Symmetric submanifolds in symmetric spaces

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Abstract

In this paper we construct new examples of symmetric non-totally geodesic submanifolds in irreducible symmetric spaces of non-compact type and of rank $\geq 2$. These symmetric spaces are characterized by the fact that they contain a reflective submanifold with one-dimensional Euclidean factor; they are listed at the end of the paper. © 2001 Published by Elsevier Science B.V.

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1. Introduction

It is well-known that symmetric spaces play a special role in Riemannian geometry. In submanifold geometry their analogues are the so-called symmetric submanifolds. A submanifold $N$ of a Riemannian manifold $M$ is called a symmetric submanifold if for each point $p$ in $N$ there exists an involutive isometry $t_p$ of $M$ which fixes $p$, leaves $N$ invariant and whose differential at $p$ fixes the normal vectors of $N$ at $p$ and reflects the tangent vectors. Any such isometry $t_p$ is called a symmetry of $N$ at $p$.

In a series of papers Ferus [4–7] studied and classified the symmetric submanifolds in Euclidean spaces. Surprisingly, in Euclidean spaces the symmetric submanifolds are essentially the symmetric spaces among the orbits of isotropy representations of semisimple symmetric spaces. These orbits are known as symmetric $R$-spaces or symmetric real flag manifolds. Further efforts by various mathematicians lead first to classifications in compact symmetric spaces of rank one, culminating eventually in the classification in simply connected symmetric spaces of compact type by Naitoh [15–17].

In symmetric spaces of non-compact type the situation is not yet clarified unless the rank of the symmetric space is one, in which case complete classifications are known. The only known examples
of symmetric submanifolds in irreducible symmetric spaces of non-compact type and higher rank, that is, rank greater than one, are totally geodesic. In fact, the totally geodesic symmetric submanifolds are precisely the reflective submanifolds. A submanifold \( N \) of a Riemannian manifold \( M \) is called reflective if the geodesic reflection of \( M \) in \( N \) is a well-defined global isometry. Since a reflective submanifold is a connected component of the fixed point set of an isometry it is necessarily totally geodesic. In a symmetric space \( M \) a totally geodesic submanifold is reflective precisely if for each of its normal spaces there exists a totally geodesic submanifold of \( M \) tangent to the normal space. Any such normal totally geodesic submanifold is also reflective. The reflective submanifolds in irreducible simply connected symmetric spaces have been classified by Leung [11–14]. The purpose of this paper is to present examples of non-totally geodesic symmetric submanifolds in irreducible symmetric spaces of non-compact type and of higher rank. Our main result is the following construction:

Let \( M \) be an irreducible symmetric space of non-compact type and of higher rank. Let \( T \) be a reflective submanifold of \( M \) for which the Euclidean factor \( \gamma \) in its de Rham decomposition is one-dimensional. Let \( o \in T \) and \( Q \) the reflective submanifold of \( M \) which is tangent to the normal space of \( T \) at \( o \). Let \( G_Q \) be the Lie subgroup of the isometry group of \( M \) which is generated by the symmetries of \( Q \). Then the orbit of \( G_Q \) through any point on \( \gamma \) different from \( o \) is a non-totally geodesic symmetric submanifold of \( M \).

We remark that not every irreducible symmetric space of non-compact type and of higher rank admits such a reflective submanifold \( T \). At the end of this paper we provide the list of such triples \((M, T, Q)\). This list follows from the classification of reflective submanifolds in Riemannian symmetric spaces obtained by Leung [12,13].

The paper is organised as follows. In Section 2 we summarize some basic results about symmetric and reflective submanifolds. In Section 3 we give the proof of our main result and in Section 4 we present the list of all possible triples \((M, T, Q)\).

It is with pleasure that I express my appreciation to Dr. Jürgen Berndt for his continuous support and help.

2. Preliminaries

In this section we give some basic introduction to the theory of symmetric submanifolds, mention the known facts and state the new result. For an introduction to symmetric submanifolds and further references see [8]. Throughout this paper we are assuming that all manifolds and submanifolds are connected.

**Definition 2.1.** A Riemannian manifold is called a symmetric space if at each point \( p \in M \) there exists an involutive isometry \( s_p \) of \( M \) and \( p \) is an isolated fixed point of this isometry. In this case \( s_p \) is called the symmetry of \( M \) at \( p \).

Any simply connected symmetric space decomposes into the Riemannian product of a Euclidean space and some simply connected, irreducible symmetric spaces. The irreducible symmetric spaces were studied and classified by Élie Cartan who has put them into four types: two compact and two non-compact ones. There exists duality between the compact and non-compact types. The fundamental book on this topic is Helgason [9], a good introduction and classification may be found in Besse [1].
In this paper we are dealing with the submanifold theory. In Riemannian manifolds there are several interesting classes of submanifolds which we will describe now.

**Definition 2.2.** Let $T$ be a submanifold of a Riemannian manifold $M$. $T$ is called a totally geodesic submanifold of $M$ if every geodesic in $T$ is also a geodesic in $M$.

B.Y. Chen and T. Nagano worked on the classification of totally geodesic submanifolds in Riemannian symmetric spaces of compact type. The list of maximal totally geodesic submanifolds in some of those spaces can be found in [2]. Due to the concept of duality the classification in the case of non-compact Riemannian symmetric spaces is easily obtained from the one in the compact case.

There exists a type of submanifolds which is an analogue of the symmetric spaces. These are so-called symmetric submanifolds.

**Definition 2.3.** A submanifold $S$ of Riemannian manifold $M$ is called symmetric if for each point $q$ in $S$ there exists an isometry $t_q$ of $M$ such that

$$
t_q(q) = q, \quad t_q(S) = S, \quad (t_q)_* X = -X, \quad (t_q)_* Y = Y$$

for all $X \in T_q S$, $Y \in \nu_q S$. Here we denote by $T_q S$ the tangent space of $S$ at $q$, by $\nu_q S$ the normal space of $S$ at $q$ and by $(t_q)_*$ the differential of $t_q$.

The study of symmetric submanifolds in Euclidean spaces began in the seventies. The origin of these studies goes back to the paper of Chern, do Carmo and Kobayashi [3], where one finds explicitly the condition that the second fundamental form $\alpha$ of any symmetric submanifold is parallel, i.e., $\nabla^\perp \alpha = 0$.

Some of the known facts about symmetric submanifolds are summarized in the following

**Proposition 2.1.** Let $S$ be a symmetric submanifold of some Riemannian manifold $M$. Then the second fundamental form of $S$ is parallel, each tangent space of $S$ is curvature-invariant, and for each point $p \in S$ there exists a totally geodesic submanifold $S_p$ of $M$ with $p \in S_p$ and $T_p S_p = \nu_p S$, i.e., tangent to each normal space there exists a totally geodesic submanifold of the ambient space.

The last class of submanifolds that is going to be mentioned is a subclass of totally geodesic submanifolds. These are reflective submanifolds and they are defined as follows.

**Definition 2.4.** Let $M$ be a Riemannian manifold and $Q$ a submanifold of $M$. When the geodesic reflection of $M$ in $Q$ is a globally well-defined isometry of $M$, then $Q$ is called a reflective submanifold.

Since any reflective submanifold is a connected component of the fixed point set of an isometry, it is totally geodesic. For symmetric spaces there is the following useful criterion.

**Proposition 2.2.** A totally geodesic submanifold of a simply connected Riemannian symmetric space is symmetric if and only if it is reflective.

As a reflective submanifold is symmetric, at each point there exists a totally geodesic submanifold normal to it. In symmetric spaces this normal submanifold is also reflective. The general result is as follows.
Proposition 2.3. Let $M$ be a Riemannian symmetric space. If $Q$ is a reflective submanifold of $M$ then $T_pQ$ and $v_pQ$ are Lie triple systems in $T_pM$ for each $p \in Q$. Moreover, the complete totally geodesic submanifold $Q^\perp$ of $M$ with $p \in Q^\perp$ and $T_pQ^\perp = v_pQ$ is also reflective. Conversely, if $M$ is simply connected, $p \in M$, and if $V$ is a Lie triple system in $T_pM$ such that the orthogonal complement $V^\perp$ of $V$ in $T_pM$ is also a Lie triple system in $T_pM$, then there exists a reflective submanifold $Q$ of $M$ with $p \in Q$ and $T_pQ = V$.

Reflective submanifolds were studied by Leung who established the complete classification of reflective submanifolds in Riemannian symmetric spaces [11–14].

The question now is: are there any non-totally geodesic symmetric submanifolds in symmetric spaces? It is known that each symmetric submanifold of a symmetric space belongs to the Grassmann geometry associated to a certain reflective submanifold. We now recall the definition of a Grassmann geometry:

Definition 2.5. An $m$-dimensional submanifold $N$ of a Riemannian manifold $M$ is said to belong to the $m$-dimensional Grassmann geometry of $M$ if all its tangent spaces lie in the same orbit of the action of the isometry group $I(M)$ on the Grassmann bundle $Gr_m(TM)$ of all $m$-dimensional linear subspaces of $T_pM$, $p \in M$.

If $N$ belongs to some Grassmann geometry, then the Grassmann geometry $Gr(N, M)$ associated to $N$ is the set of all $m$-dimensional submanifolds whose tangent spaces lie in the same orbit as those of $N$.

In the compact case Naitoh [15–17] has obtained the following result:

Proposition 2.4. Every non-totally geodesic symmetric submanifold of an irreducible simply connected compact symmetric space belongs to one of the following five Grassmann geometries:

1. $Gr(S^m, S^n)$, $1 \leq m \leq n - 1$;
2. $Gr(\mathbb{C}P^m, \mathbb{C}P^n)$, $1 \leq m \leq n - 1$;
3. $Gr(\mathbb{R}P^n, \mathbb{C}P^n)$, $n \geq 2$;
4. $Gr(\mathbb{C}P^n, \mathbb{H}P^n)$, $n \geq 2$;
5. $Gr(N, M)$, here $N$ is a symmetric real flag manifold of non-Hermitian type and $M$ is an irreducible simply connected compact symmetric space of rank greater than 1 such that $N$ is a reflective orbit of the connected component of the isotropy group at some point in $M$.

The symmetric submanifolds in these spaces were classified by different authors. In the non-compact case the classification is not known, though the symmetric spaces of rank one were studied by various mathematicians and symmetric submanifolds in these spaces were classified. Non-totally geodesic symmetric submanifolds in non-compact rank one symmetric spaces exist only in $\mathbb{R}H^n$ and $\mathbb{C}H^n$. They were classified by Naitoh [18] and Takeuchi [19]. Due to their classifications we see that the dual Grassmann geometries in cases (1) and (3) do contain non-totally geodesic symmetric submanifolds, whereas in cases (2) and (4), due to the results of Kon [10] and Tsukada [20], we do not get any non-totally geodesic examples.

Case (5) of a non-compact symmetric space of rank greater than one has not been studied and now we shall describe a method of constructing symmetric submanifolds in symmetric spaces which works also in the compact case where it gives the same examples as those obtained by Naitoh. This method
is applicable to those symmetric spaces which contain a reflective submanifold with one-dimensional Euclidean factor in its de Rham decomposition.

**Theorem 1.** Let $M$ be an irreducible symmetric space of non-compact type with rank greater than one. Let $T$ be a reflective submanifold of $M$ such that the Euclidean factor in the de Rham decomposition of $T$ is one-dimensional. Let $o \in T$, denote by $\gamma$ the exponential image of the one-dimensional component of $T$ at $o$ and assume that this geodesic $\gamma$ is parametrised by arc length so that $o = \gamma(0)$. Let $Q$ be the reflective submanifold of $M$ perpendicular to $T$ at the point $o$, and $G_Q$ the Lie subgroup of the isometry group of $M$, which is generated by the symmetries $t_q$ of $Q$.

Then $G_Q \cdot o = Q$, and the orbit of $G_Q$ through any point on $\gamma$ different from $o$ is a non-totally geodesic symmetric submanifold of $M$.

## 3. Proof of the theorem

### 3.1. The group $G_Q$

We shall now define the group which generates the reflective submanifold $Q$. By Proposition 2.2 every reflective submanifold is symmetric, therefore at each point $p \in Q$ there exists $t_p$, an isometry of $M$ satisfying (1). Denote by $G_Q$ the subgroup of the isometry group $G$ of $M$ which is generated by these symmetries.

Fix a point $o$ in $Q$. We have to prove now that the orbit $G_Q \cdot o$ of $G_Q$ through $o$ is equal to $Q$, which we shall achieve by showing two inclusions:

$Q \subset G_Q \cdot o$:

Let $q$ be a point in $Q$. We must find an isometry in $G_Q$ which maps $o$ into $q$. Since $Q$ is reflective and hence symmetric and therefore complete, there exists a geodesic in $Q$ connecting $q$ and $o$. Let $q'$ be the midpoint on this geodesic. Then $t_{q'}$ is an element in $G_Q$ and $t_{q'}(o) = q$.

$G_Q \cdot o \subset Q$:

Since $Q$ is invariant under the action of any element from $G_Q$ we know that $G_Q(Q) = Q$ and hence

$G_Q \cdot o \subset Q$. \hfill \Box$

### 3.2. The submanifold $G_Q \cdot \gamma(r)$

Now we describe the orbit of the action of the group $G_Q$ at points of the geodesic $\gamma$.

Denote this submanifold by $N_r$, i.e., $N_r = G_Q \cdot \gamma(r)$. Take a point $q$ in $N_r$. There exists an isometry $g$ in the group $G_Q$ such that $q = g \cdot \gamma(r)$. We define an isometry $\tau_q$ of $M$ in the following way.

$\tau_q := t_{g \cdot \gamma(0)} = t_{q'}$,

where $t_{q'}$ is the symmetry of $Q$ at $q' = g \cdot \gamma(0)$. This symmetry exists since $Q$ is reflective and hence symmetric. We will now prove that $\tau_q$ is a symmetry of $N_r$ at $q$, i.e., it satisfies conditions (1).
3.3. The first condition $\tau_q(q) = q$ for all $q \in N_r$

**Lemma 3.1.** Images of the geodesic $\gamma$ under the isometries of the group $G_Q$ are geodesics perpendicular to $Q$, i.e., for each $g \in G_Q$ the curve $\alpha(u) := g\gamma(u)$ is a geodesic perpendicular to $Q$ at $g(o)$.

**Proof.** The isometry $g$ maps geodesics into geodesics. Thus $\alpha(u)$ is a geodesic. Isometries preserve angles, $\gamma$ is perpendicular to $Q$ at $o$ and $g(Q) = Q$, therefore the geodesic $\alpha(u)$ is perpendicular to $Q$ at $g(o)$. $\square$

**Lemma 3.2.** The isometry $\tau_q$ leaves the geodesic $\alpha(u)$ defined in Lemma 3.1 invariant,

$$\tau_q(\alpha(u)) = \alpha(u) \quad \text{for all } u.$$

**Proof.** $\tau_q = t_q'$ maps the geodesic $\alpha(u)$ into the geodesic $t_q'(\alpha(u))$. Since $q'$ is a fixed point for $t_q'$, $t_q'(\alpha(u))$ is a geodesic through $q'$. From Proposition 3.1 $\alpha(u)$ is a geodesic perpendicular to $Q$ at the point $g(o)$, i.e., $\dot{\alpha}(o) \in \nu_{q'} Q$. Since $t_q'$ is the symmetry of $Q$ at $q'$, by (1) we get that $t_{q'}(\dot{\alpha}(o)) = \dot{\alpha}(o)$. Thus the velocity vectors of geodesics $\alpha(u)$ and $t_{q'}(\alpha(u))$ at the point $q'$ coincide and therefore these geodesics coincide. $\square$

Now for the first condition in (1) we have

$$\tau_q(q) = t_q'(q) = t_q'(g\gamma(r)) = t_q'(\alpha(r)) = \alpha(r) = g\gamma(r) = q.$$

The first condition is proved.

3.4. The second condition $\tau_q(N_r) = N_r$

This can be proved very easily using the fact that $gG_Q = G_Q$ for any element $g$ in $G_Q$:

$$\tau_q(N_r) = \tau_q(G_Q \cdot \gamma(r)) = t_q'(G_Q \cdot \gamma(r)) = G_Q \cdot \gamma(r) = N_r.$$ 

3.5. The third and the fourth conditions

Further we shall prove that

$$(\tau_q)_* X = -X, \quad (\tau_q)_* Y = Y$$

for $X \in T_q N_r$, $Y \in \nu_{q'} Q$ along $\gamma$, respectively.

We first prove it for $q = \gamma(r)$.

**Proposition 3.1.** The tangent space and the normal space of $N_r$ at $q$ is the parallel translate of $T_q Q$ and $\nu_{q'} Q$ along $\gamma$, respectively.

By definition $Q = G_Q \cdot o = G_Q \cdot \gamma(0)$, where $\gamma(u)$ is a geodesic, which is the one-dimensional factor of $T$ through $o$, where $T$ is the reflective submanifold perpendicular to $Q$.

**Lemma 3.3.** For any point $p$ in $Q$ the corresponding perpendicular submanifold $T_p$ is congruent to $T_p$, and therefore the Euclidean factor in the de Rham decomposition of $T_p$ is also one-dimensional.
Proof. Let \( p \in Q \). Then there exists an isometry \( g \) in \( G_Q \) such that \( p = g(o) \). Since any element in \( G_Q \) leaves \( Q \) invariant and since isometries preserve angles and product structure, \( g(T) \) is a submanifold perpendicular to \( Q \) at \( g(o) \) and \( g(T) = T_p \) is congruent to \( T \). \( \square \)

This one-dimensional factor at each \( p \in Q \) determines (up to sign) a unit normal vector field \( \xi \) on \( Q \), \( \xi_p \in \nu_p Q \). We choose the sign for which \( \gamma(u) = \exp(u \cdot \xi_p) \), where \( \exp \) is the exponential map of \( M \).

For any \( u \in \mathbb{R} \) we define a diffeomorphism \( F_u \) into \( M \) in the following way:

\[
F_u : Q \to M, \quad p \mapsto \exp(u \cdot \xi_p).
\]

Lemma 3.4. The submanifold \( N_r \) coincides with the image of \( Q \) under \( F_r \), i.e., \( N_r = F_r(Q) \).

Proof. \( N_r \) was defined as the orbit of the group \( G_Q \) through the point \( \gamma(r) \), \( N_r = G_Q \cdot \gamma(r) \).

We shall prove that for any \( g \in G_Q \)

\[
g(\gamma(r)) = F_r(g(o)). \tag{2}
\]

By definition of \( F_r \), \( F_r(g(o)) = \exp(r \cdot \xi_{g(o)}) \).

Thus on the right hand side of (2) we have a geodesic through the point \( g(o) \) with the initial velocity vector \( \xi_{g(o)} \). This vector \( \xi_{g(o)} \) is also the tangent vector of the geodesic from the left hand side \( g(\gamma(r)) \) at the point \( g(\gamma(0)) = g(o) \) since the vector field \( \xi \) was defined in this way. Therefore these geodesics coincide. Since both of them are parametrised by arc length,

\[
g(\gamma(r)) = \exp(r \cdot \xi_{g(o)}) = F_r(g(o)).
\]

By this we proved the proposition:

\[
N_r = G_Q \cdot \gamma(r) = F_r(G_Q \cdot o) = F_r(Q). \tag{2}
\]

Now we are interested in the tangent and normal spaces of \( N_r = F_r(Q) \) at \( \gamma(r) \), i.e., \( T_{\gamma(r)}F_r(Q), \nu_{\gamma(r)}F_r(Q) \). In order to prove that these are the parallel translates of \( T_o Q, \nu_o Q \) respectively, we turn to Jacobi vector fields.

Let \( c(s) \) be a smooth curve in \( Q \) with \( c(0) = o \). Consider \( V(s, u) = \gamma_s(u) = \exp(u \xi_{c(s)}) \), a smooth geodesic variation of \( \gamma = \gamma_0 \) with \( c(s) = \gamma_s(0) \in Q \), and \( \xi(s) = \xi_{\gamma(s)} = \dot{\gamma}(0) \), where \( \dot{\gamma}(0) \) spans the one-dimensional factor of \( T_{\gamma(s)} F_r(Q) \), the reflective submanifold perpendicular to \( Q \) at \( c(s) \). The Jacobi vector field \( Y \) along \( \gamma \) induced by this geodesic variation is determined by the initial values

\[
Y(0) = \frac{d}{ds} \bigg|_{s=0} (s \mapsto V(s, 0) = c(s)) = \dot{c}(0) \in T_o Q,
\]

\[
Y'(0) = \frac{d}{ds} \bigg|_{s=0} \left( s \mapsto \frac{d}{du} \bigg|_{u=0} (u \mapsto V(s, u)) \right) = \frac{d}{ds} \bigg|_{s=0} \left( s \mapsto \dot{\gamma}(s) = \xi(s) \right) = \xi'(0).
\]

Denote by \( \tilde{J}(Q, \gamma) \) the real vector space of the Jacobi vector fields along \( \gamma \) corresponding to the geodesic variations \( V(s, u) = \exp(u \xi(s)) \), where \( \xi \) is the normal vector field along a smooth curve \( c(s) \) in \( Q \) defined by \( \xi(s) = \xi_{c(s)} \).

The curve \( c, s \mapsto \exp(r \cdot \xi(s)) \) is a smooth curve in \( N_r = F_r(Q) \) and hence \( Y(r) = \dot{c}(0) \) is a vector in the tangent space of \( N_r \) at \( \gamma(r) \), \( Y(r) \in T_{\gamma(r)}N_r \). As any tangent vector of \( F_r(Q) \) at \( \gamma(r) \) arises in this...
manner we can see that

\[ T_{\gamma(r)}N_{r} = \{ Y(r), Y \in \hat{J}(Q, \gamma) \}. \tag{3} \]

**Proposition 3.2.** Any Jacobi vector field \( Y \) in \( \hat{J}(Q, \gamma) \) has zero derivative at 0,

\[ Y'(0) = \xi'(0) = 0. \]

**Proof.** We need to prove \( \xi'(0) = 0 \), that is, for any curve \( c(s) \) in \( Q \), the derivative of the normal vector field \( \xi_{c(s)} \) along this curve at the point \( o \) is zero, \( \xi''_{c(s)}(0) = 0 \). Note that \( \xi''_{c}(0) \) does not depend on the whole curve \( c(s) \) but only on its initial velocity vector \( c'(0) \). In any direction in \( T_{o}Q \) there exists a geodesic in \( Q \). Therefore it is enough to consider only geodesics among all possible \( c(s) \) in \( Q \). Every geodesic is uniquely determined by its initial velocity vector \( \dot{c}(0) = X \) at \( o = \gamma(0) \). Since \( Q \) is a reflective and hence a totally geodesic submanifold of \( M \), we may assume that \( X \in V_{Q} \), where \( V_{Q} \) is a Lie triple system in \( m \). As usual, we identify \( T_{o}M \) with \( m \), where \( m \) is from the Cartan decomposition at \( o \) of \( g \), \( g = k + m \), and \( g \) is the Lie algebra of \( G = I_{0}(M) \), the identity component of the isometry group of \( M \).

Denote by \( \tilde{X} \) the Killing vector field on \( M \) which is uniquely determined by \( X \):

\[ \tilde{X}_{p} = \frac{d}{ds} \bigg|_{s=0} (s \mapsto \text{Exp}(sX) \cdot p), \]

where \( \text{Exp} \) is the exponential map from \( g \) to \( G \). This vector field defines isometries of \( M \)

\[ \Phi_{\tilde{X}} : M \to M, \quad p \mapsto \alpha_{p}(s), \]

which map any point \( p \) into \( \alpha_{p}(s) \). Here \( \alpha_{p}(s) \) is the integral curve of the Killing vector field \( \tilde{X} \) through \( p \), i.e., \( \alpha_{p}(0) = p, \dot{\alpha}_{p} = \tilde{X} \circ \alpha_{p} \).

**Lemma 3.5.** The isometries \( \Phi_{\tilde{X}} \) leave the submanifold \( Q \) invariant, \( \Phi_{\tilde{X}}(Q) = Q \).

**Proof.** We shall prove two inclusions:

\[ \Phi_{\tilde{X}}(Q) \subset Q. \]

We construct a subgroup of \( G \) from the Lie triple system \( V_{Q} \) as follows. The subspace

\[ g' = [V_{Q}, V_{Q}] \oplus V_{Q} \subset \mathfrak{t} + m = g \]

is a Lie subalgebra of \( g \). Denote by \( G' \) the connected Lie subgroup of \( G \) with the Lie algebra \( g' \). \( G' \) acts transitively on \( Q \) and \( Q \) is exactly the orbit of \( G' \) through the point \( o \), \( Q = G' \cdot o \) (see [9, pp. 224–226]).

Since \( X \) belongs to the Lie triple system \( V_{Q} \), it is obvious that \( X \) belongs to the Lie algebra \( g' \), \( X \in [V_{Q}, V_{Q}] \oplus V_{Q} \in g' \) and hence \( \text{Exp}(sX) \in G' \).

Therefore for any point \( p \) in \( Q \)

\[ \Phi_{s}(p) = \alpha_{p}(s) = \text{Exp}(sX) \cdot p \in G'(Q) = Q. \]

By this

\[ \Phi_{s}(Q) \subset Q. \tag{4} \]

\( \Phi_{\tilde{X}}(Q) \supset Q \). \( \Phi_{\tilde{X}} \) is a one-parameter group of isometries of \( M \) and hence \( Q = \Phi_{\tilde{X}} \Phi_{\tilde{X}}^{-1}(Q) \subset \Phi_{\tilde{X}}(Q) \).

Here we have used (4) for \( \Phi_{-s} \). \( \square \)
We have proved that the isometries $\Phi_{\tilde{X}}$ leave the submanifold $Q$ invariant, therefore the differential $(\Phi_{\tilde{X}})_\ast$ maps the normal space $v_o Q$ into the normal space $v_{\alpha_o(s)} Q$ and $\xi_o$ into $\xi_{\alpha_o(s)}$. Since $\alpha_o(s) = c(s)$ is a geodesic in $M$ (and in $Q$) and the parallel translation along $\alpha_o$ is given by $(\Phi_{\tilde{X}})_\ast$ we conclude that $\xi_{c(s)}$ is parallel along $c(s)$, i.e., $\xi_{c(s)}'(0) = 0$. □

Now we return to the Jacobi vector fields. We have shown that

$$T_{\gamma(r)} N_r = \{ Y(r), Y \in \tilde{J}(Q, \gamma) \}.$$  

By definition of a Jacobi vector field each $Y \in \tilde{J}(Q, \gamma)$ satisfies the following differential equation:

$$Y'' + R(Y, \dot{\gamma}) \dot{\gamma} = 0$$  

with the initial conditions:

$$Y(0) = X, \quad X \in T_{\gamma(0)} Q,$$

$$Y'(0) = 0 \quad \text{(by Proposition 3.2)},$$

where $R$ denotes the Riemannian curvature tensor of $M$. Consider the action of $R$ on $T_o Q$ and $T_o T$. The Gauss equation implies that

$$R(T_o T, T_o T) T_o T \subset T_o T,$$

since $T$ is totally geodesic. Therefore $R(T_o T, \xi) \xi \subset T_o T$. The endomorphism $R(\cdot, \xi) \xi : T_o Q \to T_o Q$ is self-adjoint, therefore we have $R(T_o Q, \xi) \xi \subset T_o Q$ and we can choose an orthonormal basis $X_1, X_2, \ldots, X_k$ of $T_o Q$ consisting of eigenvectors of $R(\cdot, \xi) \xi$:

$$R(X_i, \xi) \xi = \lambda_i X_i, \quad i = 1, \ldots, k.$$  

All the eigenvalues $\lambda_i$ are non-positive because the sectional curvature of $M$ is non-positive. The Jacobi equation (5) with the initial data

$$Y_i(0) = X_i, \quad Y_i'(0) = 0$$

gives the solution

$$Y_i(u) = \cosh (\sqrt{-\lambda_i} u) E_i(u)$$

where $E_i(u)$ is the parallel vector field along $\gamma(u)$ with $E_i(0) = X_i$.

By this and (3) we see that $T_{\gamma(r)} N_r$ is spanned by $E_1(r), E_2(r), \ldots, E_k(r)$ and hence any vector $W \in T_{\gamma(r)} N_r$ can be expressed as a linear combination

$$W = \alpha_1 E_1(r) + \alpha_2 E_2(r) + \cdots + \alpha_k E_k(r).$$

Lemma 3.6. $t_o^\ast (E_i(r)) = -E_i(r)$.

Proof. $E_i(u)$ is a parallel vector field along $\gamma(u)$. Then $t_o^\ast E_i(u)$ is also a parallel vector field along $\gamma(u)$ since $t_o^\ast$ is an isometry which fixes $\gamma$ pointwise. For $u = 0$ we have $t_o^\ast E_i = -E_i$ since $E_i(0) \in T_o Q$ and $t_o^\ast$ is a symmetry of $Q$. Therefore it holds for all $u$:

$$t_o^\ast E_i(u) = -E_i(u).$$  

$\square$
Now
\[ t_\alpha W = \alpha_1 t_\alpha E_1(r) + \cdots + \alpha_k t_\alpha E_k(r) = -\alpha_1 E_1(r) - \cdots - \alpha_k E_k(r) = -W. \]

We extend \( E_1, E_2, \ldots, E_k \) to a parallel orthonormal frame field \( E_1, E_2, \ldots, E_n \) of \( TM \) along \( \gamma \). Then \( E_{k+1}(r), E_{k+2}(r), \ldots, E_n(r) \) is the basis of the normal space \( \nu_{\gamma(r)} N_r \). The same argument as for \( E_1, E_2, \ldots, E_k \) will show us that
\[ t_\alpha E_j(u) = E_j(u), \quad j = k + 1, \ldots, n, \]
and hence for any vector \( V \in \nu_{\gamma(r)} N_r \)
\[ t_\alpha V = V. \]

We have now proved Proposition 3.1 and the property (1) for \( T_q N_r, \nu_q N_r \), where \( q = \gamma(r) \).

For any other point \( q \in N_r \) we can complete the same construction with the geodesic variations around the geodesic \( \alpha(u) = q \gamma(u) \) where \( g \in G \) such that \( q = g \cdot o \).

By this we have proved that \( N_r \) is a symmetric submanifold of \( M \).  \( \square \)

3.6. Proof of the fact that \( N_r \) is not totally geodesic

It is a consequence of the Gauss formula that a submanifold is totally geodesic if and only if its second fundamental form \( \alpha \) vanishes.

We shall use this to prove that \( N_r \) is not totally geodesic. Since the shape operator \( A^r \) of \( N_r \) is related to the second fundamental form \( \alpha^r \) of \( N_r \) by the equation
\[ \{ \alpha^r(X, Y), \xi \} = \{ A^r_x X, Y \}, \]
we see that the condition \( \alpha^r(X, Y) \equiv 0 \) is equivalent to the condition that \( A^r_\eta = 0 \) for any \( \eta \in \nu N_r \), that is, the shape operator has only zero eigenvalues for any \( \eta \).

Hence to show that \( N_r \) is not totally geodesic it is enough to prove the following

Proposition 3.3. \( A^r_{\dot{\gamma}(r)} \), the shape operator of \( N_r \) in direction \( \dot{\gamma}(r) \), has at least one non-zero eigenvalue.

Proof. Let us again consider the Jacobi vector fields \( Y_1, Y_2, \ldots, Y_k \) along \( \gamma(u) \) corresponding to the eigenvectors \( X_i \) of the curvature tensor \( R(X, \dot{\gamma}) \dot{\gamma} \). We shall prove further that at each point of the geodesic \( \gamma \) they are the eigenvectors of the shape operator \( A^r_{\dot{y}(r)} \) of the corresponding submanifold \( N_r \).

Then we calculate their eigenvalues and show that there exists a non-zero one.

We first calculate the derivative of the Jacobi vector field \( Y(u) \) at the point \( \gamma(r) \):
\[ Y'(r) = \frac{d}{ds} \bigg|_{s=0} \left( s \mapsto \frac{d}{du} \bigg|_{u=r} \left( u \mapsto V(s, u) \right) \right) \]
\[ = \frac{d}{ds} \bigg|_{s=0} \left( s \mapsto \dot{\gamma}_{\cdot \cdot}(r) \right) \]
\[ = \eta_r(0) \]
\[ = -A_{\dot{\gamma}(r)}^r Y(r) + \left( \eta_r(0) \right) \perp. \]

Here \( \eta_r(s) = \dot{\gamma}_{\cdot \cdot}(r) \). Hence \( A^r_{\dot{\gamma}(r)} Y(r) = -(Y'(r)) \perp \).
We know that for the Jacobi vector fields $Y_i(u)$ with the initial data $Y_i(0) = X_i$, $Y_i'(0) = 0$ the solution of the Jacobi equation is

$$Y_i(u) = \cosh\left(\sqrt{-\kappa_i} u\right) E_i(u),$$

where $E_i$ is the parallel vector field along $\gamma$ with the initial value $E_i(0) = X_i$.

Therefore the derivative at $u = r$ is

$$Y_i'(r) = \sqrt{-\kappa_i} \sinh\left(\sqrt{-\kappa_i} r\right) E_i(r).$$

Since the derivative belongs to the tangent space of $N_r$ which is spanned by the Jacobi vector fields $Y_i(r)$ (this was proved earlier, see (3)), we may say that

$$Y_i'(r) = \left(Y_i'(r)\right)^\top$$

and hence $A_{\gamma'(r)} Y_i(r) = -Y_i'(r)$.

Now we calculate the eigenvalues of the shape operator. For them we get the following equation:

$$-\sqrt{-\kappa_i} \sinh\left(\sqrt{-\kappa_i} r\right) E_i(r) = \lambda_i \cosh\left(\sqrt{-\kappa_i} r\right) E_i(r),$$

which gives the solution

$$\lambda_i = -\sqrt{-\kappa_i} \tanh\left(\sqrt{-\kappa_i} r\right).$$

Since $\tanh(\sqrt{-\kappa_i} r)$ is zero only when $r$ is zero, the only thing left to prove is that there exists $\kappa_i \neq 0$ for some $i \in \{1, \ldots, k\}$. We will show this by contradiction.

Assume that $\kappa_i = 0$ for all $i \in \{1, \ldots, k\}$, i.e., the eigenvalues of the endomorphism $R(\cdot, \dot{\gamma}(0))\dot{\gamma}(0)$ restricted to $T_o Q$ all vanish. For an irreducible symmetric space the Ricci tensor $ric^M$ can be expressed as a product of a constant $\lambda$ with the Riemannian metric $g$ of $M$, $ric^M = \lambda \cdot g$, see [1]. In the case that $M$ is of non-compact type this constant $\lambda$ is negative. We shall calculate now $ric(\dot{\gamma}(0), \dot{\gamma}(0))$. At the point $o$ we choose the same basis of $M$ as we did in Section 3.5. $E_1, E_2, \ldots, E_k$ is the basis of $T_o Q$ consisting of the eigenvectors of $R(\cdot, \dot{\gamma}(0))\dot{\gamma}(0)$. $E_{k+1}, \ldots, E_n$ is the basis of $v_o Q = T_o T$, take $E_{k+1} = \dot{\gamma}(0)$. Now we can write:

$$\lambda = ric\left(\dot{\gamma}(0), \dot{\gamma}(0)\right) = \sum_{i=1}^{n} \langle R(E_i, \dot{\gamma}(0))\dot{\gamma}(0), E_i \rangle.$$

Since $T$ splits with one-dimensional Euclidean factor and $R(X, Y) = 0$ whenever $X$ and $Y$ are from different factors, we see that $R(E_i, \dot{\gamma}(0)) = 0$ for $i = k + 2, \ldots, n$. For $i = k + 1$ we have $E_{k+1} = \dot{\gamma}(0)$ and $R(\dot{\gamma}(0), \dot{\gamma}(0))\dot{\gamma}(0) = 0$. Therefore

$$\lambda = ric\left(\dot{\gamma}(0), \dot{\gamma}(0)\right) = \sum_{i=1}^{k} \langle R(E_i, \dot{\gamma}(0))\dot{\gamma}(0), E_i \rangle = \sum_{i=1}^{k} \langle \kappa_i E_i, E_i \rangle = \sum_{i=1}^{k} \kappa_i = 0$$

since all $\kappa_i = 0$ by assumption. By that $\lambda = 0$ and $ric^M = 0$ which is a contradiction since $M$ is not Euclidean. Therefore there exists $\kappa_j \neq 0$ and hence $\lambda_j \neq 0$. So we have found a non-zero eigenvalue for $A_{\gamma(0)}$. This proves that $N_r$ is not totally geodesic. □

4. Triples $(M, Q, T)$

In Table 1 we present the list of all possible triples $(M, Q, T)$. 
Table 1

<table>
<thead>
<tr>
<th>Symmetric space</th>
<th>Submanifold $Q$</th>
<th>Submanifold $T$</th>
<th>Remarks</th>
</tr>
</thead>
<tbody>
<tr>
<td>$SL(n, \mathbb{R})$</td>
<td>$SO^{0}(n, r - \mathbb{R})$</td>
<td>$SL(r, \mathbb{R}) \times SO(n - r)$</td>
<td>$SL(n, \mathbb{R}) \times SO(n - r)$</td>
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<td>$SO(r)$</td>
<td>$SO(n - r)$</td>
</tr>
<tr>
<td>$SU^{*}(2n)$</td>
<td>$Sp(r, n - r)$</td>
<td>$SU^{*}(2n)$</td>
<td>$Sp(r, n - r)$</td>
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<td>$SU(n)$</td>
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References