Generalized Quasi-Variational Inequalities in Locally Convex Topological Vector Spaces*

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Let $E$ be a Hausdorff topological vector space and $X \subseteq E$ an arbitrary nonempty set. Denote by $E'$ the dual space of $E$ and the pairing between $E'$ and $E$ by $\langle w, x \rangle$ for $w \in E'$ and $x \in E$. Given a point-to-set map $S: X \rightarrow 2^X$ and a point-to-set map $T: X \rightarrow E'$, the generalized quasi-variational inequality problem (GQVI) is to find a point $\hat{y} \in S(\hat{y})$ and a point $u \in T(\hat{y})$ such that $\text{Re}(\langle u, \hat{y} - x \rangle) \leq 0$ for all $x \in S(\hat{y})$. By using the Ky Fan minimax principle or its generalized version as a tool, some general theorems on solutions of the GQVI in locally convex Hausdorff topological vector spaces are obtained which include a fixed point theorem due to Ky Fan and I. L. Glicksberg, and two different multivalued versions of the Hartman-Stam-pacchia variational inequality. © 1985 Academic Press, Inc.

1

Let $E$ be a Hausdorff topological vector space, $X \subseteq E$ an arbitrary nonempty set and $2^X$ the collection of all subsets of $X$. We shall denote by $E'$ the dual space of $E$ (i.e., the vector space of all continuous linear functionals on $E$). We denote the pairing between $E'$ and $E$ by $\langle w, x \rangle$ for $w \in E'$ and $x \in E$. Given a (point-to-set) map $S: X \rightarrow 2^X$ and a (point-to-point) map $T: X \rightarrow E'$, the quasi-variational inequality problem (QVI) is to find a point $\hat{y} \in S(\hat{y})$ such that $\text{Re}(\langle T(\hat{y}), \hat{y} - x \rangle) \leq 0$ for all $x \in S(\hat{y})$. The QVI was introduced by Bensoussan and Lions in 1973 (see, e.g., [3]) in

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connection with impulse control. A recent work concerning the QVI may be found in Mosco [11]. If we consider a point-to-set map \( T: X \to 2^E \), then the generalized quasi-variational inequality problem (GQVI) is to find a point \( \hat{y} \in S(\hat{y}) \) and a point \( \hat{u} \in T(\hat{y}) \) such that \( \text{Re} \langle \hat{u}, \hat{y} - x \rangle \leq 0 \) for all \( x \in S(\hat{y}) \) (see [5]).

In the present paper we shall give some general theorems on solutions of the GQVI. Our basic tool is the Ky Fan minimax principle [7] or the following generalized version due to Yen [15].

**Theorem A.** Let \( X \) be a nonempty compact convex set in a Hausdorff topological vector space \( E \). Let \( \phi \) and \( \psi \) be two real-valued functions on \( X \times X \) having the following properties:

1. \( \phi \leq \psi \) on \( X \times X \) and \( \psi(x, x) < 0 \) for all \( x \in X \);
2. For each fixed \( x \in X \), \( \phi(x, y) \) is a lower semicontinuous function of \( y \) on \( X \);
3. For each fixed \( y \in X \), \( \psi(x, y) \) is a quasi-concave function of \( x \) on \( X \).

Then there exists a point \( \hat{y} \in X \) such that \( \phi(x, \hat{y}) \leq 0 \) for all \( x \in X \).

Here, a real-valued function \( \psi \) defined on a convex set \( X \) is said to be quasi-concave if for every real number \( \lambda \), the set \( \{ x \in X : \psi(x) > \lambda \} \) is convex.

Let \( X \) be any nonempty subset of a Hausdorff topological vector space \( E \). A set-valued map \( T: X \to 2^E \) is said to be monotone on \( X \) [4, p. 79] if for all \( x \) and \( y \) in \( X \), each \( u \) in \( T(x) \), and each \( w \) in \( T(y) \), \( \text{Re} \langle w - u, y - x \rangle \geq 0 \).

We need the following two kinds of continuity for set-valued maps. Let \( M \) and \( N \) be topological spaces, and let \( \Gamma: M \to 2^N \) be a set-valued map. We say that \( \Gamma \) is upper semicontinuous at \( x_0 \in M \) [2, p. 109] if for each open set \( G \) with \( \Gamma(x_0) \subseteq G \) there exists a neighborhood \( N(x_0) \) of \( x_0 \) such that if \( x \in N(x_0) \), then \( \Gamma(x) \subseteq G \); \( \Gamma \) is upper semicontinuous on \( M \) if it is upper semicontinuous at each point of \( M \). Also, \( \Gamma \) is lower semicontinuous at \( x_0 \in M \) [2, p. 109] if for each open set \( G \) with \( \Gamma(x_0) \cap G \neq \emptyset \) there is a neighborhood \( N(x_0) \) of \( x_0 \) such that if \( x \in N(x_0) \), then \( \Gamma(x) \cap G \neq \emptyset \); \( \Gamma \) is lower semicontinuous on \( M \) if it is lower semicontinuous at each point of \( M \). Moreover, \( \Gamma \) is said to be continuous on \( M \) if it is both upper semicontinuous and lower semicontinuous on \( M \).

Our proofs of Theorems 1 and 3 require the following lemma.

**Lemma 1.** Let \( E \) be a Hausdorff topological vector space, \( X \subseteq E \) be nonempty and \( S: X \to 2^E \) be upper semicontinuous such that for each \( x \in X \), \( S(x) \)
is nonempty and bounded. Then for \( p \in E' \) the map \( f_p : X \to \mathbb{R} \) defined by

\[
f_p(y) := \sup_{x \in S(y)} \text{Re}\langle p, x \rangle
\]

is upper semicontinuous. \( \text{Re}\langle p, x \rangle \)

Proof. Let \( y_0 \in X \) and \( \varepsilon > 0 \) be given. Let

\[
U_\varepsilon := \{ x \in E : |\langle p, x \rangle| < \varepsilon/2 \};
\]

then \( U_\varepsilon \) is an open neighborhood of 0. As \( S(y_0) + U_\varepsilon \) is an open set containing \( S(y_0) \), by upper semicontinuity of \( S \) at \( y_0 \), there exists a neighborhood \( N(y_0) \) of \( y_0 \) in \( X \) such that if \( y \in N(y_0) \) then \( S(y) \subseteq S(y_0) + U_\varepsilon \). Thus, for each \( y \in N(y_0) \),

\[
f_p(y) = \sup_{x \in S(y)} \text{Re}\langle p, x \rangle
\leq \sup_{x \in S(y_0) + U_\varepsilon} \text{Re}\langle p, x \rangle
\leq \sup_{x \in S(y_0)} \text{Re}\langle p, x \rangle + \sup_{x \in U_\varepsilon} \text{Re}\langle p, x \rangle
\leq f_p(y_0) + \varepsilon.
\]

Hence \( f_p \) is upper semicontinuous and the proof is completed. \( \square \)

THEOREM 1. Let \( E \) be a locally convex Hausdorff topological vector space and \( X \) be a nonempty compact convex subset of \( E \). Let \( S : X \to 2^X \) be upper semicontinuous such that for each \( x \in X \), \( S(x) \) is a nonempty closed convex subset of \( X \), and let \( T : X \to 2^{E'} \) be monotone such that for all \( x \in X \), \( T(x) \) is a nonempty subset of \( E' \) and for each one-dimensional flat \( L \subset E \), \( T \mid L \cap X \) is lower semicontinuous from the topology of \( E \) to the weak*-topology \( \sigma(E', E) \) of \( E' \). Suppose further that the set \( \Sigma_1 := \{ y \in X : \sup_{x \in S(y)} \text{Re}\langle u, y - x \rangle > 0 \} \) is open in \( X \). Then there exists a point \( \hat{y} \in X \) such that

(i) \( \hat{y} \in S(\hat{y}) \) and
(ii) \( \sup_{w \in T(\hat{y})} \text{Re}\langle w, \hat{y} - x \rangle \leq 0 \) for all \( x \in S(\hat{y}) \).

Proof. We divide the proof into two steps:

Step 1. There exists a point \( \hat{y} \in X \) such that \( \hat{y} \in S(\hat{y}) \) and \( \sup_{u \in T(\hat{y})} \text{Re}\langle u, \hat{y} - x \rangle \leq 0 \) for all \( x \in S(\hat{y}) \).

Suppose the assertion were false. Then for all \( y \in X \), either \( y \notin S(\hat{y}) \) or there exists a point \( x \in S(y) \) such that \( \sup_{u \in T(x)} \text{Re}\langle u, y - x \rangle > 0 \). Observe that whenever \( y \notin S(y) \), there exists \( p \in E' \) such that

\[
\text{Re}\langle p, y \rangle - \sup_{x \in S(y)} \text{Re}\langle p, x \rangle > 0
\]
by applying the Hahn-Banach separation theorem. For each $y \in X$, we set

$$\alpha(y) := \sup_{x \in S(y)} \sup_{u \in T(x)} \Re \langle u, y - x \rangle.$$  

Let

$$V_0 := \{ y \in X : \alpha(y) > 0 \}.$$  

For each $p \in E'$, we set

$$V(p) := \{ y \in X : \Re \langle p, y \rangle - \sup_{x \in S(y)} \Re \langle p, x \rangle > 0 \}.$$  

Then $X = V_0 \cup \bigcup_{p \in E'} V(p)$. By hypothesis, $V_0$ is open in $X$. By Lemma 1, $V(p)$ is open in $X$ for each $p \in E'$. Since $X$ is compact, there exist $p_1, \ldots, p_n \in E'$ such that $X = V_0 \cup \bigcup_{i=1}^n V(p_i)$ and a continuous partition of unity $\{ \beta_0, \beta_1, \ldots, \beta_n \}$ subordinated to the covering $\{ V_0, V(p_1), \ldots, V(p_n) \}$, that is, $\beta_0, \beta_1, \ldots, \beta_n$ are continuous nonnegative real-valued functions on $X$ such that $\beta_0$ vanishes on $X \setminus V_0$ and for each $1 \leq i \leq n$, $\beta_i$ vanishes on $X \setminus V(p_i)$ and $\sum_{i=0}^n \beta_i(x) = 1$ for all $x \in X$.

Define $\phi, \psi : X \times X \to \mathbb{R}$ by setting

$$\phi(x, y) := \beta_0(y) \sup_{u \in T(x)} \Re \langle u, y - x \rangle + \sum_{i=1}^n \beta_i(y) \Re \langle p_i, y - x \rangle,$$

$$\psi(x, y) := \beta_0(y) \inf_{w \in T(y)} \Re \langle w, y - x \rangle + \sum_{i=1}^n \beta_i(y) \Re \langle p_i, y - x \rangle.$$  

By monotonicity of $T$, we have

$$\sup_{u \in T(x)} \Re \langle u, y - x \rangle \leq \inf_{w \in T(y)} \Re \langle w, y - x \rangle \quad \text{for all} \ x, y \in X.$$  

It follows that $\phi \leq \psi$ on $X \times X$. Clearly $\psi(x, x) = 0$ for all $x \in X$. For each fixed $x \in X$, since $\beta_i(i = 0, 1, \ldots, n)$ are continuous nonnegative functions of $y$ on $X$ and $\sup_{u \in T(x)} \Re \langle u, y - x \rangle, \Re \langle p_i, y - x \rangle (i = 1, \ldots, n)$ are lower semicontinuous functions of $y$ on $X$, by Lemma 3 in [13, p. 177], $y \mapsto \phi(x, y)$ is lower semicontinuous on $X$. Furthermore, for each fixed $y \in X$, $x \mapsto \psi(x, y)$ is quasi-concave. Hence, all the conditions of Theorem A are satisfied, so that there exists a point $\hat{y} \in X$ such that $\phi(x, \hat{y}) \leq 0$ for all $x \in X$; that is,

$$\beta_0(\hat{y}) \sup_{u \in T(x)} \Re \langle u, \hat{y} - x \rangle + \sum_{i=1}^n \beta_i(\hat{y}) \Re \langle p_i, \hat{y} - x \rangle \leq 0$$  

for all $x \in X$. (*)
Since \( \{\beta_0, \beta_1, \ldots, \beta_n\} \) is a partition of unity, \( \beta_i(\hat{y}) > 0 \) for at least one index \( i \in \{0, 1, \ldots, n\} \). Choose any \( \hat{x} \in S(\hat{y}) \) such that

\[
\sup_{u \in T(\hat{x})} \Re \langle u, \hat{y} - \hat{x} \rangle \geq \frac{\alpha(\hat{y})}{2} \quad \text{whenever } \alpha(\hat{y}) > 0.
\]

If \( \beta_0(\hat{y}) > 0 \), then \( \hat{y} \in V_0 \) so that \( \alpha(\hat{y}) > 0 \). Hence,

\[
\sup_{u \in T(\hat{x})} \Re \langle u, \hat{y} - \hat{x} \rangle \geq \frac{\alpha(\hat{y})}{2} > 0.
\]

If \( \beta_i(\hat{y}) > 0 \) for \( i = 1, \ldots, n \), then \( \hat{y} \in V(p_i) \) and hence

\[
\Re \langle p_i, \hat{y} \rangle > \sup_{x \in S(\hat{y})} \Re \langle p_i, x \rangle \geq \Re \langle p_i, \hat{x} \rangle
\]

so that \( \Re \langle p_i, \hat{y} - \hat{x} \rangle > 0 \). It follows that

\[
\beta_0(\hat{y}) \sup_{u \in T(\hat{x})} \Re \langle u, \hat{y} - \hat{x} \rangle + \sum_{i=1}^{n} \beta_i(\hat{y}) \Re \langle p_i, \hat{y} - \hat{x} \rangle > 0,
\]

contradicting (\(*\)*). This contradiction proves Step 1.

**Step 2.** \( \sup_{u \in T(\hat{x})} \Re \langle w, \hat{y} - x \rangle \leq 0 \) for all \( x \in S(\hat{y}) \).

Let \( x \in S(\hat{y}) \) be arbitrarily fixed and let \( z_t := tx + (1 - t) \hat{y} - t(\hat{y} - x) \) for \( t \in [0, 1] \). As \( S(\hat{y}) \) is convex, we have \( z_t \in S(\hat{y}) \) for \( t \in [0, 1] \). Therefore by Step 1, we have

\[
\sup_{u \in T(z_t)} \Re \langle u, \hat{y} - z_t \rangle \leq 0 \quad \text{for all } t \in [0, 1],
\]

and it follows that

\[
\sup_{u \in T(z_t)} \Re \langle u, \hat{y} - x \rangle \leq 0 \quad \text{for all } t \in (0, 1]. \quad (**) 
\]

Let \( w_0 \in T(\hat{y}) \) be arbitrarily fixed. For each \( \varepsilon > 0 \), let

\[
U_{w_0} := \{ w \in E' : |\langle w_0 - w, \hat{y} - x \rangle| < \varepsilon \};
\]

then \( U_{w_0} \) is a \( \sigma(E', E) \)-neighborhood of \( w_0 \). Since \( T|L \cap X \) is lower semicontinuous, where \( L := \{ z_t : t \in [0, 1] \} \), and \( U_{w_0} \cap T(\hat{y}) \neq \emptyset \), there exists a neighborhood \( N(\hat{y}) \) of \( \hat{y} \) in \( L \) such that if \( z \in N(\hat{y}) \) then \( T(z) \cap U_{w_0} \neq \emptyset \). But then there exists \( \delta \in (0, 1) \) such that \( z_t \in N(\hat{y}) \) for all \( t \in (0, \delta) \). Fix any \( t \in (0, \delta) \) and \( u \in T(z_t) \cap U_{w_0} \), we have

\[
|\langle w_0 - u, \hat{y} - x \rangle| < \varepsilon.
\]
This implies
\[ \text{Re}\langle w_0, \hat{y} - x \rangle < \text{Re}\langle u, \hat{y} - x \rangle + \varepsilon. \]

By (**), we have \( \text{Re}\langle w_0, \hat{y} - x \rangle < \varepsilon. \) Since \( \varepsilon > 0 \) is arbitrary, \( \text{Re}\langle w_0, \hat{y} - x \rangle \leq 0. \) As \( w_0 \in T(\hat{y}) \) is arbitrary,

\[ \sup_{w \in T(\hat{y})} \text{Re}\langle w, \hat{y} - x \rangle \leq 0 \quad \text{for all } x \in S(\hat{y}). \]

This concludes the proof of our theorem. \( \square \)

In the first step of our proof, we follow the argument of Aubin [1, pp. 373–374]. In the second step of our proof, we use the argument of Shih and Tan [12] and Tan [14].

When \( T \equiv 0 \), Theorem 1 gives the well-known Fan–Glicksberg fixed point theorem [6, 8].

**Corollary 1 (Fan and Glicksberg).** Let \( E \) be a locally convex Hausdorff topological vector space and \( X \) a nonempty compact convex set in \( E \). Let \( S: X \to 2^X \) be upper semicontinuous such that for each \( x \in X \), \( S(x) \) is a nonempty closed convex subset of \( X \). Then there exists a point \( \hat{x} \in X \) such that \( \hat{x} \in S(\hat{x}). \)

We shall now observe that in Theorem 1, the interaction between the maps \( S \) and \( T \) (namely, \( \Sigma \) is open in \( X \)) can be achieved by imposing additional continuity conditions on \( S \) and \( T \).

**Theorem 2.** Let \( E \) be a locally convex Hausdorff topological vector space and \( X \) be a nonempty compact convex subset of \( E \). Let \( S: X \to 2^X \) be continuous such that for each \( x \in X \), \( S(x) \) is a nonempty closed convex subset of \( X \), and \( T: X \to 2^{E'} \) be monotone such that for each \( x \in X \), \( T(x) \) is a nonempty subset of \( E' \) and \( T \) is lower semicontinuous from the relative topology of \( X \) to the strong topology of \( E' \). Then there exists a point \( \hat{y} \in X \) such that

(i) \( \hat{y} \in S(\hat{y}) \) and

(ii) \( \sup_{w \in T(\hat{y})} \text{Re}\langle w, \hat{y} - x \rangle \leq 0 \) for all \( x \in S(\hat{y}). \)

**Proof.** By virtue of Theorem 1, we need only show that

\[ \Sigma_1 := \{ y \in X : \sup_{x \in S(y)} \sup_{u \in T(x)} \text{Re}\langle u, y - x \rangle > 0 \} \]

is open in \( X \). Let \( y_0 \in \Sigma_1 \); then there exist \( x_0 \in S(y_0) \) and \( f_0 \in T(x_0) \) such that

\[ x := \text{Re}\langle f_0, y_0 - x_0 \rangle > 0. \]
GENERALIZED QUASI-VARIATIONAL INEQUALITIES

Since $f_0$ is continuous at $x_0$ and at $y_0$, there exist an open neighborhood $N_1$ of $x_0$ and an open neighborhood $U_1$ of $y_0$ such that

$$
x \in N_1 \Rightarrow |\langle f_0, x_0 \rangle - \langle f_0, x \rangle| < \alpha/6.
$$

$$
y \in U_1 \Rightarrow |\langle f_0, y_0 \rangle - \langle f_0, y \rangle| < \alpha/6.
$$

Let

$$
W := \{ f \in E' : \sup_{z_1, z_2 \in X} |\langle f - f_0, z_1 - z_2 \rangle| < \alpha/6 \};
$$

then $W$ is a strongly open neighborhood of $f_0$ and $W \cap T(x_0) \neq \emptyset$ so that by lower semicontinuity of $T$ at $x_0$, there exists an open neighborhood $N_2$ of $x_0$ such that

$$
x \in N_2 \Rightarrow T(x) \cap W \neq \emptyset.
$$

Let $N := N_1 \cap N_2$; since $N$ is a neighborhood of $x_0$ and $N \cap S(y_0) \neq \emptyset$, by lower semicontinuity of $S$ at $y_0$, there exists an open neighborhood $U_2$ of $y_0$ such that

$$
y \in U_2 \Rightarrow S(y) \cap N \neq \emptyset.
$$

Let $U := U_1 \cap U_2$; then $U$ is an open neighborhood of $y_0$. For each $y_1 \in U$, choose $x_1 \in S(y_1) \cap N$ and $f_1 \in T(x_1) \cap W$; it follows that

$$
\alpha = \text{Re} \langle f_0, y_0 - x_0 \rangle
= \text{Re} \langle f_1, y_1 - x_1 \rangle + \text{Re} \langle f_0, y_0 - y_1 \rangle + \text{Re} \langle f_0 - f_1, y_1 - x_1 \rangle
+ \text{Re} \langle f_0, x_1 - x_0 \rangle
< \text{Re} \langle f_1, y_1 - x_1 \rangle + \alpha/2.
$$

Thus, $\text{Re} \langle f_1, y_1 - x_1 \rangle \geq \alpha/2 > 0$ so that $y_1 \in \Sigma_1$ for all $y_1 \in U$. Hence $\Sigma_1$ is open in $X$ and the proof is completed.

When $S(x) \equiv X$, Theorem 2 gives a multivalued version of the Hartman–Stampacchia variational inequality [9] as follows.

**Corollary 2.** Let $E$ be a locally convex Hausdorff topological vector space and $X$ be a nonempty compact convex subset of $E$. Let $T : X \to 2^E$ be monotone such that for each $x \subset X$, $T(x)$ is a nonempty subset of $E'$ and $T$ is lower semicontinuous from the relative topology of $X$ to the strong topology of $E'$. Then there exists a point $\hat{y} \in X$ such that

$$
\sup_{w \in T(\hat{y})} \text{Re} \langle w, \hat{y} - x \rangle \leq 0 \quad \text{for all } x \in X.
$$
In Theorems 1 and 2, \( T \) is assumed to be monotone together with some kind of lower semicontinuity. In this section we shall establish results for upper semicontinuous map \( T \) without monotonicity.

**Theorem 3.** Let \( E \) be a locally convex Hausdorff topological vector space and \( X \) be a nonempty compact convex subset of \( E \). Let \( S: X \to 2^X \) be upper semicontinuous such that for each \( x \in X \), \( S(x) \) is a nonempty closed convex subset of \( X \), and let \( T: X \to 2^{E'} \) be upper semicontinuous from the relative topology of \( X \) to the strong topology of \( E' \) such that for each \( x \in X \), \( T(x) \) is a nonempty compact convex subset of \( E' \). Suppose further that the set \( \Sigma_x := \{ y \in X: \sup_{z \in S(y)} \inf_{z \in T(y)} \Re \langle z, y - x \rangle = 0 \} \) is open in \( X \). Then there exists a point \( j \in X \) such that

(i) \( j \in S(j) \) and

(ii) there exists a point \( \hat{z} \in T(j) \) with \( \Re \langle \hat{z}, \hat{z} - x \rangle \leq 0 \) for all \( x \in S(j) \).

**Proof.** We divide the proof into two steps:

**Step 1.** There exists a point \( \hat{y} \in S \) such that \( \hat{y} \in S(\hat{y}) \) and \( \sup_{z \in S(\hat{y})} \inf_{z \in T(\hat{y})} \Re \langle z, \hat{y} - x \rangle \leq 0 \).

Suppose the assertion were false. Then for all \( y \in X \), either \( y \not\in S(y) \) or there exists \( x \in S(y) \) such that \( \inf_{z \in T(y)} \Re \langle z, y - x \rangle > 0 \). Observe that whenever \( y \not\in S(y) \), there exists \( p \in E' \) with

\[
\Re \langle p, y \rangle - \sup_{x \in S(y)} \Re \langle p, x \rangle > 0.
\]

For each \( y \in X \), we set

\[
\alpha(y) := \sup_{x \in S(y)} \inf_{z \in T(y)} \Re \langle z, y - x \rangle.
\]

Let

\[
V_0 := \{ y \in X: \alpha(y) > 0 \},
\]

and for each \( p \in E' \), we set

\[
V(p) := \{ y \in X: \Re \langle p, y \rangle - \sup_{x \in S(y)} \Re \langle p, x \rangle > 0 \}.
\]

Then \( X = V_0 \cup \bigcup_{p \in E'} V(p) \). By hypothesis, \( V_0 \) is open in \( X \). By Lemma 1, \( V(p) \) is open in \( X \) for each \( p \in E' \). Since \( X \) is compact, there exist \( p_1, \ldots, p_n \in E' \) such that

\[
X = V_0 \cup \bigcup_{i=1}^n V(p_i).
\]
and a continuous partition of unity \( \{\beta_0, \beta_1, \ldots, \beta_n\} \) subordinated to the covering \( \{V_0, V(p_1), \ldots, V(p_n)\} \).

Define \( \phi: X \times X \to \mathbb{R} \) by setting

\[
\phi(x, y) := -\beta_0(y) \inf_{w \in T(y)} \Re \langle w, y - x \rangle + \sum_{i=1}^{n} \beta_i(y) \Re \langle p_i, y - x \rangle.
\]

Clearly \( \phi(x, x) = 0 \) for each \( x \in X \). Note that for each fixed \( x \in X \),

\[
y \to \inf_{w \in T(y)} \Re \langle w, y - x \rangle
\]

is lower semicontinuous as can be seen within the proof of Theorem 21 in [13], so that \( y \to \phi(x, y) \) is lower semicontinuous. Also it is clear that for each fixed \( y \in X \), \( x \to \phi(x, y) \) is quasi-concave. Hence by the Ky Fan minimax principle (i.e., Theorem A with \( \phi \equiv \psi \)), there exists a point \( \hat{y} \in X \) such that \( \phi(x, \hat{y}) \leq 0 \) for all \( x \in X \). The contradiction that there is a point \( \hat{x} \in X \) with \( \phi(\hat{x}, \hat{y}) > 0 \) can be achieved by using the corresponding proof of Step 1 of Theorem 1.

**Step 2.** There exists a point \( \hat{y} \in T(\hat{y}) \) such that \( \Re \langle \hat{z}, \hat{y} - x \rangle \leq 0 \) for all \( x \in S(\hat{y}) \).

Indeed, define \( f: S(\hat{y}) \times T(\hat{y}) \to \mathbb{R} \) by

\[
f(x, z) := \Re \langle z, \hat{y} - x \rangle.
\]

Note that for each fixed \( x \in S(\hat{y}) \), \( z \to f(x, z) \) is continuous and affine, and for each \( z \in T(\hat{y}) \), \( x \to f(x, z) \) is affine. Thus by Kneser's minimax theorem [10], we have

\[
\min_{z \in T(\hat{y})} \max_{x \in S(\hat{y})} f(x, z) = \max_{x \in S(\hat{y})} \min_{z \in T(\hat{y})} f(x, z).
\]

Thus

\[
\min_{z \in T(\hat{y})} \max_{x \in S(\hat{y})} \Re \langle z, \hat{y} - x \rangle \leq 0 \quad \text{by Step 1.}
\]

Since \( T(\hat{y}) \) is compact, there exists \( \hat{z} \in T(\hat{y}) \) such that

\[
\Re \langle \hat{z}, \hat{y} - x \rangle \leq 0 \quad \text{for all } x \in S(\hat{y}).
\]

When \( E \) is a normed linear space, by imposing additional lower semicontinuity on \( S \), the interacting set \( \Sigma_2 \) in Theorem 3 is always open:

**Theorem 4.** Let \( E \) be a normed linear space and \( X \) be a nonempty compact convex subset of \( E \). Let \( S: X \to 2^X \) be continuous such that for each \( x \in X \), \( S(x) \) is a nonempty closed convex subset of \( X \), and let \( T: X \to 2^{E'} \) be upper semicontinuous such that for each \( x \in X \), \( T(x) \) is a nonempty compact convex subset of \( E' \). Then there exists a point \( \hat{y} \in X \) such that
(i) \( \hat{y} \in S(\hat{y}) \) and
(ii) there exists a point \( \hat{z} \in T(\hat{y}) \) with \( \Re \langle \hat{z}, \hat{y} - x \rangle \leq 0 \) for all \( x \in S(\hat{y}) \).

**Proof.** By virtue of Theorem 3, we need only show that the set

\[ \Sigma_2 := \{ y \in X : \sup_{x \in S(y)} \inf_{z \in T(y)} \Re \langle z, y - x \rangle > 0 \} \]

is open in \( X \). For this purpose, let \( y_0 \in \Sigma_2 \), then there exists \( x_0 \in S(y_0) \) with

\[ \alpha = \inf_{z \in T(y_0)} \Re \langle z, y_0 - x_0 \rangle > 0. \]

Let

\[ M := \max \{ \text{diam}(x), \sup_{z \in T(y_0)} \| z \| \} \quad \text{and} \quad B := \{ f \in E' : \| f \| < 1 \}. \]

Since \( T \) is upper semicontinuous at \( y_0 \), for \( \eta = \alpha/6(1 + M) > 0 \), there exists \( \delta_1 \in (0, \min \{ 1, \alpha/6(1 + M) \}) \) such that for all \( y \in X, \| y - y_0 \| < \delta_1 \) implies \( T(y) \subset T(y_0) + \eta B \). As \( S \) is lower semicontinuous at \( y_0 \), there exists \( \delta_2 \in (0, \min \{ 1, \alpha/6(1 + M) \}) \) such that for all \( y \in X, \| y - y_0 \| < \delta_2 \) implies

\[ S(y) \cap \{ x \in X : \| x - x_0 \| < \eta \} \neq \emptyset. \]

Let \( \delta := \min \{ \delta_1, \delta_2 \} \). Let \( y_1 \in X \) be such that \( \| y_1 - y_0 \| < \delta \). Then \( T(y_1) \subset T(y_0) + \eta B \) and we can choose \( x_1 \in S(y_1) \) with \( \| x_1 - x_0 \| < \eta \). It follows that

\[ \inf_{z \in T(y_1)} \Re \langle z, y_1 - x_1 \rangle \]

\[ \geq \inf_{z \in T(y_0) + \eta B} \Re \langle z, y_1 - x_1 \rangle \]

\[ \geq \inf_{z \in T(y_0)} \Re \langle z, y_1 - x_1 \rangle + \inf_{z \in \eta B} \Re \langle z, y_1 - x_1 \rangle \]

\[ \geq \inf_{z \in T(y_0)} \Re \langle z, y_0 \rangle + \inf_{z \in T(y_0)} \Re \langle z, y_0 - x_0 \rangle \]

\[ + \inf_{z \in T(y_0)} \Re \langle z, x_0 - x_1 \rangle - \eta \| y_1 - x_1 \| \]

\[ \geq - \sup_{z \in T(y_0)} \| z \| \| y_1 - y_0 \| + \alpha \]

\[ - \sup_{z \in T(y_0)} \| z \| \| x_0 - x_1 \| - \alpha/6 \]

\[ > \alpha/2 > 0. \]

Thus,

\[ \sup_{x \in S(y_1)} \inf_{z \in T(y_1)} \Re \langle z, y_1 - x \rangle > 0 \]
so that \( y_1 \in \Sigma_2 \) whenever \( y_1 \in X \) with \( \| y_1 - y_0 \| < \delta \). This shows that \( \Sigma_2 \) is open in \( X \) and the proof is completed.

When \( S(x) = X \), we obtain another multivalued version of the Hartman–Stampacchia variational inequality as follows.

**Corollary 3.** Let \( E \) be a normed linear space and \( X \subset E \) a nonempty compact convex subset of \( E \). Let \( T : X \to 2^E \) be upper semicontinuous such that for each \( x \in X \), \( T(x) \) is a nonempty compact convex subset of \( E' \). Then there exist a point \( \hat{y} \in X \) and a point \( \hat{z} \in T(\hat{y}) \) such that

\[
\text{Re} \langle \hat{z}, \hat{y} - x \rangle \leq 0 \quad \text{for all } x \in X.
\]

**References**