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More Bounds for Eigenvalues

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A method introduced by Leighton [*J. Math. Anal. Appl.* **35**, 381–388 (1971)] for bounding eigenvalues has been extended to include problems of the form $-y'' + p(x)\, y = \lambda y$, when $p(x) \geqslant 0$ on [0, 1]. The boundary conditions are the general homogeneous conditions $y(0) - ay'(0) = 0 = y(1) + by'(1)$, where $0 \leqslant a, b \leqslant \infty$. Upper and lower bounds for the eigenvalues of these problems are obtained, and these bounds may be made as close together as desired, thereby allowing λ to be estimated precisely.

1. Introduction

Leighton [4] introduced a new method for obtaining two explicit sequences of upper and lower bounds on eigenvalues for a certain class of Sturm–Liouville problems. In this paper we extend that method to include problems of the form

$$\begin{aligned} -y'' + q(x)\, y &= \lambda y, \\ y(0) - ay'(0) = 0 &= y(1) + by'(1), \end{aligned} \tag{1}$$

where $q(x) \geqslant 0$ on [0, 1] and where $0 \leqslant a, b \leqslant \infty$. The corresponding terms in the two sequences can be made extremely close together, which thereby provides precise approximations for the eigenvalues of problem (1).

The problem of bounding eigenvalues is certainly a classical one, and the most famous methods such as Rayleigh–Ritz and the Courant Minimax principle have in theory produced the desired eigenvalues. The method presented here leads to a simple systematic scheme for generating a sequence of upper bounds and a sequence of lower bounds. This method is easily adapted to the computer and consequently generates almost effortlessly excellent eigenvalue approximations.

2. The General Method

The basic idea involved here is to subdivide the unit interval into n equal subintervals and to replace $q(x)$ by its maximum (or minimum) value over each subinterval in order to obtain an upper (or lower) bound for the eigenvalue λ. This result follows by a simple extension of one of the basic comparison theorems for eigenfunctions, as found in Swanson [5, p. 26]. Although this theorem is valid only for the first eigenvalue, our method also gives appropriate bounds for any eigenvalue.

We shall first work through problem (1) for the case $a = b = \infty$ and then point out the modifications that must be made when $0 \leqslant a, b \leqslant \infty$. Also, we shall initially assume that $q(x)$ is monotonically increasing on the unit interval and then explain why this assumption is not necessary.

Initially, we shall find an upper bound for the first eiegenvalue λ_1 by identifying a_k as the maximum value of $q(x)$ in the kth interval:

$$(k-1)\,h \leqslant x \leqslant kh \qquad \text{for } k = 1, 2, \ldots, n.$$

Then the problem in the first subinterval becomes

$$-y'' + a_1 y = \lambda y \qquad \text{and} \qquad y'(0) = 0. \tag{2}$$

The solution to problem (2) is $y(x) = B_1 \cos c_1 x$, where $c_1 = (\lambda_1 - a_1)^{1/2}$. In case λ_1 turns out to be less a_1, we shall see later the appropriate change needed.

In the second subinterval, we have $y(x) = A_2 \sin c_2 x + B_2 \cos c_2 x$ where $c_2 = (\lambda_1 - a_2)^{1/2}$ and where A_2 and B_2 are determined so that y and y' are continuous at $x = h$. Consequently, we have

$$A_2 \sin c_2 h + B_2 \cos c_2 h = B_1 \cos c_1 h$$

and (3)

$$c_2 A_2 \cos c_2 h - c_2 B_2 \sin c_2 h = -B_1 c_1 \sin c_1 h.$$

For the succeeding intervals, we shall have similar continuity requirements. In the kth subinterval we have

$$y(x) = A_k \sin c_k h + B_k \cos c_k h,$$

where $c_k = (\lambda_1 - a_k)^{1/2}$ and where our continuity requirements lead to

$$A_k \sin(k-1)\, c_k h + B_k \cos(k-1)\, c_k h$$
$$= A_{k-1} \sin(k-1)\, c_{k-1} h + B_{k-1} \cos(k-1)\, c_{k-1} h$$

and (4)

$$c_k A_k \cos(k-1)\, c_k h - c_k B_k \sin(k-1)\, c_k h$$
$$= c_{k-1} A_{k-1} \cos(k-1)\, c_{k-1} h - c_{k-1} B_{k-1} \sin(k-1)\, c_{k-1} h.$$

The process is not continued for all the subintervals, however, since we shall eventually reach some jth interval where $\lambda_1 < a_j$. In this case our solution looks like

$$y(x) = A_j \sinh c_j h + B_j \cosh c_j h,$$

where now $c_j = (a_j - \lambda_1)^{1/2}$. Thus, our continuity requirements at $x = (j-1)h$ are

$$\begin{aligned} &A_j \sinh(j-1)\, c_j h + B_j \cosh(j-1)\, c_j h \\ &\qquad = A_{j-1} \sin(j-1)\, c_{j-1} h + B_{j-1} \cos(j-1)\, c_{j-1} h \\ \text{and} \\ &c_j A_j \cosh(j-1)\, c_j h + c_j B_j \cosh(j-1)\, c_j h \\ &\qquad = c_{j-1} A_{j-1} \cos(j-1)\, c_{j-1} h - c_{j-1} B_{j-1} \sin(j-1)\, c_{j-1} h. \end{aligned} \tag{5}$$

Since we are assuming that $q(x)$ is monotonically increasing, all expressions for y after the jth interval will also be linear combinations of $\sinh c_k x$ and $\cosh c_k x$, where $c_k = (a_k - \lambda_1)^{1/2}$ for $k \geqslant j$. Therefore, our continuity requirements for $j + 1 \leqslant k \leqslant n$ are

$$\begin{aligned} &A_k \sinh(k-1)\, c_k h + B_k \cosh(k-1)\, c_k h \\ &\qquad = A_{k-1} \sinh(k-1)\, c_{k-1} h + B_{k-1} \cosh(k-1)\, c_{k-1} h \\ \text{and} \\ &c_k A_k \cosh(k-1)\, c_k h + c_k B_k \sinh(k-1)\, c_k h \\ &\qquad = c_{k-1} A_{k-1} \cosh(k-1)\, c_{k-1} h + c_{k-1} B_{k-1} \sinh(k-1)\, c_{k-1} h. \end{aligned} \tag{6}$$

Our final boundary condition at $x = 1 = nh$ becomes

$$y'(1) = 0 = c_n(A_n \cosh c_n + B_n \sinh c_n) \tag{7}$$

or, since $c_n \neq 0$,

$$A_n \cosh c_n + B_n \sinh c_n = 0. \tag{8}$$

We can simplify the above system of equations by defining

$$\cos \phi_k = A_k/P_k, \qquad \sin \phi_k = B_k/P_k, \quad \text{and} \quad P_k^2 = A_k^2 + B_k^2 \tag{9}$$

for $k = 2, 3, \ldots, j-1$ and

$$\cosh \theta_k = A_k/R_k, \qquad \sinh \theta_k = B_k/R_k, \quad \text{and} \quad R_k^2 = A_k^2 - B_k^2 \tag{10}$$

for $k = j, j+1, \dots, n-1$. We shall choose

$$\sinh\theta_n = A_n/R_n\,, \qquad \cosh\theta_n = B_n/R_n\,,$$

and (11)

$$R_n^2 = B_n^2 - A_n^2.$$

Our system of equations then becomes

$$\begin{aligned}
P_2\sin(c_2h + \phi_2) &= \cos c_1h,\\
c_2P_2\cos(c_2h + \phi_2) &= -c_1\sin c_1h,\\
P_3\sin(2c_3h + \phi_3) &= P_2\sin(2c_2h + \phi_2),\\
c_3P_3\cos(2c_3h + \phi_3) &= c_2P_2\cos(2c_2h + \phi_2),\\
&\vdots\\
P_{j-1}\sin((j-2)\,c_{j-1}h + \phi_{j-1}) &= P_{j-2}\sin((j-2)\,c_{j-2}h + \phi_{j-2}),\\
c_{j-1}P_{j-1}\cos((j-2)\,c_{j-1}h + \phi_{j-1}) &= c_{j-2}P_{j-2}\cos((j-2)\,c_{j-2}h + \phi_{j-2}),\\
R_j\sinh((j-1)\,c_jh + \theta_j) &= P_{j-1}\sin((j-1)\,c_{j-1}h + \phi_{j-1}),\\
c_jR_j\cosh((j-1)\,c_jh + \theta_j) &= c_{j-1}P_{j-1}\cos((j-1)\,c_{j-1}h + \phi_{j-1}),\\
R_{j+1}\sinh(jc_{j+1}h + \theta_{j+1}) &= R_j\sinh(jc_jh + \theta_j),\\
c_{j+1}R_{j+1}\cosh(jc_{j+1}h + \theta_{j+1}) &= c_jR_j\cosh(jc_jh + \theta_j),\\
&\vdots\\
R_{n-1}\sinh((n-2)\,c_{n-1}h + \theta_{n-1}) &= R_{n-2}\sinh((n-2)\,c_{n-2}h + \theta_{n-2}),\\
c_{n-1}R_{n-1}\cosh((n-2)\,c_{n-1}h + \theta_{n-1})&\\
&= c_{n-2}R_{n-2}\cosh((n-2)\,c_{n-2}h + \theta_{n-2}),\\
R_n\cosh((n-1)\,c_nh + \theta_n) &= R_{n-1}\sinh((n-1)\,c_{n-1}h + \theta_{n-1}),\\
c_nR_n\sinh((n-1)\,c_nh + \theta_n) &= c_{n-1}R_{n-1}\cosh((n-1)\,c_{n-1}h + \theta_{n-1}),
\end{aligned}$$

and

$$\sinh(c_n + \theta_n) = 0. \tag{12}$$

Now we introduce the angles

$$\psi_k = (k-1)\,c_kh + \phi_k$$

for $k = 2, 3, \dots, j-1$ and

$$\chi_k = (k-1)\,c_kh + \theta_k$$

for $k = j, j+1, \dots, n$, and we divide the first $2n-2$ equations pairwise in system (12) to obtain

$$
\begin{aligned}
\tan \psi_2 &= -\frac{c_2}{c_1}\cot c_1 h, \\
\tan \psi_3 &= \frac{c_3}{c_2}\tan(c_2 h + \psi_2), \\
&\ \vdots \\
\tan \psi_{j-1} &= \frac{c_{j-1}}{c_{j-2}}\tan(c_{j-2} h + \psi_{j-2}), \\
\tanh \chi_j &= \frac{c_j}{c_{j-1}}\tan(c_{j-1} h + \psi_{j-1}), \\
\tanh \chi_{j+1} &= \frac{c_{j+1}}{c_j}\tanh(c_j h + \chi_j), \\
&\ \vdots \\
\tanh \chi_{n-1} &= \frac{c_{n-1}}{c_{n-2}}\tanh(c_{n-2} h + \chi_{n-2}), \\
\coth \chi_n &= \frac{c_n}{c_{n-1}}\tanh(c_{n-1} h + \chi_{n-1}),
\end{aligned} \tag{13}
$$

and

$$
\begin{aligned}
c_n + \theta_n &= nhc_n + (\chi_n - (n-1)\,c_n) \\
&= c_n h + \chi_n = 0.
\end{aligned}
$$

Finally, by setting

$$z_k = \tan \psi_k$$

for $k = 2, 3, \ldots, j-1$,

$$z_k = \tanh \chi_k$$

for $k = j, j+1, \ldots, n-1$, and

$$z_n = \coth \chi_n ,$$

we obtain the following n equations:

$$
\begin{aligned}
z_1 &= -\frac{c_2}{c_1}\cot c_1 h, \\
z_2 &= \frac{c_3}{c_2}\,\frac{z_2 + \tan c_2 h}{1 - z_2 \tan c_2 h}, \\
&\ \vdots \\
z_{j-1} &= \frac{c_{j-1}}{c_{j-2}}\,\frac{z_{j-2} + \tan c_{j-2} h}{1 - z_{j-2} \tan c_{j-2} h}, \\
z_j &= \frac{c_j}{c_{j-1}}\,\frac{z_{j-1} + \tan c_{j-1} h}{1 - z_{j-1} \tan c_{j-1} h}, \\
z_{j+1} &= \frac{c_{j+1}}{c_j}\,\frac{z_j + \tanh c_j h}{1 + z_j \tanh c_j h}, \\
&\ \vdots \\
z_{n-1} &= \frac{c_{n-1}}{c_{n-2}}\,\frac{z_{n-2} + \tanh c_{n-2} h}{1 + z_{n-2} \tanh c_{n-2} h}, \\
z_n &= \frac{c_n}{c_{n-1}}\,\frac{z_{n-1} + \tanh c_{n-1} h}{1 + z_{n-1} \tanh c_{n-1} h},
\end{aligned} \tag{14}
$$

and

$$c_n h + \coth^{-1} z_n = 0.$$

Any of the standard numerical methods may be used to solve these non-linear equations simultaneously and thereby obtain an upper bound for λ_1. The preceding derivation can be repeated verbatim except with a_k, the minimum value of $q(x)$ where $(k-1)\,h \leqslant x \leqslant kh$, to obtain a lower bound for λ_1. Of course the transitional value $x = (j-1)\,h$ will be different.

The modifications that must be made to system (14) to accommodate the general positive function $q(x)$ and the general boundary conditions $0 \leqslant a$, $b \leqslant \infty$ are embarrassingly simple. First of all, let us consider all the equations in system (14) except the first and the last. The only decision that must be made for the general $q(x)$ is whether $\lambda_1 < a_k$ or $\lambda_1 > a_k$ in the kth interval. If $\lambda_1 < a_k$, the kth equations of system (14) becomes

$$z_{k+1} = \frac{c_{k+1}}{c_k} \frac{z_k + \tanh c_k h}{1 + z_k \tanh c_k h}, \tag{15}$$

and, if $\lambda_1 > a_k$, the kth equation of system (14) is

$$z_{k+1} = \frac{c_{k+1}}{c_k} \frac{z_k + \tan c_k h}{1 - z_k \tan c_k h}. \tag{16}$$

Consequently, we are reduced to looking at the modifications required in the first and the last equations of system (14).

In the first equation, if $\lambda_1 > a_1$, we have

$$z_2 = \tan(c_1 h + \phi_1), \tag{17}$$

where

$$\cos\phi_1 = 1/P_1, \qquad \sin\phi_1 = ac_1/P_1,$$

and

$$P_1{}^2 = a^2 c_1{}^2 + 1 \tag{18}$$

with $a = \infty$ assumed to mean $\phi_1 = \pi/2$. If $\lambda_1 < a_1$, then the first equation can be either

$$z_2 = \tanh(c_1 h + \theta_1) \tag{19}$$

if $ac_1 < 1$, where

$$\cosh\theta_1 = 1/R_1, \qquad \sinh\theta_1 = ac_1/R_1,$$

and

$$R_1{}^2 = 1 - a^2 c_1{}^2, \tag{20}$$

or

$$z_2 = \coth(c_1 h + \theta_1) \tag{21}$$

if $ac_1 > 1$, where

$$\cosh\theta_1 = ac_1/R_1\,, \qquad \sinh\theta_1 = 1/R_1\,,$$

and (22)

$$R_1^2 = a^2c_1^2 - 1.$$

The case we considered in system (14) falls under the option $\lambda_1 > a_1$ and $\phi_1 = \pi/2$. Thus, our first equation is

$$z_2 = \tan(c_1h + \pi/2) = -\cot c_1h.$$

The last equation also has an option. If $a_n < \lambda_1$, then we have

$$\sin(c_n + \phi_n) + bc_n \cos(c_n + \phi_n) = 0 \tag{23}$$

for our last equation in system (14), where by $b = \infty$ we mean

$$\cos(c_n + \phi_n) = 0.$$

In the other case, $a_n > \lambda_1$, we have a second choice for the final equation:

$$\sinh(c_n + \theta_n) + bc_n \cosh(c_n + \theta_n) = 0, \tag{24}$$

if $bc_n < 1$, or

$$\cosh(c_n + \theta_n) + bc_n \sinh(c_n + \theta_n) = 0, \tag{25}$$

if $bc_n > 1$. The extreme case of $b = \infty$ is taken from Eq. (25) to be

$$\sinh(c_n + \theta_n) = 0.$$

This is the case we had in system (14). A sequence of upper and lower bounds can be obtained by further subdividing the unit interval.

The other eigenvalues, $\lambda_2 \leqslant \lambda_3 < \cdots$, can also be found by the above procedure since some of the equations in the system will be changed; for example, $\lambda_1 < a_m$ but $\lambda_2 > a_m$ for some m.

It should be noted that as we successively, bound the eigenvalues, $\lambda_1, \lambda_2, \ldots$, we shall eventually have $\lambda_i > \max q(x)$ on the unit interval, so that our last equation in system (14) will always be Eq. (23). This is the one we need, naturally, since it alone supplies an infinite number of solutions (one for each eigenvalue). For example, if $b = 0$, Eq. (23) becomes

$$\sin(c_n + \phi_n) = 0$$

or

$$c_n + \phi_n = m\pi,$$

where m assumes positive integral values and therefore accounts for the infinite supply of eigenvalues.

One final observation is that one can find very crude approximations for λ_i by replacing $q(x)$ by its maximum (or minimum) value on the unit interval and thereby obtain an upper (or lower) bound on λ_i. This is useful in making an initial guess for λ_i.

3. Examples

We shall now present results for four examples, using the above procedure.

Example I.

$$-y'' + xy = \lambda y \qquad \text{and} \qquad y'(0) = 0 = y'(1). \tag{26}$$

Since $0 \leqslant q(x) \leqslant 1$ on the unit interval, we know that λ_1 is somewhere between zero and one. With an initial guess of $\lambda = 0.4$ for a lower bound on the first eigenvalue of problem (26) and with $n = 4$, we shall then try to satisfy the following system:

$$z_2 = -\frac{c_2}{c_1} \cot \frac{c_1}{4},$$

$$z_3 = \frac{c_3}{c_2} \frac{z_2 + \tan(c_2/4)}{1 - z_2 \tan(c_2/4)},$$

$$z_4 = \frac{c_4}{c_3} \frac{z_3 + \tanh(c_3/4)}{1 + z_3 \tanh(c_3/4)},$$

and

$$c_4(\tfrac{1}{4}) + \coth^{-1} z_4 = 0,$$

where

$$c_1 = (0.4 - 0)^{1/2}, \qquad c_2 = (0.4 - 0.25)^{1/2},$$

$$c_3 = (0.5 - 0.4)^{1/2}, \qquad \text{and} \qquad c_4 = (0.75 - 0.4)^{1/2}.$$

Since it is unlikely that $\lambda = 0.4$ will exactly satisfy system (27), the common numerical techniques may be employed to adjust λ. The middle two equations of system (27) are flexible; i.e., depending on the value of λ, the right sides may involve either $\tan(c_i/4)$ or $\tanh(c_i/4)$.

A similar system would also exist for finding an upper bound for λ in problem (26), except $c_i = (|\lambda - a_i|)^{1/2}$ where now $a_i = \max q(x)$ on $(i - 1)h \leqslant x \leqslant ih$.

Table I summarizes the sequences of upper and lower bounds for the first two eigenvalues of problem (26) with $n = 2^{k+1}$ for $k = 1, 2, \ldots, 8$.

TABLE I

λ_1	$n = 4$	8	16	32	64	128	356	512
λ_L	0.3675	0.4294	0.4605	0.4761	0.4839	0.4878	0.4897	0.4907
λ_U	0.6175	0.5544	0.5230	0.5073	0.4995	0.4956	0.4936	0.4926
λ_2	$n = 4$	8	16	32	64	128	256	
λ_L	10.251	10.314	10.345	10.361	10.368	10.372	10.374	
λ_U	10.501	10.439	10.407	10.392	10.384	10.380	10.378	

EXAMPLE II. Table II summarizes the sequences of upper and lower bounds for the first two eigenvalues of the problem

$$-y'' + (x^2 + 4x + 1) y = \lambda y \qquad \text{and} \qquad y(0) = 0 = y(1) + y'(1). \tag{28}$$

TABLE II

λ_1	$n = 4$	8	16	32	64	128	256	512
λ_L	7.518	7.860	8.027	8.111	8.152	8.173	8.183	8.189
λ_U	8.836	8.519	8.357	8.276	8.235	8.214	8.204	8.199
λ_2	$n = 4$	8	16	32	64	128	256	512
λ_L	27.022	27.372	27.539	27.621	27.661	27.681	27.691	27.697
λ_U	28.291	28.009	27.858	27.780	27.741	27.721	27.711	27.706

EXAMPLE III. Table III summarizes the sequences of upper and lower bounds for the first two eigenvalues of the problem

$$-y'' + 2e^{-x} y = \lambda y \qquad \text{and} \qquad y(0) = 0 = y(1). \tag{29}$$

TABLE III

λ_1	$n = 4$	8	16	32	64	128	256	512
λ_L	10.959	11.027	11.063	11.082	11.091	11.096	11.099	11.100
λ_U	11.269	11.181	11.140	11.120	11.111	11.106	11.104	11.102
λ_2	$n = 4$	8	16	32	64	128	256	512
λ_L	40.592	40.660	40.697	40.716	40.726	40.730	40.733	40.735
λ_U	40.908	40.817	40.775	40.755	40.745	40.740	40.738	40.737

Example IV. Table IV summarizes the sequences of upper and lower bounds for the first two eigenvalues of the problem

$$-y'' + 20xe^{-2x}y = \lambda y \qquad \text{and} \qquad y(0) = 0 = y(1) + y'(1). \tag{30}$$

TABLE IV

λ_1	$n = 4$	8	16	32	64	128	256	512
λ_L	7.030	7.263	7.352	7.391	7.410	7.419	7.424	7.426
λ_U	7.642	7.549	7.493	7.462	7.445	7.437	7.432	7.430
λ_2	$n = 4$	8	16	32	64	128	256	512
λ_L	26.515	26.958	27.116	27.185	27.217	27.233	27.241	27.245
λ_U	27.581	27.447	27.357	27.305	27.277	27.263	27.256	27.252

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