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Noncommutative algebras related with Schubert calculus on Coxeter groups

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Dedicated to Alain Lascoux on the occasion of his sixtieth birthday

Abstract

For any finite Coxeter system (W, S) we construct a certain noncommutative algebra, the so-called *bracket algebra*, together with a family of commuting elements, the so-called *Dunkl elements*. The Dunkl elements conjecturally generate an algebra which is canonically isomorphic to the coinvariant algebra of the Coxeter group W. We prove this conjecture for classical Coxeter groups and $I_2(m)$. We define a "quantization" and a multiparameter deformation of our construction and show that for Lie groups of classical type and G_2, the algebra generated by Dunkl's elements in the quantized bracket algebra is canonically isomorphic to the small quantum cohomology ring of the corresponding flag variety, as described by B. Kim. For crystallographic Coxeter systems we define the so-called *quantum Bruhat representation* of the corresponding bracket algebra. We study in more detail the structure of the relations in B_n-, D_n- and G_2-bracket algebras, and as an application, discover *a Pieri-type formula* in the B_n-bracket algebra. As a corollary, we obtain a Pieri-type formula for multiplication of an arbitrary B_n-Schubert class by some special ones. Our Pieri-type formula is a generalization of Pieri's formulas obtained by A. Lascoux and M.-P. Schützenberger for flag varieties of type A. We also introduce a super-version of the bracket algebra together with a family of pairwise anticommutative elements, the so-called *flat connections with constant coefficients*, which describes "a noncommutative differential geometry on a finite Coxeter group" in the sense of S. Majid.

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0. Introduction

The study of the small quantum cohomology ring of flag varieties of type A was initiated by Di Francesco and Itzykson [5], and completed by Givental and Kim [11]. Later, results of [11] were generalized by Kim [14] to the case of flag varieties corresponding to any finite-dimensional semi-simple Lie group. A more “geometric” approach to a description of the small quantum cohomology ring of flag varieties was developed in as yet unpublished lectures by Peterson [21]. A pure algebraic approach to the study of a small quantum cohomology ring of flag varieties of type A was developed in [7, 16]. A new point of view on both classical and quantum cohomology rings of flag varieties of type A has been developed in [8]. Namely, the cohomology rings in question were realized as certain commutative subalgebras in some (noncommutative) quadratic algebras. The latter quadratic algebra corresponding to the classical cohomology ring of flag variety of type A, has many interesting combinatorial and algebraic properties, e.g. it appears to be a braided Hopf algebra over a symmetric group, see, e.g., [1, 18, 20]; its commutative quotient is isomorphic to the algebra of Heaviside’s functions of hyperplane arrangements of type A, see, e.g., [15] and the literature quoted therein; the value of Schubert polynomials on Dunkl elements in the A_n-bracket algebra can be used to describe the structural constants for the product of Schubert classes in the cohomology ring of the flag variety of type A_n [8]. The main algebraic problem related with the latter quadratic algebra is the following: Is this quadratic algebra finite dimensional or not? The main combinatorial problem related with the bracket algebra $BE(A_n)$ is to find a combinatorial description, i.e. a “positive expression” in the algebra $BE(A_n)$, for the Schubert polynomials evaluated at the Dunkl elements. It seems natural to raise a question: Does there exist for any Coxeter group W a certain algebra with properties similar to those for the algebra $BE(A_n)$?

In the present paper we are going to present partial answers to the questions stated above. We introduce and study a generalization of the quadratic algebras from [8] to the case of any finite Coxeter system (W, S). Our starting point is a remarkable result by Dunkl [6] that the algebra generated by “truncated Dunkl operators” [6] is canonically isomorphic to the coinvariant algebra of the Coxeter group W. It is an attempt to construct a “quantum coinvariant algebra” of a finite Coxeter group and find a “quantum” analog of Dunkl’s result mentioned above, that were the main motivation for the present paper.

Let us say a few words about the content of our paper.

In Section 2 we present a definition of bracket algebra $BE(W, S)$, as well as that of its super-version $BE^+(W, S)$, corresponding to any finite Coxeter system (W, S). If (W, S) is the Coxeter system of type A_n, the bracket algebra $BE(W, S)$ coincides with the quadratic algebra $\mathcal{E}_{n+1}$ introduced and studied in [8, 15], while the algebra $BE^+(W, S)$ coincides with the quadratic algebra Λ_{quad} of [18], see also [1, 20]. We note that the algebra $BE(W, S)$, as well as that $BE^+(W, S)$, is a quadratic one only if the Coxeter group W corresponds to a simply-laced semi-simple Lie group. In the case of a crystallographic Coxeter system (W, S), except G_2, we introduce a Hopf algebra structure on the twisted group ring $BE(W, S)\{W\}$ of the algebra $BE(W, S)$, and show that the latter algebra satisfies a “factorization property”, see Lemma 2.1. As a corollary, for a crystallographic Coxeter system (W, S), except G_2, we obtain a decomposition of the bracket algebra $BE(W, S)$ into the tensor product of certain algebras corresponding to the connected components of the Dynkin diagram of the Coxeter system (W, S) after removing all the simple edges. Our results may be considered as a partial generalization of the results obtained in [9] for A_n-quadratic algebras.

In Section 3 we describe two basic representations of the algebra $BE(W, S)$, namely, Calogero–Moser's and Bruhat's representations. The latter is a bridge between the algebra $BE(W, S)$ and the Schubert calculus on the Coxeter system (W, S).

In Section 4 for any $s \in S$, we introduce Dunkl elements in the algebra $BE(W, S)$, denoted by θ_s, and prove that they commute with each other, see Theorem 4.1. The commutative subalgebra generated by Dunkl elements is the main object of our study. We also remark that in the algebra $BE^+(W, S)$ the corresponding elements θ_s, $s \in S$, are pairwise anticommutative.

In Section 5 we state a "classical version" of one of the main results of our paper, namely, that for classical Coxeter groups and $I_2(m)$, the algebra generated by the Dunkl elements is canonically isomorphic to the coinvariant algebra of the corresponding Coxeter group, see Theorem 5.1. We believe that the same result holds for any finite Coxeter system. Our proof of Theorem 5.1 is based on explicit calculations in the corresponding bracket algebras, and we hope to improve our techniques to cover other cases. More specifically, using the defining relations in the algebra $BE(B_n)$, we show that all power sums $p_{2m} := \theta_1^{2m} + \cdots + \theta_n^{2m}$, $m > 0$, are equal to zero. Note that to show the equality $p_4 = 0$ in the algebra $BE(B_n)$, $n \geq 2$, we have to use the 4-term relations of degree four in the algebra $BE(B_n)$. However, in the algebra $BE(D_n)$, $n \geq 4$, the equality $p_4 = 0$ follows only from quadratic relations.

In Section 6 we construct a quantization $qBE(W, S)$ of our bracket algebra $BE(W, S)$.

From Section 7 we will assume that the Coxeter system (W, S) is a crystallographic one. Under the assumption that (W, S) is a crystallographic, we construct a representation of the quantized bracket algebra $qBE(W, S)$ in the group ring of W, Theorem 7.1. The main reason why we made such an assumption on the Coxeter system (W, S) is that the quantum Bruhat representation of the quantized bracket algebra $qBE(W, S)$, as defined in Section 7, does not work for general noncrystallographic groups, e.g. for $I_2(m)$, if $m \geq 9$. In Section 7 we also state one of the main results of the paper, Theorem 7.2, namely, that under the same assumptions as in Theorem 5.1, the subalgebra generated by the Dunkl elements in the quantized bracket algebra $qBE(W, S)$ is canonically isomorphic to the small quantum cohomology ring of the corresponding flag variety.

In Section 8 we state the "quantum Chevalley formula" and prove it for classical Lie groups as a corollary of the existence of the quantum Bruhat representation and our Theorem 7.2.

In Section 9 we describe in more detail the bracket algebras for Lie groups of type B_n, D_n and G_2. In Section 9.2 we are going to make use of an algebraic structure of relations in the algebra $BE(B_n)$ to the study of the so-called *Pieri problem* in the Schubert calculus. Remember that Pieri's problem for a finite Coxeter pair (W, S) means to find a generalization of the Chevalley formula, see Section 5, for multiplication of an arbitrary Schubert class X_w, $w \in W$, by the Schubert class X_s corresponding to a simple reflection $s \in S$, to the case of multiplication of an arbitrary Schubert class X_w by the Schubert class X_u corresponding to an element $u \in W$ which has a *unique* reduced decomposition. For the Coxeter group of type A, the solution to Pieri's problem is well known, see e.g. [17, 22, 24], and is given by the so-called *Pieri formula*. The latter formula may be interpreted as an explicit computation of the elementary $e_k(\mathbf{X}_m)$, and the complete $h_k(\mathbf{X}_m)$, symmetric polynomials in the bracket algebra $BE(A_n)$ after the substitution of the variables $\mathbf{X}_m = (x_1, \ldots, x_m)$ by the A_n-Dunkl elements, see e.g. [8, 22]. In Section 9.2 we give a partial answer to the B_n-Pieri problem stated above, namely, we give an explicit (if complicated) combinatorial formula for the value of the elementary symmetric polynomials of an arbitrary degree and the complete symmetric polynomials of degree at most two in the bracket algebra $BE(B_n)$ after the substitution of the variables by the B_n-Dunkl elements. Let us observe that if we specialize all the generators $[i] \in BE(B_n)$ to zero, we obtain a D_n-analog of Pieri's formula. If we further specialize all the generators $\overline{[i, j]} \in BE(B_n)$ to zero, we will come to the Pieri rule of type A_n. It is known that for Coxeter groups of classical type, the condition that an element $u \in W$ has a unique reduced decomposition is equivalent to the condition that modulo the ideal generated by the fundamental invariant polynomials, the Schubert class X_u is equal to either $e_k(\mathbf{X}_m)$ or $h_k(\mathbf{X}_m)$ for some k and $m \le n$, up to multiplication by some power of 2. Let us remark that our Theorem 9.1 describes Pieri's formula in the algebra $BE(B_n)$. In order to obtain a Pieri-type formula in the corresponding (quantum) cohomology ring one has to apply the (quantum) Bruhat representation, see Theorems 3.2 and 7.1. Since both the classical and the quantum Bruhat representations have a huge kernel, it is not obvious how to deduce the Pieri-type formulas of [2, 23] from the B_n-type Pieri formulas of this paper.

It seems a very interesting problem to extend our results to the cases of the Grothendieck ring and (quantum) equivariant cohomology ring of flag varieties. We will consider these problems in subsequent publications.

We expect that for simply laced Coxeter systems (W, S) the algebra $BE(W, S)$ is a finite-dimensional braided Hopf algebra over W. However, our algebra $BE(D_4)$ is different from the pointed Hopf algebra over D_4 constructed in [20]. Surprisingly, the latter Hopf algebra appears to be isomorphic to a certain quotient of the algebra $BE^+(B_2)$, see Section 9.1. For nonsimply laced Coxeter systems (W, S) the algebra $BE(W, S)$ turns out to be infinite dimensional, but it seems plausible that a certain finite-dimensional quotient of the algebra $BE(W, S)$ has a natural structure of a pointed Hopf algebra, and the algebra generated by the images of Dunkl's elements is isomorphic to that in the algebra $BE(W, S)$.

The main motivation for introducing our bracket algebra and its quantization is an intimate connection of the former and latter with classical and quantum Schubert calculi for finite Coxeter groups [3, 12]. For Coxeter systems of type A, combinatorial and algebraic study of Schubert polynomials was initiated and developed in great detail by Lascoux and Schützenberger [17]. It is our pleasure to express deep gratitude to Lascoux from whom we have learned a lot about this beautiful and deep branch of Mathematics.

1. Coxeter groups

Most part of this section can be found in Humphreys [13].

Definition 1.1. A Coxeter system is a pair (W, S) of a group W and a set of generators $S \subset W$, subject to relations

$$(ss')^{m(s,s')} = 1,$$

where $m(s, s) = 1$ and $m(s, s') = m(s', s) \geq 2$ for $s \neq s' \in S$. The group W is called a Coxeter group.

Definition 1.2. Let (W, S) be a Coxeter system. For an element $w \in W$, the number

$$l(w) = \min\{r \mid w = s_1 \cdots s_r, s_i \in S\}$$

is called the length of w. We say the expression $w = s_1 \cdots s_r (s_i \in S)$ is reduced if $r = l(w)$. The set of all reduced expressions of an element $w \in W$ is denoted by $R(w)$.

We assume S to be finite. Let V be an **R**-vector space with a basis $\Sigma = \{\alpha_s \mid s \in S\}$ and symmetric bilinear form (,) such that

$$(\alpha_s, \alpha_{s'}) = -\cos \frac{\pi}{m(s, s')}.$$

Consider the linear action σ of W on V defined by

$$\sigma(s)\lambda = \lambda - 2(\alpha_s, \lambda)\alpha_s.$$

The representation $\sigma: W \to \mathrm{GL}(V)$ is called the geometric representation of W.

Definition 1.3. We define the root system Δ of W to be the set of the all images of α_s under the action of W.

Any element $\gamma \in \Delta$ can be expressed in the form

$$\gamma = \sum_{s \in S} c_s \alpha_s \quad (c_s \in \mathbf{R}).$$

Call γ positive (resp. negative) and write $\gamma > 0$ (resp. $\gamma < 0$) if all $c_s \geq 0$ (resp. $c_s \leq 0$). Write Δ_+ (resp. Δ_-) for the set of positive (resp. negative) roots. Note that $\Delta = -\Delta$ and $\Delta = \Delta_+ \amalg \Delta_-$.

Lemma 1.1. *The representation* $\sigma: W \to \mathrm{GL}(V)$ *is faithful.*

For a given root $\gamma = w(\alpha_s)$ $(w \in W, s \in S)$, the element wsw^{-1} depends only on γ and it acts on V as a reflection sending γ to $-\gamma$. We denote it by s_γ.

Lemma 1.2. *Let* $w \in W$ *and* $\gamma \in \Delta_+$. *Then* $l(ws_\gamma) > l(w)$ *if and only if* $w(\gamma) > 0$.

Definition 1.4. The Coxeter system (W, S) is called crystallographic when its root system Δ can be normalized to satisfy the condition

$$\frac{2(\gamma, \gamma')}{(\gamma, \gamma)} \in \mathbf{Z}$$

for all $\gamma, \gamma' \in \Delta$.

In our paper, the crystallographic systems are always normalized to satisfy the condition above.

2. Bracket algebra of Coxeter group

2.1. Definition of the bracket algebra

Definition 2.1. Let (W, S) be a Coxeter system and assume W to be finite. We define the bracket algebra $BE(W, S)$ as an associative algebra over $\mathbf{R}$ with generators $[\gamma]$, $\gamma \in \Delta$, subject to the following relations:

(i) For any $\gamma \in \Delta$,

$$[-\gamma] = -[\gamma].$$

(ii) For any $\gamma \in \Delta$,

$$[\gamma]^2 = 0. \tag{1}$$

(iii) (Quadratic relations). Let $\Delta' = \{\gamma_0, \ldots, \gamma_{m-1}\} \subset \Delta_+$ be a set of positive roots such that $\mathbf{R}_{\geq 0}\langle \gamma_i, \gamma_{i+1}\rangle \cap \Delta_+ = \{\gamma_i, \gamma_{i+1}\}$ for all $i = 0, \ldots, m-2$. If Δ' forms a root system of type $I_2(m)(m \geq 2)$, then

$$\sum_{i=0}^{m} [\gamma_i][\gamma_{i+k}] = 0 \tag{2}$$

for $1 \leq k \leq m/2$, where we set by definition $\gamma_{j+m} = -\gamma_j$.

(iv) (4-Term relations for subsystems of type $I_2(m)$). Let $\Delta' \subset \Delta_+$ be a set of positive roots as in (iii). If Δ' forms a root system of type $I_2(m)$, $m \geq 4$, and $k = [m/2] - 1$, then

$$\begin{aligned}&[\gamma_k] \cdot [\gamma_0][\gamma_1] \cdots [\gamma_{2k}] + [\gamma_0][\gamma_1] \cdots [\gamma_{2k}] \cdot [\gamma_k]\\ &\quad + [\gamma_k] \cdot [\gamma_{2k}][\gamma_{2k-1}] \cdots [\gamma_0] + [\gamma_{2k}][\gamma_{2k-1}] \cdots [\gamma_0] \cdot [\gamma_k] = 0.\end{aligned}$$

Remark 2.1. (1) The defining ideal generated by the relations (i), (ii), (iii) and (iv) is stable with respect to the action of the Weyl group W. In other words, the algebra $BE(W, S)$ is a W-module.

(2) If (W, S) is a Coxeter system of type A_n, then the bracket algebra $BE(W, S)$ coincides with the quadratic algebra $\mathcal{E}_{n+1}$ introduced in [8], see also [15].

(3) All the defining relations above come from the subsystems of rank two. As for explicit descriptions of these relations in the case of type B_2, D_2 and G_2, as well as for B_n and D_n types, see Section 9.

(4) Algebra $BE(W, S)$ has a natural grading, if we consider the generators $[\gamma]$ as elements of degree one.

Problem 2.1. Find the Hilbert series of the bracket algebra $BE(W, S)$.

We expect that the algebra $BE(W, S)$ is finite dimensional for simply laced Coxeter groups.

Problem 2.2. Describe the algebra $BE(W, S)$ as a W-module, find its character, and/or the graded multiplicities of its irreducible components.

Remark 2.2. We can define the super-version $BE^+(W, S)$ of the bracket algebra by using the relation $[\gamma] = [-\gamma]$ $(\gamma \in \Delta)$ instead of (i) in Definition 2.1. If (W, S) is of type A_n, the algebra $BE^+(W, S)$ coincides with the algebra Λ_{quad} of [18], see also [1, 20]. For crystallographic groups, one can show that the left-invariant Woronowicz exterior algebra Λ_w [25] for some special choice of a differential structure on W, see [18], is a quotient of the algebra $BE^+(W, S)$. However, in a nonsimply laced case, the algebra Λ_w is a proper quotient of our algebra $BE^+(W, S)$.

2.2. Hopf algebra structure on the twisted group algebra

Since the bracket algebra $BE(W, S)$ has a W-module structure, one can construct the twisted group algebra $BE(W, S)\{W\} = \{\sum_{w \in W} c_w \cdot w \mid c_w \in BE(W, S)\}$ by putting commutation relations $w[\gamma] = [w(\gamma)]w$ for $w \in W$ and $[\gamma] \in BE(W, S)$.

Proposition 2.1. *Let (W, S) be a crystallographic Coxeter system, except G_2, the twisted group algebra $BE(W, S)\{W\}$ has a natural Hopf algebra structure with the coproduct Δ, the antipode S and the counit ϵ defined by the following formulas:*

$$\Delta([\gamma]) = [\gamma] \otimes 1 + s_\gamma \otimes [\gamma], \qquad \Delta(w) = w \otimes w,$$
$$S([\gamma]) = s_\gamma[\gamma], \qquad S(w) = w^{-1},$$
$$\epsilon([\gamma]) = 0, \qquad \epsilon(w) = 1,$$

for $[\gamma] \in BE(W, S)$ and $w \in W$.

Such a Hopf algebra structure was invented and studied in [9] for A_n-quadratic algebras.

The Hopf algebra $BE(W, S)\{W\}$ acts on itself by the adjoint action

$$w\colon x \mapsto wxw^{-1}, \qquad w \in W,$$
$$[\gamma]\colon x \mapsto [\gamma]x - s_\gamma(x)[\gamma].$$

The subalgebra $BE(W, S)$ is invariant under the adjoint action of $BE(W, S)\{W\}$. The element $[\gamma] \in BE(W, S)$ acts on $BE(W, S)$ by a twisted derivation

$$D_\gamma(x) = [\gamma]x - s_\gamma(x)[\gamma],$$

which satisfies the twisted Leibniz rule

$$D_\gamma(xy) = D_\gamma(x)y + s_\gamma(x)D_\gamma(y).$$

Lemma 2.1. *Let (W', S') be a parabolic subsystem of (W, S) and Δ' the set of roots corresponding to (W', S'). Denote by $\mathcal{A}(\Delta \backslash \Delta')$ the subalgebra of $BE(W, S)$ generated by the elements $[\gamma]$, $\gamma \in \Delta \backslash \Delta'$. Assume that $S \backslash S' = \{t\}$ and $m(s, t) \leq 3$ for any $s \in S'$. Then the subalgebra $\mathcal{A}(\Delta \backslash \Delta')$ is invariant under the adjoint action of algebra $BE(W', S')$, and the multiplication map*

$$[\gamma''] \otimes [\gamma'] \mapsto [\gamma''][\gamma'], \qquad [\gamma'] \in BE(W', S'), \qquad [\gamma''] \in \mathcal{A}(\Delta \backslash \Delta')$$

defines a $BE(W', S')$-linear isomorphism of algebras

$$\mathcal{A}(\Delta\backslash\Delta') \otimes BE(W', S') \cong BE(W, S),$$

where $BE(W', S')$-module structure on the tensor product $\mathcal{A}(\Delta\backslash\Delta') \otimes BE(W', S')$ is given by

$$[\gamma](a \otimes b) = D_\gamma(a) \otimes b + s_\gamma \otimes [\gamma]b.$$

It follows from Lemma 2.1 that the Hilbert series of algebra $BE(W', S')$ divides that of algebra $BE(W, S)$. We give a few more examples of application of Lemma 2.1 in Section 9.

Remark 2.3. It is not difficult to see that the algebra $BE(W, S)$ is a braided group in the category of W-crossed modules with braiding $\Psi([\gamma_1] \otimes [\gamma_2]) = s_{\gamma_1}[\gamma_2] \otimes [\gamma_1]$. The Hopf algebra $BE(W, S)\{W\}$ is obtained as its biproduct bosonization in the sense of Majid. For Coxeter groups of type A these results have been shown originally by Majid, see [18] and the literature quoted therein.

3. Representations of bracket algebra

In this section we are going to construct two basic representations of the algebra $BE(W, S)$.

3.1. *Calogero–Moser representation*

Given the geometric representation $\sigma : W \to \mathrm{GL}(V)$, it induces the natural action of W on the ring of polynomial functions $\mathbf{S}(V^*)$. For any positive root γ, the divided difference operator ∂_γ, or Demazure's operator [4], acting on the ring $\mathbf{S}(V^*)$ is defined by

$$\partial_\gamma = \frac{1 - s_\gamma}{\gamma}.$$

Theorem 3.1. *A map $[\gamma] \mapsto \partial_\gamma$ defines a representation of the algebra $BE(W, S)$.*

Proof. Compatibility with the relation (iv) is clear. As for the compatibility with the relation (iii), we may restrict our consideration to subsystems of rank two. It is easy to check the compatibility for A_2, B_2 and $I_2(m)$. □

3.2. *Bruhat representation*

Let us define a linear operator $\mathbf{s}_\gamma$ acting on the group ring $\mathbf{R}\langle W\rangle$ by the rule

$$\mathbf{s}_\gamma \cdot w = \begin{cases} ws_\gamma, & \text{if } l(ws_\gamma) = l(w) + 1, \\ 0, & \text{otherwise.} \end{cases}$$

Theorem 3.2. *A map $[\gamma] \mapsto \mathbf{s}_\gamma$ defines a representation of the algebra $BE(W, S)$.*

Proof. It is enough to show the compatibility with the relation (iii). We use only linear relations among the roots in the subsystem of rank two containing α and β. We may assume that α and β generate a root system of type $I_2(m)$ $(m \geq 2)$. Let $a_i = (\cos(i\pi/m), \sin(i\pi/m)) \in \mathbf{R}^2$, $i = 0, \ldots, m-1$. Then $\Delta_+ = \{a_0, \ldots, a_{m-1}\}$ and we have to check

$$\sum_{i=m-k}^{m-1} [a_{i+k-m}][a_i]w = \sum_{i=0}^{m-k-1} [a_{i+k}][a_i]w,$$

for $1 \leq k \leq (m-1)/2$. From now on, we put $k = 1$ for simplicity, but the following argument works well for all k. If $\mathbf{s}_{a_{i+1}}\mathbf{s}_{a_i}w = ws_{a_i}s_{a_{i+1}}$, then $l(w) = l(ws_{a_i}) - 1$ and $l(ws_{a_i}) = l(ws_{a_i}s_{a_{i+1}}) - 1$ From Lemma 1.2, $w(a_i) > 0$ and $ws_{a_i}(a_{i+1}) = -w(a_{i-1}) > 0$. So we have that $w(a_{m-1}) > 0$ and $ws_{a_{m-1}}(a_0) = w(a_{m-2}) < 0$, and that $w(a_j)$ and $w(a_{j-1})$ are both positive or both negative for $j \neq i$. Hence, if $\mathbf{s}_{a_{i+1}}\mathbf{s}_{a_i}w = ws_{a_i}s_{a_{i+1}}$, then $\mathbf{s}_{a_{j+1}}\mathbf{s}_{a_j}w = 0$ for $j \neq i$ and $\mathbf{s}_{a_0}\mathbf{s}_{a_{m-1}}w = ws_{a_{m-1}}s_{a_0}$. Conversely, if $\mathbf{s}_{a_0}\mathbf{s}_{a_{m-1}}w = ws_{a_{m-1}}s_{a_0}$, then there is only one i such that $\mathbf{s}_{a_{i+1}}\mathbf{s}_{a_i}w = ws_{a_i}s_{a_{i+1}}$ and $\mathbf{s}_{a_{j+1}}\mathbf{s}_{a_j}w = 0$ for $j \neq i$. $\square$

Problem 3.1. Does there exist a finite-dimensional faithful representation of the algebra $BE(W, S)$?

4. Chevalley and Dunkl elements

Definition 4.1. For each $s \in S$, the Chevalley element η_s in the algebra $BE(W, S)$ is defined by

$$\eta_s = \sum_{\gamma \in \Delta_+} \langle \omega_s, \gamma^\vee \rangle [\gamma], \tag{3}$$

where ω_s is the fundamental dominant weight corresponding to α_s and $\gamma^\vee = 2\gamma/(\gamma, \gamma)$.

Definition 4.2. For each $s \in S$, the Dunkl element θ_s in the algebra $BE(W, S)$ is defined by

$$\theta_s = \sum_{s' \in S} c_{s,s'} \eta_{s'},$$

where the coefficients $c_{s,s'}$ are defined by $c_{s,s'} = (\alpha_s, \alpha_{s'})$.

Theorem 4.1. *The Dunkl elements $\theta_s (s \in S)$ commute pairwise.*

Proof. It is enough to show that the Chevalley elements commute pairwise. First of all, let us observe that the element $\eta_s\eta_s' - \eta_s'\eta_s$ can be decomposed as a sum of contributions from root subsystems of rank two. Thus, we may assume that the root system Δ is of type $I_2(m)$. Let $S = \{a_0, a_{m-1}\}$ and

$$a_i = \lambda_1^{-1}\lambda_{i+1}a_0 + \lambda_1^{-1}\lambda_i a_{m-1}, \qquad \lambda_i = \sin\frac{i}{m}\pi, \qquad 0 \leq i \leq m-1.$$

Then $\Delta = \{a_0, a_1, \ldots, a_{m-1}\}$. We have to show that η_1 and η_2 commute, where

$$\eta_1 = \sum_{i=0}^{m-1} \lambda_{i+1}[a_i], \qquad \eta_2 = \sum_{i=0}^{m-1} \lambda_i[a_i].$$

We have

$$2(\eta_1\eta_2 - \eta_2\eta_1) = \sum_{i,j=0}^{m-1} \left(\cos \frac{i-j+1}{m}\pi + \cos \frac{i+j+1}{m}\pi \right) ([a_i][a_j] - [a_j][a_i]).$$

Here, $\cos((i+j+1)\pi/m)$ is symmetric on i and j, so

$$\sum_{i,j} \left(\cos \frac{i+j+1}{m}\pi \right) ([a_i][a_j] - [a_j][a_i]) = 0.$$

Note that $s_{a_i}s_{a_j} = s_{a_p}s_{a_q}$ if and only if $i - j \equiv p - q \mod m$. Hence the relations in Definition 2.1 (iii) imply that

$$\sum_k \sum_{i-j\equiv k\ (m)} \left(\cos \frac{k+1}{m}\pi \right) \big([a_i][a_j] - [a_j][a_i]\big) = 0. \quad \square$$

Remark 4.1. For the commutativity of the Dunkl elements θ_s it is enough to assume the validity of quadratic relations (iii) in Definition 2.1 only.

Remark 4.2. In a similar fashion one can check that the elements θ_s, $s \in S$, defined as in Definition 4.2 in the super-version $BE^+(W, S)$ of the bracket algebra $BE(W, S)$ are pairwise *anticommutative*. It is a challenging problem to describe the subalgebra in $BE^+(W, S)$ generated by the elements θ_s, $s \in S$.

5. Algebra generated by Dunkl elements

Let $|S| = n$. In case when W is a finite reflection group, it is known that the subalgebra $\mathbf{S}(V^*)^W \subset \mathbf{S}(V^*)$ of W-invariant polynomials is generated over $\mathbf{R}$ by n homogeneous, algebraically independent polynomials $f_1, \ldots, f_n$ of positive degree. We denote by $I_W \subset \mathbf{S}(V^*)$ the ideal generated by $f_1, \ldots, f_n$. The quotient ring $\mathbf{S}_W := \mathbf{S}(V^*)/I_W$ is called the coinvariant algebra of W.

An explicit construction of a linear basis of $\mathbf{S}_W$ is given by Bernstein et al. [3], and Hiller [12]. Let $w = s_{i_1} \ldots s_{i_l}$ ($s_{i_1}, \ldots, s_{i_l} \in S$) be a reduced decomposition of $w \in W$. We define the operator ∂_w acting on the algebra of polynomial functions $\mathbf{S}(V^*)$ by $\partial_w = \partial_{\alpha_{s_{i_1}}} \cdots \partial_{\alpha_{s_{i_l}}}$, where $\partial_{\alpha_{s_{i_1}}}, \ldots, \partial_{\alpha_{s_{i_l}}}$ are divided difference operators defined in Section 3. The definition of the operator ∂_w is independent of the choice of a reduced decomposition of w.

For any polynomial $f \in \mathbf{S}(V^*)$, one can define an element $[f] \in BE(W, S)$ as an image of f by the algebra homomorphism obtained by the substitution $\omega_s \mapsto \eta_s$.

Definition 5.1 (cf. [3, 12]). We define the polynomials $X_w \in \mathbf{S}(V^*)$, $w \in W$, by the following formulas:

$$X_{w_0} = |W|^{-1} \prod_{\gamma\in\Delta_+} \gamma,$$
$$X_w = \partial_{w^{-1}w_0} X_{w_0},$$

where $w_0 \in W$ is the element of maximal length.

It is known [3, 12] that the images of the polynomials $\{X_w\}$ in the coinvariant algebra $\mathbf{S}_W$ form a linear basis and satisfy the *Chevalley formula*

$$X_s X_w = \sum (\omega_s, \gamma^\vee) X_{ws_\gamma} \bmod I_W,$$

where the sum is taken over the positive roots γ such that $l(ws_\gamma) = l(w) + 1$. It is useful to note that one can obtain the Chevalley formula above by applying the Bruhat representation, see Theorem 3.2, to the equality $[X_s] = \eta_s$ in the algebra $BE(W, S)$.

We have the following statement from the Chevalley formula.

Lemma 5.1. *There exists a surjective homomorphism from* $\mathbf{S}_W$ *to the subalgebra generated by the Chevalley elements* $\mathbf{R}[\eta_s \mid s \in S] \subset BE(W, S)$*, which maps* X_s *to* η_s*.*

Theorem 5.1. *For Coxeter groups of classical type and* $I_2(m)$*, the subalgebra* $\mathbf{R}[\theta_s \mid s \in S]$ *in* $BE(W, S)$ *generated by Dunkl elements is canonically isomorphic to the coinvariant algebra of the group* W*, i.e.*

$$\mathbf{R}[\theta_s \mid s \in S] \cong \mathbf{S}_W.$$

We postpone a proof until Section 9.

Conjecture 5.1. *The statement of Theorem 5.1 is valid for any finite Coxeter group.*

Conjecture 5.2. *Let* (W, S) *be a crystallographic Coxeter system, then there exists a monomial basis* $\{b_\mu\}_\mu$ *in the algebra* $BE(W, S)$*, such that for any* $w \in W$ *the polynomial* $[X_w]$ *can be expressed as a linear combination of* b_μ*'s with non-negative coefficients.*

6. Quantization of bracket algebra

We consider the group of characters

$$C = \mathrm{Hom}(V^*, S^1),$$

and its elements $q_s = \exp(2\pi\sqrt{-1}\langle\cdot, \alpha_s^\vee\rangle)$ for $s \in S$. For $\gamma^\vee = \sum_{s\in S} n_s \alpha_s^\vee$, we set $q_{\gamma^\vee} = \prod_s q_s^{n_s}$.

Definition 6.1. The quantized bracket algebra $qBE(W, S)$ is the associative algebra over the ring $\mathbf{R}[q_s \mid s \in S]$ with generators $[\gamma]$, $\gamma \in \Delta$, subject to the relations:

(i)′ For any $\gamma \in \Delta$,

$$[-\gamma] = -[\gamma].$$

(ii)′ For $\gamma \in \Delta_+$,

$$[\gamma]^2 = q_\gamma, \qquad \text{if } \gamma \in \Sigma,$$
$$[\gamma]^2 = 0, \qquad \text{otherwise.}$$

(iii)′ The same relations as in Definition 2.1 (iii).

(iv)′ Under the same assumptions as in Definition 2.1 (iv), if in addition the following inequality $l(s_{\gamma_k}) \neq 2(\rho, \gamma_k^\vee) - 1$ holds, then

$$[\gamma_k] \cdot [\gamma_0][\gamma_1] \cdots [\gamma_{2k}] + [\gamma_0][\gamma_1] \cdots [\gamma_{2k}] \cdot [\gamma_k] + [\gamma_k] \cdot [\gamma_{2k}][\gamma_{2k-1}] \cdots [\gamma_0] + [\gamma_{2k}][\gamma_{2k-1}] \cdots [\gamma_0] \cdot [\gamma_k] = 0.$$

Definition 6.2. The Chevalley elements $\tilde{\eta}_s$ and the Dunkl elements $\tilde{\theta}_s$, $s \in S$, in the algebra $qBE(W, S)$ are defined by the same formulas as in Definitions 4.1 and 4.2.

Theorem 6.1. *The Dunkl elements $\tilde{\theta}_s$, $s \in S$, commute pairwise.*

Proof. The proof can be performed in the same manner as that of Theorem 3.1. □

Remark 6.1. It is natural to consider a multiparameter deformation of the algebra $BE(W, S)$ which is generated by elements $[\gamma]$, $\gamma \in \Delta$, with defining relations (i)′, (iii)′, (iv)′ and the additional one

(ii)″ $$[\gamma]^2 = Q_\gamma, \qquad \text{if } \gamma \in \Delta_+,$$

where Q_γ's are independent central parameters indexed by $\gamma \in \Delta_+$. The commutative algebra generated by Dunkl elements in this case may be considered as a "multiparameter" deformation of the coinvariant algebra of the Coxeter system (W, S).

Remark 6.2. In a similar fashion one can define a quantization $qBE^+(W, S)$ of the super-version $BE^+(W, S)$ of the bracket algebra $BE(W, S)$ and a family of elements $\tilde{\theta}_s \in qBE^+(W, S)$. See Remark 2.2 for the definition of the algebra $BE^+(W, S)$. It is a challenging problem to describe the subalgebra in $qBE^+(W, S)$ generated by the pairwise anticommutative elements $\tilde{\theta}_s$, $s \in S$.

7. Extended Bruhat graph and quantum Bruhat representation

Starting from this section, we assume that the Coxeter system (W, S) is a crystallographic one. Let us denote by ρ the half-sum of all positive roots, i.e.

$$\rho = \frac{1}{2} \sum_{\gamma \in \Delta_+} \gamma.$$

If $\gamma^\vee = \sum_{s \in S} n_s \alpha_s^\vee$, then

$$(\rho, \gamma^\vee) = \sum_{s \in S} n_s.$$

Lemma 7.1. *Let γ be a positive root, then*

$$2(\rho, \gamma^\vee) - 1 \geq l(s_\gamma).$$

Proof. For $\gamma \in \Delta_+$, define r as the minimal number of simple reflections $s_0, s_1, \ldots, s_{r-1}$ such that $s_\gamma = s_{r-1} \cdots s_1 s_0 s_1 \cdots s_{r-1}$. Then, we can conclude that $l(s_\gamma) = 2r - 1$. Hence, by induction on r, we have $2(\rho, \gamma^\vee) - 1 \geq 2r - 1 = l(s_\gamma)$. □

7.1. Extended Bruhat graph

Definition 7.1. The extended Bruhat graph $\Gamma(W, S)$ is a graph whose vertices are elements of W with arrows $v \to w$ in the Bruhat ordering and additional arrows $v \xrightarrow{\gamma}_e w$ which mean that $w = vs_\gamma (\gamma \in \Delta_+)$ and $l(w) = l(v) - 2(\rho, \gamma^\vee) + 1$.

Lemma 7.2. *Let (W, S) be a crystallographic Coxeter system and (W', S') its parabolic subsystem. Then the extended Bruhat graph $\Gamma(W', S')$ is a subgraph of $\Gamma(W, S)$ by the map induced by the inclusion $W' \to W$. Moreover, if there exists an arrow $v \to_e w$ with $v, w \in W'$ in $\Gamma(W, S)$, then the arrow $v \to_e w$ belongs to $\Gamma(W', S')$.*

This follows immediately from Lemmas 1.2 and 7.1.

Remark 7.1. Definition 7.1 and Lemma 7.2 were discovered originally by Peterson [21].

7.2. Quantum Bruhat representation

Let us define an operator $\tilde{\mathbf{s}}_\gamma (\gamma \in \Delta_+)$ acting on the group ring $\mathbf{Q}[q_s \mid s \in S]\langle W\rangle$, by the rule

$$\tilde{\mathbf{s}}_\gamma \cdot w = \begin{cases} ws_\gamma, & \text{if } l(w) = l(ws_\gamma) - 1, \\ q_{\gamma^\vee} ws_\gamma, & \text{if } l(w) = l(ws_\gamma) + 2(\rho, \gamma^\vee) - 1, \\ 0, & \text{otherwise.} \end{cases}$$

Theorem 7.1. *A map $[\gamma] \mapsto \tilde{\mathbf{s}}_\gamma$ defines a representation of the quantized bracket algebra $qBE(W, S)$.*

Proof. The compatibility with the relations (ii)′ in Definition 6.1 is clear. We check the compatibility with the relations (iii)′ and (iv)′. Let Δ' be as in Definition 2.1 (iii). We are considering only crystallographic root systems, so we may assume that Δ' is of type $I_2(m)$ with $m = 3, 4, 6$. Take an arbitrary element $w \in W$. If $l(ws_\beta s_\alpha) = l(w) + 2$, then relation

$$[\gamma_0][\gamma_{m-1}]w = \sum_i [\gamma_i][\gamma_{i+1}]w$$

follows from the same argument as in the proof of Theorem 3.2.

Let $\alpha = \gamma_0$, $\beta = \gamma_{m-1}$ and $A(\alpha, \beta) = \{(\gamma_i, \gamma_{i+1}) \mid i = 0, \ldots, m-2\}$. We consider the case $l(ws_\beta s_\alpha) \le l(w)$. Note that $(\rho, \gamma_i^\vee) \ge (\rho, \alpha^\vee)$ and $(\rho, \gamma_{i+1}^\vee) \ge (\rho, \beta^\vee)$ for $i = 0, \ldots, m-2$. For $(\gamma_1, \gamma_2), (\delta_1, \delta_2) \in A(\alpha, \beta)$, $(\rho, \gamma_j^\vee) = (\rho, \delta_j^\vee)$ holds if and only if $(\gamma_1, \gamma_2) = (\delta_1, \delta_2)$. Hence, if there exists a path Γ of type $w \xrightarrow{\gamma}_e * \xrightarrow{\delta}_e ws_\beta s_\alpha$, then we have

$$l(w) = l(ws_\beta s_\alpha) + 2(\rho, \gamma^\vee) + 2(\rho, \delta^\vee) - 2 \ge l(ws_\beta s_\alpha) + l(s_\alpha) + l(s_\beta). \qquad (*)$$

This means that $l(w) = l(ws_\beta s_\alpha) + l(s_\alpha) + l(s_\beta)$, and $(\gamma, \delta) = (\beta, \alpha)$. In this case, we can see that there exists unique pair $(\gamma_1, \gamma_2) \in A(\alpha, \beta)$ such that if $[\gamma_1][\gamma_2]w \neq 0$ and $(*)$ holds. Similarly, if there exists a path Γ of type $w \xrightarrow{\beta}_e * \xrightarrow{\alpha} ws_\beta s_\alpha$ or $w \xrightarrow{\beta} * \xrightarrow{\alpha}_e ws_\beta s_\alpha$, we can find unique pair $(\gamma_1, \gamma_2) \in A(\alpha, \beta)$ such that $[\gamma_1][\gamma_2]w \neq 0$. $\square$

Remark 7.2. It follows from our proof that in the extended Bruhat graph corresponding to a crystallographic Coxeter group, there exist exactly two paths connecting two vertices v_1, v_2 such that $l(v_1) - l(v_2) \equiv 0 \pmod 2$. This property does not hold in general for noncrystallographic Coxeter systems.

Now we assume that Coxeter system (W, S) comes from a connected simply connected semi-simple Lie group G. We denote by B the Borel subgroup of G. The small quantum cohomology ring of the flag variety G/B is isomorphic to the quotient ring of the polynomial ring $\mathbf{S}(V^*) \otimes \mathbf{R}[q_s]$ by the ideal $\tilde{I}_W$ generated by quantum W-invariant polynomials, which are explicitly given by Kim [14].

Theorem 7.2. *For Coxeter groups of classical type and of type G_2, the subalgebra in $qBE(W, S)$ generated by Dunkl elements, $\mathbf{R}[q_s][\tilde{\theta}_s \mid s \in S]$, is canonically isomorphic to the quantum cohomology ring $QH^*(G/B)$.*

A proof of Theorem 7.2 is based on direct computations, see Section 9. Note that for Lie algebras of type A, Theorem 7.2 was stated for the first time in [8], and has been proved later in [22].

Conjecture 7.1. *Theorem 7.2 holds for any crystallographic finite Coxeter system.*

Problem 7.1. For any finite Coxeter system (W, S), describe "a quantum coinvariant algebra" of the group W, i.e. to describe the subalgebra in $qBE(W, S)$ generated by the Dunkl elements $\tilde{\theta}_s$, $s \in S$.

8. Quantum Chevalley formula

For any polynomial $f \in \mathbf{S}(V^*) \otimes \mathbf{R}[q_s]$, one can define an element $[f]$ of $qBE(W, S)$, using the substitution $\omega_s \mapsto \tilde{\eta}_s$. We regard this element $[f]$ as an operator acting on the group ring $\mathbf{R}[q_s]\langle W\rangle$.

Proposition 8.1. *Let $w \in W$, there exists a unique polynomial $\tilde{P}_w \in \mathbf{S}(V^*) \otimes \mathbf{R}[q_s]$ characterized by the following conditions:*

$$[\tilde{P}_w](1) = w,$$
$$\tilde{P}_w = X_w + \sum_{l(v)<l(w)} c_v X_v \qquad (c_v \in \mathbf{R}[q_s]),$$

where X_w are the polynomials defined in Section 5, Definition 5.1.

Proof. If $l(w) < 2$, then $[X_w](1) = w$. In general, we have

$$[X_w](1) = w + \sum_{l(v)<l(w)} c_v v \qquad (c_v \in \mathbf{R}[q_s], v \in W). \quad \square$$

Remark 8.1. The polynomial $\tilde{P}_w$ defined in Proposition 8.1 coincides with the quantum Bernstein–Gelfand–Gelfand polynomial introduced in [19].

It follows from Theorems 7.1 and 7.2 that for classical Coxeter groups and G_2 one has **Quantum Chevalley formula** ([10, 21]). For $s \in S$ and $w \in W$, we have

$$\tilde{P}_s \tilde{P}_w = \sum_{w \xrightarrow{\gamma} w'} \langle \omega_s, \gamma^\vee \rangle \tilde{P}_{w'} + \sum_{w \xrightarrow{\gamma}_e w'} q_{\gamma^\vee} \langle \omega_s, \gamma^\vee \rangle \tilde{P}_{w'} \mod \tilde{I}_W,$$

where the sums are taken with respect to the positive roots γ.

Remark 8.2. In Proposition 8.1, we have introduced the polynomial $\tilde{P}_w$ satisfying the condition $[\tilde{P}_w](1) = w$. One can consider the action of $[\tilde{P}_w]$ on any element $u \in W$ via the quantum Bruhat representation, and obtain an expression

$$[\tilde{P}_w](u) = \sum_{v \in W} c_{wu}^v(q) \cdot v,$$

where $c_{wu}^v(q) \in \mathbf{R}[q_s]$ are polynomials whose coefficients are the so-called 3-point Gromov–Witten invariants of genus zero for the target space G/B.

Conjecture 8.1. *Let (W, S) be a crystallographic Coxeter system, then there exists a monomial basis $\{b_\mu\}_\mu$ in the algebra $qBE(W, S)$ such that for any $w \in W$ the polynomial $[\tilde{P}_w]$ can be written as a linear combination of b_μ's with nonnegative coefficients which do not depend on q_s's.*

9. Examples

Explicit description of relations and the Dunkl elements for quantized A_n-bracket algebra is given in [8]. In this section we study in more detail the cases of B_n-, D_n- and G_2-bracket algebras.

We fix an orthonormal basis $e_1, \ldots, e_n$ of n-dimensional Euclidean space.

9.1. Quantized B_n-bracket algebra

The root system of type B_n, $n \geq 2$, consists of the elements $\pm e_i \pm e_j$ and $\pm e_i$ $(1 \leq i, j \leq n)$, and we fix a set of simple roots

$$S(B_n) = \{\alpha_1 = e_1 - e_2, \ldots, \alpha_{n-1} = e_{n-1} - e_n, \alpha_n = e_n\}.$$

The quantized B_n-bracket algebra $qBE(B_n) = qBE(W(B_n), S(B_n))$ is generated by the symbols $[i, j] = [e_i - e_j]$, $\overline{[i, j]} = [e_i + e_j]$ and $[i] = [e_i]$ over $\mathbf{R}[q_1, \ldots, q_n]$ subject to the following relations:

(0) $[i, j] = -[j, i]$, $\overline{[i, j]} = \overline{[j, i]}$,

(1) $[i, i+1]^2 = q_i$, $[n]^2 = q_n$, $[i, j]^2 = 0$, if $|i - j| \neq 1$; $[i]^2 = 0$, if $i < n$; $\overline{[i, j]}^2 = 0$, if $i \neq j$,

(2) $[i, j][k, l] = [k, l][i, j]$, $\overline{[i, j]}[k, l] = [k, l]\overline{[i, j]}$, $\overline{[i, j][k, l]} = \overline{[k, l][i, j]}$, if $\{i, j\} \cap \{k, l\} = \phi$,

(3) $[i][j] = [j][i]$, $[i, j]\overline{[i, j]} = \overline{[i, j]}[i, j]$, $[i, j][k] = [k][i, j]$, $\overline{[i, j]}[k] = [k]\overline{[i, j]}$, if $k \neq i, j$,

(4) $[i,j][j,k]+[j,k][k,i]+[k,i][i,j]=0$, $\overline{[i,k]}[i,j]+[j,i]\overline{[j,k]}+\overline{[k,j][i,k]}=0$, $[i,j][i]+[j][j,i]+[i]\overline{[i,j]}+\overline{[i,j]}[j]=0$, if all i, j and k are distinct,

(5) $[i,j][i]\overline{[i,j]}[i]+\overline{[i,j]}[i][i,j][i]+[i][i,j][i]\overline{[i,j]}+[i]\overline{[i,j]}[i][i,j]=0$, if $i<j$.

The Chevalley and Dunkl elements are given by $\tilde{\eta}_{s_{\alpha_i}}=\tilde{\theta}_1+\cdots+\tilde{\theta}_i$, where

$$\tilde{\theta}_i := \tilde{\theta}_i^{B_n} = \sum_{j\neq i}([i,j]+\overline{[i,j]})+2[i], \qquad 1\le i\le n.$$

The Chevalley elements $\tilde{\eta}_{s_{\alpha_i}}$ correspond to the Pieri–Chevalley type formula, where as the Dunkl elements $\tilde{\theta}_i$ correspond to the Monk type formula in the cohomology ring of the flag variety. It is easy to see that in the formula for $\tilde{\theta}_i$ above, one can replace the term $2[i]$ by that $c[i]$ for any constant c. The resulting operators still commute pairwise.

Now we define the quantum B_n-invariant polynomials following [14]. Let $E_{i,j}\in M_{2n}(\mathbf{R})$ be a matrix such that its (i,j) entry is 1 and other entries are 0. We set $t_i=E_{i,i}-E_{i+n,i+n}$, $E_{\alpha_i^\vee}=E_{i+1,i}+E_{i+n,i+n+1}$, $E_{-\alpha_i^\vee}=E_{i,i+1}-E_{i+n+1,i+n}$ $(1\le i\le n-1)$, $E_{\alpha_n^\vee}=-2E_{2n,n}$ and $E_{-\alpha_n^\vee}=2E_{n,2n}$. Let

$$X^B(e,q)=\sum_i e_i t_i+\sum_j q_j E_{-\alpha_j^\vee}+\sum_j E_{\alpha_j^\vee}.$$

The quantum B_n-invariant polynomials $J_\nu^B(e,q)=J_\nu^B(e_1,\ldots,e_n;q_1,\ldots,q_n)$ $(1\le\nu\le n)$ are coefficients of the characteristic polynomial of $X^B(e,q)$, namely,

$$\det(tI+X^B(e,q))=t^{2n}+\sum_{\nu=1}^{n}J_\nu^B(e,q)t^{2(n-\nu)}.$$

The quantum cohomology ring of B_n-flag variety is isomorphic to the ring

$$\mathbf{C}[e_1,\ldots,e_n,q_1,\ldots,q_n]/(J_1^B,\ldots,J_n^B).$$

Proposition 9.1. *In the quantized bracket algebra $qBE(B_n)$ we have the following identities:*

$$J_\nu^B(\tilde{\theta}_1,\ldots,\tilde{\theta}_n;q)=0, \qquad 1\le\nu\le n.$$

Proof of Proposition 9.1 is based on Lemma 9.1 below.

Before stating it, let us introduce a bit of notation.

Notation. Let $\{i,j\}$ denote either generator $[i,j]$ or $\overline{[i,j]}$, and define $\overline{\overline{[i,j]}}=[i,j]$. We also define elements $A(a_1,\ldots,a_k),\overline{A}(a_1,\ldots,a_k)\in BE(B_n)$ for distinct integers $2\le a_1,\ldots,a_k\le n$ as follows:

$$A(a_1,\ldots,a_k)=\sum_{j=1}^{k}(-1)^{j-1}\left(\prod_{m=j}^{k}[1,a_m]\right)\cdot[1]\cdot\left(\prod_{m=1}^{j}\overline{[1,a_m]}\right),$$

$$\overline{A}(a_1,\ldots,a_k)=\sum_{j=1}^{k}(-1)^{k-j}\left(\prod_{m=j}^{k}\overline{[1,a_m]}\right)\cdot[1]\cdot\left(\prod_{m=1}^{j}[1,a_m]\right).$$

Lemma 9.1. *We have the following cyclic relations in the algebra $BE(B_n)$ for distinct integers $2 \leq a_1, \ldots, a_k \leq n$:*

(1) $\{1,a_1\}\{1,a_2\}\cdots\{1,a_k\}\{1,a_1\} + (\text{cyclic permutations on indices}) = 0$;
(2) $\{1,a_1\}\{1,a_2\}\cdots\{1,a_k\}\overline{\{1,a_1\}} + \{1,a_2\}\{1,a_3\}\cdots\{1,a_k\}\overline{\{1,a_1\}\{1,a_2\}} + \cdots + \{1,a_k\}\overline{\{1,a_1\}\cdots\{1,a_{k-1}\}}\{1,a_k\} = \overline{\{1,a_1\}}\{1,a_2\}\cdots\{1,a_k\}\{1,a_1\} + \overline{\{1,a_2\}}\{1,a_3\}\cdots\{1,a_k\}\overline{\{1,a_1\}}\{1,a_2\} + \cdots + \overline{\{1,a_k\}\{1,a_1\}\cdots\{1,a_{k-1}\}}\{1,a_k\}$;
(3) $[1](A(a_1,\ldots,a_k) + \overline{A}(a_1,\ldots,a_k)) + (A(a_1,\ldots,a_k) + \overline{A}(a_1,\ldots,a_k))[1] = 0$;
(4) *All the relations which are obtained from (1), (2) and (3) by the action of the Weyl group.*

Example 9.1. For $k = 3$, one can write down the relations in Lemma 9.1 as follows:

(1) $\{1,a_1\}\{1,a_2\}\{1,a_3\}\{1,a_1\}+\{1,a_2\}\{1,a_3\}\{1,a_1\}\{1,a_2\}+\{1,a_3\}\{1,a_1\}\{1,a_2\}\{1,a_3\} = 0$;
(2) $\{1,a_1\}\{1,a_2\}\{1,a_3\}\overline{\{1,a_1\}}+\{1,a_2\}\{1,a_3\}\overline{\{1,a_1\}\{1,a_2\}}+\{1,a_3\}\overline{\{1,a_1\}\{1,a_2\}\{1,a_3\}}$
$= \overline{\{1,a_1\}}\{1,a_2\}\{1,a_3\}\{1,a_1\} + \overline{\{1,a_2\}}\{1,a_3\}\overline{\{1,a_1\}}\{1,a_2\}$
$+\overline{\{1,a_3\}\{1,a_1\}\{1,a_2\}}\{1,a_3\}$;
(3) $[1][1,a_1][1,a_2][1,a_3][1]\overline{[1,a_1]} - [1][1,a_2][1,a_3][1]\overline{[1,a_1][1,a_2]}$
$+ [1][1,a_3][1]\overline{[1,a_1][1,a_2][1,a_3]} + [1]\overline{[1,a_1][1,a_2][1,a_3]}[1][1,a_1]$
$- [1]\overline{[1,a_2][1,a_3]}[1][1,a_1][1,a_2] + [1]\overline{[1,a_3]}[1][1,a_1][1,a_2][1,a_3]$
$+ [1,a_1][1,a_2][1,a_3][1]\overline{[1,a_1]}[1] - [1,a_2][1,a_3][1]\overline{[1,a_1][1,a_2]}[1]$
$+ [1,a_3][1]\overline{[1,a_1][1,a_2][1,a_3]}[1] + \overline{[1,a_1][1,a_2][1,a_3]}[1][1,a_1][1]$
$- \overline{[1,a_2][1,a_3]}[1][1,a_1][1,a_2][1] + \overline{[1,a_3]}[1][1,a_1][1,a_2][1,a_3][1] = 0.$

In the final part of this subsection we consider an application of Lemma 2.1 to the case of B_n-bracket algebra.

Let $x_i = [i,n]$, $y_i = \overline{[i,n]}$ for $1 \leq i \leq n-1$, and $z_n = [n]$ be elements of $BE(B_n)$. Denote by $\mathcal{A}_n^B = \mathcal{A}(\Delta(B_n)\backslash\Delta(B_{n-1}))$ the subalgebra of $BE(B_n)$ generated by $x_1, \ldots, x_n$, $y_1, \ldots, y_n$ and z_n.

Proposition 9.2. *Action of the twisted derivation $D_{[\gamma]}$ on the algebra $\mathcal{A}_n^B$ is determined by the following formulas:*

$$D_{[i,j]}(x_i) = -x_i x_j, \qquad D_{[i,j]}(x_j) = x_j x_i, \qquad D_{[i,j]}(y_i) = -y_i y_j,$$
$$D_{[i,j]}(y_j) = y_j y_i, \qquad D_{\overline{[i,j]}}(x_i) = x_i y_j, \qquad D_{\overline{[i,j]}}(x_j) = x_j y_i,$$
$$D_{\overline{[i,j]}}(y_i) = y_i x_j, \qquad D_{\overline{[i,j]}}(y_j) = y_j x_i, \qquad D_{[i]}(x_i) = x_i z_n - z_n y_i,$$
$$D_{[i]}(y_i) = z_n x_i - y_i z_n, \qquad D_{[i,j]}(z_n) = D_{\overline{[i,j]}}(z_n) = D_{[i]}(z_n) = 0,$$

for $1 \leq i \neq j \leq n-1$, and

$$D_{\{k,l\}}(x_i) = D_{\{k,l\}}(y_i) = D_{[k]}(x_i) = D_{[k]}(y_i) = 0, \qquad \text{if } i,k,l \text{are all distinct},$$

and the twisted Leibniz rule, see the definition of the former in Section 2.2.

Therefore, the subalgebra $\mathcal{A}_n^B$ is invariant under the twisted derivation $D_{[\gamma]}$ for any root $\gamma \in \Delta(B_{n-1})$. By applying Lemma 2.1 successively, we obtain the following decomposition of the algebra $BE(B_n)$ for $n \geq 2$:

$$BE(B_n) \cong \mathcal{A}_2^B \otimes \cdots \otimes \mathcal{A}_n^B.$$

Note that the relations in Lemma 9.1 can be obtained by applying the twisted derivations successively to the defining relations of the bracket algebra. We do not know whether or not all the relations in the bracket algebra can be obtained in such a way.

Example 9.2. By applying $D_{[a_2,a_3]}D_{[a_1,a_2]}$ to the 4-term relation

$$\begin{aligned}&[1][1,a_1][1]\overline{[1,a_1]}+[1,a_1][1]\overline{[1,a_1]}[1]\\&\quad+[1]\overline{[1,a_1]}[1][1,a_1]+\overline{[1,a_1]}[1][1,a_1][1]=0,\end{aligned}$$

we obtain the relation (3) in Example 9.1.

We conclude this subsection by a construction of one more representation of the algebra $BE(B_{n-1})$. Denote by $\mathcal{F}_n^B$ the quotient of the free associative algebra over $\mathbf{R}$ generated by $X_1,\ldots,X_{n-1}$, $Y_1,\ldots,Y_{n-1}$ and Z_n modulo the two-side ideal generated by X_i^2, Y_i^2, Z_n^2 and $Z_nX_iZ_nY_i+X_iZ_nY_iZ_n+Z_nY_iZ_nX_i+Y_iZ_nX_iZ_n$ for $1\le i\le n-1$. The Weyl group $W(B_{n-1})$ acts on the algebra $\mathcal{F}_n^B$ by the rule

$$\begin{aligned}&s_{ij}(X_i)=X_j, \qquad s_{ij}(Y_i)=Y_j, \qquad s_{\overline{ij}}(X_i)=-Y_j, \qquad s_{\overline{ij}}(Y_i)=-X_j,\\&s_i(X_i)=-Y_i, \qquad s_i(Y_i)=-X_i, \qquad s_{ij}(X_k)=s_{\overline{ij}}(X_k)=s_i(X_k)=X_k,\\&s_{ij}(Y_k)=s_{\overline{ij}}(Y_k)=s_i(Y_k)=Y_k, \qquad s_{ij}(Z_n)=s_{\overline{ij}}(Z_n)=s_i(Z_n)=Z_n\end{aligned}$$

for distinct $i,j,k\in\{1,\ldots,n-1\}$. Now define operators $\nabla_{[i,j]}$, $\nabla_{\overline{[i,j]}}$ and $\nabla_{[i]}$, $1\le i\ne j\le n-1$, which act on the algebra $\mathcal{F}_n^B$ by the same formulas as for the operators $D_{[i,j]}$, $D_{\overline{[i,j]}}$ and $D_{[i]}$ from Proposition 9.2 after replacing x_i, y_i and z_n by X_i, Y_i and Z_n respectively. Then the operators $\nabla_{[i,j]}$ and $\nabla_{\overline{[i,j]}}$ and $\nabla_{[i]}$, $1\le i\ne j\le n-1$, give rise to a representation of the algebra $BE(B_{n-1})$ in the algebra $\mathcal{F}_n^B$, and natural epimorphism $\pi_n^B:\mathcal{F}_n^B\to\mathcal{A}_n^B$ is compatible with the action of the algebra $BE(B_{n-1})$.

Problem 9.1. Describe the kernel of the epimorphism π_n^B.

9.2. Pieri formula for B_n-bracket algebra

The main goal of this subsection is to describe a B_n-analog of Pieri's formula in some cases, namely, we give an explicit formula for the value of elementary symmetric polynomials of arbitrary degree and complete symmetric polynomials of degree two in the bracket algebra $BE(B_n)$ after the substitution of variables by the B_n-Dunkl elements. Let us observe that if we specialize all the generators $[i]\in BE(B_n)$ to zero, we obtain a D_n-analog of Pieri's formula. To state our result, it is convenient to introduce a bit of notation. Let $S=\{i_1<i_2<\cdots<i_s:=r\}$ be a set of positive integers. Define inductively a family of elements $\{K_l(S)\}_{l\ge1}$ in the algebra $BE(B_r)$ by the following rules:

(i)
$$K_1(S)=\sum_{i\in S}[i]+\sum_{i\le j,i,j\in S}\overline{[i,j]};\ K_l(S)=0, \qquad \text{if } s<l;$$

(ii)
$$K_l(S)=K_l(S\backslash\{r\})+\sum_{a\in S}([a,r]+\overline{[a,r]})K_{l-1}(S\backslash\{a\})+K_{l-1}(S\backslash\{r\})\theta_{r,S},$$

where $\theta_{r,S}=\sum_{a\in S}(-[a,r]+\overline{[a,r]})+2[r]$.

Theorem 9.1. (1a) *Let $m \le n$, then*

$$e_k(\theta_1^{B_n}, \ldots, \theta_m^{B_n}) = \tilde{\sum_{(*)}} \prod_{a=1}^{k} \{i_a, j_a\} + 2 \sum_{l=1}^{k} \tilde{\sum_{(*)}} \prod_{a=1}^{k-l} \{i_a, j_a\} K_l(\{1, \ldots, m\} \backslash \{i_1, \ldots, i_{k-l}\}),$$

where the symbol $\tilde{\sum}$ means that in the corresponding sums we have to take only distinct monomials among the products $\prod_{a=1}^{k}\{i_a, j_a\}$ and $\prod_{a=1}^{k-l}\{i_a, j_a\}$; the condition $(*)$ means that $1 \le i_a \le m < j_a \le n$ and all indices i_a are distinct.

(1b) *The elements $K_l(S)$ can be expressed in the algebra $BE(B_r)$ as a linear combination of monomials in $[i]$'s and $\overline{[i, j]}$'s with non-negative integer coefficients.*

(1c) *If the number of elements in the set S is equal to l, then $K_l(S) = 0$ after the specialization $[a] = 0$ for all $a \in S$.*

(1d) $K_2(S) = (K_1(S))^2$*:*

$$K_3(S) = K_3(S\backslash\{r\}) + \sum_{a \in S} \overline{[a, r]} K_2(S\backslash\{a\}) + K_2(S\backslash\{r\}) \left(\sum_{a\in S} \overline{[a, r]} + 2[r] \right) + \sum_{a\in S} \left(\overline{[a, r]}[a] + [r]\overline{[a, r]} + \sum_{b\in S} \overline{[b, r][a, b]} \right) K_1(S\backslash\{a\}) + K_1(S\backslash\{r\}) \sum_{a\in S} \left(\overline{[a, r]}[a] + [r]\overline{[a, r]} + \sum_{b\in S} \overline{[b, r][a, b]} \right).$$

For example, the multiplicity of the monomial $\overline{[12][34][56]}$ in $e_3(\theta_1^{B_6}, \theta_2^{B_6}, \ldots, \theta_6^{B_6})$ is equal to 4.

(2) *Let $m \le n$, then*

$$h_2(\theta_1^{B_n}, \ldots, \theta_m^{B_n}) = \tilde{\sum_{(**)}} \prod_{a=1}^{2} \{i_a, j_a\} + 2 \sum_{1\le i\le m<j\le n} \{i, j\} K_1(\{1, \ldots, m\}\backslash\{i\}) + K_2(\{1, \ldots, m\}) + 2 \tilde{\sum_{1\le i_a\le m<j\le n}} (\overline{[i_1, j]}[i_2, j] + [i_1, j]\overline{[i_2, j]}) + 2 \sum_{1\le i\le m<j\le n} (\{i, j\}[i] + [i]\{i, j\}),$$

where the condition $(**)$ means that $1 \le i_a \le m < j_a \le n$ and all j_a's are distinct.

(3) $h_k(\theta_1^{B_n}, \ldots, \theta_m^{B_n}) = 0$*, if $k + m > 2n$.*

Finally, let us remark that for classical Coxeter groups $W = W(A_n)$, $W(B_n)$, and $W(D_n)$, the condition $|R(u)| = 1$, $u \in W$, is equivalent to the condition that modulo the ideal I_W, the Schubert class X_u is equal to either $e_k(\mathbf{X}_m)$ or $h_k(\mathbf{X}_m)$ for some k and $m \le n$, up to multiplication by some power of 2. In the case of symmetric groups, the permutations w such that $|R(w)| = 1$ are exactly the permutations of the following forms:

$$w = h(a, b) =: (1, 2, \ldots, a + b, a, b + 1, b + 2, \ldots),$$

or

$$w = e(a, b) =: \\ (1, 2, \ldots, a - b, a - b + 2, \ldots, a + 1, a - b + 1, a + 2, a + 3, \ldots),$$

see e.g. [24].

9.3. *Quantized B_2-algebra and quantum cohomology*

Here we give an explicit calculation of quantum cohomology ring and certain polynomial representatives for Schubert classes for B_2-flag variety. The bracket algebra $qBE(B_2)$ is generated by the symbols [12], $\overline{[12]}$, [1] and [2] subject to the following relations:

(i) $[12]^2 = q_1, \overline{[12]}^2 = 0, [1]^2 = 0, [2]^2 = q_2$,
(ii) $[12]\overline{[12]} = \overline{[12]}[12], [1][2] = [2][1]$,
(iii) $[12][1] - [2][12] + [1]\overline{[12]} + \overline{[12]}[2] = 0, [1][12] - [12][2] + \overline{[12]}[1] + [2]\overline{[12]} = 0$,
(iv) $[12][1]\overline{[12]}[1] + \overline{[12]}[1][12][1] + [1][12][1]\overline{[12]} + [1]\overline{[12]}[1][12] = 0$.

The Chevalley and Dunkl elements are $\tilde{\eta}_{s_{\alpha_1}} = \tilde{\theta}_1$ and $\tilde{\eta}_{s_{\alpha_2}} = \tilde{\theta}_1 + \tilde{\theta}_2$, where

$$\tilde{\theta}_1 = [12] + \overline{[12]} + 2[1], \qquad \tilde{\theta}_2 = -[12] + \overline{[12]} + 2[2].$$

Quantum cohomology ring of the B_2-flag variety is isomorphic to the algebra $\mathbf{C}[q_1, q_2][e_1, e_2]/I_{B_2}$, where

$$I_{B_2} = (e_1^2 + e_2^2 - 2q_1 - 4q_2, e_1^2 e_2^2 + 2q_1 e_1 e_2 - 4q_2 e_1^2 + q_1^2).$$

The subalgebra generated by $\tilde{\theta}_1$, $\tilde{\theta}_2$ in $qBE(B_2) \otimes \mathbf{C}$ is isomorphic to the quantum cohomology ring. Let us consider the quantum Bruhat representation of $qBE(B_2)$ and regard the Dunkl elements $\tilde{\theta}_i$ as operators acting on the group ring $\mathbf{R}[q_1, \ldots, q_n]\langle W(B_2)\rangle$. Denote by s_{12} and s_2 the simple reflections with respect to the simple roots $e_1 - e_2$, and e_2 respectively. Then,

$$\frac{\tilde{\theta}_1^2 - q_1}{2}(\mathrm{id.}) = s_2 s_{12}$$

$$\frac{\tilde{\theta}_1\tilde{\theta}_2 + q_1}{2}(\mathrm{id.}) = s_{12} s_2$$

$$\frac{\tilde{\theta}_1^3 - 2q_1\tilde{\theta}_1 - q_1\tilde{\theta}_2}{2}(\mathrm{id.}) = s_{12} s_2 s_{12}$$

$$\frac{\tilde{\theta}_1^2\tilde{\theta}_2 - \tilde{\theta}_1^3 + 3q_1\tilde{\theta}_1 + q_1\tilde{\theta}_2}{4}(\mathrm{id.}) = s_2 s_{12} s_2$$

$$\frac{\tilde{\theta}_1^3\tilde{\theta}_2 + q_1\tilde{\theta}_1^2 - q_1\tilde{\theta}_1\tilde{\theta}_2 - q_1^2 - 4q_1q_2}{4}(\mathrm{id.}) = s_{12} s_2 s_{12} s_2.$$

Remark 9.1. Both algebras $BE(B_2)$ and $BE^+(B_2)$ are infinite dimensional, but if we add the new relation

$$[1][12][1][12] = [12][1][12][1]$$

in the algebra $BE(B_2)$, and that

$$[1][12][1][12] + [12][1][12][1] = 0$$

in the algebra $BE^+(B_2)$, the resulting algebras appear to be finite dimensional and have the same Hilbert polynomial

$$(1+t)^4(1+t^2)^2.$$

One can check that the pointed Hopf algebra over the Coxeter group D_4 constructed in [20], is isomorphic to the quotient of the algebra $BE^+(B_2)$ by the relation of degree 4 defined above.

9.4. Quantized D_n-bracket algebra

In D_n case, $n \geq 2$, fix a set of simple roots as

$$S(D_n) = \{\alpha_1 = e_1 - e_2, \ldots, \alpha_{n-1} = e_{n-1} - e_n, \alpha_n = e_{n-1} + e_n\}.$$

The quantized D_n-bracket algebra $qBE(D_n) = qBE(W(D_n), S(D_n))$ is generated by the symbols $[i, j] = [e_i - e_j]$ and $\overline{[i, j]} = [e_i + e_j]$ over $\mathbf{R}[q_1, \ldots, q_n]$ subject to the following relations:

(0) $[i, j] = -[j, i]$, $\overline{[i, j]} = \overline{[j, i]}$,
(1) $[i, i+1]^2 = q_i$, $\overline{[n-1, n]}^2 = q_n$, $[i, j]^2 = 0$, if $|i - j| \neq 1$, $\overline{[i, j]}^2 = 0$, if $(i, j) \neq (n-1, n), (n, n-1)$,
(2) $[i, j][k, l] = [k, l][i, j]$, $\overline{[i, j]}[k, l] = [k, l]\overline{[i, j]}$, $\overline{[i, j]}\overline{[k, l]} = \overline{[k, l]}\overline{[i, j]}$, if $\{i, j\} \cap \{k, l\} = \phi$,
(3) $[i, j]\overline{[i, j]} = \overline{[i, j]}[i, j]$,
(4) $[i, j][j, k] + [j, k][k, i] + [k, i][i, j] = 0$, $\overline{[i, k]}[i, j] + [j, i]\overline{[j, k]} + \overline{[k, j]}\overline{[i, k]} = 0$, if all i, j and k are distinct.

Remark 9.2. Our construction of the quantized bracket algebra $qBE(D_n)$ is compatible with the isomorphisms between the Coxeter systems $D_2 \cong A_1 \times A_1$ and $D_3 \cong A_3$. It is easy to see that $qBE(D_2) \cong qBE(A_1) \times qBE(A_1)$ and $qBE(D_3) \cong qBE(A_3)$.

We set $t_i = E_{i,i} - E_{i+n,i+n}$, $E_{\alpha_i^\vee} = -E_{i+1,i} + E_{i+n,i+n+1}$, $E_{-\alpha_i^\vee} = E_{i,i+1} - E_{i+n+1,i+n}$ $(1 \leq i \leq n-1)$, $E_{\alpha_n^\vee} = -E_{2n-1,n} + E_{2n,n-1}$ and $E_{-\alpha_n^\vee} = E_{n,2n-1} - E_{n-1,2n}$. Let

$$X(e, q) = \sum_i e_i t_i + \sum_j q_j E_{-\alpha_j^\vee} + \sum_j E_{\alpha_j^\vee}.$$

We define the polynomials $J_\nu^D(e, q)$ using the equation

$$\det(tI + X^D(e, q)) = t^{2n} + \sum_{\nu=1}^{n} J_\nu^D(e, q) t^{2(n-\nu)}.$$

Then the quantum cohomology ring of D_n-flag variety is isomorphic to the ring

$$\mathbf{C}[e_1,\dots,e_n,q_1,\dots,q_n]/(J_1^D,\dots,J_{n-1}^D,\overline{J_n^D}),$$

where $\overline{J_n^D}$ is a polynomial such that $(\overline{J_n^D})^2 = J_n^D$.

The Chevalley and Dunkl elements are given by $\tilde{\eta}_{s_{\alpha_i}} = \tilde{\theta}_1 + \cdots + \tilde{\theta}_i$, where

$$\tilde{\theta}_i = \sum_{j\neq i}([i,j]+\overline{[i,j]}), \qquad 1 \le i \le n.$$

Proposition 9.3. *In the quantized bracket algebra $qBE(D_n)$ we have the following identities:*

$$J_\nu^D(\tilde{\theta}_1,\dots,\tilde{\theta}_n;q) = 0, \qquad 1 \le \nu \le n.$$

Proof of Proposition 9.2 follows from the following lemma.

Lemma 9.2. *Relations* (1), (2) *in Lemma* 9.1 *and all the relations obtained from them by the action of the Weyl group, hold also in the algebra $BE(D_n)$.*

Remark 9.3. The non-quantized bracket algebra $BE(D_n)$ is a quotient ring of $BE(B_n)$ obtained by putting $[i] = 0$, and the Dunkl elements of $BE(D_n)$ are images of those of $BE(B_n)$. Hence the Dunkl elements of $BE(D_n)$ satisfy the equations coming from the B_n case. However, $qBE(D_n)$ is not a quotient of $qBE(B_n)$.

Example 9.3. Quantum D_n-invariants for $n = 4$,

$$\begin{aligned}
J_1^D &= -e_1^2 - e_2^2 - e_3^2 - e_4^2 + 2q_1 + 2q_2 + 2q_3 + 2q_4,\\
J_2^D &= q_3^2 + 2q_1e_1e_2 + q_4^2 - 2q_4e_3e_4 + 2q_2q_4 + 2q_3e_3e_4 + q_2^2 - 2q_3q_4 + 2q_1q_2\\
&\quad + 4q_1q_3 + 2q_2q_3 + q_1^2 + 4q_1q_4 + 2q_2e_2e_3 - 2q_1e_3^2 - 2q_2e_4^2 - 2q_4e_1^2 + e_1^2e_3^2\\
&\quad - 2q_2e_1^2 + e_3^2e_4^2 + e_1^2e_2^2 - 2q_1e_4^2 - 2q_3e_1^2 + e_1^2e_4^2 + e_2^2e_4^2 + e_2^2e_3^2 - 2q_4e_2^2\\
&\quad - 2q_3e_2^2,\\
J_3^D &= -2q_2q_3e_1^2 - e_1^2e_3^2e_4^2 - 2q_2q_4e_1^2 + 2q_4e_1^2e_2^2 - 2q_1q_2e_4^2 - e_2^2e_3^2e_4^2 + 2q_3e_1^2e_2^2\\
&\quad + 2q_1e_3^2e_4^2 - e_1^2e_2^2e_4^2 + 4q_1q_3e_3e_4 - 4q_1q_3q_4 + 2q_1q_3^2 + 2q_2e_1^2e_4^2 - e_1^2e_2^2e_3^2\\
&\quad + 2q_3q_4e_1^2 + 2q_1q_2q_3 + 2q_1q_2q_4 + 2q_3q_4e_2^2 - q_1^2e_4^2 - q_1^2e_3^2 + 2q_4q_1^2 - q_4^2e_1^2\\
&\quad + 2q_4^2q_1 + 2q_3q_1^2 - q_3^2e_1^2 - q_3^2e_2^2 - q_2^2e_1^2 - q_2^2e_4^2 - 2q_1q_2e_1e_3 - 2q_2q_3e_2e_4\\
&\quad - 2q_1e_1e_2e_3^2 + 4q_1q_4e_1e_2 + 2q_4e_1^2e_3e_4 - 2q_2e_1^2e_2e_3 + 4q_1q_3e_1e_2\\
&\quad - 4q_1q_4e_3e_4 - q_4^2e_2^2 - 2q_3e_2^2e_3e_4 - 2q_1e_1e_2e_4^2 + 2q_4e_2^2e_3e_4 - 2q_3e_1^2e_3e_4\\
&\quad + 2q_2q_4e_2e_4 - 2q_2e_2e_3e_4^2,\\
\overline{J_4^D} &= e_1e_2e_3e_4 + q_1e_3e_4 + q_2e_1e_4 + q_3e_1e_2 - q_4e_1e_2 + q_1q_3 - q_1q_4.
\end{aligned}$$

Remark 9.4. We do not know whether or not the algebra $BE(D_4)$ is finite dimensional. However, the commutative quotient of the algebra $BE(D_4)$ is finite dimensional and has the following Hilbert polynomial:

$$\begin{aligned}
1 + 12t + 50t^2 + 84t^3 + 48t^4 &= (1+2t)(1+4t)(1+6t+6t^2)\\
&= (1+t)(1+3t)^2(1+5t) + 3t^4.
\end{aligned}$$

Let us remark that the polynomial $(1+t)(1+3t)^2(1+5t)$ coincides with the Hilbert polynomial of the cohomology ring of the pure braid group of type D_4.

It was a big surprise for us to find that the Hilbert polynomial of the commutative quotient of the algebra $BE(D_5)$ is equal to

$$1+20t+150t^2+520t^3+824t^4+480t^5 = (1+2t)(1+4t)(1+6t) \times (1+8t+10t^2).$$

However, the obvious generalization of the above formulas for the Hilbert polynomial of the commutative quotient of the algebra $BE(D_n)$ is false.

Similar to the case of B_n-bracket algebra, the subalgebra $\mathcal{A}_n^D = \mathcal{A}(\Delta(D_n)\backslash\Delta(D_{n-1}))$ generated by $x_i = [i,n]$ and $y_i = \overline{[i,n]}$, $i = 1,\ldots,n-1$, in the algebra $BE(D_n)$ is invariant under the twisted derivation $D_{[\gamma]}$ for any root $\gamma \in \Delta(D_{n-1})$. By applying Lemma 2.1 successively, we obtain the following decomposition of the algebra $BE(D_n)$ for $n \geq 2$:

$$BE(D_n) \cong \mathcal{A}_2^D \otimes \cdots \otimes \mathcal{A}_n^D.$$

9.5. *Quantized G_2-bracket algebra*

Fix a set of positive roots of type G_2 as

$$\{a, b = 3a + f, c = 2a + f, d = 3a + 2f, e = a + f, f\}.$$

Then quantized G_2-bracket algebra is generated by the symbols a, b, c, d, e, f with the following relations:

(1) $a^2 = q_1,\ f^2 = q_2,\ b^2 = c^2 = d^2 = e^2 = 0,$
(2) $ea = ce + ac,\ ae = ec + ca,\ fb = df + bd,\ bf = fd + db,\ eb = be,\ cf = fc,$ $ad = da,\ af = ba + cb + dc + ed + fe,\ fa = ab + bc + cd + de + ef,$
(3) $bcdefd + dbcdef + fedcbd + dfedcb = 0,\ fabcdb + bfabcd + dcbafb + bdcbaf = 0,\ defabf + fdefab + bafedf + fbafed = 0.$

The Chevalley elements are defined by

$$\tilde{\eta}_{s_a} = a + 3b + 2c + 3d + e,$$
$$\tilde{\eta}_{s_f} = b + c + 2d + e + f.$$

Let $\tilde{\theta}_1 = \tilde{\eta}_{s_a} - \tilde{\eta}_{s_f}$ and $\tilde{\theta}_2 = \tilde{\eta}_{s_f}$ be the corresponding Dunkl elements, then we have the relations $g_2(\tilde{\theta}_1, \tilde{\theta}_2) = g_6(\tilde{\theta}_1, \tilde{\theta}_2) = 0$ in the algebra $qBE(G_2)$, where

$$g_2(\xi_1, \xi_2) := \xi_1^2 + \xi_2^2 - \xi_1\xi_2 - q_1 - 3q_2,$$
$$g_6(\xi_1, \xi_2) := \xi_1^3\xi_2^3 - 3q_2\xi_1^2\xi_2^2 + q_1\xi_1\xi_2^3 + q_1\xi_2^4 + q_1q_2\xi_1^2 + 3q_2(q_1+q_2)\xi_1\xi_2 + 2q_1q_2\xi_2^2 + q_1^2q_2 - 6q_1q_2^2 - q_2^3.$$

The small quantum cohomology ring of G_2-flag variety is isomorphic to the ring

$$\mathbf{R}[q_1, q_2][\xi_1, \xi_2]/(g_2, g_6).$$

One can check the latter representation for the small quantum cohomology ring of G_2-flag variety is equivalent to that given by Kim [14].

9.6. Dunkl elements and fundamental invariant polynomials for $I_2(m)$

Let $a_i = \mu_i e_1 + \lambda_i e_2$, where $\mu_i = \cos(i\pi/m)$ and $\lambda_i = \sin(i\pi/m)$ for $i = 0, 1, \ldots, m-1$. Then $\Delta_+ = \{a_1, \ldots, a_{m-1}\}$ forms the set of positive roots of type $I_2(m)$. The set of simple roots is $S = \{a_0, a_{m-1}\}$ and

$$a_i = \lambda_1^{-1}\lambda_{i+1}a_0 + \lambda_1^{-1}\lambda_i a_{m-1}.$$

The Chevalley elements are given by

$$\eta_{s_{a_1}} = \sum_{i=0}^{m-1} \lambda_1^{-1}\lambda_{i+1}[a_i],$$

$$\eta_{s_{a_{m-1}}} = \sum_{i=0}^{m-1} \lambda_1^{-1}\lambda_i[a_i].$$

Let $\theta_1 = \eta_{s_{a_0}} + \lambda_1\eta_{s_{a_{m-1}}}$ and $\theta_2 = (\lambda_1^{-1}\mu_1 + 1)\eta_{s_{a_0}} + (\lambda_1^{-1} + \mu_1)\eta_{s_{a_{m-1}}}$ be the Dunkl elements of type $I_2(m)$. The fundamental invariant polynomials are

$$f_2(\xi_1, \xi_2) = \xi_1^2 + \xi_2^2,$$

and

$$f_m(\xi_1, \xi_2) = \sum_{i=0}^{[m/2]} (-1)^i \binom{m}{2i} \xi_1^{2i}\xi_2^{m-2i},$$

where ξ_1 and ξ_2 are variables corresponding to the orthonormal basis e_1 and e_2.

Proposition 9.4. *In the algebra $BE(I_2(m))$ one has*

$$f_2(\theta_1, \theta_2) = 0, \quad f_m(\theta_1, \theta_2) = 0.$$

We can check that the algebra generated by the Dunkl elements θ_1 and θ_2 in the algebra $BE(I_2(m))$ is isomorphic to the quotient of the polynomial ring $\mathbf{R}[\xi_1, \xi_2]/(f_2, f_m)$.

Remark 9.5. In this subsection, all roots are normalized to satisfy the condition $(a_i, a_i) = 1$. The root systems of type $I_2(4)$ and $I_2(6)$ can be identified with the crystallographic systems of type B_2 and G_2, but the choice of the normalization is different. Hence, the Dunkl elements for $I_2(4)$ and $I_2(6)$ in this subsection have a different expression from the ones defined in Sections 9.3 and 9.5.

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