ and $\hat{A}$ satisfy clauses (T1) and (T3) in the definition of a tester. So $\hat{A}$ is a tester for $\tilde{y}$ and $G$.

Let $\eta$ be an infinite and co-infinite subset of $\sigma$. Then

$$\tilde{w} := (\tilde{y} \mid (\sigma \setminus \eta)) \cup (\hat{A}(1) \mid \eta)$$

is not convergent. So $\tilde{w} \sim^C \tilde{y}$. That is, $\tilde{x}$ does not fulfill $E_1$. This proves (i), and hence Claim 1 is proved. □

For $x \subseteq X$ let $G_x$ denote the stabilizer of $x$ in $G$, that is, $G_x = \{g \in G \mid g \cdot x = \text{Id}\}$. Let $E2(\tilde{x}, \tilde{y})$ be the following property of sequences $\tilde{x}, \tilde{y} \subseteq X_0$:

$$E2(\tilde{x}, \tilde{y}) \equiv \text{for every subsequence } \tilde{z} \text{ of } \tilde{y}, \text{ a tester } \hat{A}(t) \text{ for } \tilde{z} \text{ and } G_x$$

and

$$\text{a subset } \eta \subseteq \sigma_2, (\tilde{z} \mid (\sigma_2 \setminus \eta)) \cup (\hat{A}(1) \mid \eta) \sim^C \tilde{z}.$$ 

Note that $E1(\tilde{x})$ is expressed using only the relations and operations of $\text{MPO}(X_0, G)$. So for each $\chi : \text{MPO}(X_0, G) \cong \text{MPO}(Y_0, H)$ and $\tilde{x}, \tilde{y} \subseteq X_0$; $E1(\chi(\tilde{x}), \chi(\tilde{y}))$ holds.

**Claim 2.** Let $\tilde{x}, \tilde{y} \subseteq X_0$ be sequences which are discrete in $X_0$ and convergent in $X$. Then $\lim \tilde{x} = \lim \tilde{y}$ iff $E2(\tilde{x}, \tilde{y})$ holds.

**Proof.** Set $\lim \tilde{x} = x^*$ and $\lim \tilde{y} = y^*$. Suppose that $x^* \neq y^*$. Let $\tilde{z}$ be a subsequence of $\tilde{y}$ and $\tilde{A}(t)$ be a tester for $\tilde{z}$ and $G_x$. Denote $\sigma_2$ by $\sigma$.

Let $\{t_i \mid i \in \sigma\} \subseteq \sigma_2$ be a monotonous sequence. Then there is $g \in G$ such that $g \cdot \tilde{x} = \text{Id}$ and $g(\tilde{z}) = \tilde{A}(t)$. So

$$\lim \tilde{A}(t) = \lim g(\tilde{z}) = \lim \tilde{g}(\tilde{x}) = \lim \tilde{x} = x^*.$$ 

(1)

Next suppose that $x^* = y^*$. Denote $\sigma_2$ by $\sigma$. Let $\varphi \in \Phi$ and $(u, v) \in B_{n, t}$ be such that $\varphi((u, v)) = y^*$. Let $r > 0$ be such that $\varphi(B_{n, t}(\varphi(u, v), r)) \cap \text{Rng}(\tilde{x}) = \emptyset$. For simplicity, assume that $((u, v) = (0, 0)$ and $r = 1$. By removing finitely many members of $\tilde{y}$ we obtain a subsequence $\tilde{y}'$ of $\tilde{y}$ such that $\tilde{y}' \neq \varphi(B_{n, t})$. We may assume that $\tilde{y}' = \tilde{y}$.

Let $y^0 = \varphi^{-1}(\tilde{y})$. For $\eta \in \sigma$, write $y^0 = (u_i, v_i)$, where $u_i \in B_{r} \text{ and } v_i \in B_{n, t}$. Set $\tilde{u} = \{u_i \mid i \in \sigma\}$ and $\tilde{v} = \{v_i \mid i \in \sigma\}$. Since $\varphi((0, 0)) \not\subseteq X_0$ and $X_0$ is a union of leaves, $\varphi(B_{r} \times \{0\}) \cap X_0 = \emptyset$. So for every $i \in \sigma, v_i \neq 0$. Since $\lim v_i = 0$, there is an infinite $\rho \subseteq \sigma$ such that $\tilde{v} \neq \rho$ is 1–1. Let $z^0 = y^0 \setminus \rho$ and $\tilde{z} = \varphi^{-1}(z^0)$. Since $\lim \tilde{u} \subseteq B_{r}$, there is $w \in B_{r} \setminus \{0\}$ such that for all $i \in \rho, u_i + w \in B_{r}$. For $i \in \rho$ let $g_i$ be the function

$$t \mapsto (u_i + t \cdot w_i, v_i), \text{ } t \in I,$$

and $A_i(t) := \varphi \circ \gamma_i(t), \tilde{A}(t) := \{A_i(t) \mid i \in \rho\}$ and $A_i := \text{Rng}(A_i(t))$. By Proposition 3.7(a), each $A_i$ is a good arc. That is, $\tilde{A}(t)$ satisfies clause (T2) in the definition of a tester. It is easy to check that clauses (T1) and (T3) also hold for $\tilde{A}(t)$, $\tilde{z}$ and $G_x$. Thus $\tilde{A}(t)$ is a tester for $\tilde{z}$ and $G_x$.

However, if $\eta$ is an infinite and co-infinite subset of $\rho$, then $\tilde{w} := (\tilde{z} \mid (\rho \setminus \eta)) \cup (\tilde{A}(1) \mid \eta)$ is not convergent. So $\tilde{w} \sim^C \tilde{z}$. That is, $\tilde{x}$ and $\tilde{y}$ do not fulfill $E2$. We have proved Claim 2. □

Let $E3(\tilde{x})$ be the following property of a sequence $\tilde{x} \subseteq X_0$.

1. $\tilde{x}$ is discrete in $X_0$.
2. There is no subsequence $\tilde{y}$ of $\tilde{x}$ fulfilling $E1$.
3. For every two infinite sets $\eta, \rho \subseteq \sigma_2$ there are infinite sets $\eta' \subseteq \eta$ and $\rho' \subseteq \rho$ such that $\tilde{x} \mid \eta' \eta$ and $\tilde{x} \mid \rho' \rho$ fulfill $E2$.

Note that $E3$ is preserved under any $\chi : \text{MPO}(X_0, G) \cong \text{MPO}(Y_0, H)$.
Claim 3. Let \( \bar{x} \subseteq X_0 \) be a 1–1 sequence. Then the following are equivalent:

(i) \( \bar{x} \) converges to a member of \( X \setminus X_0 \).

(ii) \( E3(\bar{x}) \) holds.

Proof. Note that by Claim 1, the conjunction of \( E3(1) \) and \( E3(2) \) is equivalent to the fact that \( \bar{x} \) has no subsequences which are discrete in \( X \).

Let \( \bar{x} \subseteq X_0 \) be a 1–1 sequence. Suppose first that \( \bar{x} \) converges to a member of \( X \setminus X_0 \). Then \( \bar{x} \) has no subsequence which is discrete in \( X \). So \( E3(1) \) and \( E3(2) \) hold. Let \( \eta, \rho \subseteq \sigma_\gamma \) be infinite. Choose \( \eta' = \eta \) and \( \rho' = \rho \). Then by Claim 2, \( \bar{x} \upharpoonright \eta' \) and \( \bar{x} \upharpoonright \rho' \) fulfill \( E2 \). We have shown that \( \bar{x} \) fulfills \( E3 \).

Suppose now that \( \bar{x} \) does not converge to a member of \( X \setminus X_0 \). If \( \bar{x} \) has a subsequence which is discrete in \( X \), then \( \bar{x} \) does not fulfill the conjunction of \( E3(1) \) and \( E3(2) \).

So suppose that \( \bar{x} \) has no discrete subsequences. Assume that \( E1(\bar{x}) \) holds. Then \( \bar{x} \) has two convergent subsequences \( \bar{y} := \bar{x} \upharpoonright \eta \) and \( \bar{z} := \bar{x} \upharpoonright \rho \) such that \( \lim \bar{y} \neq \lim \bar{z} \) and \( \lim \bar{y}, \lim \bar{z} \in X \setminus X_0 \). Hence by Claim 2, for every subsequence \( \bar{y}' \) of \( \bar{y} \) and \( \bar{z}' \) of \( \bar{z} \), \( E2(\bar{y}', \bar{z}') \) does not hold. Hence \( \bar{x} \) does not fulfill \( E3(2) \). We have proved Claim 3.

Let \( \chi : MPO(X_0, G) \cong MPO(Y_0, H) \) and \( \bar{x} \subseteq X_0 \) be 1–1. Then \( \bar{x} \) converges to a member of \( X \setminus X_0 \) iff \( E3(\bar{x}) \) holds iff \( E3(\chi(\bar{x})) \) holds iff \( \chi(\bar{x}) \) converges to a member of \( Y \setminus Y_0 \).

A similar argument, using now \( E2 \), shows that 1–1 sequences converging to the same member of \( X \setminus X_0 \) are sent by \( \chi \) to sequences converging to the same member of \( Y \setminus Y_0 \).

Proof of Theorem 3.3. Let \( X \) and \( Y \) be countably paracompact foliated manifolds, \( H_0(X) \subseteq G \subseteq H_1(X) \) and \( H_0(Y) \subseteq H \subseteq H_1(Y) \) and \( \zeta : G \cong H \).

By Proposition 3.5, there are a dense \( G \)-invariant set \( X_0 \subseteq X \), a dense \( H \)-invariant set \( Y_0 \subseteq Y \) and \( \tau_0 : X_0 \cong Y_0 \) such that

\[
\zeta \cup \tau_0 : MPO(X_0, G) \cong MPO(Y_0, H).
\]

Denote \( \zeta \cup \tau_0 \) by \( \chi \). We define \( \tau : X \rightarrow Y \). For \( x \in X_0 \), \( \tau(x) = \tau_0(x) \). Let \( x \in X \setminus X_0 \). Choose a 1–1 sequence \( \bar{x} \subseteq X_0 \) such that \( \lim \bar{x} = x \). Then by Lemma 3.8(1), \( \tau_0(\bar{x}) \) converges to a member of \( Y \setminus Y_0 \). Define \( \tau(x) = \lim \tau_0(\bar{x}) \). By Lemma 3.8(2) the definition of \( \tau(x) \) does not depend on the choice of \( \bar{x} \).

We leave it to the reader to check that \( \tau \) is a bijection between \( X \) and \( Y \), and we show that \( \tau \) is continuous. It is an immediate conclusion from the definition of \( \tau \) that for every \( x \in X \setminus X_0 \), \( \tau \upharpoonright (X_0 \cup \{x\}) \) is continuous. A trivial general lemma says that this implies that \( \tau \) is continuous. The same proof shows that \( \tau^{-1} \) is continuous. So \( \tau : X \cong Y \).

It remains to show that for every \( g \in G \) and \( x \in X \), \( \zeta(g)(\tau(x)) = \tau(g(x)) \). This fact is true for every \( x \in X_0 \), and since \( X_0 \) is dense in \( X \), it is true for every \( x \in X \).

The uniqueness of \( \tau \) follows from the fact that \( G \) is centerless.

4. Further observations

4.1. Smooth foliations

If \( X \) is an \( r \)-smooth foliated manifold and \( k \leq r \), denote by \( C^k_r(X) \) the group of \( k \)-smooth foliation-preserving homeomorphisms of \( X \) and by \( C^k_r(\mathbb{H}) \) the group of \( k \)-smooth leaf-fixing homeomorphisms of \( X \). We consider the class \( K_{SM} \) of all space-group pairs of the form \( \langle X, C^k_r(X) \rangle \). The faithfulness of \( K_{SM} \) is considered in [8]. Officially, [8] deals only with the case that \( k = \infty \), but the faithfulness proof there, in reality, assumes only \( k \geq 1 \). It seems though that the faithfulness proof in [8] does not apply to all countably paracompact manifolds. This stems from the fact that in that proof the foliated manifolds in question are assumed to have the following property:

(R) If \( g \in C^k_r(X) \), \( g(x) = x \) and \( \text{Det}(Dg(x)) > 0 \) with respect to some local chart which includes \( x \), then for every \( U \in \text{Nbr}(x) \)

there are \( V \in \text{Nbr}(x) \) and \( h \in C^k_r(\mathbb{H}) \) such that \( h \upharpoonright V = g \upharpoonright V \) and \( h \upharpoonright (X \setminus U) = \text{Id} \).

Whereas many foliated manifolds have property (R), there are also some countably paracompact foliated manifolds that do not have this property.

In a subsequent work we shall show that \( K_{SM} \) is faithful. However, the faithfulness of a certain subclass of \( K_{SM} \) can be deduced from Corollary 2.6 in this work.

Let \( K_1 \) be the class of all \( \langle (X, G) \in K_{SM} \rangle \) such that the orbit of every \( x \in X \) under \( G \) is somewhere dense. Then by Corollary 2.6, \( K_1 \) is faithful. Note that \( \langle B_{n, r}, C^k_r(B_{n, r}) \rangle \in K_1 \) for every \( 1 \leq k \leq \infty \). Let \( X \) be the torus with a foliation consisting of lines with a fixed irrational angle. Then \( \langle X, C^k_r(X) \rangle \in K_1 \) for every \( 1 \leq k \leq \infty \) and \( i = 0, 1 \).
4.2. A factorizable space-group pair which does not belong to Rybicki's or Ling's classes

We construct a non-fixing factorizable space-group pair \( \langle X, G \rangle \) such that

1. For every \( x \in X \), \( G(x) \) is nowhere dense,
2. For every \( g \in G \), \( \text{fix}(g) \) has no isolated points.

This space-group pair satisfies the assumptions of Theorem 2.1, so every automorphism of \( G \) is induced by a homeomorphism of \( X \). However, \( \langle X, G \rangle \) does not satisfy clauses (B1) and (B2) from [9] which are quoted in the introduction. So the theorem of Rybicki does not apply to \( \langle X, G \rangle \). Also, \( \langle X, G \rangle \) does not satisfy the assumptions of Ling from [1, Theorem 2.1]. So [1, Theorem 2.1] does not apply to \( \langle X, G \rangle \) either.

**Example 4.1.** Let \( \langle Y, H \rangle \) be any non-fixing factorizable space-group pair such that \( Y \) is compact, and let \( C \) be the Cantor set, (or in fact, any 0-dimensional compact space). Define \( X = Y \times C \). Let

\[
\mathcal{H} = \{ h \mid h : C \to H \text{ and } h^{-1}(h) \text{ is clopen for every } h \in H \}.
\]

(So \( \text{Rng}(h) \) is finite for every \( h \in \mathcal{H} \).) For every \( h \in \mathcal{H} \) define \( g_h : X \to X \) as follows: \( g_h((y, c)) = (h(c)(y), c) \), and let \( G = \{ g_h \mid h \in \mathcal{H} \} \).

It is left to the reader to check that \( \langle X, G \rangle \) is factorizable and non-fixing, and that it satisfies (1) and (2) above.

4.3. The assumptions of 2.5 do not ensure the existence of an inducing homeomorphism

The following example shows that the assumptions of Theorem 2.5 do not ensure the existence of \( \tau : X \cong Y \).

**Example 4.2.** Let \( X \) be \( \mathbb{R}^2 \) with the foliation consisting of the lines parallel to the \( y \)-axis and \( G = H_0(X) \). Let \( Y = X \setminus ([0] \times \mathbb{R}) \) and \( H = \{ g \mid Y \mid g \in G \} \).

Obviously, \( X \not\cong Y \) and \( G \cong H \). Let \( G^* = C_{c0}^\infty(X) \) be the group of all \( C^\infty \) homeomorphisms which are isotopic to the identity through a compactly supported \( C^\infty \) isotopy. Notice that \( G^* \) is a factorizable non-fixing subgroup of \( G \). Let \( Xr = X \setminus ([t-r] \times \mathbb{R}) \), \( H^* = \{ g \mid Y \mid g \in G^* \} \), and \( H^r = H^*|Xr \} \). Notice that \( \bigcup_{r \in \mathbb{R}} H^r \) is a factorizable non-fixing subgroup of \( H \).

(The proof is identical to the proof that \( C_{c0}^\infty(X) \) is factorizable.) Hence, the assumptions of Theorem 2.5 are met, whereas \( X \not\cong Y \).

4.4. A weakly factorizable space-group pair which is not factorizable

**Example 4.3.** There is a space-group pair \( \langle X, G \rangle \) such that:

1. \( X \) is compact.
2. \( G \) is a locally moving group of \( X \).
3. \( \langle X, G \rangle \) is weakly factorizable.
4. \( \langle X, G \rangle \) is not factorizable.
5. For every \( x \in X \), \( G(x) \) is nowhere dense.

Let \( X = \{0,1\}^{\mathbb{N}_0} \) for every \( x, y \in X \) define \( d(x, y) \) to be \( 1/(n + 1) \), where \( n \) is the length of the maximal common initial segment of \( x \) and \( y \). Let \( \rho \) be an infinite and co-infinite subset of \( \mathbb{N} \). Let \( G \) be the group of all isometries \( g \) of \( X \) such that \( g(x) \mid \rho = x \mid \rho \) for every \( x \in X \). Then \( \langle X, G \rangle \) satisfies clauses (1)–(5).

Note that the results of this work do not imply that every automorphism of \( G \) is induced by a homeomorphism of \( X \). However, define \( G_1 \) to be the group of all isometries \( g \) of \( X \) such that for every \( x, y \in X \), \( g(x) \mid \rho = g(y) \mid \rho \) if and only if \( x \mid \rho = y \mid \rho \). Then by Corollary 2.6, every automorphism of \( G_1 \) is inner.

4.5. Applications to non-metrizable spaces

Let \( X = [0,1]^\lambda \), where \( \lambda \) is any infinite cardinal. Let \( \rho \subseteq \lambda \) be an infinite set. Define \( G_1(X, \rho) \) to be the group of all \( g \in H(X) \) such that for every \( x, y \in X \), \( x \mid \rho = y \mid \rho \) if and only if \( g(x) \mid \rho = g(y) \mid \rho \). The group \( G_1(X, \rho) \) fulfills the conditions of Corollary 2.6. Hence every automorphism of \( G_1(X, \rho) \) is inner.
4.6. A foliation that has just one leaf

We construct a 3-dimensional foliated manifold that has just one leaf, and this leaf is 2-dimensional.

We first construct a space $Y$ which is embeddable in $\mathbb{R}^2$ as an open set, and which can be pictured as a binary tree. We use one building block which we denote by $Z$.

$Z$ is the union of two parallelograms $Z_0, Z_1$. The vertices of $Z_0$ are $(-1,0), (1,0), (-1,2), (-3,2)$ and those of $Z_1$ are $(-1,0), (1,0), (1,2), (3,2)$. The top and bottom sides of each parallelogram without their endpoints are subsets of $Z$, and the left and right sides are not. Let $Z_{\text{bot}} = ((-1,0), (1,0))$ be the open line segment. Similarly, $Z_{\text{fl}} = ((-3,2), (-1,2))$ and $Z_{\text{rt}} = ((3,2), (1,2))$. Hence we assume that $Z_{\text{bot}}, Z_{\text{fl}}, Z_{\text{rt}} \subseteq Z$. On the other hand, the closed line segments $[-1,0), (-3,2), [1,0), (-1,2), [(-1,0), (1,2)], [(1,0), (2,3)]$ are disjoint from $Z$.

We glue copies of $Z$ according to the following sketch.

Let $\Lambda$ denote the empty sequence and $\eta$ denote the set of finite $\{0,1\}$-sequences. The members of $\eta$ are the nodes of the binary tree, and to each we assign a copy of $Z$. For $\eta \in \eta \setminus \{\Lambda\}$ let $Z_{\eta} = Z \times \{\eta\}$, $Z_{\eta,0} = Z_0 \times \{\eta\}$ and $Z_{\eta,1} = Z_1 \times \{\eta\}$. The copy of $Z$ assigned to $\Lambda$ is taken without its bottom. That is, we define $Z_{\Lambda} = (Z \setminus Z_{\text{bot}}) \times \{\Lambda\}, Z_{\Lambda,0} = (Z_0 \setminus Z_{\text{bot}}) \times \{\Lambda\}$ and $Z_{\Lambda,1} = (Z_1 \setminus Z_{\text{bot}}) \times \{\Lambda\}$. Set $\tilde{Z} = \bigcup_{\eta \in \eta} Z_{\eta}$. For $\eta \in \eta$ and $\ell \in \{0,1\}$ denote by $\eta^{\sim}(\ell)$ the sequence $\eta$ with $\ell$ added at the end. For every $\eta \in \eta$ we identify $Z_{\eta,0} \times \{\eta\}$ with $Z_{\text{bot}} \times \{\eta^{\sim}(0)\}$. That is, the point $(x,2,\eta)$ is identified with $(x+2,0,\eta^{\sim}(0))$. Similarly, we identify $Z_{\text{fl}} \times \{\eta\}$ with $Z_{\text{bot}} \times \{\eta^{\sim}(1)\}$. Officially, we make the following definitions. We equip $\tilde{Z}$ with the sum topology; that is, $U \subseteq \tilde{Z}$ is open, if $U \cap Z_{\eta}$ is open for every $\eta \in \eta$. Let $E$ be the equivalence relation on $\tilde{Z}$ generated by the set of pairs $E_0 \cup E_1$, where

$$E_0 = \{((x,2,\eta), (x+2,0,\eta^{\sim}(0))) \mid \eta \in \eta \text{ and } x \in [-3, -1]\}$$

and

$$E_1 = \{((x,2,\eta), (x+2,0,\eta^{\sim}(1))) \mid \eta \in \eta \text{ and } x \in [1, 3]\}.$$ 

Define $\overline{Y} := \tilde{Z}/E$. Clearly, $Y$ is homeomorphic to an open subset of $\mathbb{R}^2$.

Let $V$ denote the set of infinite $\{0,1\}$-sequences. For $v \in V$ and $n \geq 0$ denote by $v \upharpoonright n$ the initial segment of $v$ with length $n$. So $v \upharpoonright n \in \eta$. With every $v \in V$ we associate an open subset of $\overline{Y}$ which we call the strip of $v$:

$$\text{Strip of } v = (Z_{v,0} \cup Z_{v,1,\eta(1)} \cup \cdots)$/E.$$

To the top of each strip we add a boundary homeomorphic to $(-1,1)$. The official description follows. Set $Y_{\infty} := (-1,1) \times V$ and $Y = \overline{Y} \cup Y_{\infty}$. We define a topology on $Y$. For $I = (a,b) \subseteq (-1,1)$ define the left parallelogram $I[0]$ and the right parallelogram $I[1]$. $I[0]$ is the parallelograms whose vertices are $(0,a), (0,b), (b-2,2), (a-2,2)$, and the vertices of $I[1]$ are $(0,a), (0,b), (b+2,2), (a+2,2)$ respectively. So $I[\ell] \subseteq Z_{\ell}$. We assume that the top edge of $I[\ell]$ without its endpoints is a subset of $I[\ell]$, but all the other edges are disjoint from $I[\ell]$. 
For \( I \) as above, \( n \geq 0 \) and \( v \in V \) define \( \text{strp}(I, n, v) \) to be the union of all parallelograms starting from the \( n \)-th one along the strip of \( v \). That is,

\[
\text{strp}(I, n, v) := \bigcup_{m \geq n} I[v(m)] \times \{v \mid m\}/E.
\]

We define a topology on \( Y \). An open base for \( Y \) consists of all open subsets of \( \hat{Y} \) together with all sets of the form

\[
\text{strp}(I, n, v) \cup I \times \{v\},
\]

where \( I \subseteq (-1, 1) \) is open interval, \( v \in V \) and \( n \geq 0 \).

Note that \( Y \) is a 2-dimensional manifold with boundary, and \( \partial Y = Y_\infty \). Also note that \( Y \) is not embeddable in \( \mathbb{R}^2 \), since \( \{0\}, \{v \mid v \in V\} \) is an uncountable discrete subset of \( Y \).

Let \( \tilde{X} = Y \times (0, 1) \). Then \( \tilde{X} \) is a foliated manifold with boundary. The leaves of \( \tilde{X} \) are the sets of the form \( Y \times \{a\} \), where \( a \in (0, 1) \).

Next we glue the leaves of \( \tilde{X} \) so that they form a single leaf. In this way we shall obtain the final foliated manifold \( X \).

For \( v \in V \) and \( \ell \in (0, 1) \) denote by \( (\ell) \hat{\sim} v \) the sequence \( \rho \in V \) such that \( \rho(0) = \ell \), and for every \( n > 0 \), \( \rho(n) = v(n - 1) \). Let \( \{f_v \mid v \in V\} \) be an enumeration of all order-preserving homeomorphisms of \( (0, 1) \). For every \( v \in V \), \( a \in (0, 1) \) and \( x \in (-1, 1) \) we identify \( (x, (0) \hat{\sim} v, a) \) with \( (x, (1) \hat{\sim} v, f_v(a)) \). Hence the line segments \( (-1, 1) \times \{(0) \hat{\sim} v\} \times \{a\} \) and \( (-1, 1) \times \{(1) \hat{\sim} v\} \times \{f_v(a)\} \) are glued together. This makes the leaves \( Y \times \{a\} \) and \( Y \times \{f_v(a)\} \) unite to one leaf. Since for every \( a, b \in (0, 1) \) there is an order-preserving homeomorphism of \( (0, 1) \) taking \( a \) to \( b \), all leaves of \( \tilde{X} \) unite to a single leaf of \( X \).

Formally, define \( F \) to be the equivalence relation generated by

\[
\{((x, (0) \hat{\sim} v, a), (x, (1) \hat{\sim} v, f_v(a))) \mid v \in V, a \in (0, 1), x \in (-1, 1)\},
\]

and define \( X := \tilde{X}/F \).

The formal verification that \( X \) is \((3, 2)\)-foliated manifold, that has a single leaf is left to the reader.

References