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Nowhere-zero 4-flow in almost Petersen-minor free graphs

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Abstract

Tutte [W.T. Tutte, On the algebraic theory of graph colorings, J. Combin. Theory 1 (1966) 15–20] conjectured that every bridgeless Petersen-minor free graph admits a nowhere-zero 4-flow. Let $(P_{10})_{\bar{\mu}}$ be the graph obtained from the Petersen graph by contracting μ edges from a perfect matching. In this paper we prove that every bridgeless $(P_{10})_{\bar{3}}$-minor free graph admits a nowhere-zero 4-flow.

Keywords: Integer flow; 4-flow; Edge-3-coloring; Petersen minor; Almost Petersen-minor free graphs

1. Introduction

The concept of integer flow was introduced by Tutte as a generalization of map coloring problem. The following conjecture is one of the major open problems in graph theory.

Conjecture 1.1 (*Tutte [14]*). *Every bridgeless graph without a Petersen minor admits a nowhere-zero* 4*-flow.*

For planar graphs, admitting a nowhere-zero 4-flow is equivalent to having a face 4-coloring. Hence, by the 4-Color Theorem [1–3,10], Conjecture 1.1 has been verified for all planar graphs. Furthermore, it was also announced that Conjecture 1.1 was verified for all cubic graphs [11,12]. By the Kuratowski Theorem, a graph is planar if and only if it contains neither K_5-minor nor $K_{3,3}$-minor. By applying the 4-Color Theorem, Conjecture 1.1 was further verified for $K_{3,3}$-minor free graphs [15], K_5-minor free graphs [7], and P_{10}^--minor free graphs [13]. Each of these families contains the family of all planar graphs and may not necessarily be cubic. Graphs K_5, $K_{3,3}$, P_{10} and P_{10}^- are illustrated in Figs. 1–5.

Let P_{10} be the Petersen graph with the exterior pentagon $1'2'3'4'5'1'$, interior pentagon $1''3''5''2''4''1''$ and a perfect matching $M = \{e_i = i'i'' : i = 1, 2, 3, 4, 5\}$. Let $(P_{10})_{\bar{\mu}}$ be the graph obtained from P_{10} by contracting F, where $F \subseteq M$ and $|F| = \mu$.

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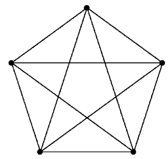


Fig. 1.

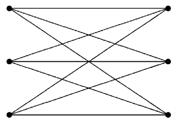


Fig. 2.

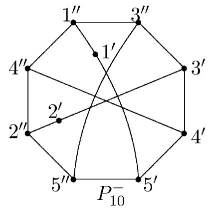


Fig. 3.

Remark. It is not hard to see that if M and M' are two perfect matchings of P_{10}, $F \subseteq M$, $F' \subseteq M'$ and $|F| = |F'|$, then $P_{10}/F \cong P_{10}/F'$. Hence $(P_{10})_{\bar{\mu}}$ is well defined.

The following is our main theorem.

Theorem 1.2. *Let G be a bridgeless graph. If G does not have a* $(P_{10})_{\bar{3}}$*-minor, then G admits a nowhere-zero* 4*-flow.*

2. Notation and terminologies

For terms that are not defined here, readers can refer to textbooks [4,8,16] (for flows).

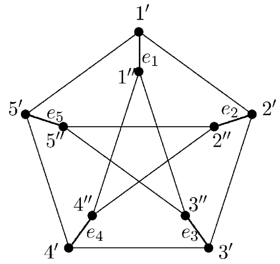
Let $G = (V, E)$ be a graph with vertex set V and edge set E and let D be an orientation of G. For a vertex $v \in V(G)$, let $E^+(v)$ (or $E^-(v)$) be the set of all arcs of $D(G)$ with their tails (or heads, respectively) at the vertex v. G is said to *admit a nowhere-zero k-flow* if there exists an ordered pair (D, f), where $f : E(G) \to \{\pm 1, \pm 2, \ldots \pm (k-1)\}$, such that

$$\sum_{e \in E^+(v)} f(e) = \sum_{e \in E^-(v)} f(e)$$

for every vertex $v \in V(G)$. A graph G is a 4-*flow snark* if it is bridgeless and does not admit a nowhere-zero 4-flow. Let G and H be two graphs. If G contains a subgraph which is contractible to H, then H is a *minor* of G and we say G contains an H-minor. A 4-flow snark G is *minor-prime* if no proper minor of G is a 4-flow snark. With the definitions above, Conjecture 1.1 can be restated as follows.

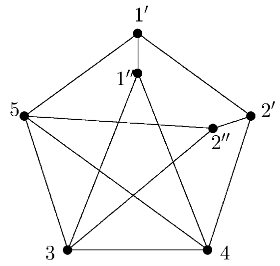
Conjecture 2.1. *The Petersen graph is the only minor-prime* 4*-flow snark.*

Let H be a minor of a connected graph G. Then there is an onto mapping $f : V(G) \mapsto V(H)$ such that $f^{-1}(v)$ induces a connected subgraph $G[f^{-1}(v)]$ of G for every $v \in V(H)$ and H can be obtained from a spanning subgraph of G by contracting the edges of $G[f^{-1}(v)]$ for all $v \in V(H)$. Here f is called a *minor mapping* and $f^{-1}(v)$ is called a *v-domain* of f. A *k-separator* of a graph G is an ordered triple $(H_1, H_2; T)$ such that $H_1 \cup H_2 = G$ and $V(H_1 \cap H_2) = T$, where T is a vertex subset of G and $|T| = k$. Sometimes we say T is a k-separator if there is no



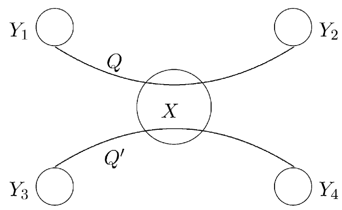
Petersen graph P_{10}

Fig. 4.



$(P_{10})_3 = P_{10}/\{e_3, e_4, e_5\}$

Fig. 5.



X is $\{\{1,2\},\{3,4\}\}$-splittable

Fig. 6.

confusion. A k-separator $(H_1, H_2; T)$ of G is *trivial* if one of H_1 and H_2, say H_1, is acyclic. G is *quasi-k-connected* if G is 3-connected and every t-separator of G with $t \leq k$ is trivial. Let x be a vertex of G. The vertex x *separates* G into q parts $H_1, \ldots, H_q$ if $G = H_1 \cup \cdots \cup H_q$ and $V(H_i \cap H_j) = \{x\}$ for every pair of $i \neq j$.

Let X be a connected subgraph of G and Y_1, Y_2, Y_3, Y_4 be four disjoint connected subgraphs of $G - V(X)$ and $X \cap N(Y_i) \neq \emptyset$ for $i = 1, 2, 3, 4$ where $N(Y_i)$ denotes the set of neighbors of Y_i. Let $J = \{Y_1, Y_2, Y_3, Y_4\}$. For each 2×2-partition $P = \{\{a, b\}, \{c, d\}\}$ of $\{1, 2, 3, 4\}$, X is *P-splittable* if X contains two disjoint paths Q and Q' such that Q joins $X \cap N(Y_a)$ and $X \cap N(Y_b)$, Q' joins $X \cap N(Y_c)$ and $X \cap N(Y_d)$, $i \in \{a, b, c, d\}$. An example of a $\{\{1, 2\}, \{3, 4\}\}$-splittable subgraph is illustrated in Fig. 6. X is *k-splittable* with respect to J if there are k distinct 2×2 partitions $P_1, \ldots, P_k$ of $\{1, 2, 3, 4\}$ such that X is P_i-splittable for each $i = 1, \ldots, k$. (Remark: $k \leq 3$.) An example of a 2-splittable subgraph is illustrated in Fig. 7.

3. Lemmas

Lemma 3.1 (*Catlin [5]*). *If G is a minor-prime 4-flow snark, then the girth of G is at least 5.*

Lemma 3.2 (*Lai, Li and Poon [7]*). *If a bridgeless graph G does not admit a nowhere-zero 4-flow, then G has a K_5-minor.*

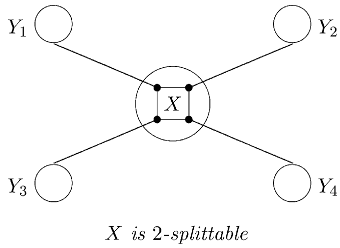


Fig. 7.

Lemma 3.3 (*Thomas and Thomson, Lemma 4.4 of [13]*). *If G is a minor-prime 4-flow snark, then G is quasi-4-connected (that is, every k-separator of G is trivial for each $k \le 3$).*

Obviously, Lemma 3.3 generalizes Lemma 3.1 and Theorem 3.7.15 of [16].

Proposition 3.4. *Let X be a connected subgraph of G and Y_1, Y_2, Y_3, Y_4 be four disjoint connected subgraphs of $G - V(X)$ where $V(X) \cap N(Y_i) \neq \emptyset$ for $i = 1, 2, 3, 4$. Let k be the greatest integer such that X is k-splittable with respect to $J = \{Y_1, Y_2, Y_3, Y_4\}$.*

(i) *If $k \le 1$, say, X is {{1,2},{3,4}}-splittable or 0-splittable, then X has a 1-separator $(H_1, H_2; \{x\})$ such that $V(X) \cap [N(Y_1) \cup N(Y_2)] \subseteq V(H_1)$ and $V(X) \cap [N(Y_3) \cup N(Y_4)] \subseteq V(H_2)$.*
(ii) *If $k = 0$, then there exists a cut vertex x of X that separates X into four parts H_1, H_2, H_3, H_4 such that $V(X) \cap N(Y_i) \subseteq V(H_i)$ for each i.*

Proof. (i) Let G_1 be the graph induced by $X \cup Y_1 \cup Y_2 \cup Y_3 \cup Y_4$. Let G_2 be the graph obtained from G_1 by contracting each Y_i into a single vertex y_i for $i = 1, 2, 3, 4$, and deleting all edges between y_i and y_j for all $\{i, j\} \subset \{1, 2, 3, 4\}$. Note that G_2 is connected since $V(X) \cap N(Y_i) \neq \emptyset$ for $i = 1, 2, 3, 4$.

Since X is neither {{1,3},{2,4}}-splittable nor {{1,4},{2,3}}-splittable, it is impossible that there is a pair of disjoint paths joining $\{y_1, y_2\}$ and $\{y_3, y_4\}$. By Menger's theorem, there is a cut vertex $x \in V(G_2)$ that separates $\{y_1, y_2\}$ and $\{y_3, y_4\}$. It is obvious that $x \in V(X)$. That is, X has a 1-separator $(H_1, H_2; x)$ that $N_{G_2}(y_1) \cup N_{G_2}(y_2) \subseteq V(H_1)$ and $N_{G_2}(y_3) \cup N_{G_2}(y_4) \subseteq V(H_2)$.

(ii) Continue from (i). Assume that there is a path P_1 joining y_1 and y_2 in the graph $G_2 - \{x\}$ (without passing through x). Note that x is a cut vertex that separates $\{y_1, y_2\}$ and $\{y_3, y_4\}$. Thus, this path P_1 is contained in the induced subgraph $G_2[V(H_1 - x) \cup \{y_1, y_2\}]$ and there is another path P_2 joining y_3 and y_4 in the induced subgraph $G_2[H_2 \cup \{y_3, y_4\}]$ since H_2 is connected. This contradicts that X is 0-splittable. So every path from y_1 to y_2 must go through x. Symmetrically, every path from y_3 to y_4 must go through x as well. That implies each component of $X - x$ is adjacent to at most one of $\{y_1, y_2, y_3, y_4\}$. ■

4. Proof of the main theorem

Let G be a minor-prime 4-flow snark. By Lemma 3.3, G is quasi-4-connected. By Lemma 3.2, K_5 is a minor of G. Let $V(K_5) = \{v_1, v_2, v_3, v_4, v_5\}$, and $f : V(G) \mapsto V(K_5)$ be a minor mapping.

If G does not contain a $(P_{10})_{\bar{4}}$-minor, then v_a-domain $f^{-1}(v_a)$ is at most 0-splittable with respect to $\{f^{-1}(v_{i_j}) : j = 1, 2, 3, 4\}$ for every $\{a, i_1, i_2, i_3, i_4\} = \{1, 2, 3, 4, 5\}$. By Proposition 3.4(ii), each $f^{-1}(v_a)$ has a cut vertex v_a^* that separates $N(f^{-1}(v_{i_j}))$ for $j = 1, 2, 3, 4$. Hence $\{v_i^*, v_j^*\}$ is a 2-separator of G. Since $G \neq K_5$, there exist $\{i, j\} \subseteq \{1, 2, 3, 4, 5\}$ such that $\{v_i^*, v_j^*\}$ is a non-trivial 2-separator. This contradicts the fact that G is quasi-4-connected. Hence G contains $(P_{10})_{\bar{4}}$ as a minor.[1]

Let $f : V(G) \to (P_{10})_{\bar{4}}$ be a minor mapping where the vertex set of $(P_{10})_{\bar{4}}$ is $\{v_{1'}, v_{1''}, v_2, v_3, v_4, v_5\}$, the contraction of the edge $v_{1'}v_{1''}$ yields a K_5, $v_{1'}$ is adjacent to v_2 and v_5, and $v_{1''}$ is adjacent to v_3 and v_4. Let $U_i = f^{-1}(v_i)$ for $i \in \{1', 1'', 2, 3, 4, 5\}$ (see Fig. 8). Define $U_1 = U_{1'} \cup U_{1''}$ and choose a minor mapping f such that $|U_1|$ is as small as possible. Now assume that G does not contain a $(P_{10})_{\bar{3}}$-minor.

[1] It was suggested by a referee that this part of the proof can be obtained directly by applying the Splitter Theorem (see [6,9]). Here, for the purpose of completeness, we include this short proof.

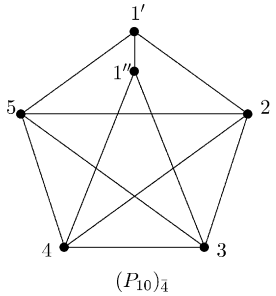


Fig. 8.

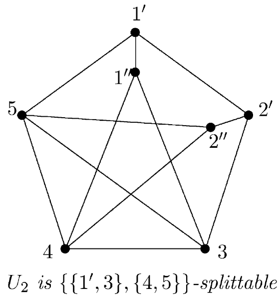


Fig. 9.

Claim 1. $|U_{1'}| = |U_{1''}| = 1$.

Proof. It is clear that $|U_{1'}| \geq 1$ and $|U_{1''}| \geq 1$.

Let $e = u_{1'}u_{1''}$ be an edge between $U_{1'}$ and $U_{1''}$ where $u_{1'} \in U_{1'}$, $u_{1''} \in U_{1''}$. Since each of $U_{1'}$ and $U_{1''}$ is connected, there are spanning trees T_1 and T_2 of $U_{1'}$ and $U_{1''}$, respectively. Let $T = T_1 \cup T_2 \cup \{e\}$. T is a spanning tree of U_1.

Assume there exist $w_2 \in N(U_2) \cap U_{1'}$ and $w_5 \in N(U_5) \cap U_{1'}$ such that $w_2 \neq w_5$. Since T_1 is a spanning tree of $U_{1'}$, there is a unique path P_2 from $u_{1'}$ to w_2 in T_1, and a path P_5 from $u_{1'}$ to w_5 in T_1. Without loss of generality, we may assume that P_2 is not shorter than P_5. Since $w_2 \neq w_5$, P_5 does not contain w_2. Let C_2 be the set of vertices of the component of $T_1 \setminus P_5$ that contains w_2. Now we define a new minor mapping f_1 by $f_1^{-1}(v_i) = f^{-1}(v_i)$ for $i = 1'', 3, 4, 5$, $f_1^{-1}(v_{1'}) = f^{-1}(v_{1'}) \setminus C_2$ and $f_1^{-1}(v_2) = f^{-1}(v_2) \cup C_2$. We call this operation *moving* w_2 *from* $U_{1'}$ *to* U_2.

$$|f_1^{-1}(v_{1'}) \cup f_1^{-1}(v_{1''})| = |f^{-1}(v_{1'}) \cup f^{-1}(v_{1''})| - |C_2| < |f^{-1}(v_{1'}) \cup f^{-1}(v_{1''})|.$$

That contradicts the choice of f. So we have $N(U_2) \cap U_{1'} = N(U_5) \cap U_{1'} = \{u\}$ for some u. Similarly, $N(U_3) \cap U_{1''} = N(U_4) \cap U_{1''} = \{v\}$ for some v.

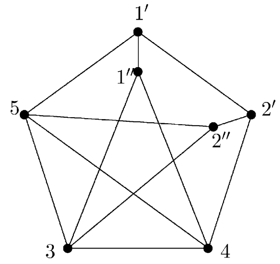
Since G is quasi-4-connected, if $\{u, v\}$ is a 2-separator, then $|U_1| = 2$. If $|U_1| \geq 3$, then $\{u, v\}$ is not a 2-separator and there exists $w \in U_1 \setminus \{u, v\}$ such that $w \in N(U_i)$ for some $i = 2, 3, 4, 5$. Without loss of generality, we can assume $w \in N(U_2)$.

Since $N(U_2) \cap U_{1'} = \{u\}$, $w \notin U_{1'}$. If the path P from u to v in T passes through w, then we can move w from $U_{1''}$ to $U_{1'}$, which contradicts $w \notin U_{1'}$. If P does not pass through w, then we can move w from $U_{1''}$ to U_2, which contradicts the choice of f. ■

From Claim 1, we can let $U_{1'} = \{u_{1'}\}$ and $U_{1''} = \{u_{1''}\}$.

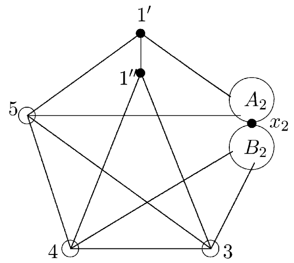
Claim 2. *U_2 is at most* 1*-splittable with respect to* $J = \{1', 3, 4, 5\}$ *with a possible partition* $\{\{1', 5\}, \{3, 4\}\}$.

Proof. U_2 is neither $\{\{1', 3\}, \{4, 5\}\}$-splittable nor $\{\{1', 4\}, \{3, 5\}\}$-splittable. Otherwise we can have the $(P_{10})_{\bar{3}}$-minors illustrated in Figs. 9 and 10, respectively. ■



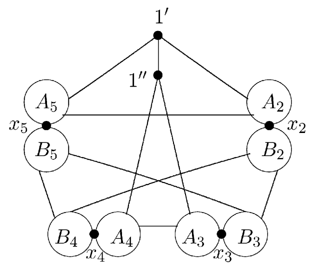
U_2 *is* $\{\{1', 4\}, \{3, 5\}\}$-*splittable*

Fig. 10.



U_2 *is at most* 1-*splittable*
w.r.t. $J = \{1', 3, 4, 5\}$

Fig. 11.



Each U_i *is at most* 1-*splittable*
for $i = 2, 3, 4, 5$

Fig. 12.

By Proposition 3.4(i), U_2 has a 1-separator $(A_2, B_2; x_2)$ such that $[N(U_{1'}) \cup N(U_5)] \cap U_2 \subseteq A_2$ and $[N(U_3) \cup N(U_4)] \cap U_2 \subseteq B_2$, as we can see in Fig. 11.

Symmetrically, we have the following conclusions (as shown in Fig. 12):

(i) U_5 is at most 1-splittable with respect to $J = \{1', 2, 3, 4\}$ with the only possible (2×2)-partition $\{\{1', 2\}, \{3, 4\}\}$ and it has a 1-separator $(A_5, B_5; x_5)$ such that $[N(U_{1'}) \cup N(U_2)] \cap U_5 \subseteq A_5$, and $[N(U_3) \cup N(U_4)] \cap U_5 \subseteq B_5$.

(ii) U_3 is at most 1-splittable with respect to $J = \{1'', 2, 4, 5\}$ with the only possible (2×2)-partition $\{\{1'', 4\}, \{2, 5\}\}$ and it has a 1-separator $(A_3, B_3; x_3)$ such that $[N(U_{1''}) \cup N(U_4)] \cap U_3 \subseteq A_3$, and $[N(U_2) \cup N(U_5)] \cap U_3 \subseteq B_3$.

(iii) U_4 is at most 1-splittable with respect to $J = \{1'', 2, 3, 5\}$ with the only possible (2×2)-partition $\{\{1'', 3\}, \{2, 5\}\}$ and it has a 1-separator $(A_4, B_4; x_4)$ such that $[N(U_{1''}) \cup N(U_3)] \cap U_4 \subseteq A_4$, and $[N(U_2) \cup N(U_5)] \cap U_4 \subseteq B_4$.

Claim 3. $\{N(u_{1''}) \cap A_2 - \{x_2\}\} \cup \{N(u_{1''}) \cap A_5 - \{x_5\}\} \neq \emptyset$.

Proof. Otherwise, $T = \{u_{1'}, x_2, x_5\}$ is a non-trivial 3-separator of G that separates G with $A_2 \cup A_5 \cup U_{1'}$ as one part. By Lemma 3.3, G is quasi-4-connected; therefore, $A_2 \cup A_5 \cup U_{1'}$ is trivial, but it is not acyclic. ■

Similarly, $\{N(u_{1'}) \cap A_3 - \{x_3\}\} \cup \{N(u_{1'}) \cap A_4 - \{x_4\}\} \neq \emptyset$.

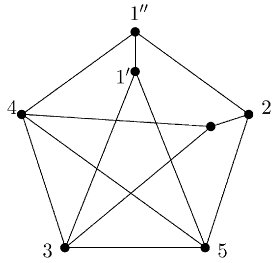


Fig. 13.

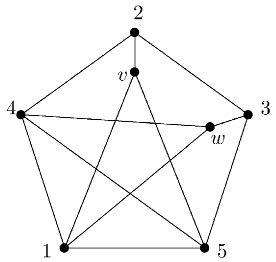


Fig. 14. A $(P_{10})_{\bar{3}}$-minor.

Without loss of generality, we assume that

$$\{N(u_{1'}) \cap A_3 - \{x_3\}\} \neq \emptyset, \qquad \{N(u_{1''}) \cap A_2 - \{x_2\}\} \neq \emptyset. \tag{1}$$

Claim 4. *U_2 is not $\{\{1'', 5\}, \{3, 4\}\}$-splittable.*

Proof. Otherwise G has a $(P_{10})_{\bar{3}}$-minor as in Fig. 13 (note that the edge between $U_{1'}$ and U_3 is given by (1)). ■

Symmetrically, U_5 is not $\{\{1'', 2\}, \{3, 4\}\}$-splittable.

Claim 5. *U_2 is at most 0-splittable with respect to $J = \{1', 3, 4, 5\}$.*

Proof. By way of contradiction, assume U_2 is not 0-splittable with respect to $J = \{1', 3, 4, 5\}$. By Claim 2, U_2 is $\{\{1', 5\}, \{3, 4\}\}$-splittable.

Let $\{P_{1',5}, P_{3,4}\}$ be a pair of vertex disjoint paths in U_2 that P_{ij} joins $N(U_i) \cap U_2$ and $N(U_j) \cap U_2$ for $i, j \in \{1', 3, 4, 5\}$.

It is obvious that $P_{3,4}$ must contain the cut vertex x_2 for otherwise A_2 contains a path joining $N(u_{1''})$ and $N(U_5)$. This contradicts Claim 4. Therefore, $N(U_{1'}) \cap (A_2 - x_2) \neq \emptyset$, $N(U_5) \cap (A_2 - x_2) \neq \emptyset$ and both of them are contained in the same component of $A_2 - x_2$, called C_2, while $N(U_{1''}) \cap (A_2 - x_2)$ is contained in another component of $A_2 - x_2$.

Symmetrically, $A_5 - x_5$ has a component C_5 that contains $N(U_{1'}) \cap (A_5 - x_5)$ and $N(U_2) \cap (A_5 - x_5)$ and is disjoint with $N(U_{1''})$.

Here we have obtained a 3-separator $(H_1, H_2; T)$ with $T = \{u_{1'}, x_2, x_5\}$ as the cut and $H_1 = C_2 \cup C_5 \cup \{u_{1'}, x_2, x_5\}$. Note that neither H_1 nor H_2 is trivial. This contradicts Lemma 3.3. ■

Similarly, U_3 is at most 0-splittable with respect to $J = \{1'', 3, 4, 5\}$.

Final Step:

By Claim 5 and Proposition 3.4(ii), x_2 separates U_2 into four parts $U_2(1')$, $U_2(5)$, $U_2(4)$ and $U_2(3)$ such that $N(U_i) \cap U_2 \subseteq U_2(i)$ for $i \in \{1', 3, 4, 5\}$.

By Claim 3, $N(u_{1''}) \cap A_2 - \{x_2\} \neq \emptyset$. Assume that $N(u_{1''}) \cap A_2 - \{x_2\} \subseteq U_2(1') - x_2$. Then $\{u_{1'}, u_{1''}, x_2\}$ is a 3-separator of G with $U_2(1') \cup U_1$ as a part. Both parts of G separated by $\{u_{1'}, u_{1''}, x_2\}$ contain cycles. This contradicts Lemma 3.3. So, there exists a vertex $v \in U_2(5) \cap N(u_{1''}) - \{x_2\}$ since $A_2 = U_2(1') \cup U_2(5)$.

Similarly, there is a vertex $w \in N(u_{1'}) \cap A_3 - \{x_3\}$, from which we deduce $w \in U_3(4)$. Now we have a $(P_{10})_{\bar{3}}$-minor as in Fig. 14.

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