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Polymatroid Greedoids*

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This paper discusses polymatroid greedoids, a superclass of them, called local poset greedoids, and their relations to other subclasses of greedoids. Polymatroid greedoids combine in a certain sense the different relaxation concepts of matroids as polymatroids and as greedoids. Some characterization results are given especially for local poset greedoids via excluded minors. General construction principles for intersection of matroids and polymatroid greedoids with shelling structures are given. Furthermore, relations among many subclasses of greedoids which are known so far, are demonstrated. © 1985 Academic Press, Inc.

1. INTRODUCTION

Matroids on a finite ground set E can be defined by a rank function $r: 2^E \to \mathbb{Z}_+$, which is subcardinal, monotone, and submodular. Polymatroids are relaxations of them. The rank function of a polymatroid is monotone and submodular. Subcardinality is required only for the empty set by normalizing $r(\emptyset)=0$. One way of defining greedoids is also via a rank function. Here, the rank function is subcardinal, monotone, but only locally submodular, which means that if the rank of a set is not increased by adding one element as well as another element to it, then the rank of the set will not increase by adding both elements to it. Again, this is a direct relaxation of submodularity and hence of the matroid definition.

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In this paper we introduce polymatroid greedoids, which in a certain sense combine these different relaxation concepts. Given a polymatroid and its rank function, we get the associated polymatroid greedoid as the collection of all words (i.e., ordered subsets of E) such that for every beginning section of these words the rank function equals its cardinality. For a special case, this way of deriving a language from a polymatroid was described by Faigle [6].

Polymatroid greedoids are a very important and, in a sense, central subclass of greedoids, as it will be shown in this paper. In Section 2 we mention briefly some basic definitions and facts about greedoids. However, for more detail about structural properties and subclasses of greedoids, which are used in this paper, we have to refer to some earlier papers about greedoids, especially Korte and Lovász [9–11].

Section 3 gives some examples of polymatroid greedoids, among which poset greedoids and undirected branching greedoids are notable.

In Section 4 we give the main results about polymatroid greedoids and introduce local poset greedoids, a proper superclass of them, which contains the classical (directed) branchings. Some elementary properties of the polymatroid rank function are derived, and we relate it to the locally submodular rank function of the associated greedoid. Furthermore, we show that a polymatroid greedoid has some interesting local properties, namely a local intersection property, a local union property, and a local augmentation property. We prove that the first two of these properties characterize local poset greedoids. We have been unable to give a similar characterization for polymatroid greedoids. Like in the case of matroid theory we could try to characterize subclasses of greedoids via excluded minors. A first result of this kind is given for local poset greedoids, but again not for polymatroid greedoids.

Section 5 gives some constructions for more general local poset greedoids as intersections of polymatroid greedoids with certain shelling structures. These greedoids are called polymatroid branchings. A further generalization which is obtained by the intersection of matroids and quasi-modular set systems leads to a class of greedoids which we call trimmed matroids. These are not local posets anymore.

Finally, in Section 6 we give a complete chart of many subclasses of greedoids, which were discovered so far and demonstrate the relations among these classes. We believe that this is helpful for a further and deeper understanding of greedoidal structures.

2. Preliminaries: Matroids, Polymatroids, and Greedoids

We assume that the reader is familiar with the basic facts of matroid theory. In general our notation is in accordance with the standard matroid terminology.

A *set system* over a finite ground set E is a pair $(E, \mathscr{F})$ with $\mathscr{F} \subseteq 2^E$. A set system is a *matroid* if the following axioms hold:

(M1) $\varnothing \in \mathscr{F}$,

(M2) $X \subseteq Y \in \mathscr{F}$ implies $X \in \mathscr{F}$,

(M3) if $X, Y \in \mathscr{F}$ and $|X| > |Y|$, then there exists an $x \in X - Y$ such that $Y \cup x \in \mathscr{F}$.

A set system which satisfies only (M1) and (M2) is called *independence system* or *hereditary set system*. For an arbitrary set system we define its *hereditary closure* as

$$\mathscr{H}(\mathscr{F}) := \mathscr{H} = \{X \subseteq Y \colon Y \in \mathscr{F}\}.$$

and its *accessible kernel* as

$$\mathscr{K}(\mathscr{F}) := \mathscr{K} = \{X \in \mathscr{F} \colon X = \{x_1, ..., x_k\} \text{ and } \{x_1, ..., x_i\} \in \mathscr{F}$$

for all $1 \leqslant i \leqslant k\}$.

Sets belonging to $\mathscr{F}$ are called *feasible sets* (or in case of a hereditary set system *independent sets*). Elements of $2^E - \mathscr{F}$ are *nonfeasible sets* (or *dependent sets*). For $X \subseteq E$ a maximal feasible subset is called a *basis* of X. Another way of defining matroids is via a rank function $r \colon 2^E \to \mathbb{Z}_+$ which is

(R1) $r(X) \leqslant |X|$ (subcardinal),

(R2) $X \subseteq Y \subseteq E$ implies $r(X) \leqslant r(Y)$ (monotone),

(R3) for $X, Y \subseteq E$: $r(X \cup Y) + r(X \cap Y) \leqslant r(X) + r(Y)$ (submodular).

Since it is a basic fact of matroid theory that these two concepts are equivalent, we sometimes denote a matroid by (E, r).

Polymatroids are straightforward generalizations of matroids. Let E be a finite ground set and $r \colon 2^E \to \mathbb{Z}_+$. We call (E, r) a *polymatroid* if

(R1′) $r(\varnothing) = 0$

together with (R2) and (R3) holds. As in the case of matroids there are many equivalent axiomatic definitions of polymatroids. But they are not needed for the purpose of this paper.

Greedoids were introduced in Korte and Lovász [7]. They are other generalizations (or in a sense ordered versions) of matroids. The reader is referred to Korte and Lovász [9, 14] for more details and structural properties of greedoids. However, to make this paper selfcontained, we state here some basic facts about greedoids.

A *language* $\mathscr{L}$ over a finite ground set E (which is called the *alphabet*) is a collection of finite sequences $x_1 \cdots x_k$ of elements $x_i \in E$ for $1 \leqslant i \leqslant k$. We call these sequences *strings* or *words*. A language is called *simple* if no letter is repeated in any of its words. A letter $x \in E$ is *dummy* if it does not appear in any word of $\mathscr{L}$. A language is called *normal* if it has no dummy letters. A language is *full* if it has at least one word containing all letters.

$(E, \mathscr{L})$ is called a *hereditary language* if

(G1) $\varnothing \in \mathscr{L}$,

(G2) if $x_1 \cdots x_k \in \mathscr{L}$ then $x_1 \cdots x_i \in \mathscr{L}$ for $1 \leqslant i \leqslant k$.

A simple hereditary language is called a *greedoid* if in addition the following holds:

(G3) if $x_1 \cdots x_k \in \mathscr{L}$ and $y_1 \cdots y_l \in \mathscr{L}$ with $k > l$, then there exists a j with $1 \leqslant j \leqslant k$ such that $y_1 \cdots y_l x_j \in \mathscr{L}$.

Apart from this definition of hereditary languages and greedoids as collections of *ordered* sets, we can also define them in an *unordered* version by considering the underlying sets of strings. Then an *accessible set-system* $(E, \mathscr{F})$ is a set system $\mathscr{F} \subseteq 2^E$ with

(H1) $\varnothing \in \mathscr{F}$,

(H2) for all $X \in \mathscr{F}$ there exists an $x \in X$ such that $X - x \in \mathscr{F}$.

A set system $(E, \mathscr{F})$ is a greedoid if (H1), (H2) and the following hold:

(H3) if $X, Y \in \mathscr{F}$ and $|X| = |Y| + 1$, then there exists an $x \in X - Y$ such that $Y \cup x \in \mathscr{F}$.

Given a greedoid $(E, \mathscr{L})$ defined by (G1), (G2), and (G3), let

$$\mathscr{F} := \{\{x_1, ..., x_k\}: x_1 \cdots x_k \in \mathscr{L}\}$$

then it is immediate that $(E, \mathscr{F})$ satisfies (H1), (H2), and (H3). Conversely, if a set system $(E, \mathscr{F})$ satisfies the unordered axioms (H1), (H2) and (H3), then one can show that

$$\mathscr{L} = \{x_1 \cdots x_k \colon \{x_1, ..., x_i\} \in \mathscr{F} \text{ for all } 1 \leqslant i \leqslant k\}$$

defines uniquely a greedoid $(E, \mathscr{L})$ which fulfils (G1), (G2), and (G3). Thus, the ordered and unordered definitions are equivalent and we will use them in the following concurrently. It is an easy observation that (H2) and (H3) are equivalent to (M3). Then $(E, \mathscr{F})$ is a greedoid iff (M1) and (M3) hold, which shows that greedoids are direct relaxations of matroids.

For a greedoid we can define the (*independence*) *rank* of a set $X \subseteq E$ as:

$$r(X) := \max\{|A| : A \subseteq X, A \in \mathscr{F}\}.$$

This function has the following properties for $X, Y \subseteq E$ and $x, y \in E$:

(R0) $r(\varnothing) = 0$,

(R1) $r(X) \leqslant |X|$,

(R2) if $X \subseteq Y$ then $r(X) \leqslant r(Y)$,

(R3′) if $r(X) = r(X \cup \{x\}) = r(X \cup \{y\})$ then $r(X) = r(X \cup \{x\} \cup \{y\})$.

Conversely, a function $r: 2^E \to \mathbb{Z}$ satisfying (R0), (R1), (R2), and (R3′) defines uniquely a greedoid (cf. Korte and Lovász [4]). These axioms are again direct relaxations of the rank definition of matroids, for which in addition we have the *unit increase* property

$$r(X \cup \{x\}) \leqslant r(X) + 1 \qquad \text{for} \quad X \subseteq E, x \in E.$$

From (R0)–(R3′) and the unit increase property one derives in matroid theory that the rank function is *submodular*, i.e., $r(X \cap Y) + r(X \cup Y) \leqslant r(X) + r(Y)$. This fails to hold for greedoids in general; but the property (R3′), which we call *local submodularity*, is often a reasonable substitute.

In contrast to matroids, the intersection of a set with a basis of a greedoid may have larger cardinality than the rank of this set. Therefore we define the *basis rank* of a set $X \subseteq E$ as

$$\beta(X) := \max\{|X \cap B| : B \in \mathscr{F}\}.$$

Clearly, $\beta(X) \geqslant r(X)$. A set $X \subseteq E$ is called *rank feasible* if $\beta(X) = r(X)$. We denote the family of all rank feasible sets by $\mathscr{R} = \mathscr{R}(E, \mathscr{F})$. Clearly, $\mathscr{F} \subseteq \mathscr{R}$ and $\mathscr{F} = \mathscr{R}$ for a full greedoid. In general $(E, \mathscr{R})$ is not a greedoid and $\mathscr{R}$ is not closed under union.

We recall here some facts about rank-feasibility (cf. Korte and Lovász [9]): A greedoid is a matroid iff $\mathscr{R} = 2^E$. For $A, B \subseteq E$ we have

$$\beta(A \cup B) + r(A \cap B) \leqslant \beta(A) + \beta(B)$$

and consequently, if $A, B \in \mathscr{R}$ then

$$r(A \cup B) + r(A \cap B) \leqslant r(A) + r(B),$$

i.e., r is submodular on $\mathscr{R}$. This can be also derived from the fact that $A \in \mathscr{R}$ iff $r(A \cup X) \leqslant r(A) + |X|$ for all $X \subseteq E - A$.

A fundamental concept in matroid theory is the closure operator.

Therefore we define analogously for greedoids the (*rank*) *closure* of a set $X \subseteq E$ as

$$\sigma(X) := \{x \in E: r(X \cup \{x\}) = r(X)\}.$$

This operator is not monotone, but it has the following properties:

(C1) $X \subseteq \sigma(X)$ for all $X \subseteq E$,

(C2) if $X \subseteq Y \subseteq \sigma(X)$ then $\sigma(X) = \sigma(Y)$,

(C3) if $X \subseteq E$ and $x \in E - X$ such that for all $z \in X \cup x$, $z \notin \sigma(X \cup x - z)$, and $x \in \sigma(X \cup y)$, then $y \in \sigma(X \cup x)$.

It was shown in Korte and Lovász [9] that a mapping $\sigma: 2^E \to 2^E$ satisfying (C1), (C2), and (C3) uniquely defines a greedoid.

The closure axioms for greedoids are again relaxations of the closure for matroids. (C1) is trivial, (C2) follows from monotonicity and idempotence, and (C3) is a weakening of the Steinitz–McLane axiom for matroids. It can be shown that (C2) implies idempotence, but of course not monotonicity.

A set $X \subseteq E$ is called *closed* if $X = \sigma(X)$. An easy construction leads to a *monotone* closure operator, namely

$$\mu(X) := \bigcap \{Y: X \subseteq Y \text{ and } Y \text{ closed}\}.$$

μ does not determine the greedoid uniquely. In fact, for a full greedoid we have $\mu = id$.

We call a set *closure-feasible* if $X \subseteq \sigma(A)$ implies $X \subseteq \mu(A)$, or—which is equivalent—if $X \subseteq \sigma(A)$ implies $X \subseteq \sigma(B)$ for $A \subseteq B \subseteq E$. The family of all closure feasible sets will be denoted by $\mathscr{C} = \mathscr{C}(E, \mathscr{F})$. The family $\mathscr{C}$ is closed under union and we have $\mathscr{C} \subseteq \mathscr{R}$. Further, $\mathscr{C}$ with inclusion as a partial order forms a lattice with the operation $A \vee B := A \cup B$ and $A \wedge B := \bigcup \{C \in \mathscr{C}: C \subseteq A \cap B\}$. The rank function r is submodular on this lattice. $(E, \mathscr{C})$ is not a greedoid in general, but the accessible kernel $\mathscr{K} = \mathscr{K}(\mathscr{C})$ of $\mathscr{C}$ defines trivially a greedoid. The rank function does not have the unit increase property on $\mathscr{C}$. But since $\mathscr{K} \subseteq \mathscr{C}$ is also a lattice, the rank function is also submodular on $\mathscr{K}$.

A very substantial subclass of greedoids are *interval greedoids*. We call a greedoid $(E, \mathscr{F})$ an interval greedoid if for all $A, B, C \in \mathscr{F}$ with $A \subseteq B \subseteq C$ and $x \in E - C$ such that $A \cup x \in \mathscr{F}$ and $C \cup x \in \mathscr{F}$, it follows that $B \cup x \in \mathscr{F}$. In Korte and Lovász [9] it was shown that a greedoid is an interval greedoid iff $\mathscr{C} = \mathscr{R}$ and iff $\mathscr{F} \subseteq \mathscr{C}$. Generally, no inclusion relation holds between $\mathscr{F}$ and $\mathscr{C}$. Furthermore, if $(E, \mathscr{F})$ is an interval greedoid, then already $(E, \mathscr{C})$ is a greedoid.

We call a normal interval greedoid a *shelling structure* if the *interval property* mentioned above holds *without upper bounds*, i.e., if for all $A \subseteq B$ and $x \in E - B$ such that $A \cup x \in \mathscr{F}$ it follows $B \cup x \in \mathscr{F}$. Shelling structures are full greedoids. They were introduced by Edelman [17] and Jamison-Waldner [18] as combinatorial abstractions of convexity. They are studied in greater detail in Korte and Lovász [10]. There are many equivalent ways to define shelling structures. For the purpose of this paper we give here two other definitions:

Let $\mathscr{A}$ denote the family of subsets of E which can be represented as unions of elements of $\mathscr{F}$. Then a normal greedoid $(E, \mathscr{F})$ is a shelling structure iff $\mathscr{A} = \mathscr{F}$. In fact if $(E, \mathscr{F})$ is any greedoid then $(E, \mathscr{A})$ is a shelling structure. (cf. Korte and Lovász [11]).

A third equivalent definition is the following: Let for each $e \in E$ a set system $\mathscr{H}_e \subseteq 2^E$ of *alternative precedences* be given. Let

$$\begin{aligned}\mathscr{L} = \{x_1 \cdots x_k\colon &\text{for all } 1 \leqslant i \leqslant k,\ x_i \in E \text{ and there} \\ &\text{is a set } U \in \mathscr{H}_{x_i} \text{ such that } U \subseteq \{x_1,..., x_{i-1}\}\}\end{aligned}$$

(in words: x_i may occur in a word if at least one "alternative precedence set" occurs before x_i), and assume that $\mathscr{L}$ has no dummy elements. Then $(E, \mathscr{L})$ is a shelling structure. (Clearly we could assume that $\mathscr{H}_e \subseteq 2^{E-e}$, since the members of $\mathscr{H}_e$ containing e play no role as alternative precedences. But it will be more convenient to allow $\mathscr{H}_e \subseteq 2^E$.)

A subclass of shelling structures are *poset greedoids*. Let $(E, \leqslant)$ be a finite poset. *(Lower) ideals* of this poset are all sets $X \subseteq E$ with the property that $x \in X$ and $y \leqslant x$ implies that $y \in X$. For $B \subseteq E$ we call the set $I(B) := \{x \in E\colon \text{there exists a } b \in B \text{ such that } x \leqslant b\}$ the *(lower) ideal generated by B*. If B contains a single element b, we write $I(b)$. Let $\mathscr{F}$ be all (lower) ideals of $(E, \leqslant)$. Then $(E, \mathscr{F})$ is a poset greedoid.

Another interesting subclass of shelling structures is *convex shelling*. Let E be a finite set of vectors in $\mathbb{R}^n$ and let

$$\begin{aligned}\mathscr{L} = \{x_1 \cdots x_k\colon &x_i \text{ is a vertex of the convex} \\ &\text{hull of } E - x_1 \cdots x_{i-1} \text{ for } 1 \leqslant i \leqslant k\}.\end{aligned}$$

Then $(E, \mathscr{L})$ is a convex shelling. For further details see Korte and Lovász [10].

We need some further definitions of subclasses of greedoids which are also related to posets. Faigle [6] has extended matroidal structures to partially ordered sets. It turns out that these structures, which we call *F-geometries*, are special greedoids. The relations of F-geometries and greedoids are studied in Korte and Lovász [11]. We give here only some definitions.

Let $(E, \mathscr{L})$ be a greedoid and $\leqslant$ a partial order of the ground set E. Then $(E, \mathscr{L}, \leqslant)$ is called an F-geometry if in addition the following two conditions are fulfilled:

(F1) For every $B \in \mathscr{F}$: $I(B) \subseteq \sigma(B)$,

(F2) every ideal in $(E, \leqslant)$ is closure-feasible.

For every F-geometry there exists on the same poset a unique largest, as well as a unique smallest F-geometry, for which the rank function is the same on ideals. Given an F-geometry $(E, \mathscr{L})$, let

$$\mathscr{L}_0 = \{x_1 \cdots x_k \in \mathscr{L}\text{: for all } 1 \leqslant i \leqslant k \text{ and all } y < x_i,\ x_1 \cdots x_{i-1} y \notin \mathscr{L}\}$$

(i.e., $\mathscr{L}_0$ is the set of *lexicographically minimal* words in $\mathscr{L}$) and

$$\mathscr{L}^0 = \{x_1 \cdots x_k\colon r(I(x_1 \cdots x_i)) = i \text{ for all } 1 \leqslant i \leqslant k\}.$$

We call $(E, \mathscr{L}_0)$ a *minimal F-geometry* and $(E, \mathscr{L}^0)$ a *maximal F-geometry.*

Supermatroids were introduced by Dunstan, Ingleton, and Welsh [3]. They are F-geometries in which every feasible set is an ideal. For a direct definition, we refer to Welsh [16].

Analogously to matroids, we can define *minor operations* for greedoids: Let $(E, \mathscr{F})$ be a greedoid and $T \subseteq E$. The *restriction* of $(E, \mathscr{F})$ is the greedoid $(T, \mathscr{F}|_T)$, where

$$\mathscr{F}|_T = \{X \in \mathscr{F} : X \subseteq T\}.$$

If $U := E - T$ we say that the restriction to T is obtained by the *deletion* of U. We set

$$\mathscr{F} \backslash U = \mathscr{F} \backslash (E - T) := \mathscr{F}|_T.$$

It is obvious that $(T, \mathscr{F}|_T) = (E - U, \mathscr{F} \backslash U)$ is a greedoid.

The definition of *contraction* is not so straightforward and will be meaningful for general greedoids only if the contracted set is independent. Let $(E, \mathscr{F})$ be a greedoid and $U \in \mathscr{F}$. Then the *contraction* of U results in the greedoid $(E - U, \mathscr{F}/U)$ where

$$\mathscr{F}/U := \{X \subseteq E - U\colon X \cup U \in \mathscr{F}\}.$$

It is again obvious that $(E - U, \mathscr{F}/U)$ is a greedoid.

We define a *k-truncation* of a greedoid $(E, \mathscr{L})$ by $(E, \mathscr{L}_k)$, where $\mathscr{L}_k$ contains all words in $\mathscr{L}$ of length of at most k. Obviously $(E, \mathscr{L}_k)$ is also a greedoid.

Finally, we have to define *polymatroid greedoids*, the main topic of this paper: Given a polymatroid (E, f), let

$$\mathscr{L}_f = \mathscr{L} = \{x_1 \cdots x_k\colon f(x_1, ..., x_i) = i \text{ for all } 1 \leqslant i \leqslant k\}.$$

Then $(E, \mathscr{L}_f)$ is called a *polymatroid greedoid* and we prove:

THEOREM 2.1. *$(E, \mathscr{L}_f)$ is an interval greedoid.*

Proof. Properties (G1) and (G2) are obvious. To prove property (G3) we take two words $x_1 \cdots x_i$, $y_1 \cdots y_j \in \mathscr{L}_f$ with $i > j$. Take the first k such that

$$f(y_1, ..., y_j, x_1, ..., x_k) > f(y_1, ..., y_j) = j.$$

Such a k exists, since if not, then

$$f(y_1, ..., y_j, x_1, ..., x_i) \leqslant j < i$$

which contradicts monotonicity of f. By submodularity we have

$$f(y_1, ..., y_j, x_1, ..., x_{k-1}) + f(x_1, ..., x_k)$$
$$\leqslant f(x_1, ..., x_{k-1}) + f(y_1, ..., y_j, x_1, ..., x_k).$$

This implies $f(y_1, ..., y_j, x_1, ..., x_k) \leqslant j+1$, thus it is equal to $j+1$. Again, by submodularity $f(y_1, ..., y_j, x_k) + f(y_1, ..., y_j, x_1, ..., x_{k-1}) \geqslant f(y_1, ..., y_j) + f(y_1, ..., y_j, x_1, ..., x_k)$, which gives $f(y_1, ..., y_j, x_k) \geqslant j+1$. By monotonicity of f we have

$$f(y_1, ..., y_j, x_k) \leqslant f(y_1, ..., y_j, x_1, ..., x_k) = j+1.$$

Hence, $f(y_1, ..., y_j, x_k) = j+1$ and therefore $y_1 \cdots y_j x_k \in \mathscr{L}_f$.

It remains to show that $(E, \mathscr{L}_f)$ has the interval property. Since f is submodular, we have that $f(X \cup z) - f(X)$ is monotone decreasing for subsets of $E - z$; if it is equal to 1 for A, C with $A \subseteq C$, then it is also equal to 1 for all B with $A \subseteq B \subseteq C$. ∎

Note that $\mathscr{L}_f = \{\varnothing\}$ if $f(\{x\}) \geqslant 2$ for all $x \in E$.

It is also worth pointing out the difference from, and similarity with, the notion of the *matroid induced by* (E, f) (Edmonds [17]), which is defined by

$$\mathscr{M} = \{X \subseteq E: f(Y) \geqq |Y| \text{ for all } Y \subseteq X\}.$$

3. Examples of Polymatroid Greedoids

Before we discuss some structural properties of polymatroid greedoids, it might be worthwhile to discuss some special polymatroids, which lead to interesting greedoids.

(1) If the polymatroid is a matroid, then trivially the associated greedoid is the matroid itself.

(2) Let G be a graph, $r \in V(G)$, and $E = E(G)$. Define, for $X \subseteq E$, $f(X)$ as the number of points in $V(G) - r$ covered by X. It is straightforward to see that (E, f) is a polymatroid. Let $(E, \mathscr{F})$ be the greedoid associated with this polymatroid. Then a set $X \subseteq E$ belongs to $\mathscr{F}$ iff X is a subtree of G containing the root r. We call these greedoids *undirected branching*

greedoids. We remark that *directed branching greedoids*, as defined in Korte and Lovász [14] are not polymatroid greedoids (see Korte and Lovász [10]).

(3) Let $(E, \leqslant)$ be a poset and for each $X \subseteq E$, let $f(X)$ denote the size of the ideal generated by X. Again, it is easy to see that (E, f) is a polymatroid. Let $(E, \mathscr{F})$ be the greedoid associated with (E, f). Then $X \in \mathscr{F}$ iff X is an ideal in $(E, \leqslant)$. Hence (E, f) is a *poset* (or *schedule*) *greedoid* as defined in Korte and Lovász [14].

(4) Faigle [6] showed that if r is the rank function of an F-geometry, then $f(X) = r(I(X))$ defines a polymatroid function. Hence maximal F-geometries, as defined in Section 2, are polymatroid greedoids.

4. On the Structure of Polymatroid Greedoids and Local Poset Greedoids

Throughout this section, if not noted otherwise, let (E, f) be a polymatroid and $(E, \mathscr{F})$ be associated greedoid. It is clear from the definition of $\mathscr{F}$ that if $X \in \mathscr{F}$ then $f(X) = |X|$. The converse is not necessarily true, but if we add an accessibility hypothesis, then a converse can be easily shown:

Lemma 4.1. *Let $X \subseteq E$ and $f(X) = |X|$. Suppose that $X - x \in \mathscr{F}$ for some $x \in X$. Then $X \in \mathscr{F}$.*

Proof. By definition, $X - x$ has an ordering $(x_1, ..., x_j)$ such that $f(\{x_1, ..., x_i\}) = i$ for $i = 1, ..., j$. Then the ordering $(x_1, ..., x_j, x)$ shows that $X \in \mathscr{F}$. ∎

The next lemma shows that a similar converse statement also holds for subsets of feasible sets.

Lemma 4.2. *Let $A \in \mathscr{F}$ and $X \subseteq A$, then*

(1) $f(X) \geqslant |X|$;

(2) $f(X) = |X|$ *iff* $X \in \mathscr{F}$.

Proof. (1) We use induction on $|X|$. Let A' be a maximal feasible subset of A not containing X. Then $A' \cup X \in \mathscr{F}$. Indeed, by (M3), there exists an element $a \in A - A'$ such that $A' \cup a \in \mathscr{F}$. But by the maximality of A', we have $X \subseteq A' \cup a$ and hence $A' \cup X = A' \cup a \in \mathscr{F}$.

By the submodularity of f, $f(A') + f(X) \geqslant f(A' \cap X) + f(A' \cup X)$. Here $A', A' \cup X \in \mathscr{F}$ and so $f(A') = |A'|$ and $f(A' \cup X) = |A' \cup X| = |A'| + 1$. Further $f(A' \cap X) \geqslant |A' \cap X| = |X| - 1$ by the induction hypothesis. Hence $f(X) \geqslant |X|$.

(2) "If" is trivial by the definition of $\mathscr{F}$. To prove the "only if" part, we use induction on $|X|$ again. Let A' be as in part (1) of the proof. Then equality holds throughout, in particular $f(A' \cap X) = |A' \cap X|$. So by the induction hypothesis, $A' \cap X \in \mathscr{F}$, and then by Lemma 4.1 we have that $X \in \mathscr{F}$. ∎

LEMMA 4.3. *Suppose that* $(E, \mathscr{F})$ *is normal. Then* $f(E) = r(E)$.

Proof. Let B be a basis of $(E, \mathscr{F})$. Then $f(E) \geqslant f(B) = |B| = r(E)$. Let E' be a maximal set such that $B \subseteq E' \subseteq E$ and $f(B) = f(E')$. We show that $E' = E$. Suppose not. Then since $(E, \mathscr{F})$ is normal, there exists a feasible set $A \not\subseteq E'$. Choose A to be a minimal such set, then clearly $A - E' = \{a\}$ for some a. By the submodularity of f, $f(A) + f(E') \geqslant f(A \cap E') + f(A \cup E')$. Here $f(A) = |A|$, $f(E') = |B|$, and $f(A \cap E') \geqslant |A \cap E| = |A| - 1$ by Lemma 4.2. So $f(A \cup E') \leqslant |B| + 1$. By the maximality of E', we have $f(A \cup E') = |B| + 1$.

On the other hand, again by submodularity,

$$f(B \cup a) + f(E') \geqslant f(E' \cup a) + f(B).$$

Here $f(E') = f(B) = |B|$ and hence $f(B \cup a) \geqslant f(E' \cup a)$. Since the reverse is trivial, we have $f(B \cup a) = f(E' \cup a) = f(E' \cup A) = |B| + 1$. But then by Lemma 4.1, $B \cup a \in \mathscr{F}$, which contradicts the assumption that B is a basis. ∎

COROLLARY 4.4. *If* $X \in \mathscr{A}$ *then* $r(X) = f(X)$.

Proof. Apply Lemma 4.3 to the restriction of $(E, \mathscr{F})$ to X. ∎

Now we define the following properties of a greedoid $(E, \mathscr{F})$:

(A) (*Local intersection property*). If $A \subseteq E$, $x, y \in A$, and A, $A - x$, $A - y$ are feasible then $A - x - y$ is feasible.

(B) (*Local union property*). If $A \subseteq E$, x, y, $z \in E - A$, and A, $A \cup x$, $A \cup y$, $A \cup x \cup y \cup z$ are feasible then $A \cup x \cup y$ is feasible.

(C) (*Local augmentation property*). If $A \subseteq E$, x, y, $z \in E - A$, and A, $A \cup x$, $A \cup y \cup z$ are feasible, then one of $A \cup z$, $A \cup x \cup z$, and $A \cup x \cup y \cup z$ is feasible.

Remark. Figure 1 illustrates these properties; it shows the relevant part of the Boolean algebra 2^E. If the full points are all feasible then one of the light points must also be feasible.

THEOREM 4.5. *Every polymatroid greedoid has properties A, B, and C.*

Proof. (1) (Local intersection). By submodularity,

$$f(A - x - y) + f(A) \leqslant f(A - x) + f(A - y).$$

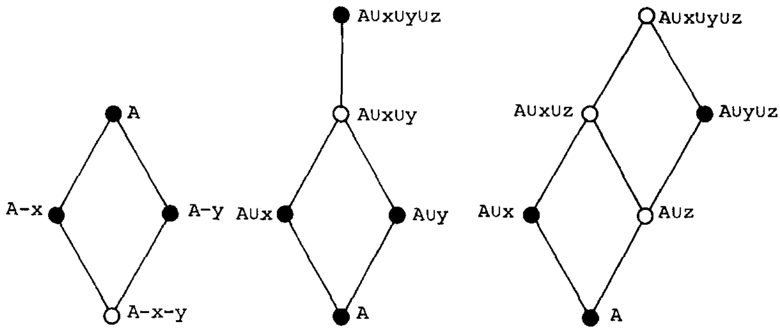

FIG. 1. Property A, property B, and property C.

Here $f(A)=|A|$, $f(A-x)=|A|-1$, and $f(A-y)=|A|-1$ by definition, and $f(A-x-y)\geqslant|A|-2$ by Lemma 4.2. Hence equality must hold, in particular $f(A-x-y)=|A|-2$. Again by Lemma 4.2, this implies that $A-x-y\in\mathscr{F}$.

(2) (Local union). Again by submodularity,

$$f(A)+f(A\cup x\cup y)\leqslant f(A\cup x)+f(A\cup y).$$

Here $f(A)=|A|$, $f(A\cup x)=|A|+1$, $f(A\cup y)=|A|+1$ by hypothesis, and $f(A\cup x\cup y)\geqslant|A|+2$ by Lemma 4.2, since $A\cup x\cup y\subset A\cup x\cup y\cup z\in\mathscr{F}$. Hence we conclude like above.

(3) (Local augmentation). By Lemma 4.2, $f(A\cup z)\geqslant|A|+1$. If equality holds, then $A\cup z\in\mathscr{F}$ by Lemma 4.1. So suppose that $f(A\cup z)\geqslant|A|+2$. Hence $f(A\cup x\cup z)\geqslant|A|+2$. If equality holds, then $A\cup x\cup z\in\mathscr{F}$ by Lemma 4.1, so suppose that $f(A\cup x\cup z)\geqslant|A|+3$. Then $f(A\cup x\cup y\cup z)\geqslant|A|+3$. On the other hand by submodularity

$$f(A\cup x\cup y\cup z)+f(A)\leqslant f(A\cup y\cup z)+f(A\cup x).$$

Since $f(A)=|A|$, $f(A\cup y\cup z)=|A|+2$, $f(A\cup x)=|A|+1$, we get $f(A\cup x\cup y\cup z)\leqslant|A|+3$. So $f(A\cup x\cup y\cup z)=|A|+3$. Lemma 4.1 then yields $A\cup x\cup y\cup z\in\mathscr{F}$. ∎

We state some consequences of these three exchange properties.

LEMMA 4.6. *Every greedoid with properties* (*A*) *and* (*B*) *is an interval greedoid.*

Proof. Let $A\subseteq B\subseteq C\subseteq E$, $x\in E-C$, and suppose that $A\cup x\in\mathscr{F}$ and $C\cup x\in\mathscr{F}$. We claim that $B\cup x\in\mathscr{F}$. Without loss of generality, assume that $B=A\cup b$, and also that C is a minimal feasible superset of B with this property. Let $C = B \cup \{c_1, ..., c_k\}$ so that $B \cup \{c_1, ..., c_i\} \in \mathscr{F}$ for all $1 \leqslant i \leqslant k$. We show by induction on i that $A \cup x \cup \{c_1, ..., c_i\} \in \mathscr{F}$. For suppose this holds for some $0 \leqslant i < k$, then augment $A \cup x \cup \{c_1, ..., c_i\}$ from $B \cup \{c_1, ..., c_{i+1}\}$. The element to add must be either b or c_{i+1}. But if $A \cup x \cup \{c_1, ..., c_i\} \cup b \in \mathscr{F}$ then $C' = A \cup b \cup \{c_1, ..., c_i\} = B \cup \{c_1, ..., c_i\}$ contradicts the minimality of C. So $A \cup x \cup \{c_1, ..., c_{i+1}\} \in \mathscr{F}$. In particular, $A \cup x \cup \{c_1, ..., c_k\} = C \cup x - b \in \mathscr{F}$. Thus $C \in \mathscr{F}$, $C \cup x \in \mathscr{F}$, and $C \cup x - b \in \mathscr{F}$, and so by property (A), $C - b \in \mathscr{F}$. But also $C - c_k \in \mathscr{F}$, so again by property (A), $C - c_k - b \in \mathscr{F}$. But also $(C - c_k - b) \cup b = C - c_k \in \mathscr{F}$ and $(C - c_k - b) \cup x = A \cup x \cup \{c_1, ..., c_{k-1}\} \in \mathscr{F}$, and $(C - c_k - b) \cup x \cup b \cup c_k = C \cup x \in \mathscr{F}$. Hence by property (B), $C - c_k \cup x \in \mathscr{F}$. So $C' = C - c_k$ contradicts the minimality of C again. ∎

LEMMA 4.7. *Let* $(E, \mathscr{F})$ *be a greedoid with properties* (A) *and* (B). *Let* $X, Y, Z \in \mathscr{F}$, $X, Y \subseteq Z$. *Then* $X \cap Y \in \mathscr{F}$ *and* $X \cup Y \in \mathscr{F}$.

Proof. Let U be a maximal feasible set such that $X \subseteq U \subseteq X \cup Y$. We claim that $U = X \cup Y$. Suppose not. Let y be the first element of Y not in U in some feasible ordering of Y; thus there exists a feasible set $Y' \subseteq Y \cap U$ such that $Y' \cup y \in \mathscr{F}$. By augmenting U repeatedly from Z, we obtain a feasible set Z' such that $U \subseteq Z' \subseteq Z - y$ and $Z' \cup y \in \mathscr{F}$. Thus $Y' \subseteq U \subseteq Z'$ are feasible, $Y' \cup y \in \mathscr{F}$, $Z' \cup y \in \mathscr{F}$, and thus by the interval property, $U \cup y \in \mathscr{F}$. But this contradicts the maximality of U. Hence $X \cup Y \in \mathscr{F}$.

To prove the other assertion, we use induction on $|X \cup Y|$. We may assume that $X \not\subseteq Y$ and $Y \not\subseteq X$. Augment X from $X \cup Y$ to get a feasible set X' such that $X \subseteq X' \subseteq X \cup Y$ and $|X'| = |X \cup Y| - 1$. Let $X' = X \cup Y - y$. Similarly, augment Y to a set $Y' = X \cup Y - x \in \mathscr{F}$. Then by property (A), $U := X' \cap Y' = X \cup Y - x - y \in \mathscr{F}$. But then by induction hypothesis, $X \cap U = X - x \in \mathscr{F}$. Again by the induction hypothesis, $(X - x) \cap Y = X \cap Y \in \mathscr{F}$. ∎

Remark. Observe that, conversely, properties A and B follow trivially from the assertion of the above lemma.

COROLLARY 4.8. *For a full greedoid* $(E, \mathscr{F})$ *the following properties are equivalent*:

(a) $(E, \mathscr{F})$ *is a poset greedoid*;

(b) $\mathscr{F}$ *is closed under union and intersection*;

(c) $(E, \mathscr{F})$ *has properties* (A) *and* (B);

(d) $(E, \mathscr{F})$ *is a polymatroid greedoid.*

Proof. We have seen that (a) $\Rightarrow$ (d) $\Rightarrow$ (c) $\Rightarrow$ (b). To show that

(b) $\Rightarrow$ (a), let for each $x \in E$, $I(x) = \bigcap\{A \in \mathscr{F} : x \in A\}$. By hypothesis, $I(x) \in \mathscr{F}$.

Define a relation $\leqslant$ by $x \leqslant y$ iff $x \in I(y)$. Trivially, $x \in I(y)$ is equivalent to $I(x) \subseteq I(y)$. Hence $\leqslant$ is transitive. Furthermore, $x \leqslant y$ and $y \leqslant x$ implies $x = y$, because there exits a feasible ordering of E, and if $x \neq y$ and say x comes before y in this ordering then the beginning section of this ordering up to x is a feasible set containing x but not y and so $y \geqslant x$.

So $\leqslant$ is a partial order. It remains to show that $\mathscr{F}$ is just the family of ideals in $(E, \leqslant)$. First, let $U \subseteq E$ be an ideal. Then clearly

$$U = \bigcup \{I(u) : u \in U\} \in \mathscr{F}.$$

Conversely, let $V \in \mathscr{F}$. To show that V is an ideal, let $x \in V$ and $y \leqslant x$. Then $y \in I(x) \subseteq V$. So V is an ideal. ∎

The results of Corollary 4.8 give rise to another definition: We call a greedoid $(E, \mathscr{F})$ a *local poset greedoid* if it has properties (A) and (B). As a direct consequence of Corollary 4.8 we get that $(E, \mathscr{F})$ is a local poset greedoid iff for $A, B, C \in \mathscr{F}$ with $A, B \subseteq C$ we have $A \cup B$, $A \cap B \in \mathscr{F}$. This means that the restriction of a local poset greedoid to any feasible set C results in a poset greedoid. This is the motivation for the name.

The following theorem gives an *excluded minor* characterization of local poset greedoids. (Note that both local poset greedoids and polymatroid greedoids are closed under taking minors.) So far we characterized subclasses of greedoids by structural properties. This is the first characterization via forbidden minors. It might increase the structural knowledge about greedoids to get similar excluded minor results for other classes of greedoids.

THEOREM 4.9. *A greedoid $(E, \mathscr{F})$ is a local poset greedoid iff it does not contain any of the following three types of greedoids as a minor*:

(1) $E = \{x, y, z\}$, $\mathscr{F} = 2^E - \{\{z\}\}$,

(2) $E = \{x, y, z\}$, $\mathscr{F} = 2^E - \{\{x, y\}\}$,

(3) *Let $(E', \mathscr{F}')$ be a poset greedoid, $x, y \notin E'$, $E := E' \cup \{x, y\}$, $\mathscr{F} = \{\emptyset, E\} \cup \{A \cup x, A \cup y : A \in \mathscr{F}'\}$.*

Proof. (only if) In the case of minors (1) and (3) we have E, $E - x$, $E - y \in \mathscr{F}$, but $E - x - y \notin \mathscr{F}$ which contradicts property (A). In the case of minor (2), we have $\{x\}$, $\{y\}$, $\{x, y, z\} \in \mathscr{F}$ but $\{x, y\} \notin \mathscr{F}$, contradicting property (B).

(if) Suppose that $(E, \mathscr{F})$ is not a local poset greedoid. We may assume that all minors of $(E, \mathscr{F})$ are local poset greedoids. According to properties (A) and (B) which characterize local poset greedoids, we have two cases:

Case 1 (Property B). There exists $A \in \mathscr{F}$ with $A \cup x$, $A \cup y$, $A \cup x \cup y \cup z \in \mathscr{F}$ but $A \cup x \cup y \notin \mathscr{F}$. By minimality we may assume $A = \varnothing$, $E = \{x, y, z\}$, i.e., $\{x\}$, $\{y\}$, $\{x, y, z\} \in \mathscr{F}$, but $\{x, y\} \notin \mathscr{F}$. By greedoid property we have $\{x, z\}$, $\{y, z\} \in \mathscr{F}$. So either $\mathscr{F} = 2^E - \{\{x, y\}\}$ or $\mathscr{F} = 2^E - \{\{z\}, \{x, y\}\}$. In the second case we have minor (3) with $E' = \{z\}$ and $\mathscr{F}' = \{\varnothing, \{z\}\}$.

Case 2 (Property A). There exists $A \in \mathscr{F}$ and $x, y \in A$ such that $A - x \in \mathscr{F}$, $A - y \in \mathscr{F}$ but $A - x - y \notin \mathscr{F}$. By minimality we may assume $A = E$. Also by minimality we assume for all $z \in E - x - y$ that $\{z\} \notin \mathscr{F}$, since otherwise we can contract z and get a smaller example. Thus, we assume for the rank $r(E - x - y) = 0$. But since $E - y \in \mathscr{F}$, we have that $E - y$ contains a feasible element, and so $\{x\} \in \mathscr{F}$. Similarly $\{y\} \in \mathscr{F}$. Now, we distinguish two subcases:

Case 2a. There exists a $z \in E - x - y$ such that $E - z \in \mathscr{F}$. Then $E - x - z \in \mathscr{F}$, for otherwise we have E, $E - x$, $E - z \in \mathscr{F}$ but $E - x - z \notin \mathscr{F}$ and $y \in E - x - z$ with $\{z\} \in \mathscr{F}$, which contradicts the above, if y and z are interchanged. Similarly, $E - y - z \in \mathscr{F}$. But then $E - x - y - z \in \mathscr{F}$, since otherwise $E - z$, $E - z - x$, $E - z - y \in \mathscr{F}$ but $E - z - x - y \notin \mathscr{F}$. This means that deletion of z leads to a smaller example $(E - z, \mathscr{F} \backslash z)$. But since we know that $r(E - x - y) = 0$, we have $E - x - y - z = \varnothing$, i.e., $E = \{x, y, z\}$. So $\mathscr{F} = 2^E - \{\{z\}\}$, and we have the case of minor (1).

Case 2b. For all $z \in E - x - y$ we have $E - z \notin \mathscr{F}$. Then no feasible set other than E contains both x and y, since otherwise this set would have a feasible superset of the form $E - z$, $z \neq x, y$. Consider $E' := E - x - y$ and $\mathscr{F}' := \mathscr{F}/x \backslash y$. Then $(E', \mathscr{F}')$ is a full greedoid, since $E - y \in \mathscr{F}$. Furthermore, $(E', \mathscr{F}')$ is a local poset greedoid by the minimality of $(E, \mathscr{F})$. So $(E', \mathscr{F}')$ is a poset greedoid.

Claim. $\mathscr{F}/y \backslash x = \mathscr{F}/x \backslash y$. For, let $A \in \mathscr{F}/x \backslash y$, this means that $A \cup x \in \mathscr{F}$, $y \notin A$. Augment y from $A \cup x$ to get a set $B \cup y \in \mathscr{F}$ with $|B \cup y| = |A \cup x|$. Then $B \cup y \neq E$, so $x \notin B \cup y$, and hence $B = A$. Thus $A \cup y \in \mathscr{F}$, i.e., $A \in \mathscr{F}/y \backslash x$. This proves the claim.

Let

$$\bar{\bar{\mathscr{F}}} := \{\varnothing, E\} \cup \{A \cup x, A \cup y \colon A \in \mathscr{F}'\}.$$

We claim that $\mathscr{F} = \bar{\bar{\mathscr{F}}}$. First, we show for $X \in \mathscr{F}$ that $X \in \bar{\bar{\mathscr{F}}}$. If $X = \varnothing, E$ we are done. So, suppose $X \neq \varnothing, E$. Then $|X \cap \{x, y\}| = 1$ by the above. Let, say, $X \cap \{x, y\} = \{x\}$. Then $X - x \in \mathscr{F}/x \backslash y = \mathscr{F}'$. So $X = (X - x) \cup \{x\} \in \bar{\bar{\mathscr{F}}}$. Conversely, let $X \in \bar{\bar{\mathscr{F}}}$. Again, the cases $X = \varnothing, E$ are trivial. So suppose that $X = A \cup x$, $A \in \mathscr{F}'$ (say). Then since $\mathscr{F}' = \mathscr{F}/x \backslash y$, if follows that $X \in \mathscr{F}$. Thus, $\mathscr{F} = \bar{\bar{\mathscr{F}}}$ and we have the case of minor (3). ∎

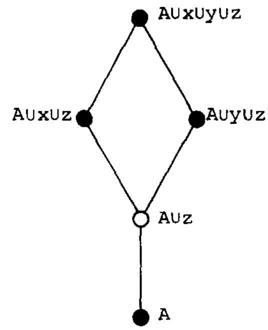

FIGURE 2

COROLLARY 4.10. *An interval greedoid is a local poset greedoid iff it does not contain the greedoid* (1) *in Theorem* 4.9 *as a minor.*

Proof. Greedoids (2) and (3) do not have the interval property, while the interval property is trivially preserved under taking minors. ∎

COROLLARY 4.11. *For an interval greedoid* $(E, \mathscr{F})$, *the following are equivalent*:

(a) $(E, \mathscr{F})$ *is a local poset greedoid*;

(b) $(E, \mathscr{F})$ *has the local intersection property A.*

(c) $(E, \mathscr{F})$ *has the following property* A′:

(A′) *If* $A, A \cup \{x, z\}, A \cup \{y, z\}, A \cup \{x, y, z\} \in \mathscr{F}$ *then* $A \cup \{z\} \in \mathscr{F}$ (*see Fig.* 2).

One may wish to get a similar excluded minor characterization for polymatroid greedoids. It follows from property (C) that no polymatroid greedoid can have the following minor:

(4) $E = \{x, y, z\}, \mathscr{F} = \{\varnothing, \{x\}, \{y\}, \{x, y\}, \{x, z\}\}$

(this is the directed branching greedoid of a transitively oriented triangle). One may conjecture that the exclusion of minors (1)–(4) already characterizes polymatroid greedoids. It is easy to see that a greedoid without minors (1)–(4) has properties (A), (B), and (C).

THEOREM 4.12. *Let* (E, f) *be a polymatroid,* $(E, \mathscr{F})$ *the associated greedoid and* $(E, \mathscr{M})$ *the matroid induced by* (E, f). *Then*

$$\mathscr{F} = \mathscr{A} \cap \mathscr{M}.$$

If $\mathscr{L}_{\mathscr{F}}$, $\mathscr{L}_{\mathscr{A}}$, and $\mathscr{L}_{\mathscr{M}}$ denote the hereditary languages corresponding to $\mathscr{F}$, $\mathscr{A}$, and $\mathscr{M}$ then

$$\mathscr{L}_{\mathscr{F}} = \mathscr{L}_{\mathscr{A}} \cap \mathscr{L}_{\mathscr{M}}.$$

Proof. By Lemma 4.2, $\mathscr{F} \subseteq \mathscr{M}$ and hence $\mathscr{F} \subseteq \mathscr{A} \cap \mathscr{M}$. Next we show that if $X, Y \in \mathscr{F}$ and $X \cup Y \in \mathscr{M}$ then $X \cup Y \in \mathscr{F}$. Let $y \in Y$ be such that $Y - y \in \mathscr{F}$, then by induction we may assume that $X \cup (Y - y) \in \mathscr{F}$, and also that $X \cup (Y - y) \neq X \cup Y$, i.e., $y \notin X$. By submodularity,

$$f(X \cup Y) + f(X \cap Y) \leqslant f(X) + f(Y).$$

Here $f(X \cup Y) \geqslant |X \cup Y|$ and $f(X \cap Y) \geqslant |X \cap Y|$ as $X \cup Y \in \mathscr{M}$. On the other hand, $f(X) = |X|$ and $f(Y) = |Y|$ as $X, Y \in \mathscr{F}$. So equality must hold, in particular $f(X \cup Y) = |X \cup Y|$. Since $X \cup Y - y \in \mathscr{F}$, this implies that $X \cup Y \in \mathscr{F}$ by Lemma 4.1.

Now $\mathscr{A} \cap \mathscr{M} \subseteq \mathscr{F}$ follows easily: let $A \in \mathscr{A} \cap \mathscr{M}$. Then $A = A_1 \cup \cdots \cup A_k$, where $A_1, ..., A_k \in \mathscr{F}$. But then it follows by induction that $A_1 \cup \cdots \cup A_i \in \mathscr{F}$ for all $i = 1, ..., k$, in particular $A \in \mathscr{F}$. The second assertion of the theorem follows trivially from the first. ∎

Remark. Theorem 4.12 shows that every polymatroid greedoid is the intersection of a matroid and a shelling structure. Crapo (personal communication) asked if every interval greedoid has such a representation. This is, however, not true. Consider the following example:

$$E = \{a, b, c, u, v\},$$
$$\mathscr{F} = \{X \subseteq E \colon |X| \leqslant 3, X \neq \{u\}, \{v\}, \{u, v\}, \{a, u, v\}, \{b, u, v\}\}.$$

It is trivial to check that $(E, \mathscr{F})$ is an interval greedoid. However, $\mathscr{F}$ cannot be represented as $\mathscr{F} = \mathscr{M} \cap \mathscr{F}'$, where $(E, \mathscr{M})$ is a matroid and $(E, \mathscr{F}')$ is a shelling structure. In fact, consider the set $\{a, u, v\}$. Since $\{a, u\} \in \mathscr{F} \subseteq \mathscr{F}'$ and $\{a, v\} \in \mathscr{F} \subseteq \mathscr{F}'$, we have $\{a, u, v\} \in \mathscr{F}'$. But $\{a, u, v\} \notin \mathscr{F}$ and so $\{a, u, v\} \notin \mathscr{M}$. Similarly, $\{b, u, v\} \notin \mathscr{M}$. But $\{u, v\} \subseteq \{c, u, v\} \in \mathscr{F} \subseteq \mathscr{M}$ and hence $\{u, v\} \in \mathscr{M}$. Thus $\{a, b, u, v\}$ is contained in the $\mathscr{M}$-closure of $\{u, v\}$. But this is a contradiction since $\{a, b, u\} \in \mathscr{F} \subseteq \mathscr{M}$.

It seems to be an interesting open problem to characterize those greedoids which arise as the intersection of a matroid with a shelling structure. Some examples of such greedoids follow in the next section.

A polymatroid is not uniquely determined by its associated greedoid. In fact, the next theorem shows that every polymatroid greedoid can be defined by a 2-polymatroid. Let (E, f) be a polymatroid and $k \in \mathbb{Z}_+$. Let the *k-truncation* $f^{(k)}$ of f be defined by

$$f^{(k)}(X) = \min_{Y \subseteq X} \{f(Y) + k\,|X - Y|\}$$

It is easy to see (cf. Lovász [15]) that $f^{(k)}$ is a polymatroid function for all $k \geqslant 0$. Note that the 1-truncation of f gives just the matroid induced by f. Theorem 4.12 describes how $(E, \mathscr{F})$ can be obtained from the 1-truncation of f. The next result shows that the situation is even simpler if we consider 2-truncation.

THEOREM 4.13. *Let (E, f) be a polymatroid and $(E, f^{(2)})$ its 2-truncation. Then the greedoid associated with (E, f) is the same as the greedoid associated with $(E, f^{(2)})$.*

Proof. Let $(E, \mathscr{F})$ and $(E, \mathscr{F}^{(2)})$ be these two greedoids. First, let $X \in \mathscr{F}$. Then by Lemma 4.2,

$$\begin{aligned} f^{(2)}(X) &= \min_{Y \subseteq X} \{f(Y) + 2\,|X - Y|\} \\ &\geqslant \min_{Y \subseteq X} \{|Y| + 2\,|X - Y|\} \geqslant |X|. \end{aligned}$$

But trivially,

$$f^{(2)}(X) \leqslant f(X) = |X|,$$

so $f^{(2)}(X) = |X|$. Furthermore, X has an ordering $\{x_1,..., x_k\}$ such that $\{x_1,..., x_i\} \in \mathscr{F}$ for all $1 \leqslant i \leqslant k$ and hence $f^{(2)}(\{x_1,..., x_i\}) = i$. Thus $\{x_1,..., x_k\} \in \mathscr{F}^{(2)}$.

Second, let $X \in \mathscr{F}^{(2)}$. Then

$$\begin{aligned} |X| = f^{(2)}(X) &= \min_{Y \subseteq X} \{f(Y) + 2\,|X - Y|\} \\ &\geqslant \min_{Y \subseteq X} \{f^{(2)}(Y) + 2\,|X - Y|\} \\ &\geqslant \min_{Y \subseteq X} \{|Y| + 2\,|X - Y|\} \\ &\geqslant |X|. \end{aligned}$$

Hence equality holds throughout, in particular the minimum is attained for $Y = X$ only, and we have $f(X) = f^{(2)}(X) = |X|$. Hence $X \in \mathscr{F}$ follows as above. ∎

Remarks. (1) For poset greedoids, the polymatroid function we con-

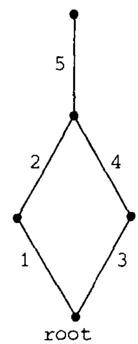

FIGURE 3

structed is not a 2-polymatroid. In fact, it is the largest among all polymatroid functions representing the given poset greedoid. For let $I(X)$ denote the ideal generated by $X \subseteq E$. Then, for any polymatroid (E, f) whose associated greedoid is the given poset greedoid, we have

$$f(X) \leqslant f(I(X)) = |I(X)|,$$

since $I(X)$ is feasible.

(2) We do not know whether there is a unique maximal one among all polymatroid functions defining a given greedoid. The following upper bound on any such function can easily be derived from Corollary 4.4:

$$f(X) \leqslant \min\{r(A): X \subseteq A \in \mathscr{A}\}.$$

Unfortunately, the right-hand side does not define a submodular set function in general. For example, consider the undirected branching greedoid of the graph in Fig. 3. Setting

$$g(X) = \min\{r(A): X \subseteq A \in \mathscr{A}\},$$

we can easily calculate that $g(125) = 3$, $g(345) = 3$, $g(12345) = 4$, $g(5) = 3$. Hence g is not submodular.

We conclude this section with a few further properties of polymatroid greedoids. Most of these will follow already from properties A, B, and C established above, and we formulate our results accordingly. This slightly more general approach may be useful in a possible proof of the conjecture that A, B, and C characterize polymatroid greedoids.

LEMMA 4.14. *Let (E, f) be a polymatroid, $A \in \mathscr{F}$ and $x \in E - A$ such that $f(A \cup x) = f(A)$ and $f(x) > 0$. Then A has a subset $A' \in \mathscr{F}$ such that $A' \cup x \in \mathscr{F}$.*

Proof. Let $(a_1, ..., a_n)$ be a feasible ordering of A. The difference

$$d_i = f(\{a_1, ..., a_i, x\}) - f(\{a_1, ..., a_i\}) = f(\{a_1, ..., a_i, x\}) - i$$

is monotone decreasing in i, as f is submodular. Furthermore, it decreases by at most one:

$$d_i - d_{i+1} = f(\{a_1, ..., a_i, x\}) - f(\{a_1, ..., a_{i+1}, x\}) + 1 \leqslant 1.$$

Since $d_n = 0$ and $d_0 > 0$, we must have an i with $d_i = 1$. But then $\{a_1, ..., a_i, x\} \in \mathscr{F}$ by Lemma 4.1. ∎

LEMMA 4.15. *Let $(E, \mathscr{F})$ be a greedoid with properties* A, B, *and* C. *Let $a_1 \cdots a_n \in \mathscr{L}$, $x \in E$, and assume that there exists a subset $B \subseteq \{a_1, ..., a_n\}$ such that $B \cup x \in \mathscr{F}$. Then there exists an i, $0 \leqslant i \leqslant n$, such that $\{a_1, ..., a_i, x\} \in \mathscr{F}$.*

Proof. Let i be the last index such that $\{a_1, ..., a_i\}$ has a subset B' such that $B' \cup x \in \mathscr{F}$. Choose $B' \in \mathscr{F}'$ maximal. If $B' = \{a_1, ..., a_i\}$, we are done. Suppose that $B' \neq \{a_1, ..., a_i\}$, and let j be the least index with $a_j \notin B'$. Let $A = B' \cap \{a_1, ..., a_{i-1}\} = B' - a_i$, then $A \in \mathscr{F}$ by Lemma 4.7, and similarly $A \cup a_j = A \cup \{a_1, ..., a_j\} \in \mathscr{F}$, $A \cup a_j \cup a_i = B' \cup \{a_1, ..., a_j\} \in \mathscr{F}$, and $A \cup a_i \cup x = B' \cup x \in \mathscr{F}$. Hence by property C, one of the sets $A \cup x$, $A \cup a_j \cup x$, and $A \cup a_i \cup a_j \cup x$ is feasible. But the first two cannot be feasible by the minimality of i, and the last cannot be feasible by the maximality of B'. ∎

LEMMA 4.16. *Let $(E, \mathscr{F})$ be a greedoid with properties* A, B, *and* C. *Let B_1 and B_2 be two bases. Then for every $b_1 \in B_1$ there exists a $b_2 \in B_2$ such that $B_2 - b_2 \cup b_1$ is a basis.*

Proof. Let C be a common feasible subset of B_1 and B_2. We use induction on $|B_2 - C|$. If $C = B_2$ then the assertion is clear. So suppose $C \neq B_2$ and choose $b \in B_2 - C$ such that $C \cup b \in \mathscr{F}$. Then augment $C \cup b$ from B_1 to a basis B_3. If $b_1 \in B_3$ then we are done since $C \cup b \subseteq B_2 \cap B_3$ and so the induction hypothesis can be applied. So suppose that $b_1 \notin B_3$. Consider a feasible ordering $\beta = c_1 \cdots c_k b d_1 \cdots d_t$ of B_3, where $C = \{c_1, ..., c_k\}$. Then $\{c_1, ..., c_k, d_1, ..., d_t, b_1\} = B_1 \in \mathscr{F}$, so by Lemma 4.14, b_1 can be added to some beginning section of β.

Case 1. $c_1 \cdots c_i b_1 \in \mathscr{L}$ for some $0 \leqslant i \leqslant k$. Then augment $\{c_1, ..., c_i, b_1\}$ from B_2 to get a basis $B_4 = B_2 \cup b_1 - b_2$ with some $b_2 \in B_2$.

Case 2. $c_1 \cdots c_k b d_1 \cdots d_j b_1 \in \mathscr{L}$ for some $0 \leqslant j \leqslant t-1$. Then augment $\{c_1, ..., c_k, b, d_1, ..., d_j, b_1\}$ from B_2 to get a basis B_5 which contains $C \cup b \cup b_1$. Replace B_3 by B_5 and conclude as above. ∎

Remark. Note that the assertion of Lemma 4.16 is not equivalent to the dual basis exchange property of matroids. In the latter we require that for every $b_1 \in B_1 \backslash B_2$ there is a $b_2 \in B_2 \backslash B_1$ such that $B_2 - b_2 \cup b_1$ is a basis.

COROLLARY 4.17. *Let* $(E, \mathscr{F})$ *be a normal greedoid with properties* A, B, *and* C, *and* $b_1 \cdots b_n$ *a basic word and* $x \in E$. *Then there exists an* i, $0 \leqslant i \leqslant n-1$ *such that* $b_1 \cdots b_i x \in \mathscr{L}$.

Proof. Lemma 4.16 implies that $\{b_1, \ldots, b_n\}$ has a subset A such that $A \cup x \in \mathscr{F}$. Hence Lemma 4.15 can be applied. ∎

THEOREM 4.18. *Let* $(E, \mathscr{F})$ *be a normal greedoid with properties* A, B, *and* C. *Then* $\mathscr{A} = \mathscr{R}$.

Proof. $\mathscr{A} \subseteq \mathscr{R}$, since $\mathscr{F} \subseteq \mathscr{R}$ and $\mathscr{R}$ is closed under union by the interval property.

Conversely, let $X \in \mathscr{R}$ and A a basis of X. Extend A to a basis B and let $a_1 \cdots a_k b_1 \cdots b_t$ be a feasible ordering of B such that $\{a_1, \ldots, a_k\} = A$. Let $x \in X$, then by Corollary 4.17, either there exists an i, $0 \leqslant i \leqslant k$ such that $a_1 \cdots a_i x \in \mathscr{L}$ or there exists a j, $1 \leqslant j \leqslant t$ such that $a_1 \cdots a_k b_1 \cdots b_j x \in \mathscr{L}$. But the second possibility cannot occur since then

$$|\{a_1, \ldots, a_k, b_1, \ldots, b_j, x\} \cap X| \geqslant k + 1 > r(X),$$

contradicting the assumption that $X \in \mathscr{R}$. So $A_x = \{a_1, \ldots, a_i, x\}$ is a feasible subset of X containing x. Hence $X = \bigcup \{A_x : x \in X\} \in \mathscr{A}$. ∎

5. INTERSECTIONS OF SHELLING STRUCTURES WITH MATROIDS AND POLYMATROIDS

As it was shown in the last section, local poset greedoids are generalizations of polymatroid greedoids. In this section we discuss some general constructions of greedoids which among others yield some local poset greedoids which are not polymatroid greedoids. It is interesting to remark that these examples are all closely related to matroids in the sense that they arise by intersecting matroids with certain shelling structures.

(1) Let D be a digraph, $r \in V(D)$ and $E = V(D)$. Let $\mathscr{F}$ be the set of all arborescences in D rooted at r. Then $(E, \mathscr{F})$ is a greedoid, which we call the *directed branching greedoid.* Note that obviously $(E, \mathscr{F})$ is a local poset greedoid.

(2) A natural way of generalization of the above example is to

replace the digraph by a hypergraph $H=(V, E)$ (where E is a family of subsets of V), in which a *head* $h(e)\in e$ is assigned to each edge $e\in E$. Define

$$\mathscr{L}=\left\{e_1\cdots e_k\text{: for all } 1\leqslant j\leqslant k,\ e_j-\left(\bigcup_{i=1}^{j-1} e_i\right)=\{h(e_j)\}\right\}.$$

Then $(E, \mathscr{L})$ is a local poset greedoid; this fact will follow from the more general constructions below. We call $(E, \mathscr{L})$ a *hypergraph branching greedoid.*

(3) Let (E_0, f) be a polymatroid, $E\subseteq E_0$, and assume that we have assigned an element $t(e)\in E_0$ to every $e\in E$ so that $f(t(e), e)=f(e)$. Let

$$\begin{aligned}\mathscr{L}=\{e_1\cdots e_k\text{: for all } 1\leqslant j\leqslant k,\\ f(e_1,..., e_j)=j \text{ and } f(e_1,..., e_{j-1}, t(e_j))=j-1\}.\end{aligned}$$

Then $(E, \mathscr{L})$ is a local poset greedoid, which we call a *polymatroid branching greedoid.*

LEMMA 5.1. *The polymatroid branching greedoid is a greedoid.*

Proof. Let $e_1\cdots e_k\in\mathscr{L}$, $f_1\cdots f_l\in\mathscr{L}$, and $k>l$. Just like in the proof of Theorem 2.1, we find a j, $1\leqslant j\leqslant k$, such that $f(f_1,..., f_l, e_j)=l+1$ but $f(f_1,..., f_l, e_1,..., e_{j-1})=l$. Then by monotonicity and submodularity,

$$\begin{aligned}f(f_1,..., f_l, t(e_j))&\leqslant f(f_1,..., f_l, e_1,..., e_{j-1}, t(e_j))\\ &\leqslant f(f_1,..., f_l, e_1,..., e_{j-1})+f(e_1,..., e_{j-1}, t(e_j))-f(e_1,..., e_{j-1})\\ &=l+(j-1)-(j-1)=l.\end{aligned}$$

So $f_1\cdots f_l e_j\in\mathscr{L}$. ∎

The next results shows a connection between polymatroid branching greedoids and polymatroid greedoids.

LEMMA 5.2. *The restriction of a polymatroid branching greedoid $(E, \mathscr{F})$ to any feasible set is identical with the restriction of the corresponding polymatroid greedoid $(E, \mathscr{F}_f)$.*

Proof. Let $A\in\mathscr{F}$ and $B\subseteq A$. Trivially, if $B\in\mathscr{F}$ then $B\in\mathscr{F}_f$. Conversely, suppose that $B\in\mathscr{F}_f$, we claim that $B\in\mathscr{F}$. Let $e_1\cdots e_n\in\mathscr{L}$ be an ordering of A and let $B=\{e_{i_1},..., e_{i_k}\}$ $(i_1<\cdots<i_k)$. Since the restriction of $\mathscr{L}_f$ to A is a greedoid defined by a partial ordering of A, B is an ideal of this partial order, and $e_{i_1}\cdots e_{i_k}$ does not conflict with this partial order, we have $e_{i_1}\cdots e_{i_k}\in\mathscr{L}_f$, i.e.,

$$f(e_{i_1},..., e_{i_\nu})=\nu \qquad (\nu=1,..., k).$$

Furthermore, we have by submodularity

$$f(e_{i_1},\dots, e_{i_{\nu-1}}, t(e_{i_\nu}), e_{i_\nu}) + f(e_1,\dots, e_{i_\nu - 1}, t(e_{i_\nu}))$$
$$\geqslant f(e_1,\dots, e_{i_\nu - 1}, t(e_{i_\nu}), e_{i_\nu}) + f(e_{i_1},\dots, e_{i_{\nu-1}}, t(e_{i_\nu})).$$

Here

$$f(e_{i_1},\dots, e_{i_{\nu-1}}, t(e_{i_\nu}), e_{i_\nu}) = \nu,$$
$$f(e_1,\dots, e_{i_\nu - 1}, t(e_{i_\nu})) = i_\nu - 1,$$
$$f(e_1,\dots, e_{i_\nu - 1}, t(e_{i_\nu}), e_{i_\nu}) = i_\nu.$$

So

$$f(e_{i_1},\dots, e_{i_{\nu-1}}, t(e_{i_\nu})) \leqslant \nu + (i_\nu - 1) - i_\nu = \nu - 1.$$

By monotonicity we have equality. Hence $e_{i_1},\dots, e_{i_k} \in \mathscr{L}$, and so $B \in \mathscr{F}$. ∎

COROLLARY 5.3. *Every polymatroid branching greedoid is a local poset greedoid.*

We mention some special cases of this construction. First, let (E, f) be a polymatroid and $E_0 = E \cup \{t\}$. Define $f(X) = f(X \cap E)$ for all $X \subseteq E_0$. Let $t(e) = t$ for all $e \in E$. Then the polymatroid branching greedoid is just the polymatroid greedoid defined by (E, f).

Second, let H be a hypergraph as in example (2). Define $t(e) = e - h(e)$ for $e \in E$, $E_0 = E \cup \{t(e)\colon e \in E\}$, and $f(X) = |\bigcup_{e \in X} e|$ for $X \subseteq E_0$. Then the polymatroid branching greedoid defined by (E_0, f) is the hypergraph branching greedoid.

Third, we obtain an interesting special case when E is a set of subspaces of a linear space, and for each $e \in E$, $t(e)$ is a subspace of e with $\dim t(e) = \dim e - 1$. Then $e_1 \cdots e_k \in \mathscr{L}$ iff for all $1 \leqslant j \leqslant k$,

$$e_j \cap \operatorname{Span}(e_1,\dots, e_{j-1}) = t(e_j).$$

A further class of polymatroid branchings is given by *minimal F-geometries.* Let $(E, \mathscr{F}, \leqslant)$ be a minimal F-geometry. For each $X \subseteq E$, let $I(X)$ denote the ideal generated by X. Let $\rho(X) = r(I(X))$ be the Faigle rank function. Then (E, ρ) is a polymatroid. Extend this polymatroid by a new element y_I parallel to I for each subset I (cf. Lovász [15]). More precisely, let

$$E_0 = E \cup \{y_I\colon I \subseteq E\}$$

and

$$f(X)=\rho\left((X \cap E) \cup\left(\bigcup\{I: y_I \in X\}\right)\right).$$

Define, for each $e \in E$, $t(e)=y_{I(e)-e}$. Then t and f define a polymatroid branching greedoid on E, which we denote temporarily by $(E, \mathscr{F}')$.

We show that $\mathscr{F}'=\mathscr{F}$. First, let $x_1 \cdots x_k$ be a feasible word in $\mathscr{F}$. Then by the results of Faigle [6], $f(\{x_1,..., x_j\})=\rho(\{x_1,..., x_j\})=j$ for all $1 \leqslant j \leqslant k$. Furthermore

$$\begin{aligned} f(\{x_1,..., x_{j-1}, t(x_j)\}) &= \rho(\{x_1,..., x_{j-1}\} \cup (I(x_j)-x_j)) \\ &= r(I(x_1,..., x_j)-x_j). \end{aligned}$$

We want to show that this rank is $j-1$. If not, then $r(I(x_1,..., x_j)-x_j)=j$. Hence, $x_1 \cdots x_{j-1}$ can be augmented by some $y \in I(x_1,..., x_j)-x_j$, to get a feasible word $x_1 \cdots x_{j-1} y$. Obviously, $y<x_j$, which contradicts the definition of minimal F-geometry.

Thus $r(I(x_1 \cdots x_j)-x_j)=j-1$ for all $1 \leqslant j \leqslant k$, and hence $x_1 \cdots x_j$ is feasible in $(E, \mathscr{F}')$. So $\mathscr{F} \subseteq \mathscr{F}'$. The reverse inclusion follows similarly. This proves that minimal F-geometries are indeed polymatroid branchings.

It is interesting to remark that by Section 3, Example 4, maximal F-geometries are polymatroid greedoids and hence also polymatroid branchings. One might expect that all F-geometries are polymatroid branchings. This is, however, not the case, as shown in Korte and Lovász [13].

We saw in Theorem 4.12 that every polymatroid greedoid can be represented as the intersection of a matroid with a shelling structure. Next we construct a rather general class of greedoids based on this idea. However, they are not local posets any more.

Let $(E, \mathscr{F})$ be a greedoid with closure operator σ and $\mathscr{H} \subseteq 2^E$. We say that $\mathscr{H}$ is *quasimodular* (with respect to $(E, \mathscr{F})$) if the following conditions hold:

(P1) $X \in \mathscr{H}$, $X \subseteq Y$ implies $Y \in \mathscr{H}$ ($\mathscr{H}$ is co-hereditary)

(P2) $\sigma(X) \in \mathscr{H}$ implies $X \in \mathscr{H}$.

The set-system $\mathscr{H}$ is called *modular* (with respect to $(E, \mathscr{F})$) if in addition to (P1) and (P2), the following holds:

(P3) If $A \subseteq E$, $x, y \in E-A$, $x \neq y$ and A, $A \cup x$, $A \cup y$, $A \cup x \cup y \in \mathscr{F}$, $A \cup x \in \mathscr{H}$, $A \cup y \in \mathscr{H}$, then $A \in \mathscr{H}$.

If $(E, \mathscr{F})$ is a matroid, then a modular system is essentially equivalent to a *modular cut* as defined by Crapo [2]. A modular cut is a family $\mathscr{D}$ of flats of the matroid such that

(D1) if $X \subseteq Y$ are flats and $X \in \mathscr{D}$ then $Y \in \mathscr{D}$;

(D2) if $X, Y \in \mathscr{D}$ and $r(X) + r(Y) = r(X \cup Y) + r(X \cap Y)$ then $X \cap Y \in \mathscr{D}$.

It is easy to see that if $\mathscr{D}$ satisfies (D1) then $\mathscr{H} = \{X \subseteq E: \sigma(X) \in \mathscr{D}\}$ satisfies (P1) and (P2). If in addition, $\mathscr{D}$ satisfies (D2) then $\mathscr{H}$ satisfies (P3) as well. Conversely, if $\mathscr{H}$ is a set-system satisfying (P1) and (P2) then $\mathscr{D} = \{\sigma(X): X \in \mathscr{H}\}$ satisfies (D1) (note that $\mathscr{D} \subseteq \mathscr{H}$). If in addition $\mathscr{H}$ satisfies (P3) then $\mathscr{D}$ satisfies (D2). Suppose not, then select flats X and Y violating (D2) with $X \cap Y$ maximal. Let $x \in X - Y$, $y \in Y - X$, and let $X_1 = \sigma(X \cup y)$. Then $r(X_1) = r(X) + 1$ and $r(X_1 \cap Y) \geqslant r(X \cap Y) + 1$, $r(X_1 \cup Y) \geqslant r(X \cup Y)$. Hence

$$\begin{aligned} r(X_1 \cup Y) + r(X_1 \cap Y) &\geqslant r(X \cup Y) + r(X \cap Y) + 1 \\ &= r(X) + r(Y) + 1 = r(X_1) + r(Y). \end{aligned}$$

By submodularity we have equality here. Finally, X_1, $Y \in \mathscr{D}$ and $|X_1 \cap Y| > |X \cap Y|$, so by the choice of X and Y, we have that $X_1 \cap Y \in \mathscr{D}$. It also follows that $r(X_1 \cap Y) = r(X \cap Y) + 1$. Let A be a basis of $X \cap Y$, then $A \cup y$ is a basis of $X_1 \cap Y$. Since $X_1 \cap Y \in \mathscr{D} \subseteq \mathscr{H}$ if follows that $A \cup y \in \mathscr{H}$. Similarly, $A \cup x \in \mathscr{H}$. Since $\sigma(A \cup y) = X_1 \cap Y \subseteq Y$ but $x \notin Y$, we also have that $A \cup x \cup y \in \mathscr{F}$. So (P3) implies that $A \in \mathscr{H}$ and so by definition, $X \cap Y = \sigma(A) \in \mathscr{D}$.

(4) Using these definitions, we can introduce the following general construction: Let $(E, \mathscr{L})$ be a greedoid and let, for each $e \in E$, $\mathscr{H}(e) \subseteq 2^E$ be a quasimodular set-system. Define

$$\mathscr{L}[\mathscr{H}] = \{e_1 \cdots e_k \in \mathscr{L}: \text{for all } 1 \leqslant j \leqslant k,\ \{e_1, ..., e_{j-1}\} \in \mathscr{H}(e_j)\}.$$

Then $(E, \mathscr{L}[\mathscr{H}])$ is a greedoid, as we shall prove below. We call $(E, \mathscr{L}[\mathscr{H}])$ the *trimming* of $(E, \mathscr{L})$ by the system of alternative precedences $\mathscr{H}$ or, for short, $\mathscr{L}_{\mathscr{H}}$ a *trimmed greedoid.* Note that if $(E, \mathscr{L}_{\mathscr{H}})$ is the shelling structure defined by the alternative precedences $\mathscr{H}$, then $\mathscr{L}[\mathscr{H}] = \mathscr{L} \cap \mathscr{L}_{\mathscr{H}}$.

LEMMA 5.4. *Trimming of a greedoid results in a greedoid.*

Proof. Let $e_1 \cdots e_k \in \mathscr{L}[\mathscr{H}]$, $f_1 \cdots f_l \in \mathscr{L}[\mathscr{H}]$, $l < k$. By the exchange property, there exists a j, $1 \leqslant j \leqslant k$ such that $\{e_1, ..., e_{j-1}\} \subseteq \sigma(f_1, ..., f_l)$ but $e_j \notin \sigma(f_1, ..., f_l)$. We also have that $\{e_1, ..., e_{j-1}\} \in \mathscr{H}(e_j)$. So by (P1), $\sigma(f_1, ..., f_l) \in \mathscr{H}(e_j)$ and by (P2) we have $\{f_1, ..., f_l\} \in \mathscr{H}(e_j)$. So $f_1 \cdots f_l e_j \in \mathscr{L}[\mathscr{H}]$. This proves that $(E, \mathscr{L}[\mathscr{H}])$ is a greedoid. ∎

LEMMA 5.5. *Trimming an interval greedoid results in an interval greedoid.*

Proof. The shelling structure defined by $\mathscr{H}$ is an interval greedoid, and this property is preserved by intersection. ∎

LEMMA 5.6. *Trimming a local poset greedoid by a modular system results in a local poset greedoid.*

Proof. Let $(E, \mathscr{L})$ be a local poset greedoid and $\mathscr{H}$ a modular system of alternative precedences. By Lemma 5.5, $\mathscr{L}[\mathscr{H}]$ has the interval property. Thus it suffices to verify that it has property (A′) as defined in Section 4. To this end, suppose that A, $A \cup \{x, z\}$, $A \cup \{y, z\}$, and $A \cup \{x, y, z\} \in \mathscr{F}[\mathscr{H}]$, we claim that $A \cup \{z\} \in \mathscr{F}[\mathscr{H}]$. Suppose not. Since $\mathscr{F}$ is a local poset greedoid, we know that $A \cup \{z\} \in \mathscr{F}$. So it suffices to show that $A \in \mathscr{H}(z)$.

By the greedoid property of $\mathscr{F}[\mathscr{H}]$ we know that $A \cup \{x\} \in \mathscr{F}[\mathscr{H}]$ and so by $A \cup \{x, z\} \in \mathscr{F}[\mathscr{H}]$, it follows that $A \cup \{x\} \in \mathscr{H}(z)$. Similarly $A \cup \{y\} \in \mathscr{H}(z)$.

Furthermore, it follows from $A \in \mathscr{F}$, $A \cup \{y\} \in \mathscr{F}$, $A \cup \{x, z\} \in \mathscr{F}$, $A \cup \{x, y, z\} \in \mathscr{F}$, and $A \cup \{x\} \in \mathscr{F}$ by the interval property that $A \cup \{x, y\} \in \mathscr{F}$.

Thus the conditions of (P3) are satisfied and we obtain that $A \in \mathscr{H}(z)$. ∎

The trimming operation is a quite general procedure. Thus it is a natural problem to represent various classes of greedoids as trimmings of more special greedoids. The following results show that many subclasses of greedoids can be obtained already by the trimming of matroids.

LEMMA 5.7. *Every polymatroid branching greedoid is a trimmed matroid.*

Proof. Let (E_0, f) be the polymatroid in the definition, and let $(E_0, \mathscr{M})$ be the matroid induced by this polymatroid, i.e.,

$$\mathscr{M} = \{X \subseteq E_0\colon\ f(Y) \geqslant |Y| \text{ for all } Y \subseteq X\}.$$

Let $(E, \mathscr{M}')$ be the restriction of $\mathscr{M}$ on E. Define

$$\mathscr{H}(e) = \{X \subseteq E - e\colon f(X \cup t(e)) = f(X), f(X \cup e) \leqslant f(X) + 1\}.$$

Then $\mathscr{H}$ satisfies (P1) and (P2). (P1) is easy. To show (P2), let $X \subseteq E$, $\sigma(X) \in \mathscr{H}(e)$.

First we show that $f(\sigma(X)) = f(X)$. In fact, the rank function of the induced matroid can be expressed by the following formula (Edmonds [4])

$$r_{\mathcal{M}}(U) = \min_{V \subseteq U} \{ f(V) + |U - V| \}.$$

Hence there exists a $V \subseteq \sigma(X)$ such that $r_{\mathcal{M}}(\sigma(X)) = f(V) + |\sigma(X) - V|$. But

$$\begin{aligned} r_{\mathcal{M}}(X) &\leqslant f(V \cap X) + |X - V| \\ &\leqslant f(V) + |\sigma(X) - V| = r_{\mathcal{M}}(\sigma(X)). \end{aligned}$$

Since we have equality, it follows that $f(V \cap X) = f(V)$ and $X - V = \sigma(X) - V$, $\sigma(X) = X \cup V$. Hence

$$\begin{aligned} f(\sigma(X)) = f(X \cup V) &\leqslant f(X) + f(V) - f(X \cap V) \\ &= f(X). \end{aligned}$$

Hence $f(\sigma(X)) = f(X)$ by monotonicity.

Now by monotonicity and submodularity,

$$\begin{aligned} f(X \cup t(e)) \leqslant f(\sigma(X) \cup t(e)) &\leqslant f(\sigma(X)) + f(X \cup t(e)) - f(X) \\ &= f(\sigma(X)) = f(X) \end{aligned}$$

and hence $f(X \cup t(e)) = f(X)$. It follows by the same argument that $f(X \cup e) \leqslant f(X) + 1$. So $X \in \mathcal{H}(e)$.

It remains to be shown that the trimmed matroid $(E, \mathcal{L}_{\mathcal{M}}[\mathcal{H}])$ is just the polymatroid branching greedoid $(E, \mathcal{L}')$ defined by (E_0, f) and t.

First, let $e_1 \cdots e_k \in \mathcal{L}'$. Then $e_1 \cdots e_k \in \mathcal{L}_f$ and hence $\{e_1, ..., e_k\} \in \mathcal{M}$. Furthermore, for all $1 \leqslant j \leqslant k$, $f(e_1, ..., e_j) = f(e_1, ..., e_{j-1}) + 1$ and $f(e_1, ..., e_{j-1}, t(e_j)) = f(e_1, ..., e_{j-1})$ by definition, and hence $\{e_1, ..., e_{j-1}\} \in \mathcal{H}(e_j)$. So $e_1 \cdots e_k \in \mathcal{L}_{\mathcal{M}}[\mathcal{H}]$.

Second, let $e_1 \cdots e_k \in \mathcal{L}_{\mathcal{M}}[\mathcal{H}]$, then for all $1 \leqslant j \leqslant k$, $f(e_1, ..., e_j) \leqslant f(e_1, ..., e_{j-1}) + 1$ and hence $f(e_1, ..., e_j) \leqslant j$. But by $\{e_1, ..., e_j\} \in \mathcal{M}$, we have $f(e_1, ..., e_j) = j$. Also by the definition of $\mathcal{L}$, $f(e_1, ..., e_{j-1}, t(e_j)) = f(e_1, ..., e_{j-1}) = j - 1$. Hence $e_1 \cdots e_k \in \mathcal{L}'$. ▌

Remark. One could verify that the system $\mathcal{H}$ as constructed in the proof also satisfies (P3) with respect to $(E, \mathcal{F})$ (but not necessarily with respect to $(E, \mathcal{M})$). This would yield another proof of Corollary 5.3.

LEMMA 5.8. *Every F-geometry is a trimmed matroid.*

Proof. Let $(E, \leqslant)$ be a poset and $(E, \mathcal{L})$ an F-geometry on it. Let r be the rank function of $(E, \mathscr{L})$ and $\rho(X) := r(I(X))$ for all $X \subseteq E$. Then (E, ρ) is a polymatroid which induces a matroid $(E, \mathscr{M})$. Define

$$\mathscr{H}(e) = \{X \subseteq E - e \colon \exists A \subseteq \sigma_{\mathscr{M}}(X) \colon A \in \mathscr{F},\ A \cup e \in \mathscr{F}\}.$$

Then $\mathscr{H}$ trivially satisfies (P1) and (P2). We show that $(E, \mathscr{L}_{\mathscr{M}}[\mathscr{H}]) = (E, \mathscr{L})$.

Let $e_1 \cdots e_k \in \mathscr{L}$. Then for all $1 \leqslant j \leqslant k$, $\rho(e_1, ..., e_j) = j$ and hence $e_1 \cdots e_k \in \mathscr{L}_\rho$. Hence $\{e_1, ..., e_k\} \in \mathscr{M}$. Furthermore, $\{e_1, ..., e_{j-1}\} \in \mathscr{H}(e_j)$ and so $e_1 \cdots e_k \in \mathscr{L}_{\mathscr{M}}[\mathscr{H}]$.

Conversely, let $e_1 \cdots e_k \in \mathscr{L}_{\mathscr{M}}[\mathscr{H}]$, we want to show that $e_1 \cdots e_k \in \mathscr{L}$. Without loss of generality we may assume that $e_1 \cdots e_{k-1} \in \mathscr{L}$. Furthermore, $\{e_1, ..., e_k\} \in \mathscr{M}$ and hence $\rho(e_1, ..., e_k) = r(I(e_1, ..., e_k)) \geqslant k$. Hence, there exists an element $e'_k \in I(e_1, ..., e_k)$ such that $e_1 \cdots e_{k-1} e'_k \in \mathscr{L}$. It follows by (F1) that $e'_k \notin I(e_1, ..., e_{k-1})$ and hence $e'_k \leqslant e_k$.

Now by $\{e_1, ..., e_{k-1}\} \in \mathscr{H}(e_k)$, it follows that there exists a set $A \subseteq \sigma_{\mathscr{M}}(\{e_1, ..., e_{k-1}\})$ such that $A \in \mathscr{F}$ and $A \cup e_k \in \mathscr{F}$. Augment $A \cup e_k$ from $\{e_1, ..., e_{k-1}, e'_k\}$ to a set $B \in \mathscr{F}$, $|B| = k$. Again by (F1), $e'_k \notin B$, so $B = A \cup e_k \cup C$, where $C \subseteq \{e_1, ..., e_{k-1}\}$. Augment $e_1 \cdots e_{k-1}$ by an element $b \in B$, to get $e_1 \cdots e_{k-1} b \in \mathscr{L}$. Then clearly $b \notin C$, and also $b \notin A$ since $\{e_1, ..., e_{k-1}, b\} \in \mathscr{M}$ but $A \subseteq \sigma_{\mathscr{M}}(\{e_1, ..., e_{k-1}\})$. Hence $b = e_k$, and so $e_1 \cdots e_k \in \mathscr{L}$. ∎

Remark. Minimal and maximal F-geometries can be represented as trimmed matroids with simpler alternative precedences. Namely,

$$\mathscr{H}^0(e) = \{X \subseteq E - e \colon I(e) \subseteq \sigma_{\mathscr{M}}(X \cup e)\}$$

and

$$\mathscr{H}_0(e) = \{X \subseteq E - e \colon I(e) - e \subseteq \sigma_{\mathscr{M}}(X)\}$$

define the maximal and minimal F-geometries.

The following construction gives a more handy description of a kind of trimming a greedoid:

(5) Let $(E, \mathscr{L})$ be a greedoid. For each $e \in E$, let $\mathscr{T}(e) \subseteq \mathscr{C}$ be a set of closure-feasible sets. Define

$$\mathscr{L}\langle\mathscr{T}\rangle = \{e_1 \cdots e_k \in \mathscr{L} \colon \text{for all } 1 \leqslant j \leqslant k, \text{ there exists a } T \in \mathscr{T}(e_j) \text{ with } T \subseteq \sigma(e_1 \cdots e_{j-1})\}.$$

Then $(E, \mathscr{L}\langle\mathscr{T}\rangle)$ is called a *strong trimming* of $(E, \mathscr{L})$.

LEMMA 5.9. *Every strong trimming is a trimming.*

Proof. Define, for $e \in E$,

$$\mathscr{H}(e) = \{X \subseteq E: \text{there exists } T \in \mathscr{T}(e),\ T \subseteq \sigma(X)\}.$$

Then $\mathscr{H}(e)$ satisfies (P1) and (P2), since each $T \in \mathscr{T}(e)$ is closure feasible. It is straight forward to verify that $\mathscr{L}[\mathscr{H}] = \mathscr{L}\langle \mathscr{T} \rangle$. ∎

LEMMA 5.10. *Every trimming of an interval greedoid is a strong trimming.*

Proof. Define, for $e \in E$,

$$\mathscr{T}(e) = \mathscr{H}(e) \cap \mathscr{F}.$$

Then by the interval property, $\mathscr{T}(e) \subseteq \mathscr{C}$. It is straight forward to verify that $\mathscr{L}\langle \mathscr{T} \rangle = \mathscr{L}[\mathscr{H}]$. ∎

Remark. The interval property cannot be dropped from the hypothesis of Lemma 5.10. In fact, consider the following greedoid $(E, \mathscr{F})$:

$$E = \{a, b, c, d\}, \qquad \mathscr{F} = 2^E - \{\{a, b\}, \{c, d\}\},$$

and the following alternative precedences:

$$\mathscr{H}(a) = 2^{E-a}, \qquad \mathscr{H}(b) = 2^{E-b},$$
$$\mathscr{H}(c) = \{X \subseteq E - c: X \cap \{a, b\} \neq \varnothing\}$$
$$\mathscr{H}(d) = \{X \subseteq E - d: X \cap \{a, b\} \neq \varnothing\}.$$

Then $\mathscr{H}$ satisfies (P1) and (P2), and the basic words of $\mathscr{L}[\mathscr{H}]$ are the words in $\{a, b\} \times \{c, d\} \times E \times E$ having no repeated letters.

We claim that $(E, \mathscr{L}[\mathscr{H}])$ cannot be represented as a strong trimming of $(E, \mathscr{F})$. For, suppose that $\mathscr{L}[\mathscr{H}] = \mathscr{L}\langle \mathscr{T} \rangle$ for some $\mathscr{T}$. Consider $\mathscr{T}(c)$. Since $c \notin \mathscr{L}[\mathscr{H}]$, we have $\varnothing \notin \mathscr{T}(c)$. Since $ac \in \mathscr{L}[\mathscr{H}]$, there exists a $T \in \mathscr{T}(c)$ such that $T \subseteq \sigma(a) = \{a, b\}$. But no subset of $\{a, b\}$ other than $\varnothing$ is closure feasible.

One useful feature of strong trimmings is that it is easy to formulate a condition on $\mathscr{T}$ under which the property of being a local poset greedoid is preserved.

LEMMA 5.11. *If $(E, \mathscr{L})$ is a local poset and $|\mathscr{T}(e)| = 1$ for all $e \in E$, then the strong trimming of $(E, \mathscr{L})$ by $\mathscr{T}$ is a local poset greedoid.*

Proof. Let $\mathscr{T}(e) = \{T(e)\}$. We show that the system $\mathscr{H}(e) = \{X \subseteq E: T(e) \subseteq \sigma(X)\}$, as constructed in the proof of Lemma 5.9, is modular.

In fact, let A, $A \cup x$, $A \cup y$, $A \cup x \cup y \in \mathscr{F}$, $A \cup x$, $A \cup y \in \mathscr{H}(e)$. Suppose

$A \notin \mathscr{H}(e)$. Then $T(e) \not\subseteq \sigma(A)$ and hence there exists a $t \in T(e)$ such that $A \cup t \in \mathscr{F}$. Augment $A \cup t$ from $A \cup x \cup y$, to get that (say) $A \cup t \cup x \in \mathscr{F}$. Then $t \notin \sigma(A \cup x)$ and so $A \cup x \notin \mathscr{H}(e)$, a contradiction. ∎

In Section 5 of Korte and Lovász [12], three similar procedures were introduced to get so-called "slimmed matroids." It is not difficult to see that the first and the third of these are special cases of trimmed matroids and the second one is also a trimmed matroid if the greedoid in question is a shelling structure.

6. Relations between Subclasses of Greedoids

In this and previous papers many different classes and constructions of greedoids have been defined. Inclusion relationships between these classes are quite complicated. In this section we exhibit an inclusion chart (Fig. 4) for these classes of greedoids which are related to the topic of this paper. In Fig. 4, a line showing downwards means set-wise inclusion. We could show that these inclusions are proper, i.e., we could exhibit a "typical member" for each class which is not contained in its subclasses. Moreover, with one exception, we could demonstrate, that two classes are not contained in

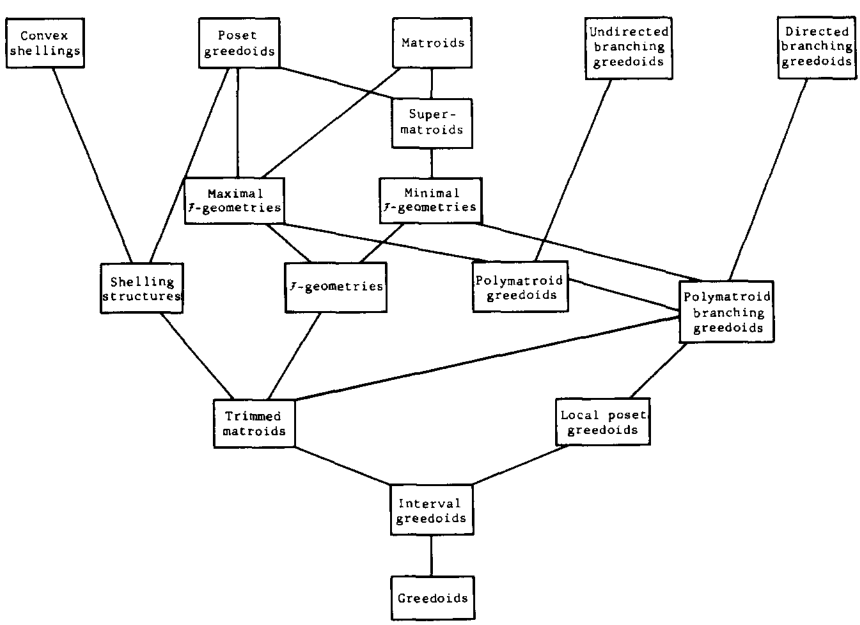

Figure 4

each other unless implied by the chart of Fig. 4. For this we could exhibit members for all classes of the chart which belong only to this class and its superclasses.

The construction of typical members of each class for the above-mentioned purposes was sometimes very elaborate. Thus, we do not present the examples here. However, these examples give some additional structural insight for different classes of greedoids. They might be of value to those readers who are interested in it. For this purpose we have documented them in an additional working paper (Korte and Lovász [13]).

Note added in proof. We have shown by a rather elaborate counterexample that properties (A), (B) and (C) do not characterize polymatroid greedoids.

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