Elastic stability of Euler columns with a continuous elastic restraint using variational iteration method

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\textbf{ABSTRACT}

In this paper, analysis of elastic stability for continuously restrained Euler columns is conducted by means of variational iteration method (VIM). A uniform homogeneous column is assumed to be restrained along its length. The restraint considered in this study is an elastic foundation model in engineering practice and it is of great interest to foundation engineers. Hence, the variation of the critical buckling loads with the stiffness of elastic restraint is investigated using VIM for the restrained columns with different end conditions. Obtaining analytical solutions for these types of problems is not a simple procedure since the equations of stability criteria are highly nonlinear. This study presents the application of VIM for obtaining exact solutions for continuously restrained Euler columns. The study proves that VIM is a very efficient and promising approach in the elastic stability analysis of specified problems.

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1. Introduction

The problem of buckling of non-uniform columns has been the topic of extensive studies because of the fact that it bears strong relevance to structural, mechanical and aeronautical engineering fields. Determination of their practical load carrying capacity requires a stability analysis. Columns are basic structural forms and there are extensive studies related to the elastic stability of columns, and their static and dynamic behaviors. Many types of structures and structural members can be simplified as a uniform and/or non-uniform column with different end conditions for buckling analysis studies. However, it is generally difficult to find the exact analytical solutions for the buckling problems of various column types with arbitrary distributions of flexural stiffness and various end conditions. Exploration and studying of buckling of columns has become the focal point of many researchers and studying this subject become more and more systematic during last decades. As a foundation of this field of research, Euler's pioneering study of buckling of columns under their own weight \cite{1} can be considered. Afterwards, Greenhill \cite{2} made important contributions to this field. This problem often is called as Greenhill's problem in the related literature. The closed form solutions are extremely rare. However, solutions for simple cases are given by Dinnik \cite{3}, Karman and Biot \cite{4} and Timoshenko and Gere \cite{5} and others. Wang et al. \cite{6} give exact solutions for buckling of structural members including various cases of columns, beams, arches, rings, plates and shells. On the other hand, the columns with variable cross section, some exact solutions are given in terms of logarithmic and trigonometric functions by Bleich \cite{7}, in terms of Bessel functions by Dinnik \cite{8} and in terms of Lomel functions by Elishakoff and Pelligrini \cite{9–11}. Exact solution in terms of series for buckling load for variable cross section columns with variable axial forces was found by Eisenberger \cite{12}.
Exact buckling solutions for several special types of tapered columns with simple boundary conditions were obtained by Gere and Carter [13] in terms of Bessel functions. Solutions for the problem of the buckling of elastic columns with step varying thicknesses are given by Arbabei and Li [14]. Siginer [15] studied the stability of a column whose flexural rigidity has a continuous linear variation along the column. Furthermore, the exact analytical solutions of a one-step bar and multi-step bar with varying cross section under the action of concentrated and variably distributed axial loads were obtained by Li et al. [16–18].

Sampaio et al. [19] established the solution for the problem of buckling behavior of inclined beam–column using energy method. They formulated the exact solution using generalized hyper-geometric functions. Also some of the researchers who studied the mechanical behavior of beams/columns are Keller [20], Tadjbakhsh and Keller [21] and Taylor [22].

A solution technique called variational iteration method (VIM) which was originally proposed by He [23–25] has been given great importance for solving linear and nonlinear differential equations in recent years. The method can solve various classes of linear and nonlinear equations [26–33]. VIM is a kind of variational based analytical technique efficient for finding solutions of nonlinear differential equations including boundary value and initial value problems, nonlinear system of differential equations, nonlinear partial differential equations. These successful applications of the method to the various linear and nonlinear types of problems in Physics, Mathematics and Engineering fields encourage the use of VIM in the present problem.

2. Problem definition

A uniform homogeneous column with flexural rigidity $EI$, length $L$ is considered. The column is assumed as continuously restrained along its length as shown in Fig. 1. The restraint is uniformly distributed lateral springs of stiffness $k$ per unit length. Such a model is called as elastic foundation in foundation engineering practice and it is of great importance for the researchers in this field.

Governing equation for the buckling of column in Fig. 1 is given by

$$\frac{d^2}{dx^2}\left( EI \frac{d^2 w}{dx^2} \right) + P \frac{d^2 w}{dx^2} + kw = 0. \quad (1)$$

Non-dimensional form of governing equation becomes

$$\frac{d^4 \tilde{w}}{d\tilde{x}^4} + \alpha \frac{d^2 \tilde{w}}{d\tilde{x}^2} + \beta \tilde{w} = 0 \quad (2)$$

where $\tilde{x} = x/L$, $\tilde{w} = w/L$, normalized critical load $\alpha = P L^2 / EI$ and normalized restraint stiffness parameter $\beta = k L^4 / EI$.

The general solution to this equation and stability criteria for the columns with different end conditions are given in Wang et al. [6]. These end conditions can be seen in Fig. 2.

The stability criteria for the columns considered in this study are as follows [6]:

- P–P Column: $\sin T = 0 \quad (3)$
- C–P Column: $T \cos T \sin S - S \cos T = 0 \quad (4)$
- C–C Column: $2ST[\cos T \cos S - 1] + (T^2 + s^2) \sin T \sin S = 0 \quad (5)$
- C–S Column: $T \sin T \cos S - S \cos T \sin S = 0 \quad (6)$
- C–F Column: $[\alpha (S^2 + T^2) - 2S^2T^2] \cos T \cos S - \alpha (S^2 + T^2) + (S^4 + T^4) + ST[2\alpha - (S^2 + T^2)] \sin T \sin S = 0 \quad (7)$
where

\[ S = \sqrt{\frac{\alpha}{2} - \sqrt{\left(\frac{\alpha}{2}\right)^2 - \beta}} \]  \hspace{1cm} (8)

\[ T = \sqrt{\frac{\alpha}{2} + \sqrt{\left(\frac{\alpha}{2}\right)^2 - \beta}}. \]  \hspace{1cm} (9)

Non-dimensional end conditions for the columns shown in Fig. 2 are also given in [6] as below:

- Pin support: \( \ddot{w} = 0 \) and \( \frac{d^2 \ddot{w}}{dx^2} = 0 \).

- Clamped support: \( \ddot{w} = 0 \) and \( \frac{d \ddot{w}}{dx} = 0 \).

- Free end: \( \frac{d^2 \ddot{w}}{dx^2} = 0 \) and \( \frac{d^3 \ddot{w}}{dx^3} + \alpha \frac{d \ddot{w}}{dx} = 0 \).

- Sliding restraint: \( \frac{d \ddot{w}}{dx} = 0 \) and \( \frac{d^3 \ddot{w}}{dx^3} + \alpha \frac{d \ddot{w}}{dx} = 0 \).

The stability criteria between Eqs. (3)–(7) are highly nonlinear and it is not easy to find an analytical solution for any assumed stiffness parameter. In the numerical solution of these equations, initial guess is very important because, the equations have more than one roots. Even though a mathematical software is used, finding the smallest root of the equations which is the critical buckling load is not an easy process. In the following sections, it can be found how to use VIM to obtain analytical solutions for the specified problem and how advantageous it is in obtaining analytical solutions for these types of problems which are very important for researchers working related to this field.
3. VIM formulation of the problem

According to VIM, a nonlinear differential equation may be considered as:
\[ Lw + Nw = g(x) \]  
\[ (14) \]
where \( L \) is a linear operator, and \( N \) is a nonlinear operator, and \( g(x) \) is an inhomogeneous term.

Based on VIM, a correct functional can be constructed as follows:
\[ w_{n+1} = w_n + \int_0^x \lambda(\xi) \left( Lw_n(\xi) + N\tilde{w}_n(\xi) - g(\xi) \right) d\xi \]  
\[ (15) \]
where \( \lambda \) is a general Lagrangian multiplier, which can be identified optimally via the variational theory, the subscript \( n \) denotes the \( n \)th-order approximation, \( \tilde{w} \) is considered as a restricted variation i.e. \( \delta \tilde{w} = 0 \). More detailed explanations for obtaining \( \lambda \) is given in [23–33]. By solving the differential equation for \( \lambda \) obtained for Eq. (15) in view of \( \delta \tilde{w} = 0 \) with respect to its boundary conditions, Lagrangian multiplier \( \lambda(\xi) \), is obtained as follows:
\[ \lambda(\xi) = \frac{1}{6}(\xi^3 - 3x\xi^2 + 3x^2\xi - x^3). \]  
\[ (16) \]
If the above VIM formulation is applied to Eq. (1), following iteration formula can be obtained accordingly:
\[ w_{n+1}(\xi) = w_n(\xi) + \int_0^x \lambda(\xi) \left\{ \begin{array}{c} w(\xi)^{\prime\prime} + 2\left[ \frac{EI(\xi)}{EI(\xi)} \right] \tilde{w}(\xi)^{\prime\prime} + \left\{ \frac{EI(\xi)}{EI(\xi)} \right\}^\prime \tilde{w}(\xi)^{\prime\prime} + \frac{k}{EI(\xi)} \tilde{w}(\xi)^{\prime}\end{array} \right\} d\xi. \]  
\[ (17) \]
The iteration formula given in Eq. (17), is a simple approximation which can be applied to columns considered in this study and it is expected to be an important contribution of VIM to the current problem.

4. VIM analysis and results

In the VIM analyses, a cubic polynomial may be chosen as an initial approximation which is given below.
\[ \tilde{w}_0 = Ax^3 + Bx^2 + Cx + D. \]  
\[ (18) \]
This approximation includes four unknown coefficients which are supposed to be found by imposing boundary conditions of the problem considered.

Computations are conducted up to ninth iteration and four end conditions for each column are written by using the last iteration. Each boundary condition produces an equation containing four unknowns coming from the initial approximation. Hence four equations may be written with respect to the boundary conditions of the problem. These equations can be interpreted as a matrix equation which defines an eigenvalue problem as follows:
\[ [M(\alpha)] [A] = [0] \]  
\[ (19) \]
where \([A] = (A B C D)^T\). For a nontrivial solution, determinant of coefficient matrix must be zero. Determinant of the coefficient matrix yields a characteristic equation in terms of \( \alpha \) and the smallest positive real root of this equation is the normalized critical buckling load for the case considered.

Table 1 demonstrates how VIM efficiently produces the critical buckling loads which are in very good agreement with analytical results obtained from the stability criteria between Eqs. (3)–(7).

In order to illustrate the efficiency of VIM, a considerable number of analyses have been performed for \( 0 < \beta < 100 \) for each case. Afterwards, these results are depicted on figures with analytical solutions from the stability criteria of different
columns. The results are shown between Figs. 3–7. From these figures it can be seen that, a very good agreement exists between VIM results and exact results. These figures successfully summarize that VIM is a powerful tool for the elastic
stability analysis of continuously restrained columns. VIM successfully handles the difficulty of finding the smallest root of stability criteria and give almost exact results in all the cases considered.

A comparison between the variations of critical buckling loads for all the cases considered in this study also can be observed in Fig. 8. From the figure, it can be easily seen that how end conditions effect the critical loads required for the buckling of restrained columns related to these conditions.

5. Conclusion

In this study, the use of VIM in the elastic stability analysis of continuously restrained Euler columns is realized. Analyses have been performed for a homogeneous column with various end conditions. Elastic restraint is taken into account as uniformly distributed lateral springs along the column length. Although stability criteria exist for various boundary conditions, it is not easy to find the critical buckling load from these equations. Since the stability criteria are nonlinear, it is not a straightforward process to find the smallest root of the equations. However, these difficulties related with the problem can be handled easily by the use of VIM. Analysis via VIM leads to a characteristic equation which is the stability criteria for the problem considered. Since VIM is a kind of analytical based method, the results of the analysis using VIM are in very good agreement with analytical results. Comparisons with those analytical solutions pointed out that VIM is very efficient and powerful in the analysis of buckling problems of continuously restrained Euler columns.

References

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