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GENUS EMBEDDINGS FOR SOME COMPLETE TRIPARTITE GRAPHS *

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The voltage graph construction of Gross is extended to the case where the base graph is non-orientably embedded. An easily applied criterion is established for determining the orientability character of the derived embedding. These methods are then applied to derive both orientable and non-orientable genus embeddings for some families of complete tripartite graphs.

1. The general theory

The graph-theoretical terminology of this paper agrees with that of [12] and [3]. In particular, a *pseudograph* admits loops and multiple edges and the vertex and edge sets of the pseudograph G are denoted by $V(G)$ and $E(G)$ respectively. With each edge uv of G we associate two directed edges $e = (u,v)$ and $e^{-1} = (v,u)$ of G. The set of directed edges of G is denoted by $D(G)$.

A voltage pseudograph is a triple (G,φ,Γ), where G is a pseudograph, Γ is a group, and the map $\varphi : D(G) \to \Gamma$ is subject to the unique restriction

$$[\varphi(e)]^{-1} = \varphi(e^{-1}) \quad \text{for all } e \in D(G)\,.$$

Given a voltage graph (G,φ,Γ), the *covering pseudograph* $G \times_\varphi \Gamma$ is defined as follows: its vertex set is $V(G) \times \Gamma$ and each edge $e = uv$ of G generates the edges $(u,g)\,(v,g\varphi(e))$ of $G \times_\varphi \Gamma$, where g ranges over all the elements of the group Γ. It is easy to see that if pseudographs are regarded as topological spaces, then $G \times_\varphi \Gamma$ is in fact a covering space of G. Moreover, the authors of [10] assert that every regular covering

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space of G can be obtained in this manner. For more details the reader is referred to [6], where this construction originated.

Given a voltage pseudograph (G, φ, Γ) and a walk c: $e_1, e_2, \ldots, e_n$ at a vertex v of G, we define

$$\varphi(c) = \prod_{i=1}^{n} \varphi(e_i) .$$

The *local group at v*, denoted by Γ_v, is defined as

$$\Gamma_v = \{\varphi(c)\colon c \text{ is a closed walk at } v\} \quad \text{for all } v \in V(G).$$

It is easily verified that Γ_v is in fact a subgroup of Γ. Moreover, if u and v are two vertices that belong to the same component of G, then Γ_u and Γ_v are conjugate subgroups of Γ; for if c is a u–v walk, then $\Gamma_v = [\varphi(c)]^{-1}\, \Gamma_u\, [\varphi(c)]$. Thus the index of Γ_v in Γ is independent of v if the pseudograph G is connected. The following theorem, which relates the index of Γ_v to the components of $G \times_\varphi \Gamma$, is the voltage version of a theorem originally proved for current graphs in [8]. As the proof of the original version is easily modified to apply to voltage pseudographs, no details are given here.

Theorem 1.1. *Given a connected voltage pseudograph (G, φ, Γ), the number of components of the covering graph $G \times_\varphi \Gamma$ equals the index of Γ_v in Γ for any vertex v of G.*

For the definition of an *embedding* of a pseudograph, as well as other related concepts, the reader is referred to [21] and [18]. The latter is particularly recommended for a discussion of embeddings that are not necessarily orientable. The *orientable* (*non-orientable*) genus of a pseudograph G is defined as the least integer n such that G can be 2-cell embedded on the orientable (*non-orientable*) closed surface of genus n. These parameters are denoted by $\gamma(G)$ and $\tilde{\gamma}(G)$ respectively. The orientable and non-orientable closed surfaces of genus n ($n \geqslant 0$) are denoted by S_n and $\tilde{S}_n$ respectively. We adopt here the convention that $S_0 = \tilde{S}_0 =$ the sphere.

It is shown in [21] that every orientable 2-cell embedding of the graph G can be described in terms of a rotation system P which assigns to every vertex v of G a cyclic permutation P_v of the vertices adjacent to v. Since our subject matter here is pseudographs, we modify this system slightly by defining P_v to be a cyclic permutation of all the directed edges of G whose terminal (head) vertex is v. Thus, if $\ldots -e_1 - e_2$ is a sequence of directed edges which describes the boundary of some region R, and if

$e_2 = uv$, then the next directed edge on the boundary of R is $e_3 = [P_v(e_2)]^{-1}$.

In a series of papers [4–10] Gross et al. have shown that many interesting embeddings can be constructed by "lifting" embeddings of pseudographs to their covering pseudographs. Suppose (G,φ,Γ) is a voltage pseudograph with values in Γ. If e is an arc of G at v, then for any $g \in \Gamma$ we denote the lift of e at (v,g) by $\tilde{e}^g$. For any rotation system P of G we define the *lift* P^φ of P to $G \times_\varphi \Gamma$ by specifying that if $P_v(e) = f$, then

$$P^\varphi_{(v,g)}(\tilde{e}^g) = \tilde{f}^g.$$

The relationship between the embeddings defined by P^φ and P is an example of a branched covering projection. For our purposes here it is sufficient to say that the map $p : \tilde{S} \to S$ is a *branched covering projection* if there exists a discrete set B of points of S such that the restriction

$$p : \tilde{S} - p^{-1}(B) \to S - B$$

is a covering projection. The points of B are the *branch points*. If b is a branch point, then for some sufficiently small open neighborhood U of b, the restricted map $p : \tilde{U} \to U - \{b\}$ is n-fold, where n is some cardinal and $\tilde{U}$ is a component of $p^{-1}(U - \{b\})$ in $\tilde{S}$. We refer to n as the *multiplicity of branching at b*. For example, the map $z \to z^3$ defines a branched covering projection of the extended complex plane onto itself with the branch points 0 and ∞; the multiplicity of branching is 3 at both branch points. For more details the reader is referred to [1] and [13]. The following notation will prove helpful in trying to describe the location of branch points. If R is a region of the embedding of G on S induced by the rotation system P, and φ is a voltage assignment from G to Γ, then $|R|_\varphi$ is the order of $\varphi(c)$ in Γ, where c is the closed walk in G consisting of the boundary of R. It is easily verified that $|R|_\varphi$ is independent of the specific orientation of R and of the initial vertex of c. The following theorem summarizes information in [6,7,9] and shows that the regions of $G \times_\varphi \Gamma$ are in fact easily computed.

Theorem 1.2. (Gross and Alpert). *Let (G,φ,Γ) be a voltage pseudograph with a rotation system P, and let P^φ be the lift of P to $G \times_\varphi \Gamma$. Let P and P^φ determine embeddings of G and $G \times_\varphi \Gamma$ on S and S^φ respectively. Then there exists a branched covering projection $p : S^\varphi \to S$ such that*

(a) $p^{-1}(G) = G \times_\varphi \Gamma$;

(b) *if b is a branch point of p of multiplicity n, then b is in the interior of a region R such that $|R|_\varphi = n$;*

(c) *if R is a region of G which is a k-gon, then $p^{-1}(R)$ has $|\Gamma|/|R|_\varphi$ components, each of which is a $k|R|_\varphi$-gon.*

Generalized embedding schemes which describe graph embeddings on surfaces which are not necessarily orientable have been anounced in the fairly recent past by several mathematicians [2,11,16,18]. While the proof techniques used to justify these algorithms vary considerably, the schemes themselves are very much alike. Using the terminology of [18], an embedding scheme is a pair (P,λ) where P is a conventional Heffter–Edmonds type rotation system, and $\lambda : D(G) \to \mathbf{Z}_2$ defines a voltage pseudograph. The regions of this embedding are computed much the same way as is done in the orientable case, with one exception – sometimes $P_v^{-1}(u)$ must be used instead of $P_v(u)$.

Specifically, if ... – u – v – w is the portion of the boundary of some region, and if $w = P_v^\delta(u)$ ($\delta \in \{1, -1\}$), then the vertex following w on this boundary is $P_w^\epsilon(v)$, where $\epsilon = \delta \quad 2\delta\lambda(vw)$. It is convenient to present such embedding schemes by means of a plane drawing of the graphs. The rotations are to be read off the diagram in the counterclockwise sense and a "~" marks those edges for which $\lambda = 1$. Thus Fig. 3 represents an embedding of K_4 in which $\lambda(e) = 1$ iff $e = uv$ or vw, and

$$P_u = (x, w, v), \qquad P_v = (u, x, w)\,,$$

$$P_w = (v, u, x)\,, \qquad P_x = (w, v, u)\,.$$

We now show that the construction of Theorem 1.2 can be extended to generalized embedding schemes as well. Again (G,φ,Γ) is a voltage pseudograph with the generalized embedding scheme (P,λ). Let P^φ be the lift of P to $G \times_\varphi \Gamma$. In addition, define $\lambda^\varphi : D(G \times_\varphi \Gamma) \to \mathbf{Z}_2$ by setting $\lambda^\varphi(\tilde{e}) = \lambda(e)$ for any lift $\tilde{e}$ of an arc e of G. We define $(P^\varphi, \lambda^\varphi)$ as the lift of (P,λ) to $G \times_\varphi \Gamma$. We draw the reader's attention to the fact that the "raison d'être" of λ is that if (P,λ) defines an embedding of G on S, then there exists a two sheeted covering projection p_1 : $(\tilde{S}, G \times_\lambda Z_2) \to (S,G)$, where $\tilde{S}$ is *orientable*.

Theorem 1.3. *Let (G,φ,Γ) be a voltage pseudograph with the gener-*

alized embedding scheme (P,λ), and let $(P^{\varphi},\lambda^{\varphi})$ denote the lift of (P,λ) to $G\times_{\varphi}\Gamma$. If (P,λ) and $(P^{\varphi},\lambda^{\varphi})$ determine embeddings of G and $G\times_{\varphi}\Gamma$ on S and S^{φ} respectively, then there exists a branched covering projection $p: S^{\varphi}\to S$ such that

(a) $p^{-1}(G)=G\times_{\varphi}\Gamma$;

(b) *if b is a branch point of multiplicity n, then b is in the interior of a region R such that $|R|_{\varphi}=n$;*

(c) *if R is a region of G which is a k-gon, then $p^{-1}(R)$ has $|\Gamma|/|R|_{\varphi}$ components each of which is a $k|R|_{\varphi}$-gon.*

Proof. Let $p_1:\tilde{S}\to S$ and $p_2:\tilde{S}^{\varphi}\to S^{\varphi}$ be twofold orientable covering projections. Then there exist lifted orientable embeddings of $G\times_{\lambda}\mathbf{Z}_2$ and $(G\times_{\varphi}\Gamma)\times_{\lambda^{\varphi}}\mathbf{Z}_2$ on $\tilde{S}$ and $\tilde{S}^{\varphi}$ respectively. A voltage assignment $\tilde{\varphi}: D(G\times_{\lambda}\mathbf{Z}_2)\to\Gamma$ is defined by setting $\tilde{\varphi}(\tilde{e})=\varphi(e)$ whenever $\tilde{e}$ is the lift of an arc e of G. Let $\bar{p}:\bar{S}\to\tilde{S}$ be the branched covering projection whose existence is guaranteed by Theorem 1.2 (here the voltage pseudograph is $(G\times_{\lambda}\mathbf{Z}_2,\tilde{\varphi},\Gamma)$). Thus we have an orientable embedding of $G_1=(G\times_{\lambda}\mathbf{Z}_2)\times_{\tilde{\varphi}}\Gamma$ on $\bar{S}$ and an orientable embedding of $G_2=(G\times_{\varphi}\Gamma)\times_{\lambda^{\varphi}}\mathbf{Z}_2$ on $\tilde{S}^{\varphi}$. However, these two pseudographs have an obvious isomorphism Φ which carries a vertex $((v,i),g)$ of G_1 into the vertex $((v,g),i)$ of G_2 (see Fig. 1). This isomorphism, moreover, conforms with their embeddings on $\bar{S}$ and $\tilde{S}^{\varphi}$. To see this we note that both em-

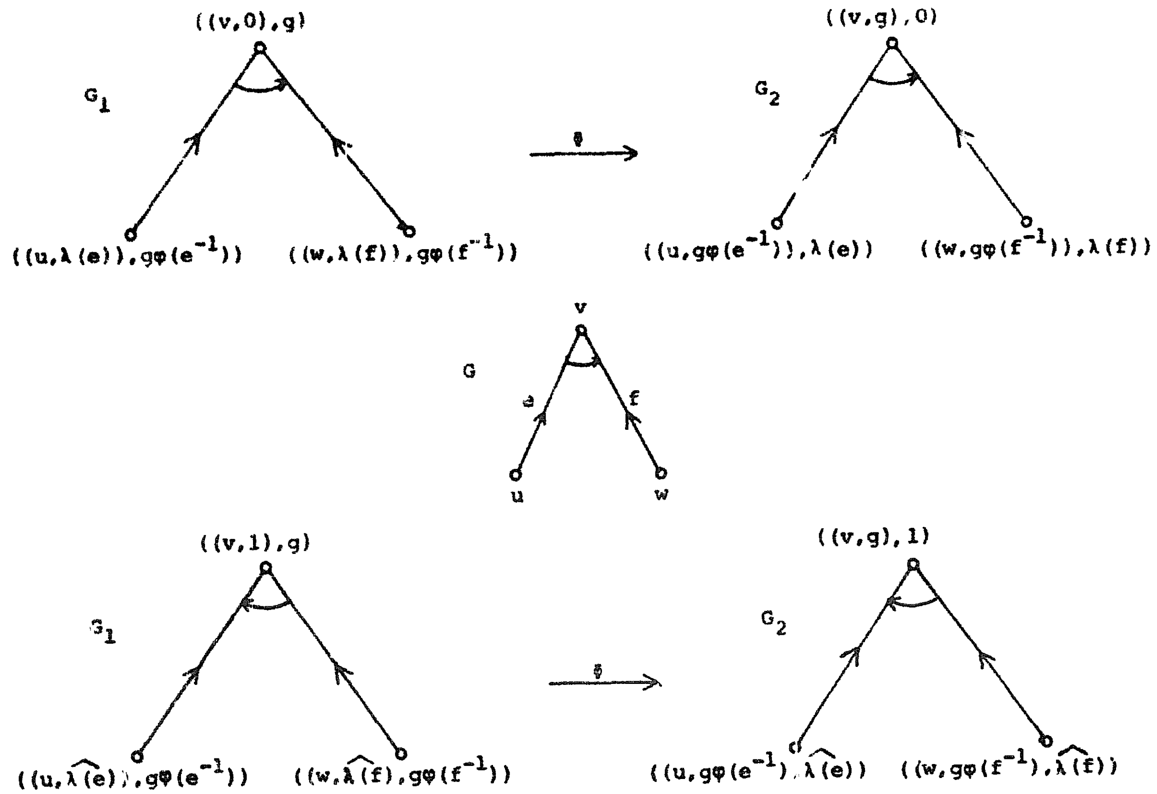

Fig. 1.

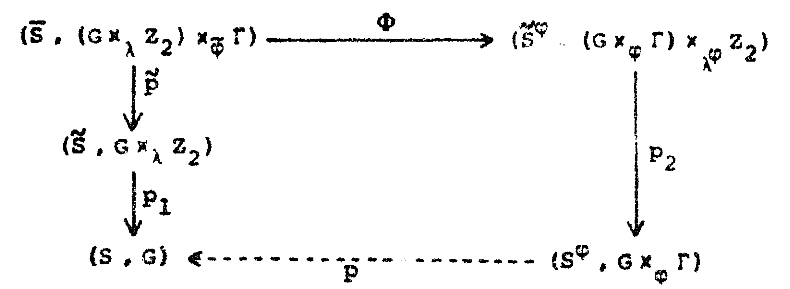

Fig. 2.

beddings are orientable and hence it suffices to show that Φ preserves the rotations. Both rotation systems for G_1 and G_2, however, are lifted from G as described in Fig. 1, and it is clear from this figure that Φ does indeed preserve rotations. Note that the value of λ were *not* inserted into this figure. Also, $\hat{0} = 1$ and $\hat{1} = 0$.

Hence Φ may be considered as a homeomorphism from $\bar{S}$ onto $\tilde{S}^\varphi$. We combine all the above maps into Fig. 2, where the maps are maps of pairs (A,B) with $A \supseteq B$, and define $p = p_1 \circ \tilde{p} \circ \Phi^{-1} \circ p_2^{-1}$. Note that if R is any region of $G \times_\varphi \Gamma$ on S^φ, then $p_2^{-1}(R)$ consists of two regions which are again identified by $p_1 \circ \tilde{p} \circ \Phi^{-1}$. We illustrate with an example. Suppose $G = K_4$ with $\lambda : D(G) \to Z_2$ and $\varphi : D(G) \to \mathbf{Z}_4$ as indicated in Fig. 3. To avoid confusion we use the symbols "+" and "−" to denote the elements of $\mathbf{Z}_2$ with the first denoting the identity element.

Now the region R of $G \times_\varphi \Gamma$ with boundary

$$(u,0) \sim (v,1) \sim (w,3) - (u,2) \sim (v,3) \sim (w,1) - (u,0)$$

lifts to two regions on $\tilde{S}$ with boundaries

$$((u,0),+) - ((v,1),-) - ((w,3),+) - ((u,2),+) - ((v,3),-) - ((w,1),+)\,,$$

$$((u,0),-) - ((v,1),+) - ((w,3),-) - ((u,2),-) - ((v,3),+) - ((w,1),-)\,.$$

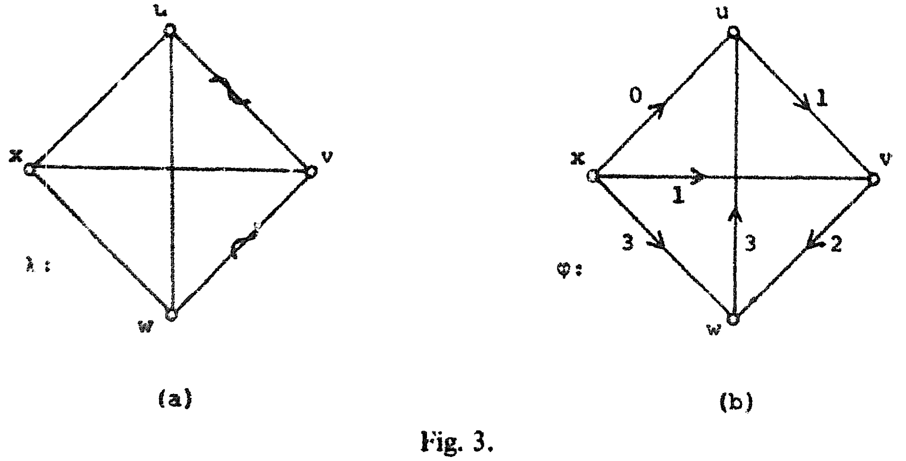

Fig. 3.

The map Φ^{-1} maps these regions to the following regions of $\bar{S}$:

$$((u,+),0)-((v,-),1)-((w,+),3)-((u,+),2)-((v,-),3)-((w,+),1)\,,$$

$$((u,-),0)-((v,+),1)-((w,-),3)-((u,-),2)-((v,+),3)-((w,-),1)\,.$$

Next, $\tilde{p}$ maps these regions onto the regions of $\tilde{S}$:

$$(u,+)-(v,-)-(w,+) \quad \text{and} \quad (u,-)-(v,+)-(w,-)\,.$$

Finally, the effect of p_1 is to map both of these regions onto the region $u \sim v \sim w$ of G on S.

Thus the map p is well defined. That p does indeed possess properties (a), (b) and (c) follows from the fact that $\tilde{p}$ possesses the analogous properties for the orientable case. This too is well illustrated by the above example. This concludes the proof of the theorem.

The surface S^φ need not be non-orientable. Following a definition we give a rule for determining the orientability character of S^φ. Given a voltage graph (G,η,Γ), we say that the closed walk c of G is *η-trivial* if $\eta(c)$ is the identity element of Γ.

Theorem 1.4. *Under the hypotheses of Theorem 1.3, the derived surface S^φ is orientable if and only if every φ-trivial closed walk in G is also λ-trivial.*

Proof. We know from [18, Theorem 3.4] that S^φ is orientable if and only if every closed walk of $G \times_\varphi \Gamma$ is λ-trivial. Now, suppose S^φ is orientable, and let c be a φ-trivial closed walk of G. Then c lifts to a circuit c^φ, of the same length as c, in $G \times_\varphi \Gamma$. Since S^φ is orientable, it follows that c^φ must be λ^φ-trivial. Consequently, c itself must be λ-trivial. The converse is proved in a similar manner.

Example 1.5. Fig. 4 exhibits an embedding of a pseudograph G with one vertex and $\frac{1}{2}m$ loops ($m \equiv 2 \pmod 4$) in the projective plane (diametrically opposite points on the circumference of the circle are identified). It follows from the final discussion in [18] that $\lambda = 1$ on every edge of G. Now define a voltage φ of $1 \in \mathbf{Z}_n$ (n even) on each arc of G in the direction indicated by the arrowhead in the figure. Here $G \times_\varphi \mathbf{Z}_n$ is a graph with the vertices $\{(v,i)\colon\ i = 0, 1, ..., n-1\}$ in which (v,i) and $(v,i+1)$ are joined by $\frac{1}{2}m$ edges for each $i = 0,1, ..., n-1$. For each region R of G, we have $|R|_\varphi = \frac{1}{2}n$. Hence the regions of $G \times_\varphi \mathbf{Z}_n$

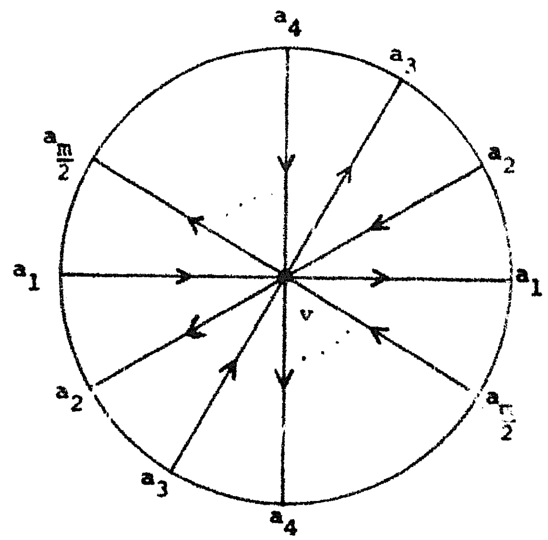

Fig. 4.

are all n-gons (by Theorem 1.3). In fact, it is easily verified that the sequence of vertices along the boundary of each region of $G \times_\varphi \mathbf{Z}_n$ is $(v,0) - (v,1) - ... - (v, n-1)$. Thus the boundary of each region of $G \times_\varphi \mathbf{Z}_n$ is a *hamiltonian cycle* in the sense that it contains each vertex of $G \times_\varphi \mathbf{Z}_n$ exactly once. There are $(\frac{1}{2}m)2 = m$ such regions in the derived embedding of $G \times_\varphi \mathbf{Z}_n$. Now place a new vertex in the interior of each such region, join it by non-intersecting edges to all the vertices on its boundary, and delete all the original edges of $G \times_\varphi \mathbf{Z}_n$. The result is a quadrilateral embedding of $K(m,n)$. This device originates in [4]. Now, because n is even and each arc of G carries a voltage of $\pm 1 \in \mathbf{Z}_n$, and moreover, for each arc of G, $\lambda = 1 \in \mathbf{Z}_2$, it is clear that every φ-trivial closed walk of G is also λ-trivial. Thus we have obtained an orientable quadrilateral embedding of $K(m,n)$, where n is even and $m \equiv 2 \pmod 4$. This, of course, is not new. The first such embedding was given in [14].

2. Lower bounds

In Section 3 the foregoing discussion will be used to produce some graph embeddings. This section is devoted to the construction of the machinery needed to show that these embeddings are in fact genus embeddings.

Let the pseudograph G be 2-cell embedded on the surface S with V vertices, E edges, and F regions. Let V_i and F_i denote the number of vertices of degree i and regions which are i-gons respectively. It is clear that $F_0 = F_1 = F_2 = 0$, and we assume that likewise $V_0 = V_1 = V_2 = 0$. If the surface S is orientable, then the Euler–Poincaré formula asserts that

$$V - E + F = 2 - 2\gamma(S)\,.$$

Hence,

$$\gamma(S) = 1 - \tfrac{1}{2}(E - V - F)\,.$$

Since

$$F = \sum_{i \geqslant 3} F_i\,, \qquad V = \sum_{i \geqslant 3} V_i\,,$$

$$2E = \sum_{i \geqslant 3} iF_i = \sum_{i \geqslant 3} iV_i\,,$$

it follows that

(1) $$\gamma(G) \leqslant \gamma(S) = 1 + \tfrac{1}{8}(4E - 4V - 4F) = 1 + \tfrac{1}{8}(2E - 4V + 2E - 4F)$$

$$= 1 + \tfrac{1}{8}\Big(\sum_{i \geqslant 3} iV_i - 4\sum_{i \geqslant 3} V_i + \sum_{i \geqslant 3} iF_i - 4\sum_{i \geqslant 3} F_i\Big)$$

$$= 1 + \tfrac{1}{8}\sum_{i \geqslant 3}(i-4)(F_i + V_i)\,.$$

Similarly, if S is non-orientable, then

(2) $$\tilde{\gamma}(G) \leqslant 2 + \tfrac{1}{4}\sum_{i \geqslant 3}(i-4)(F_i + V_i)\,.$$

We now specialize to the case where G is a complete tripartite graph.

The graph $K(p,q,r)$ has $(p + q + r)$ vertices, which are partitioned into three sets P, Q and R, containing p, q and r vertices, respectively. We assume throughout this paper that $p \geqslant q \geqslant r \geqslant 1$. The edges of $K(p,q,r)$ are precisely those edges which join a vertex in one of the three sets to a vertex in one of the other two sets. In order to distinguish the three types of edges which occur, we make the following definition.

Definition 2.1. An edge of the graph $K(p,q,r)$ which joins a vertex in the set R with a vertex in the set Q is called an edge of *type I*. Similarly, an edge joining sets R and P is called an edge of *type II*, and one joining sets Q and P an edge of *type III*.

Since there are qr edges of type I, pr edges of type II and pq edges of type III, the total number of edges is $E = qr + pr + pq$.

Lower bounds for genus formulae are ordinarily obtained by the use of the Euler–Poincaré formula and certain properties of the graph in question. Theorem 2.2, which follows, can be established in this manner, but a simpler proof is presented which uses Ringel's results for the genus of complete bipartite graphs in [14] and [15].

Theorem 2.2. *The orientable and non-orientable genera of the graph $K(p,q,r)$ are bounded below by:*

$$\gamma(K(p,q,r)) \geqslant \{\tfrac{1}{4}(p-2)(q+r-2)\},$$

$$\tilde{\gamma}(K(p,q,r)) \geqslant \{\tfrac{1}{2}(p-2)(q+r-2)\}.$$

Proof. Consider any orientable genus embedding of $K(p,q,r)$ in a surface S. By the removal of all edges of type I from this embedding, we obtain an embedding of $K(p,q+r)$ in the same surface S. Hence by Ringel's formula,

$$\gamma(K(p,q,r)) = \gamma(S) \geqslant \gamma(K(p,q+r)) = \{\tfrac{1}{4}(p-2)(q+r-2)\}.$$

Similarly, if $K(p,q,r)$ has a non-orientable genus embedding on $\tilde{S}$, we use the formula $\tilde{\gamma}(K(p,q+r)) = \{\tfrac{1}{2}(p-2)(q+r-2)\}$ to derive the second result.

Much of the remainder of this paper is dedicated to showing that equality holds in the above theorem for several families of graphs, and we conjecture that it holds for all complete tripartite graphs.

Conjecture 2.3.

$$\gamma(K(p,q,r)) = \{\tfrac{1}{4}(p-2)(q+r-2)\},$$

$$\tilde{\gamma}(K(p,q,r)) = \{\tfrac{1}{2}(p-2)(q+r-2)\}.$$

The result of Ringel and Youngs [17] that $\gamma(K(p,p,p)) = \frac{1}{2}(p-1)(p-2)$ is seen to be consistent with this conjecture. It was also shown by White in [19] that $\gamma(K(mn,n,n)) = \frac{1}{2}(mn-2)(n-1)$, which likewise agrees with this conjecture.

To show that equality holds it is sufficient to construct embeddings of $K(p,q,r)$ in the surfaces S_n and $\tilde{S}_m$, where $n = \{\frac{1}{4}(p-2)(q+r-2)\}$ and $m = \{\frac{1}{2}(p-2)(q+r-2)\}$. The following lemmas will assist us in investigating the region distribution of such embeddings.

Lemma 2.4. *In any embedding of $K(p,q,r)$, $F_3 \leqslant 2qr$.*

Proof. Any 3-cycle in $K(p,q,r)$ must be composed of one edge of each of the three types, since otherwise two vertices in the same vertex set would be adjacent, a contradiction. Hence any triangle in an embedding of this graph contains one edge of each type, and in particular an edge of type I. But there are only qr edges of type I, and each edge appears in at most two regions in any embedding of the graph. Hence $F_3 \leqslant 2qr$.

Lemma 2.5. *If any embedding of $K(p,q,r)$ has $F_3 = 2qr$, then $F_{2i+1} = 0$, for $i = 2,3, \ldots$.*

Proof. If $F_3 = 2qr$, the qr edges of type I each appear in two triangular regions. Any other region must then include only edges of type II or of type III. Since the vertices of $K(p,q,r)$ are partitioned into three sets P, Q and R of p, q and r vertices, respectively, the boundary of any non-triangular region is a subgraph of the bipartite graph $K(p,q+r)$ which has its vertex set partitioned into the sets P and $Q \cup R$. Any such subgraph is itself a bipartite graph and hence cannot contain any odd cycles. We observe that a region could contain a given vertex more than once, but in this case each cycle formed must be even, implying that the region has an even number of sides.

Theorem 2.6. *If $F_3 = 2qr$ in a 2-cell embedding of $K(p,q,r)$ in an orientable surface S, then*

$$\gamma(K(p,q,r)) \leqslant \gamma(S) = \tfrac{1}{4}(p-2)(q+r-2) + \tfrac{1}{4}\sum_{i \geqslant 3}(i-2)F_{2i}\,.$$

If S is non-orientable, then

$$\tilde{\gamma}(K(p,q,r)) \leqslant \tilde{\gamma}(S) = \tfrac{1}{2}(p-2)(q+r-2) + \tfrac{1}{2}\sum_{i \geqslant 3}(i-2)F_{2i}\,.$$

Proof. For the orientable case we use version (1) of the Euler–Poincaré formula discussed above. In particular, for $G = K(p,q,r)$, since $V_{p+q} = r$, $V_{p+r} = q$, and $V_{q+r} = p$, and since we are assuming that $F_3 = 2qr$, we have, using Lemma 2.5,

$$\begin{aligned}\gamma(K(p,q,r)) &\leqslant \gamma(S)\\ &= 1 + \tfrac{1}{8}(-2rq + (p+q-4)r + (p+r-4)q\\ &\qquad + (q+r-4)p) + \tfrac{1}{8}\sum_{i\geqslant 5}(i-4)F_i\\ &= \tfrac{1}{4}(p-2)(q+r-2) + \tfrac{1}{4}\sum_{i\geqslant 3}(i-2)F_{2i}\,.\end{aligned}$$

If S is non-orientable, we use version (2) of the Euler–Poincaré formula to derive the second part of the theorem.

As a result of the above theorem it is possible to show that equality holds in the orientable part of Theorem 2.2, provided we produce an orientable 2-cell embedding of $K(p,q,r)$ for which $F_3 = 2qr$ and

$$\tfrac{1}{4}\sum_{i\geqslant 3}(i-2)F_{2i} = \{\tfrac{1}{4}(p-2)(q+r-2)\} - \tfrac{1}{4}(p-2)(q+r-2),$$

with all other regions being quadrilateral, for then $\gamma(K(p,q,r)) \leqslant \{\tfrac{1}{4}(p-2)(q+r-2)\}$. In particular, if $\tfrac{1}{4}(p-2)(q+r-2)$ is an integer, we seek a 2-cell embedding with $F_3 = 2qr$ and all other regions quadrilaterals. It is easy to see that a non-orientable embedding which satisfies these two conditions is also necessarily a genus embedding. Thus, we have derived the following corollary.

Corollary 2.7. *An embedding of $K(p,q,r)$ which satisfies the equations $F_3 = 2qr$ and $F_4 = F - F_3$ is a genus embedding.*

We are also in position to prove the following characterization.

Corollary 2.8. *An orientable minimal embedding of $K(p,q,r)$ is triangular if and only if $p = q = r$.*

Proof. As mentioned above, Ringel and Youngs have shown in [1] that $K(p,p,p)$ does indeed possess an orientable triangular embedding. Conversely, suppose $K(p,q,r)$ has a triangular embedding. This embedding is therefore minimal, by a result of Youngs [21], and hence is a 2-cell embedding. Then $F = F_3 = 2qr$; and each edge of type I lies in exactly two triangular regions for this embedding, so that $F_3 \geqslant 2qr$ also. Hence, by Theorems 2.2 and 2.6, $\gamma(K(p,q,r)) = \frac{1}{4}((p-2)(q+r-2))$. Now, from the Euler–Poincaré formula,

$$2qr = F = -V + E + 2(1-\gamma)$$
$$= -(p+q+r) + (pq+pr+qr) + 2 - \tfrac{1}{2}(p-2)(q+r-2),$$

so that $pq + pr = 2qr$. Since $p \geqslant q \geqslant r$, it follows that $pq \geqslant qr$ and $pr \geqslant qr$. It follows that $pq = qr = pr$, and so $p = q = r$.

3. Minimal embeddings

We now proceed to construct both orientable and non-orientable embeddings of complete tripartite graphs which satisfy the hypotheses of Corollary 2.7.

Theorem 3.1. $\tilde{\gamma}(K(n,n,n-2)) = (n-2)^2$ *for* $n \geqslant 3$.

Proof. Let G be the multigraph which consists of n edges joining two vertices. Suppose G is embedded in the projective plane, as described in Fig. 5 (with diametrically opposed points on the circumference identified). This embedding clearly consists of $(n-2)$ 2-gons and one quadrilateral. Let $\varphi : D(G) \to \mathbf{Z}_n$ be the voltage assignment given in the figure. It is clear that $G \times_\varphi \mathbf{Z}_n = K(n,n)$, and that for any region R of G we have $|R|_\varphi = 1$ or n according as R is a quadrilateral or a 2-gon. Thus the lifted embedding of $G \times_\varphi \mathbf{Z}_n$ on S^φ consists of n quadrilaterals and

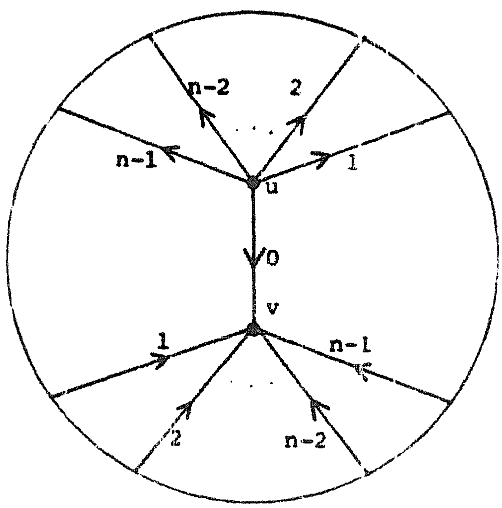

Fig. 5.

$(n-2)$ $2n$-gons (the $2n$-gons lift the 2-gons of G). It is easily verified that the boundary of each $2n$-gon is in fact a hamiltonian cycle of $G \times_\varphi Z_n$; thus by placing a new vertex inside each $2n$-gon and joining it to all the vertices of $G \times_\varphi Z_n$ we obtain an embedding of $K(n,n,n-2)$ on S^φ. To see that S^φ is in fact non-orientable, observe that if n is odd then the closed walk which consists of n repetitions of the cycle $u \xrightarrow{1} v \xleftarrow{0} u$ is φ-trivial but not λ-trivial; if n is even then the circuit

$$u \xrightarrow{(n/2)-1} v \xleftarrow{n-1} u \xrightarrow{(n/2)} v \xleftarrow{0} u$$

is also φ-trivial but not λ-trivial. Thus, in either case S^φ is non-orientable. We apply the Euler–Poincaré formula to derive the genus of S^φ. The graph $G \times_\varphi Z_n$ has $2n$ vertices, n^2 edges, and $2n-2$ regions on S^φ. Hence

$$2n - n^2 + (2n-2) = 2 - \tilde{\gamma}(S^\varphi)\,,$$

or

$$\tilde{\gamma}(S^\varphi) = (n-2)^2\,.$$

It is clear that the derived embedding of $K(n,n,n-2)$ has $2n(n-2)$ triangular regions and that the remaining regions are all quadrilateral. It follows from Corollary 2.7 that this is a genus embedding and hence $\tilde{\gamma}(K(n,n,n-2)) = (n-2)^2$.

A slight modification of the above construction yields a non-orientable genus formula for $K(n,n,n-4)$ when n is even.

Theorem 3.2. $\tilde{\gamma}(K(n,n,n-4)) = n^2 - 5n + 6$ *for* $n \geq 4$ *and* n *even.*

Proof. We use the same embedding as in the previous theorem but we change the voltage assignment φ as indicated in Fig. 6 (here $m = \frac{1}{2}n$). Each of the $(n-4)$ 2-gons but R_2 and R_3 lifts to a single $2n$-gon whose boundary is a hamiltonian cycle of $G \times_\varphi Z_n$. Since $|R_2|_\varphi = |R_3|_\varphi = 2$, R_2 and R_3 (as well as R_1) lift to quadrilaterals on S^φ. This embedding is again easily modified into an embedding of $K(n,n,n-4)$ with $2n(n-4)$ triangular and $n + \frac{1}{2}n + \frac{1}{2}n = 2n$ quadrilateral regions. The rest of the proof will be omitted here as well as in subsequent theorems, since it does not differ materially from the conclusion of the proof of the previous theorem.

We conclude with the construction of several orientable genus embeddings. All but one of these in fact give rise to new genus formulae.

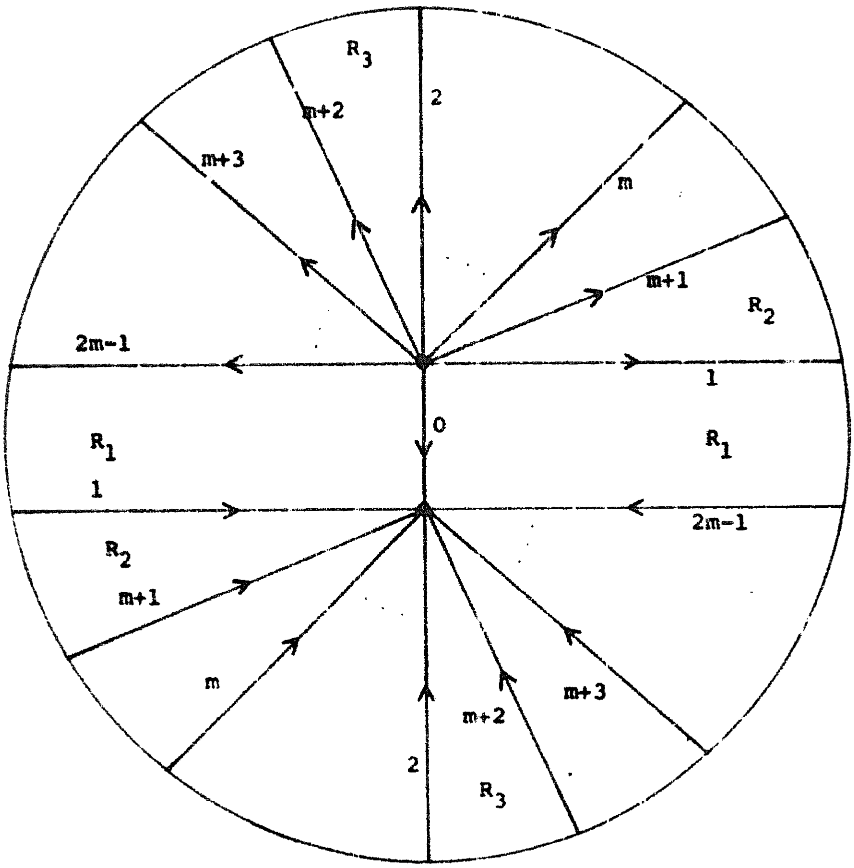

Fig. 6.

Fig. 7 exhibits a plane embedding of the above multigraph G together with a voltage assignment $\varphi : D(G) \to \mathbf{Z}_n$. Here $G \times_\varphi \mathbf{Z}_n = K(n,n)$ and the derived embedding has n regions each of which is bounded by a hamiltonian cycle. Again, this embedding is easily modified to produce a triangular embedding of $K(n,n,n)$. Such an embedding, as was mention-

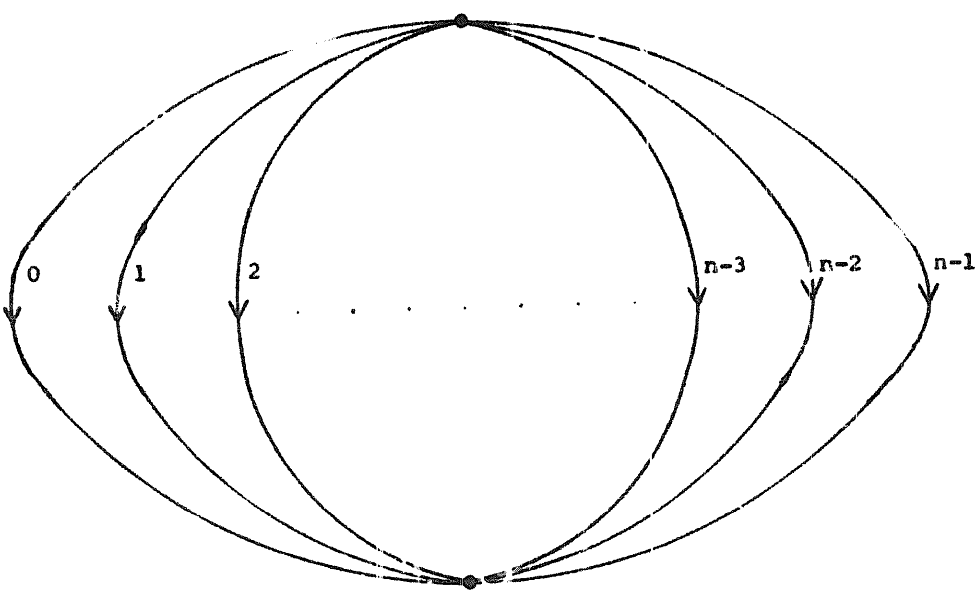

Fig. 7.

ed above, is of course not new. The authors do believe, however, that this construction is the easiest to verify.

At this point we digress to consider some genus embeddings of graphs which are not complete tripartite. The above embeddings of $K(n,n)$ can be used to extend some results of White in [20] regarding the genus of the repeated cartesian product of $K(2m,2m)$. Working with $2s$ copies of $K(s,s)$, embedded as above, we choose one of the two possible orientations for s of these embeddings, and the reverse orientation for the others. Since $K(s,s)$ is 1-factorable (see [20, Theorem 12.4]), the construction of Theorem 1 of [20] is applicable, except that here each tube carries $2s$ edges and $2s$ quadrilaterals. Now replace the $2m$ of [20, Theorem 2] by s and apply the proof verbatim. We thus obtain a quadrilateral embedding for $K(s,s)^{(n)}$ (the cartesian product of n copies of $K(s,s)$) and hence

$$\gamma(K(s,s)^{(n)}) = 1 + 2^{n-3}s^n(ns-4) \quad \text{for } n \geq 2 \text{ and } s \geq 1.$$

A slight modification of this embedding of $K(s,s)$ yields another orientable genus formula for complete tripartite graphs.

Theorem 3.3. *$\gamma(K(n,n,n-2)) = \frac{1}{4}(n-2)^2$ if n is even and $n \geq 2$.*

Proof. We set $m = \frac{1}{2}n$ and modify the previous voltage assignment as indicated in Fig. 8. All the 2-gons except R_1 and R_2 lift to $2n$-gons of $G \times_\varphi Z_n$ with hamiltonian boundaries. On the other hand, $|R_1|_\varphi = |R_2|_\varphi = 2$ and so they lift to m quadrilaterals each. Thus the derived embedding of $G \times_\varphi Z_n = K(n,n)$ can be modified into an embedding of $K(n,n,n-2)$ with $(n-2)2n$ triangular and n quadrilateral regions. That this embedding is orientable follows from the fact that we started out with an orientable embedding. Again an application of Corollary 2.7 shows that this is in fact a genus embedding.

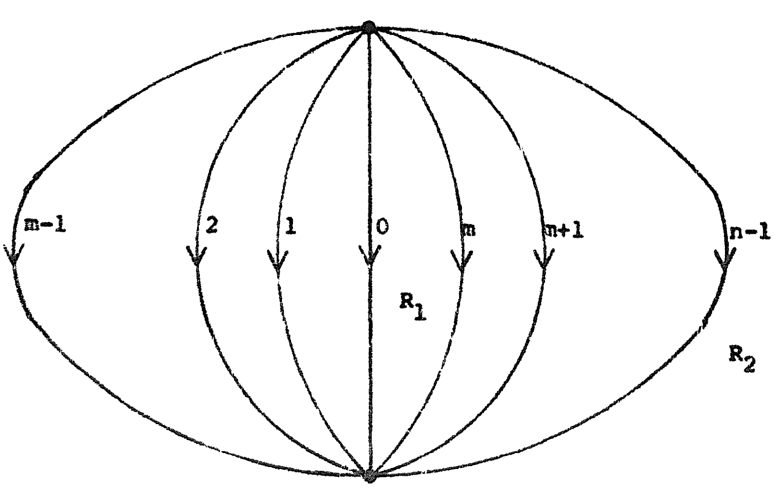

Fig. 8.

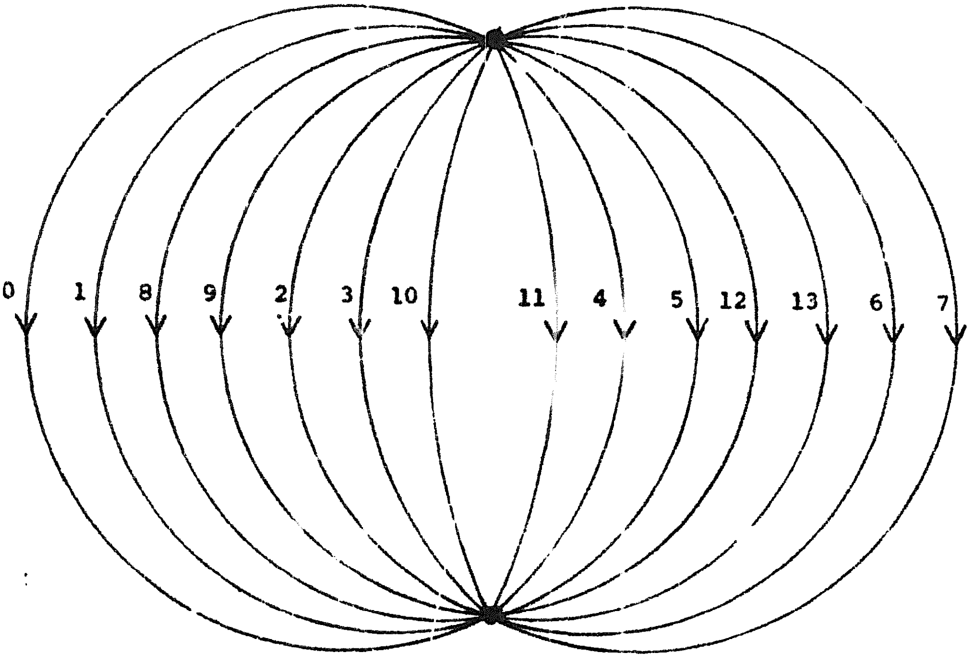

Fig. 9.

Theorem 3.4. $\gamma(K(2n, 2n, n)) = \frac{1}{2}(3n - 2)(n - 1)$ *for* $n \geqslant 1$.

Proof. If n is odd then it is possible to list the elements of $\mathbf{Z}_{2n}$ as:

(3) $\quad 0, 1, n+1, n+2;\quad 2, 3, n+3, n+4;\quad \ldots\quad n-3, n-2, 2n-2, 2n-1;\quad n-1, n.$

Again we use the multigraph G with $2n$ edges joining 2 vertices and embed it in the plane. We assign the elements of $\mathbf{Z}_{2n}$ to the arcs of G starting with the first arc on the left, and proceeding to the right, and making the assignments in the order indicated by the above sequence. Fig. 9 exhibits this assignment for the case $n = 7$. It is easily verified that for n of the 2-gons we have $|R|_\varphi = 2n$, while for the other n 2-gons we have $|R|_\varphi = 2$. Hence $G \times_\varphi \mathbf{Z}_{2n}$ has n regions with hamiltonian boundaries and n^2 quadrilaterals. This is easily transformed into an embedding of $K(2n, 2n, n)$ with $4n^2$ triangles and n^2 quadrilaterals. A straightforward computation shows that the derived surface has the required genus.

If n is even we replace sequence (3) above by

$$0, 1, n+1, n+2;\quad 2, 3, n+3, n+4;\quad \ldots\quad n-4, n-3, 2n-3, 2n-2;\quad n-2, n-1, 2n-1, n.$$

From this point the proof proceeds as above.

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