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A modularity criterion for Klein forms, with an application to modular forms of level 13 ☆

Ick Sun Eum, Ja Kyung Koo, Dong Hwa Shin *

Department of Mathematical Sciences, KAIST, Daejeon 373-1, Republic of Korea

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ABSTRACT

We find some modularity criterion for a product of Klein forms of the congruence subgroup $\Gamma_1(N)$ (Theorem 2.6) and, as its application, construct a basis of the space of modular forms for $\Gamma_1(13)$ of weight 2 (Example 3.4). In the process we face with an interesting property about the coefficients of certain theta function from a quadratic form and prove it conditionally by applying Hecke operators (Proposition 4.3).

1. Introduction

The *Dedekind eta-function* $\eta(\tau)$ is defined to be the infinite product

$$\eta(\tau) = q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1-q^n) \quad (\tau \in \mathfrak{H}) \tag{1.1}$$

where $q = e^{2\pi i\tau}$ and $\mathfrak{H} = \{\tau \in \mathbb{C}\colon \operatorname{Im}(\tau) > 0\}$. This function plays an important role of building block which constitutes various modular forms of integral or half-integral weight. For example, the classical theta function

$$\Theta(\tau) = \sum_{n=-\infty}^{\infty} q^{n^2} \quad (\tau \in \mathfrak{H}),$$

which is a modular form for $\Gamma_0(4)$ of weight $1/2$ [3], can be written as

$$\Theta(\tau) = \prod_{n=1}^{\infty} \frac{(1-q^{2n})^5}{(1-q^n)^2(1-q^{4n})^2} = \frac{\eta(2\tau)^5}{\eta(\tau)^2\eta(4\tau)^2} \tag{1.2}$$

by the Jacobi's Triple Product Identity [2, §17]

$$\prod_{n=1}^{\infty} (1-q^{2n})(1+aq^{2n-1})(1+a^{-1}x^{2n-1}) = \sum_{m=-\infty}^{\infty} a^m q^{m^2}. \tag{1.3}$$

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* Corresponding author.
E-mail addresses: zandc@kaist.ac.kr (I.S. Eum), jkkoo@math.kaist.ac.kr (J.K. Koo), shakur01@kaist.ac.kr (D.H. Shin).

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And, every modular form for $\mathrm{SL}_2(\mathbb{Z})$ is known to be expressed as a rational function in $\eta(\tau)^8$, $\eta(2\tau)^4$ and $\eta(4\tau)^8$ [8, Theorem 1.67].

On the other hand, we are further required to present more building blocks to construct modular forms of integral weight for modular groups of higher level. To this end we focus on the following Klein forms.

For $(r_1, r_2) \in \mathbb{Q}^2 - \mathbb{Z}^2$ the *Klein form* $\mathfrak{k}_{(r_1,r_2)}(\tau)$ is defined by the following infinite product expansion

$$\mathfrak{k}_{(r_1,r_2)}(\tau) = e^{\pi i r_2(r_1-1)} q^{\frac{1}{2} r_1(r_1-1)} (1-q_z) \prod_{n=1}^{\infty} \bigl(1-q^n q_z\bigr)\bigl(1-q^n q_z^{-1}\bigr)\bigl(1-q^n\bigr)^{-2} \quad (\tau \in \mathfrak{H}) \tag{1.4}$$

where $q_z = e^{2\pi i z}$ with $z = r_1\tau + r_2$. We see from Example 3.5 that the Klein forms seem to be a variation of $\eta(\tau)^{-2}$. (In the original definition [6, Chapter 2, §1] there is an extra factor $i/2\pi$.) Furthermore, we know directly from the definition that it is a holomorphic function which has no zeros and poles on $\mathfrak{H}$.

In this paper we shall investigate some modularity criterions for products of Klein forms of modular groups $\Gamma_1(N)$ of arbitrary level (Theorems 2.6 and 2.8). As applications we shall express theta functions associated with quadratic forms in view of Klein forms and find a basis of the space of modular forms for $\Gamma_1(13)$ of weight 2 (Examples 3.3 and 3.4).

Let $\Theta_Q(\tau) = \sum_{n=0}^{\infty} r_Q(n) q^n$ be the theta function associated with the quadratic form $Q(x_1, x_2, x_3, x_4) = x_1^2 + 2x_2^2 + x_3^2 + x_4^2 + x_1x_3 + x_1x_4 + x_2x_4$ where $r_Q(n)$ is the cardinality of the solution set $\{\mathbf{x} \in \mathbb{Z}^4 : Q(\mathbf{x}) = n\}$ for $n \geqslant 0$. We shall find some primes p which satisfy an interesting relation

$$r_Q\bigl(p^2 n\bigr) = \frac{r_Q(p^2) r_Q(n)}{r_Q(1)} \quad \text{for any integer } n \geqslant 1 \text{ prime to } p$$

by applying Hecke operators to $\Theta_Q(\tau)$ (Proposition 4.3 and Remark 4.4).

Cho, Kim and Koo recently performed in [1] a similar work about modularity of Klein forms and constructed bases of certain spaces of modular forms by describing the Fourier coefficients of some finite products of Klein forms in terms of divisor functions. For the purpose they adopted some useful nine identities between the q-products and the q-series from the basic hypergeometric series [2]. Thus, due to this technical restriction they could hardly find examples of higher level, from which our work was motivated to improve modularity criterion for $\Gamma_1(N)$.

2. Modularity criterions

First, we start with recalling some necessary transformation formulas investigated in [6].

Proposition 2.1.

(i) *For* $(r_1, r_2) \in \mathbb{Q}^2 - \mathbb{Z}^2$ *and* $(s_1, s_2) \in \mathbb{Z}^2$ *we get*

$$\mathfrak{k}_{(-r_1,-r_2)}(\tau) = -\mathfrak{k}_{(r_1,r_2)}(\tau),$$
$$\mathfrak{k}_{(r_1,r_2)+(s_1,s_2)}(\tau) = (-1)^{s_1s_2+s_1+s_2} e^{-\pi i(s_1r_2-s_2r_1)} \mathfrak{k}_{(r_1,r_2)}(\tau).$$

(ii) *For* $(r_1, r_2) \in \mathbb{Q}^2 - \mathbb{Z}^2$ *and* $\alpha = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ *we derive*

$$\mathfrak{k}_{(r_1,r_2)}(\tau) \circ \alpha = \mathfrak{k}_{(r_1,r_2)}\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^{-1} \mathfrak{k}_{(r_1,r_2)\alpha}(\tau) = (c\tau+d)^{-1} \mathfrak{k}_{(r_1a+r_2c, r_1b+r_2d)}(\tau).$$

(iii) *Let* $\mathbf{B}_2(X) = X^2 - X + 1/6$ *be the second Bernoulli polynomial and* $\langle X \rangle$ *be the fractional part of* $X \in \mathbb{R}$ *so that* $0 \leqslant \langle X \rangle < 1$. *For* $(r_1, r_2) \in \mathbb{Q}^2 - \mathbb{Z}^2$ *we have*

$$\mathrm{ord}_q\ \mathfrak{k}_{(r_1,r_2)}(\tau) = \frac{1}{2}\left(\mathbf{B}_2\bigl(\langle r_1 \rangle\bigr) - \frac{1}{6}\right) = \frac{1}{2}\langle r_1 \rangle \bigl(\langle r_1 \rangle - 1\bigr).$$

Proof. See [6, Chapter 2, §1]. □

For every integer k, $\alpha = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ and a function $f(\tau)$ on $\mathfrak{H}$ we write

$$f(\tau)|[\alpha]_k = (c\tau+d)^{-k}\bigl(f(\tau) \circ \alpha\bigr).$$

And, we mainly consider the three following congruence subgroups

$$\Gamma(N)=\left\{\alpha\in \mathrm{SL}_2(\mathbb{Z})\colon\ \alpha\equiv\begin{pmatrix}1&0\\0&1\end{pmatrix}\ (\mathrm{mod}\ N)\right\},$$
$$\Gamma_1(N)=\left\{\alpha\in \mathrm{SL}_2(\mathbb{Z})\colon\ \alpha\equiv\begin{pmatrix}1&*\\0&1\end{pmatrix}\ (\mathrm{mod}\ N)\right\},$$
$$\Gamma_0(N)=\left\{\alpha\in \mathrm{SL}_2(\mathbb{Z})\colon\ \alpha\equiv\begin{pmatrix}*&*\\0&*\end{pmatrix}\ (\mathrm{mod}\ N)\right\}$$

for an integer $N\geqslant 2$. When Γ is one of the above congruence subgroups and k is any integer, we say that a holomorphic function $f(\tau)$ on $\mathfrak{H}$ is a *modular form for Γ of weight k* if

(i) $f(\tau)|[\gamma]_k=f(\tau)$ for all $\gamma\in\Gamma$;
(ii) $f(\tau)$ is holomorphic at every cusp [9, Definition 2.1].

We denote by $M_k(\Gamma)$ the $\mathbb{C}$-vector space of modular forms for Γ of weight k. If we replace (ii) by

(ii)′ $f(\tau)$ is meromorphic at every cusp,

then we call $f(\tau)$ a *nearly holomorphic modular form for Γ of weight k*.

Kubert and Lang [6] gave the following modularity condition for $\Gamma(N)$.

Proposition 2.2. *For an integer $N\geqslant 2$, let $\{m(r)\}_{r\in\frac{1}{N}\mathbb{Z}^2-\mathbb{Z}^2}$ be a family of integers such that $m(r)=0$ except finitely many r. Then the product of Klein forms*

$$\prod_{r=(r_1,r_2)\in\frac{1}{N}\mathbb{Z}^2-\mathbb{Z}^2}\mathfrak{k}_r(\tau)^{m(r)}$$

is a nearly holomorphic modular form for $\Gamma(N)$ of weight $-\sum_r m(r)$ if and only if

$$\sum_r m(r)(Nr_1)^2\equiv\sum_r m(r)(Nr_2)^2\equiv 0\quad\big(\mathrm{mod}\ \gcd(2,N)\cdot N\big),$$
$$\sum_r m(r)(Nr_1)(Nr_2)\equiv 0\quad(\mathrm{mod}\ N).$$

Proof. See [6, Chapter 3, Theorem 4.1]. □

Remark 2.3. Let $N\geqslant 2$ and $r\in\frac{1}{N}\mathbb{Z}^2-\mathbb{Z}^2$. Then $\mathfrak{k}_r(\tau)$ (respectively, $\mathfrak{k}_r(\tau)^{2N}$) is a nearly holomorphic modular form for $\Gamma(2N^2)$ (respectively, $\Gamma(N)$) of weight -1 (respectively, $-2N$).

Now we shall develop a modularity criterion for the congruence subgroup $\Gamma_1(N)$.

Lemma 2.4. *For an integer $N\geqslant 2$ let t be an integer with $t\not\equiv 0\ (\mathrm{mod}\ N)$. Then we have the relation*

$$\mathfrak{k}_{(\frac{t}{N},0)}(N\tau)=Ne^{\frac{\pi it}{2}(\frac{1}{N}-1)}\prod_{n=0}^{N-1}\mathfrak{k}_{(\frac{t}{N},\frac{n}{N})}(\tau)\prod_{n=1}^{N-1}\mathfrak{k}_{(0,\frac{n}{N})}(\tau)^{-1}.$$

Proof. By the definition (1.4) we have

$$\mathfrak{k}_{(\frac{t}{N},0)}(N\tau)=q^{\frac{t}{2}(\frac{t}{N}-1)}\big(1-q^t\big)\prod_{m=1}^{\infty}\big(1-q^{Nm+t}\big)\big(1-q^{Nm-t}\big)\big(1-q^{Nm}\big)^{-2},$$

and

$$\prod_{n=0}^{N-1}\mathfrak{k}_{(\frac{t}{N},\frac{n}{N})}(\tau)\prod_{n=1}^{N-1}\mathfrak{k}_{(0,\frac{n}{N})}(\tau)^{-1}$$
$$=\prod_{n=0}^{N-1}\left(e^{\frac{\pi in}{N}(\frac{t}{N}-1)}q^{\frac{t}{2N}(\frac{t}{N}-1)}\big(1-e^{\frac{2\pi in}{N}}q^{\frac{t}{N}}\big)\prod_{m=1}^{\infty}\big(1-e^{\frac{2\pi in}{N}}q^{m+\frac{t}{N}}\big)\big(1-e^{-\frac{2\pi in}{N}}q^{m-\frac{t}{N}}\big)\big(1-q^m\big)^{-2}\right)$$

$$\times \prod_{n=1}^{N-1}\left(e^{-\frac{\pi in}{N}}\left(1-e^{\frac{2\pi in}{N}}\right)\prod_{m=1}^{\infty}\left(1-e^{\frac{2\pi in}{N}}q^m\right)\left(1-e^{-\frac{2\pi in}{N}}q^m\right)\left(1-q^m\right)^{-2}\right)^{-1}$$

$$= e^{\frac{\pi i(N-1)}{2}(\frac{t}{N}-1)}q^{\frac{t}{2}(\frac{t}{N}-1)}\left(1-q^t\right)\prod_{m=1}^{\infty}\left(1-q^{Nm+t}\right)\left(1-q^{Nm-t}\right)\left(1-q^m\right)^{-2N}$$

$$\times\left(e^{-\frac{\pi i(N-1)}{2}}N\prod_{m=1}^{\infty}\left(1-q^{Nm}\right)\left(1-q^m\right)^{-1}\left(1-q^{Nm}\right)\left(1-q^m\right)^{-1}\left(1-q^m\right)^{-2(N-1)}\right)^{-1}$$

$$= e^{\frac{\pi it}{2}(1-\frac{1}{N})}N^{-1}q^{\frac{t}{2}(\frac{t}{N}-1)}\left(1-q^t\right)\prod_{m=1}^{\infty}\left(1-q^{Nm+t}\right)\left(1-q^{Nm-t}\right)\left(1-q^{Nm}\right)^{-2}$$

by using the identity

$$1-X^N=(1-\zeta_N X)\left(1-\zeta_N^2X\right)\cdots\left(1-\zeta_N^NX\right) \quad \text{where } \zeta_N=e^{\frac{2\pi i}{N}}. \tag{2.1}$$

Hence we get the assertion. □

Lemma 2.5. *For $y \in \mathbb{Q}$ and an integer $D \geqslant 1$ we have*

$$\sum_{\substack{x \ (\mathrm{mod}\ \mathbb{Z})\\ Dx\equiv y\ (\mathrm{mod}\ \mathbb{Z})}} \mathbf{B}_2\left(\langle x\rangle\right)=D^{-1}\mathbf{B}_2\left(\langle y\rangle\right).$$

Proof. See [5, Lemma 6.3]. □

Theorem 2.6. *For an integer $N \geqslant 2$, let $\{m(t)\}_{t=1}^{N-1}$ be a family of integers. Then the product*

$$\prod_{t=1}^{N-1}\mathfrak{k}_{(\frac{t}{N},0)}(N\tau)^{m(t)}$$

is a nearly holomorphic modular form for $\Gamma_1(N)$ of weight $k=-\sum_{t=1}^{N-1}m(t)$ if

$$\sum_{t=1}^{N-1}m(t)t^2\equiv 0 \quad \left(\mathrm{mod}\ \gcd(2,N)\cdot N\right). \tag{2.2}$$

Furthermore, for $\alpha=\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in \mathrm{SL}_2(\mathbb{Z})$ we achieve

$$\mathrm{ord}_q\left(\prod_{t=1}^{N-1}\mathfrak{k}_{(\frac{t}{N},0)}(N\tau)^{m(t)}|[\alpha]_k\right)=\frac{\gcd(c,N)^2}{2N}\sum_{t=1}^{N-1}m(t)\left\langle\frac{at}{\gcd(c,N)}\right\rangle\left(\left\langle\frac{at}{\gcd(c,N)}\right\rangle-1\right). \tag{2.3}$$

Proof. By Lemma 2.4 we may prove the assertions for the function

$$\mathfrak{k}(\tau)=\prod_{t=1}^{N-1}\left(\prod_{n=0}^{N-1}\mathfrak{k}_{(\frac{t}{N},\frac{n}{N})}(\tau)\prod_{n=1}^{N-1}\mathfrak{k}_{(0,\frac{n}{N})}(\tau)^{-1}\right)^{m(t)}.$$

Assume the condition (2.2) holds and set

$$\mathfrak{k}(\tau)=\prod_{r=(r_1,r_2)\in\frac{1}{N}\mathbb{Z}^2-\mathbb{Z}^2}\mathfrak{k}_r(\tau)^{\ell(r)}.$$

Then we get that

$$\sum_r \ell(r)(Nr_1)^2=N\sum_{t=1}^{N-1}m(t)t^2\equiv 0 \quad \left(\mathrm{mod}\ \gcd(2,N)\cdot N\right),$$

$$\sum_r \ell(r)(Nr_2)^2=0\equiv 0 \quad \left(\mathrm{mod}\ \gcd(2,N)\cdot N\right),$$

$$\sum_r \ell(r)(Nr_1)(Nr_2)=\frac{N(N-1)}{2}\sum_{t=1}^{N-1}m(t)t\equiv 0 \quad (\mathrm{mod}\ N)$$

by the condition (2.2) and the fact $\sum_t m(t)t \equiv \sum_t m(t)t^2 \pmod 2$. This shows that $\mathfrak{k}(\tau)$ is a nearly holomorphic modular form for $\Gamma(N)$ of weight $k = -\sum_{t=1}^{N-1} m(t)$ by Proposition 2.2.

On the other hand, we know that $\Gamma_1(N)$ is generated by $\Gamma(N)$ and $T = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$. Thus, we derive that

$$\begin{aligned}
\mathfrak{k}(\tau)|[T]_k &= \prod_{t=1}^{N-1}\left(\prod_{n=0}^{N-1}\mathfrak{k}_{(\frac{t}{N},\frac{t+n}{N})}(\tau)\prod_{n=1}^{N-1}\mathfrak{k}_{(0,\frac{n}{N})}(\tau)^{-1}\right)^{m(t)} \quad \text{by Proposition 2.1(ii)}\\
&= \prod_{t=1}^{N-1}\left(\prod_{n=0}^{N-1-t}\mathfrak{k}_{(\frac{t}{N},\frac{t+n}{N})}(\tau)\prod_{n=N-t}^{N-1}\mathfrak{k}_{(\frac{t}{N},\frac{t+n-N}{N})+(0,1)}(\tau)\prod_{n=1}^{N-1}\mathfrak{k}_{(0,\frac{n}{N})}(\tau)^{-1}\right)^{m(t)}\\
&= \prod_{t=1}^{N-1}\left(\prod_{n=0}^{N-1-t}\mathfrak{k}_{(\frac{t}{N},\frac{t+n}{N})}(\tau)\prod_{n=N-t}^{N-1}\left(-e^{-\pi i\frac{t}{N}}\right)\mathfrak{k}_{(\frac{t}{N},\frac{t+n-N}{N})}(\tau)\prod_{n=1}^{N-1}\mathfrak{k}_{(0,\frac{n}{N})}(\tau)^{-1}\right)^{m(t)} \quad \text{by Proposition 2.1(i)}\\
&= (-1)^{\sum_t m(t)t}e^{-\pi i\frac{1}{N}\sum_t m(t)t^2}\mathfrak{k}(\tau)\\
&= \mathfrak{k}(\tau) \quad \text{by the condition (2.2) and the fact } \sum_t m(t)t \equiv \sum_t m(t)t^2 \pmod 2.
\end{aligned}$$

Therefore $\mathfrak{k}(\tau)$ is a nearly holomorphic modular form for $\Gamma_1(N)$ of weight $k = -\sum_t m(t)$.

Now, for $\alpha = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ we deduce that

$$\begin{aligned}
\mathrm{ord}_q\big(\mathfrak{k}(\tau)|[\alpha]_k\big) &= \mathrm{ord}_q\left(\prod_{t=1}^{N-1}\left(\prod_{n=0}^{N-1}\mathfrak{k}_{(\frac{at+cn}{N},\frac{bt+dn}{N})}(\tau)\prod_{n=1}^{N-1}\mathfrak{k}_{(\frac{cn}{N},\frac{dn}{N})}(\tau)^{-1}\right)^{m(t)}\right) \quad \text{by Proposition 2.1(ii)}\\
&= \sum_{t=1}^{N-1} m(t)\left\{\sum_{n=0}^{N-1}\frac{1}{2}\left(\mathbf{B}_2\left(\left\langle\frac{at+cn}{N}\right\rangle\right)-\frac{1}{6}\right)-\sum_{n=1}^{N-1}\frac{1}{2}\left(\mathbf{B}_2\left(\left\langle\frac{cn}{N}\right\rangle\right)-\frac{1}{6}\right)\right\} \quad \text{by Proposition 2.1(iii)}\\
&= \frac{1}{2}\sum_{t=1}^{N-1} m(t)\left\{\sum_{n=1}^{N}\mathbf{B}_2\left(\left\langle\frac{at+cn}{N}\right\rangle\right)-\sum_{n=1}^{N}\mathbf{B}_2\left(\left\langle\frac{cn}{N}\right\rangle\right)\right\} \quad \text{by the fact } \mathbf{B}_2(0)=\frac{1}{6}\\
&= \frac{\gcd(c,N)}{2}\sum_{t=1}^{N-1} m(t)\left\{\sum_{n=1}^{D}\mathbf{B}_2\left(\left\langle\frac{at/\gcd(c,N)}{D}+\frac{c/\gcd(c,N)}{D}n\right\rangle\right)-\sum_{n=1}^{D}\mathbf{B}_2\left(\left\langle\frac{c/\gcd(c,N)}{D}n\right\rangle\right)\right\}
\end{aligned}$$

where $D = \dfrac{N}{\gcd(c,N)}$.

If we apply Lemma 2.5 with $D = \frac{N}{\gcd(c,N)}$, $y = \frac{at}{\gcd(c,N)}$ and $x = \frac{y}{D} + \frac{c/\gcd(c,N)}{D}n$ with $1 \leqslant n \leqslant D$, then we obtain

$$\sum_{n=1}^{D}\mathbf{B}_2\left(\left\langle\frac{at/\gcd(c,N)}{D}+\frac{c/\gcd(c,N)}{D}n\right\rangle\right) = \frac{\gcd(c,N)}{N}\mathbf{B}_2\left(\left\langle\frac{at}{\gcd(c,N)}\right\rangle\right).$$

Likewise, if we set $D = \frac{N}{\gcd(c,N)}$, $y = 0$ and $x = \frac{c/\gcd(c,N)}{D}$ with $1 \leqslant n \leqslant D$ in Lemma 2.5, then we get

$$\sum_{n=1}^{D}\mathbf{B}_2\left(\left\langle\frac{c/\gcd(c,N)}{D}n\right\rangle\right) = \frac{\gcd(c,N)}{N}\mathbf{B}_2(0). \tag{2.4}$$

Hence we achieve

$$\begin{aligned}
\mathrm{ord}_q\big(\mathfrak{k}(\tau)|[\alpha]_k\big) &= \frac{\gcd(c,N)^2}{2N}\sum_{t=1}^{N-1} m(t)\left\{\mathbf{B}_2\left(\left\langle\frac{at}{\gcd(c,N)}\right\rangle\right)-\mathbf{B}_2(0)\right\}\\
&= \frac{\gcd(c,N)^2}{2N}\sum_{t=1}^{N-1} m(t)\left\langle\frac{at}{\gcd(c,N)}\right\rangle\left(\left\langle\frac{at}{\gcd(c,N)}\right\rangle-1\right),
\end{aligned}$$

as desired. □

Corollary 2.7. *Let $N \geqslant 2$ be a square integer. Then the function*

$$\mathfrak{k}_{(\frac{\sqrt{N}}{N},0)}(N\tau)^{-2}$$

belongs to $M_2(\Gamma_1(N))$.

Proof. Let $\mathfrak{k}(\tau)$ be the above function. Since $\mathfrak{k}(\tau)$ satisfies the condition (2.2), it is a nearly holomorphic modular form for $\Gamma_1(N)$ of weight 2 by Theorem 2.6. For any $\alpha = \left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ we then get by the order formula (2.3)

$$\mathrm{ord}_q\big(\mathfrak{k}(\tau)|[\alpha]_2\big) = \frac{\gcd(c,N)^2}{N}\left\langle \frac{a\sqrt{N}}{\gcd(c,N)}\right\rangle\left(1-\left\langle \frac{a\sqrt{N}}{\gcd(c,N)}\right\rangle\right),$$

which is nonnegative. This implies that the order of $\mathfrak{k}(\tau)$ at every cusp is nonnegative; hence $\mathfrak{k}(\tau)$ is indeed a modular form. □

Next we find a family of modular forms for $\Gamma_0(N)$ which are in fact quotients of the Dedekind eta-functions.

Theorem 2.8. *For an integer $N \geqslant 2$ the function*

$$\prod_{n=1}^{N-1} \mathfrak{k}_{(0,\frac{n}{N})}(\tau)^{\frac{-12}{\gcd(12,N-1)}}$$

is a modular form for $\Gamma_0(N)$ of weight $\frac{12(N-1)}{\gcd(12,N-1)}$.

Proof. Let $\mathfrak{k}(\tau)$ be the above function, $k = \frac{12(N-1)}{\gcd(12,N-1)}$ and $\alpha = \left(\begin{smallmatrix} a & b \\ Nc & d\end{smallmatrix}\right)$ with $a,b,c,d \in \mathbb{Z}$ such that $ad - Ncb = 1$. Then we achieve

$$\begin{aligned}
\mathfrak{k}(\tau)|[\alpha]_k &= \prod_{n=1}^{N-1} \mathfrak{k}_{(cn,\frac{dn}{N})}(\tau)^{\frac{-12}{\gcd(12,N-1)}} \quad \text{by Proposition 2.1(ii)}\\
&= \prod_{n=1}^{N-1} \mathfrak{k}_{(0,\frac{dn}{N})+(cn,0)}(\tau)^{\frac{-12}{\gcd(12,N-1)}}\\
&= \prod_{n=1}^{N-1} \left(\mathfrak{k}_{(0,\frac{dn}{N})}(\tau)(-1)^{cn}e^{-\pi i\frac{cdn^2}{N}}\right)^{\frac{-12}{\gcd(12,N-1)}} \quad \text{by Proposition 2.1(i)}\\
&= \left(\prod_{n=1}^{N-1} \mathfrak{k}_{(0,\frac{dn}{N})}(\tau)^{\frac{-12}{\gcd(12,N-1)}}\right)\left((-1)^{\frac{c(N-1)N}{2}}e^{-\pi i\frac{cd(N-1)(2N-1)}{6}}\right)^{\frac{-12}{\gcd(12,N-1)}}\\
&= \prod_{n=1}^{N-1} \mathfrak{k}_{(0,\langle\frac{dn}{N}\rangle)+(0,\frac{dn}{N}-\langle\frac{dn}{N}\rangle)}(\tau)^{\frac{-12}{\gcd(12,N-1)}}\\
&= \prod_{n=1}^{N-1} \left(\mathfrak{k}_{(0,\langle\frac{dn}{N}\rangle)}(\tau)(-1)^{\frac{dn}{N}-\langle\frac{dn}{N}\rangle}\right)^{\frac{-12}{\gcd(12,N-1)}} \quad \text{by Proposition 2.1(i)}\\
&= \left(\prod_{n=1}^{N-1} \mathfrak{k}_{(0,\langle\frac{dn}{N}\rangle)}(\tau)^{\frac{-12}{\gcd(12,N-1)}}\right)\left((-1)^{\sum_n\frac{dn}{N}-\sum_n\langle\frac{dn}{N}\rangle}\right)^{\frac{-12}{\gcd(12,N-1)}}\\
&= \left(\prod_{n=1}^{N-1} \mathfrak{k}_{(0,\langle\frac{n}{N}\rangle)}(\tau)^{\frac{-12}{\gcd(12,N-1)}}\right)\left((-1)^{\sum_n\frac{dn}{N}-\sum_n\langle\frac{n}{N}\rangle}\right)^{\frac{-12}{\gcd(12,N-1)}}\\
&= \left(\prod_{n=1}^{N-1} \mathfrak{k}_{(0,\frac{n}{N})}(\tau)\right)^{\frac{-12}{\gcd(12,N-1)}}\left((-1)^{\frac{(d-1)(N-1)}{2}}\right)^{\frac{-12}{\gcd(12,N-1)}}\\
&= \mathfrak{k}(\tau).
\end{aligned}$$

Hence $\mathfrak{k}(\tau)$ is a nearly holomorphic modular form for $\Gamma_0(N)$ of weight $k = \frac{12(N-1)}{\gcd(12,N-1)}$.

Now let $\beta = \left(\begin{smallmatrix} x & y \\ z & w \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$. Then we obtain that

$$\begin{aligned}
\mathrm{ord}_q\big(\mathfrak{k}(\tau)|[\beta]_k\big) &= \frac{-12}{\gcd(12, N-1)}\mathrm{ord}_q\left(\prod_{n=1}^{N-1} \mathfrak{k}_{(\frac{nz}{N}, \frac{nw}{N})}(\tau)\right) \quad \text{by Proposition 2.1(ii)} \\
&= \frac{-12}{\gcd(12, N-1)} \sum_{n=1}^{N-1} \frac{1}{2}\left(\mathbf{B}_2\left(\left\langle\frac{nz}{N}\right\rangle\right) - \frac{1}{6}\right) \quad \text{by Proposition 2.1(iii)} \\
&= \frac{-6}{\gcd(12, N-1)}\left(\sum_{n=1}^{N} \mathbf{B}_2\left(\left\langle\frac{nz}{N}\right\rangle\right) - \frac{N}{6}\right) \quad \text{by the fact } \mathbf{B}_2(0) = \frac{1}{6} \\
&= \frac{-6}{\gcd(12, N-1)}\left(\gcd(z, N) \sum_{n=1}^{\frac{N}{\gcd(z,N)}} \mathbf{B}_2\left(\left\langle\frac{nz}{N}\right\rangle\right) - \frac{N}{6}\right) \\
&= \frac{-6}{\gcd(12, N-1)}\left(\frac{\gcd(z, N)^2}{N}\mathbf{B}_2(0) - \frac{N}{6}\right) \quad \text{by the same argument as (2.4)} \\
&= \frac{N^2 - \gcd(z, N)^2}{\gcd(12, N-1)\cdot N} \geqslant 0,
\end{aligned}$$

which yields that $\mathfrak{k}(\tau)$ is holomorphic at every cusp. This completes the proof. □

Remark 2.9. Using the identity (2.1) one is readily able to verify that the function in Theorem 2.8 can be written as

$$\mathfrak{k}(\tau) = \left(N\frac{\eta(N\tau)^2}{\eta(\tau)^{2N}}\right)^{\frac{-12}{\gcd(12,N-1)}}$$

by the definitions (1.4) and (1.1). So, we may regard the following general theorem about the Dedekind eta-function as the first part of the above proof.

Theorem 2.10. *Let N be a positive integer. If $f(\tau) = \prod_{\delta|N} \eta(\delta\tau)^{r_\delta}$ is an eta-quotient with $k = \frac{1}{2}\sum_{\delta|N} r_\delta \in \mathbb{Z}$, with the additional properties that*

$$\sum_{\delta|N} \delta r_\delta \equiv 0 \pmod{24} \quad \textit{and} \quad \sum_{\delta|N} \frac{N}{\delta} r_\delta \equiv 0 \pmod{24},$$

then $f(\tau)$ satisfies

$$f\left(\frac{a\tau + b}{c\tau + d}\right) = \chi(d)(c\tau + d)^k f(\tau)$$

for every $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(N)$. Here the character χ is defined by

$$\chi(d) = \textit{the Kronecker symbol}\left(\frac{(-1)^k \prod_{\delta|N} \delta^{r_\delta}}{d}\right).$$

Proof. See [8, Theorem 1.64]. □

3. Theta functions

Let $N \geqslant 1$ and k be integers. For a Dirichlet character χ modulo N we define a character of $\Gamma_0(N)$, also denoted by χ, by

$$\chi(\gamma) = \chi(d) \quad \text{for } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N).$$

If we let $M_k(\Gamma_0(N), \chi)$ be the space

$$\big\{f(\tau) \in M_k\big(\Gamma_1(N)\big)\colon\ f(\tau)|[\gamma]_k = \chi(\gamma) f(\tau) \text{ for all } \gamma \in \Gamma_0(N)\big\},$$

then we have the following decomposition.

Proposition 3.1. *Let $N \geqslant 1$ and k be integers. We have*

$$M_k(\Gamma_1(N)) = \bigoplus_{\chi} M_k(\Gamma_0(N), \chi)$$

where χ runs over all Dirichlet characters modulo N. If $\chi(-1) \neq (-1)^k$, then $M_k(\Gamma_0(N), \chi) = \{0\}$.

Proof. See [7, Lemmas 4.3.1 and 4.3.2]. $\square$

Let A be an $r \times r$ positive definite symmetric matrix over $\mathbb{Z}$ with even diagonal entries and Q be its associated quadratic form, namely

$$Q = Q(\mathbf{x}) = \frac{1}{2}\mathbf{x}A\mathbf{x}^t \quad \text{for } \mathbf{x} = (x_1, \ldots, x_r) \in \mathbb{Z}^r.$$

Now, define the theta function $\Theta_Q(\tau)$ on $\mathfrak{H}$ associated with Q by

$$\Theta_Q(\tau) = \sum_{\mathbf{x} \in \mathbb{Z}^r} e^{2\pi i Q(\mathbf{x})\tau} = \sum_{n=0}^{\infty} r_Q(n) q^n$$

where

$$r_Q(n) = \#\{\mathbf{x} \in \mathbb{Z}^r\colon\ Q(\mathbf{x}) = n\}.$$

We take a positive integer N such that NA^{-1} is an integral matrix with even diagonal entries.

Proposition 3.2. *With the notations as above we further assume that r is even. Then $\Theta_Q(\tau)$ is a modular form for $\Gamma_1(N)$ of weight $r/2$. More precisely, $\Theta_Q(\tau)$ belongs to $M_{r/2}(\Gamma_0(N), \chi)$ where χ is a Dirichlet character defined by*

$$\chi(d) = \text{the Kronecker symbol } \left(\frac{(-1)^{\frac{r}{2}} \det(A)}{d}\right) \quad \text{for } d \in \mathbb{Z} - N\mathbb{Z}.$$

Proof. See [7, Corollary 4.9.5]. $\square$

Example 3.3. Let $A = \left(\begin{smallmatrix} 2 & 0 \\ 0 & 2 \end{smallmatrix}\right)$. Its associated quadratic form is $Q = x_1^2 + x_2^2$ and

$$\begin{aligned}\Theta_Q(\tau) &= \sum_{n=0}^{\infty} \#\{(x_1, x_2) \in \mathbb{Z}^2\colon\ x_1^2 + x_2^2 = n\} q^n = 1 + 4q + 4q^2 + \cdots \\ &= \left(\sum_{x_1=-\infty}^{\infty} q^{x_1^2}\right)\left(\sum_{x_2=-\infty}^{\infty} q^{x_2^2}\right) = \Theta(\tau)^2.\end{aligned}$$

It follows from Proposition 3.2 that $\Theta_Q(\tau) = \Theta(\tau)^2$ belongs to $M_1(\Gamma_1(4))$. On the other hand, since $M_1(\Gamma_1(4))$ is of dimension 1 [9, §2.6] and the function

$$\mathfrak{k}_{(\frac{1}{4},0)}(4\tau)^{-4}\mathfrak{k}_{(\frac{2}{4},0)}(4\tau)^3 = 1 + 4q + 4q^2 + \cdots$$

is in $M_1(\Gamma_1(4))$ by Theorem 2.6, we obtain $\Theta(\tau)^2 = \mathfrak{k}_{(\frac{1}{4},0)}(4\tau)^{-4}\mathfrak{k}_{(\frac{2}{4},0)}(4\tau)^3$.

Furthermore, we derive from the definition (1.4) that

$$\begin{aligned}\mathfrak{k}_{(\frac{1}{4},0)}(4\tau)^{-4}\mathfrak{k}_{(\frac{2}{4},0)}(4\tau)^3 &= \left(q^{\frac{1}{2}(\frac{1}{4}-1)}(1-q)\prod_{n=1}^{\infty}(1-q^{4n+1})(1-q^{4n-1})(1-q^{4n})^{-2}\right)^{-4} \\ &\quad \times \left(q^{\frac{2}{2}(\frac{2}{4}-1)}(1-q^2)\prod_{n=1}^{\infty}(1-q^{4n+2})(1-q^{4n-2})(1-q^{4n})^{-2}\right)^{3} \\ &= \prod_{n=1}^{\infty}\left((1-q^{4n-3})^{-4}(1-q^{4n-1})^{-4}\right)\left((1-q^{4n-2})^2(1-q^{4n})^2\right)(1-q^{4n-2})^4 \\ &= \prod_{n=1}^{\infty}(1-q^{2n-1})^{-4}(1-q^{2n})^2(1-q^{2n-1})^4(1+q^{2n-1})^4\end{aligned}$$

$$= \prod_{n=1}^{\infty}(1-q^{2n})^2(1+q^{2n-1})^4$$
$$= \prod_{n=1}^{\infty}\bigl((1-(-q)^{2n})^4(1-(-q)^{2n-1})^4\bigr)(1-(-q)^{2n})^{-2}$$
$$= \prod_{n=1}^{\infty}(1-(-q)^n)^4(1-(-q)^n)^{-2}(1+(-q)^n)^{-2}$$
$$= \prod_{n=1}^{\infty}\left(\frac{1-(-q)^n}{1+(-q)^n}\right)^2.$$

Therefore, we get an infinite product formula for $\Theta(\tau)^2$ which might be a simple example of how to use Klein forms. However, one can directly obtain this result from (1.2).

Example 3.4. If $A = \begin{pmatrix} 2&0&1&1\\ 0&4&0&1\\ 1&0&2&0\\ 1&1&0&2 \end{pmatrix}$, then A has positive eigenvalues $\frac{5}{2} \pm \frac{\sqrt{9\pm4\sqrt{3}}}{2}$, which shows that A is positive definite. Its associated quadratic form is $Q = x_1^2 + 2x_2^2 + x_3^2 + x_4^2 + x_1x_3 + x_1x_4 + x_2x_4$ and the theta function $\Theta_Q(\tau)$ has the expansion

$$\Theta_Q(\tau) = 1 + 12q + 14q^2 + 48q^3 + 36q^4 + 56q^5 + 56q^6 + 84q^7 + 70q^8 + 156q^9 + 48q^{10} + 140q^{11} + 144q^{12} + \cdots. \tag{3.1}$$

If $N = 13$, then $NA^{-1} = \begin{pmatrix} 14&2&-7&-8\\ 2&4&-1&-3\\ -7&-1&10&4\\ -8&-3&4&12 \end{pmatrix}$ has even diagonal entries. Hence $\Theta_Q(\tau)$ belongs to $M_2(\Gamma_1(13))$ by Proposition 3.2.

We know that $M_2(\Gamma_1(13))$ is of dimension 13 [9, §2.6] and all the inequivalent cusps for $\Gamma_1(13)$ are given by

$$\frac{a}{c} = \frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \frac{1}{13}, \frac{2}{13}, \frac{3}{13}, \frac{4}{13}, \frac{5}{13}, \frac{6}{13} \tag{3.2}$$

[9, §1.6]. Consider a function

$$\mathfrak{k}(\tau) = \prod_{t=1}^{6}\mathfrak{k}_{(\frac{t}{13},0)}(13\tau)^{m(t)} \quad \text{with } m(1),\dots,m(6) \in \mathbb{Z}.$$

(By Proposition 2.1(i) we confine ourselves to the case $1 \leqslant t \leqslant 6$.) For each cusp a/c we take a matrix $\alpha_{a/c} \in \mathrm{SL}_2(\mathbb{Z})$ so that $\alpha_{a/c}(\infty) = a/c$, for example

$$\alpha_{1/1} = \begin{pmatrix}1&0\\1&1\end{pmatrix}, \quad \alpha_{1/2} = \begin{pmatrix}1&0\\2&1\end{pmatrix}, \quad \alpha_{1/3} = \begin{pmatrix}1&0\\3&1\end{pmatrix}, \quad \alpha_{1/4} = \begin{pmatrix}1&0\\4&1\end{pmatrix},$$
$$\alpha_{1/5} = \begin{pmatrix}1&0\\5&1\end{pmatrix}, \quad \alpha_{1/6} = \begin{pmatrix}1&0\\6&1\end{pmatrix}, \quad \alpha_{1/13} = \begin{pmatrix}1&0\\13&1\end{pmatrix}, \quad \alpha_{2/13} = \begin{pmatrix}2&1\\13&7\end{pmatrix},$$
$$\alpha_{3/13} = \begin{pmatrix}3&-1\\13&-4\end{pmatrix}, \quad \alpha_{4/13} = \begin{pmatrix}4&-1\\13&-3\end{pmatrix}, \quad \alpha_{5/13} = \begin{pmatrix}5&-2\\13&-5\end{pmatrix}, \quad \alpha_{6/13} = \begin{pmatrix}6&-1\\13&-2\end{pmatrix}.$$

Note that $\alpha_{a/c} = \begin{pmatrix}a&*\\c&*\end{pmatrix}$. We then obtain a criterion by Theorem 2.6 for $\mathfrak{k}(\tau)$ to belong to the space $M_2(\Gamma_1(13))$, namely

$$\sum_{t=1}^{6}m(t) = -2, \qquad \sum_{t=1}^{6}m(t)t^2 \equiv 0 \pmod{13} \quad \text{and}$$
$$\mathrm{ord}_q\bigl(\mathfrak{k}(\tau)|[\alpha_{a/c}]_2\bigr) = \frac{\gcd(c,13)^2}{26}\sum_{t=1}^{6}m(t)\left\langle\frac{at}{\gcd(c,13)}\right\rangle\left(\left\langle\frac{at}{\gcd(c,13)}\right\rangle - 1\right) \geqslant 0 \quad \text{for all } \frac{a}{c} \text{ in (3.2)}.$$

Thus one can readily find such $\mathfrak{k}(\tau)$'s as in the following Table 1. Here we use the notation

$$\prod_{t=1}^{6}(t)^{m(t)} := \prod_{t=1}^{6}\mathfrak{k}_{(\frac{t}{13},0)}(13\tau)^{m(t)}.$$

Table 1
Modular forms for $\Gamma_1(13)$ of weight 2.

$\mathfrak{k}(\tau)$	$\mathrm{ord}_q(\mathfrak{k}(\tau)\vert[\alpha_{6/13}]_2)$
$\mathfrak{k}_1(\tau) := (1)^{-3}(2)^{-2}(3)^5(4)^{-2}(5)^{-1}(6)^1$ $= 1+3q+8q^2+11q^3+17q^4+17q^5+28q^6+26q^7+39q^8+27q^9+48q^{10}+35q^{11}+59q^{12}+\cdots$	0
$\mathfrak{k}_2(\tau) := (1)^{-4}(2)^1(3)^3(4)^{-5}(5)^5(6)^{-2}$ $= 1+4q+9q^2+13q^3+18q^4+24q^5+31q^6+31q^7+36q^8+44q^9+54q^{10}+46q^{11}+47q^{12}+\cdots$	1
$\mathfrak{k}_3(\tau) := (1)^{-4}(3)^4(4)^{-2}$ $= 1+4q+10q^2+16q^3+21q^4+24q^5+30q^6+36q^7+42q^8+46q^9+54q^{10}+60q^{11}+59q^{12}+\cdots$	2
$\mathfrak{k}_4(\tau) := (1)^{-4}(2)^{-1}(3)^5(4)^1(5)^{-5}(6)^2$ $= 1+4q+11q^2+19q^3+25q^4+26q^5+27q^6+36q^7+49q^8+59q^9+59q^{10}+57q^{11}+66q^{12}+\cdots$	3
$\mathfrak{k}_5(\tau) := (1)^{-4}(3)^3(4)^1(5)^{-3}(6)^1$ $= 1+4q+10q^2+17q^3+22q^4+25q^5+28q^6+35q^7+44q^8+51q^9+56q^{10}+57q^{11}+59q^{12}+\cdots$	4
$\mathfrak{k}_6(\tau) := (1)^{-4}(2)^1(3)^1(4)^1(5)^{-1}$ $= 1+4q+9q^2+15q^3+20q^4+24q^5+28q^6+33q^7+40q^8+47q^9+52q^{10}+53q^{11}+53q^{12}+\cdots$	5
$\mathfrak{k}_7(\tau) := (1)^{-4}(2)^1(3)^1(4)^2(5)^{-4}(6)^2$ $= 1+4q+9q^2+15q^3+19q^4+23q^5+29q^6+35q^7+42q^8+43q^9+45q^{10}+53q^{11}+56q^{12}+\cdots$	6
$\mathfrak{k}_8(\tau) := (1)^{-5}(2)^2(3)^2(4)^2(5)^{-5}(6)^2$ $= 1+5q+13q^2+23q^3+29q^4+30q^5+33q^6+43q^7+59q^8+67q^9+66q^{10}+71q^{11}+79q^{12}+\cdots$	7
$\mathfrak{k}_9(\tau) := (1)^{-5}(2)^3(4)^2(5)^{-3}(6)^1$ $= 1+5q+12q^2+20q^3+26q^4+29q^5+34q^6+42q^7+51q^8+60q^9+64q^{10}+68q^{11}+72q^{12}+\cdots$	8
$\mathfrak{k}_{10}(\tau) := (1)^{-5}(2)^4(3)^{-2}(4)^2(5)^{-1}$ $= 1+5q+11q^2+17q^3+24q^4+29q^5+32q^6+40q^7+48q^8+53q^9+61q^{10}+64q^{11}+62q^{12}+\cdots$	9
$\mathfrak{k}_{11}(\tau) := (1)^{-5}(2)^5(3)^{-3}(4)^1(5)^{-2}(6)^2$ $= 1+5q+10q^2+13q^3+19q^4+28q^5+34q^6+40q^7+41q^8+40q^9+53q^{10}+60q^{11}+54q^{12}+\cdots$	10
$\mathfrak{k}_{12}(\tau) := (1)^{-5}(2)^5(3)^{-4}(4)^3(5)^{-2}(6)^1$ $= 1+5q+10q^2+14q^3+22q^4+28q^5+29q^6+42q^7+47q^8+39q^9+58q^{10}+60q^{11}+47q^{12}+\cdots$	11
$\mathfrak{k}_{13}(\tau) := (1)^{-6}(2)^6(3)^{-3}(4)^3(5)^{-3}(6)^1$ $= 1+6q+15q^2+23q^3+30q^4+36q^5+39q^6+50q^7+63q^8+65q^9+76q^{10}+84q^{11}+81q^{12}+\cdots$	12

Since $\mathrm{ord}_q(\mathfrak{k}_m(\tau)|[\alpha_{6/13}]_2)$ $(m=1,\ldots,13)$ are all distinct, the set $\{\mathfrak{k}_1(\tau),\ldots,\mathfrak{k}_{13}(\tau)\}$ forms a basis of $M_2(\Gamma_1(13))$ over $\mathbb{C}$. Hence $\Theta_Q(\tau)$ is a linear combination of these $\mathfrak{k}_m(\tau)$ over $\mathbb{C}$, namely

$$\Theta_Q(\tau)=\sum_{n=0}^{\infty} r_Q(n)q^n=\sum_{m=1}^{13} y_m\mathfrak{k}_m(\tau)\quad\text{for some } y_1,\ldots,y_{13}\in\mathbb{C}. \tag{3.3}$$

If we set

$$\mathfrak{k}_m(\tau)=\sum_{n=0}^{\infty} c_{n,m}q^n\quad\text{for } m=1,2,\ldots,13,$$

then the relation (3.3) can be rewritten as

$$r_Q(n)=\sum_{m=1}^{13} c_{n,m}y_m\quad\text{for } n\geqslant 0.$$

In particular, from the above relations we have the linear system for $n=0,1,\ldots,12$

$$\begin{pmatrix} c_{0,1} & c_{0,2} & \cdots & c_{0,13}\\ c_{1,1} & c_{1,2} & \cdots & c_{1,13}\\ & & \vdots & \\ c_{12,1} & c_{12,2} & \cdots & c_{12,13}\end{pmatrix}\begin{pmatrix} y_1\\ y_2\\ \vdots\\ y_{13}\end{pmatrix}=\begin{pmatrix} r_Q(0)\\ r_Q(1)\\ \vdots\\ r_Q(12)\end{pmatrix}.$$

So, by using Table 1 and (3.1) we are able to determine

$$\begin{aligned}\Theta_Q(\tau)={}&-\mathfrak{k}_1(\tau)-10\mathfrak{k}_2(\tau)-19\mathfrak{k}_3(\tau)+9\mathfrak{k}_4(\tau)-24\mathfrak{k}_5(\tau)+36\mathfrak{k}_6(\tau)+21\mathfrak{k}_7(\tau)\\ &-37\mathfrak{k}_8(\tau)+35\mathfrak{k}_9(\tau)-9\mathfrak{k}_{10}(\tau)-17\mathfrak{k}_{11}(\tau)-\mathfrak{k}_{12}(\tau)+18\mathfrak{k}_{13}(\tau).\end{aligned}$$

Then, by the product expansion formula (1.4) we can easily get the Fourier expansion of $\Theta_Q(\tau)$ as follows:

$$\begin{aligned}
&1 + 12q + 14q^2 + 48q^3 + 36q^4 + 56q^5 + 56q^6 + 84q^7 + 70q^8 + 156q^9 + 48q^{10} + 140q^{11} + 144q^{12} + 168q^{13} \\
&\quad + 72q^{14} + 224q^{15} + 132q^{16} + 216q^{17} + 182q^{18} + 252q^{19} + 168q^{20} + 336q^{21} + 120q^{22} + 288q^{23} + 280q^{24} \\
&\quad + 252q^{25} + 170q^{26} + 480q^{27} + 252q^{28} + 360q^{29} + 192q^{30} + 420q^{31} + 294q^{32} + 560q^{33} + 252q^{34} + 288q^{35} \\
&\quad + 468q^{36} + 504q^{37} + 216q^{38} + 672q^{39} + 240q^{40} + 560q^{41} + 288q^{42} + 528q^{43} + 420q^{44} + 728q^{45} + 336q^{46} \\
&\quad + 644q^{47} + 528q^{48} + 516q^{49} + 294q^{50} + 864q^{51} + 504q^{52} + 648q^{53} + 560q^{54} + 480q^{55} + 360q^{56} + 1008q^{57} \\
&\quad + 420q^{58} + 812q^{59} + 672q^{60} + 744q^{61} + 360q^{62} + 1092q^{63} + 516q^{64} + 680q^{65} + 480q^{66} + 924q^{67} + 648q^{68} \\
&\quad + 1152q^{69} + 336q^{70} + 980q^{71} + 910q^{72} + 1008q^{73} + 432q^{74} + 1008q^{75} + 756q^{76} + 720q^{77} + 680q^{78} + 960q^{79} \\
&\quad + 616q^{80} + 1452q^{81} + 480q^{82} + 1148q^{83} + 1008q^{84} + 1008q^{85} + 616q^{86} + 1440q^{87} + 600q^{88} + 1232q^{89} \\
&\quad + 624q^{90} + 1020q^{91} + 864q^{92} + 1680q^{93} + 552q^{94} + 864q^{95} + 1176q^{96} + 1344q^{97} + 602q^{98} + 1820q^{99} \\
&\quad + 756q^{100} + 1224q^{101} + 1008q^{102} + 1248q^{103} + 850q^{104} + 1152q^{105} + 756q^{106} + 1296q^{107} + 1440q^{108} \\
&\quad + 1512q^{109} + 560q^{110} + 2016q^{111} + 924q^{112} + 1368q^{113} + 864q^{114} + 1344q^{115} + 1080q^{116} + 2184q^{117} \\
&\quad + 696q^{118} + 1512q^{119} + 960q^{120} + 1332q^{121} + 868q^{122} + 2240q^{123} + 1260q^{124} + 1456q^{125} + 936q^{126} \\
&\quad + 1536q^{127} + 1190q^{128} + 2112q^{129} + 672q^{130} + 1584q^{131} + 1680q^{132} + 1296q^{133} + 792q^{134} + 2240q^{135} \\
&\quad + 1260q^{136} + 1904q^{137} + 1344q^{138} + 1680q^{139} + 864q^{140} + 2576q^{141} + 840q^{142} + 1700q^{143} + 1716q^{144} \\
&\quad + 1680q^{145} + 864q^{146} + 2064q^{147} + 1512q^{148} + 2072q^{149} + 1176q^{150} + 2100q^{151} + 1080q^{152} + 2808q^{153} \\
&\quad + 840q^{154} + 1440q^{155} + 2016q^{156} + 1896q^{157} + 1120q^{158} + 2592q^{159} + 1008q^{160} + 2016q^{161} + 1694q^{162} \\
&\quad + 2268q^{163} + 1680q^{164} + 1920q^{165} + 984q^{166} + 2324q^{167} + 1440q^{168} + 2196q^{169} + 864q^{170} + 3276q^{171} \\
&\quad + 1584q^{172} + 2088q^{173} + 1680q^{174} + 1764q^{175} + 1540q^{176} + 3248q^{177} + 1056q^{178} + 2160q^{179} + 2184q^{180} \\
&\quad + 2184q^{181} + 1008q^{182} + 2976q^{183} + 1680q^{184} + 1728q^{185} + 1440q^{186} + 2520q^{187} + 1932q^{188} + 3360q^{189} \\
&\quad + 1008q^{190} + 2304q^{191} + 2064q^{192} + 2688q^{193} + 1152q^{194} + 2720q^{195} + 1548q^{196} + 2744q^{197} + 1560q^{198} \\
&\quad + 2400q^{199} + 1470q^{200} + 3696q^{201} + 1428q^{202} + 2520q^{203} + 2592q^{204} + 1920q^{205} + 1456q^{206} + 3744q^{207} \\
&\quad + 1848q^{208} + 2160q^{209} + 1344q^{210} + 2544q^{211} + 1944q^{212} + 3920q^{213} + 1512q^{214} + 2464q^{215} + 2800q^{216} \\
&\quad + 2160q^{217} + 1296q^{218} + 4032q^{219} + 1440q^{220} + 3024q^{221} + 1728q^{222} + 3108q^{223} + 1512q^{224} + 3276q^{225} \\
&\quad + 1596q^{226} + 3164q^{227} + 3024q^{228} + 3192q^{229} + 1152q^{230} + 2880q^{231} + 2100q^{232} + 2808q^{233} + 2210q^{234} \\
&\quad + 2208q^{235} + 2436q^{236} + 3840q^{237} + 1296q^{238} + 3332q^{239} + 2464q^{240} + 3360q^{241} + 1554q^{242} + 4368q^{243} \\
&\quad + 2232q^{244} + 2408q^{245} + 1920q^{246} + 3060q^{247} + 1800q^{248} + 4592q^{249} + 1248q^{250} + 3024q^{251} + 3276q^{252} \\
&\quad + 3360q^{253} + 1792q^{254} + 4032q^{255} + 2052q^{256} + 3096q^{257} + 2464q^{258} + 2592q^{259} + 2040q^{260} + 4680q^{261} \\
&\quad + 1848q^{262} + 3168q^{263} + 2400q^{264} + 3024q^{265} + 1512q^{266} + 4928q^{267} + 2772q^{268} + 3240q^{269} + 1920q^{270} \\
&\quad + 3780q^{271} + 2376q^{272} + 4080q^{273} + 1632q^{274} + 2940q^{275} + 3456q^{276} + 3336q^{277} + 1960q^{278} + 5460q^{279} \\
&\quad + 1680q^{280} + 3920q^{281} + 2208q^{282} + 3408q^{283} + 2940q^{284} + 3456q^{285} + 1680q^{286} + 2880q^{287} + 3822q^{288} \\
&\quad + 3684q^{289} + 1440q^{290} + 5376q^{291} + 3024q^{292} + 4088q^{293} + 2408q^{294} + 2784q^{295} + 2160q^{296} + 5600q^{297} \\
&\quad + 1776q^{298} + 4032q^{299} + 3024q^{300} + 3696q^{301} + 1800q^{302} + 4896q^{303} + 2772q^{304} + 3472q^{305} + 3276q^{306} \\
&\quad + 4284q^{307} + 2160q^{308} + 4992q^{309} + 1680q^{310} + 3744q^{311} + 3400q^{312} + 3768q^{313} + 2212q^{314} + 3744q^{315} \\
&\quad + 2880q^{316} + 4424q^{317} + 3024q^{318} + 4200q^{319} + 2408q^{320} + 5184q^{321} + 1728q^{322} + 4536q^{323} + 4356q^{324} \\
&\quad + 3528q^{325} + 1944q^{326} + 6048q^{327} + 2400q^{328} + 3312q^{329} + 2240q^{330} + 4620q^{331} + 3444q^{332} + 6552q^{333} \\
&\quad + 1992q^{334} + 3168q^{335} + 3696q^{336} + 4056q^{337} + 2198q^{338} + 5472q^{339} + 3024q^{340} + 3600q^{341} + 2808q^{342} \\
&\quad + 4200q^{343} + 3080q^{344} + 5376q^{345} + 2436q^{346} + 4176q^{347} + 4320q^{348} + 4872q^{349} + 1512q^{350} + \cdots.
\end{aligned}$$

From this expansion we happen to numerically find some interesting identities which will be conditionally proved in Section 4, and so we would like to pose it as a question.

$$r_Q\left(p^2n\right)=\frac{r_Q(p^2)r_Q(n)}{r_Q(1)}\quad\text{for any prime } p\neq 13 \text{ and any integer } n\geqslant 1 \text{ prime to } p. \tag{3.4}$$

Suppose that (3.4) is true. Let $\ell\geqslant 2$ be a square-free integer which is not divisible by 13 and has the prime factorization $\ell=p_1\cdots p_m$. If n is a positive integer prime to ℓ, then we derive that

$$\begin{aligned}r_Q\left(\ell^2n\right)&=\frac{r_Q(p_1^2)r_Q(p_2^2\cdots p_m^2n)}{r_Q(1)}=\cdots=\frac{r_Q(p_1^2)\cdots r_Q(p_m^2)r_Q(n)}{r_Q(1)^m}\\&=\frac{r_Q(p_1^2p_2^2)r_Q(p_3^2)\cdots r_Q(p_m^2)r_Q(n)}{r_Q(1)^{m-1}}=\cdots=\frac{r_Q(p_1^2\cdots p_m^2)r_Q(n)}{r_Q(1)}=\frac{r_Q(\ell^2)r_Q(n)}{r_Q(1)}.\end{aligned}$$

Hence we can allow p to be a square-free positive integer not divisible by 13 in the question (3.4).

However, unfortunately, the general relation $r_Q(mn)=r_Q(m)r_Q(n)/r_Q(1)$ for relatively prime positive integers m and n does not hold because $r_Q(2\cdot 5)=48$ and $r_Q(2)r_Q(5)/r_Q(1)=196/3$.

Example 3.5. By Remark 2.3 any product of Klein forms is of integral weight. So we cannot express $\eta(\tau)$ in terms of Klein forms. However, as described in [6, Chapter 3, Lemma 5.1] we have the relation

$$\eta(\tau)^2=\sqrt{\frac{2}{3}}(1-i)\mathfrak{k}_{(\frac12,0)}(\tau)\mathfrak{k}_{(0,\frac12)}(\tau)\mathfrak{k}_{(\frac12,\frac12)}(\tau)\left(\mathfrak{k}_{(\frac13,0)}(\tau)\mathfrak{k}_{(0,\frac13)}(\tau)\mathfrak{k}_{(\frac13,\frac13)}(\tau)\mathfrak{k}_{(\frac13,-\frac13)}(\tau)\right)^{-1}.$$

Let $k\geqslant 0$ be an integer and $M_{k/2}(\Gamma_0(4))$ stand for the space of modular forms for $\Gamma_0(4)$ of weight $k/2$ [4, Chapter IV, §1]. Let

$$F(\tau)=\sum_{n\geqslant 1\text{ odd}}\left(\sum_{d>0,\,d|n}d\right)q^n=q+4q^3+6q^5+8q^7+\cdots.$$

Then one can assign the weight 1/2 to $\Theta(\tau)=\eta(2\tau)^5/\eta(\tau)^2\eta(4\tau)^2$ and 2 to $F(\tau)$, respectively. And, as is well known $M_{k/2}(\Gamma_0(4))$ is the space of all polynomials in $\mathbb{C}[\Theta(\tau),F(\tau)]$ having pure weight $k/2$ [4, Chapter IV, Proposition 4]. By Theorem 2.6 we see that the functions

$$\begin{aligned}&\mathfrak{k}_{(\frac24,0)}(4\tau)^{-2}=q+4q^3+6q^5+8q^7+\cdots,\\&\mathfrak{k}_{(\frac14,0)}(4\tau)^{-8}\mathfrak{k}_{(\frac24,0)}(4\tau)^{6}=1+8q+24q^2+32q^3+24q^4+48q^5+96q^6+64q^7+\cdots\end{aligned}$$

belong to $M_2(\Gamma_1(4))=M_2(\Gamma_0(4))$ (Proposition 3.1) which is of dimension 2 [8, Theorem 1.49]. Thus they form a basis of $M_2(\Gamma_0(4))$, from which we get the following infinite product expansion

$$F(\tau)=\mathfrak{k}_{(\frac24,0)}(4\tau)^{-2}=q\prod_{n=1}^{\infty}\left(\frac{1-q^{4n}}{1-q^{4n-2}}\right)^4=\frac{\eta(4\tau)^8}{\eta(2\tau)^4}.$$

4. Hecke operators

Throughout this section by $\Theta_Q(\tau)=\sum_{n=0}^{\infty}r_Q(n)q^n$ we mean the theta function associated with the quadratic form $Q=x_1^2+2x_2^2+x_3^2+x_4^2+x_1x_3+x_1x_4+x_2x_4$ studied in Example 3.4. We shall answer, under some condition, the question raised in (3.4) by making use of Hecke operators on $\Theta_Q(\tau)$.

Let $N\geqslant 1$ and k be integers, and let $f(\tau)=\sum_{n=0}^{\infty}a(n)q^n\in M_k(\Gamma_0(N),\chi)$ for a Dirichlet character χ modulo N. For a positive integer m, the *Hecke operator* $T_{m,k,\chi}$ on $f(\tau)$ is defined by

$$f(\tau)|T_{m,k,\chi}=\sum_{n=0}^{\infty}\left(\sum_{d>0,\,d|\gcd(m,n)}\chi(d)d^{k-1}a\left(mn/d^2\right)\right)q^n. \tag{4.1}$$

Here we set $\chi(d)=0$ if $\gcd(N,d)\neq 1$. As is well known, the operator $T_{m,k,\chi}$ preserves the space $M_k(\Gamma_0(N),\chi)$ [4, Propositions 36 and 39].

From now on, we let χ be the Dirichlet character defined by

$$\chi(d)=\left(\frac{13}{d}\right)\quad\text{for } d\in\mathbb{Z}-13\mathbb{Z}.$$

Lemma 4.1. *The functions $\Theta_Q(\tau)$ and $\Theta_Q(\tau)|T_{13,2,\chi}$ form a basis of $M_2(\Gamma_0(13),\chi)$ over $\mathbb{C}$.*

Proof. Note that $M_2(\Gamma_0(13),\chi)$ is of dimension 2 over $\mathbb{C}$ [8, Theorem 1.34 and Remark 1.35]. We see from (3.1) and the definition of Hecke operator that

$$\Theta_Q(\tau)=1+12q+14q^2+\cdots,$$
$$\Theta_Q(\tau)|T_{13,2,\chi}=1+168q+170q^2+\cdots.$$

Since they are linearly independent over $\mathbb{C}$, these form a basis of $M_2(\Gamma_0(13),\chi)$. □

Remark 4.2. If $f(\tau)=\sum_{n=0}^{\infty}a(n)q^n\in M_2(\Gamma_0(13),\chi)$, then it can be written as $c_1\Theta_Q(\tau)+c_2\Theta_Q(\tau)|T_{13,2,\chi}$ for some $c_1,c_2\in\mathbb{C}$. Since $\left(\begin{smallmatrix}1&1\\12&168\end{smallmatrix}\right)$ is invertible, c_1 and c_2 can be determined only by $a(0)$ and $a(1)$. In particular, $a(1)=12a(0)=r_Q(1)a(0)$ if and only if $f(\tau)=a(0)\Theta_Q(\tau)$.

Proposition 4.3. *If p is a prime which satisfies*

$$r_Q(p)=r_Q(1)\bigl(1+\chi(p)p\bigr)\quad or \tag{4.2}$$
$$r_Q\bigl(p^2\bigr)=r_Q(1)\bigl(1+\chi(p)p+p^2\bigr), \tag{4.3}$$

then

$$r_Q\bigl(p^2n\bigr)=\frac{r_Q(p^2)r_Q(n)}{r_Q(1)}\quad \textit{for any integer } n\geqslant 1 \textit{ prime to } p.$$

Proof. Let p be such a prime. We get from the definition (4.1) and the fact $r_Q(0)=1$ that

$$\Theta_Q(\tau)|T_{p,2,\chi}=\bigl(1+\chi(p)p\bigr)+r_Q(p)q+\cdots,$$
$$\Theta_Q(\tau)|T_{p^2,2,\chi}=\bigl(1+\chi(p)p+p^2\bigr)+r_Q\bigl(p^2\bigr)q+\cdots.$$

By Remark 4.2 we deduce the assertions

$$r_Q(p)=r_Q(1)\bigl(1+\chi(p)p\bigr)\quad\Longleftrightarrow\quad \Theta_Q(\tau)|T_{p,2,\chi}=\bigl(1+\chi(p)p\bigr)\Theta_Q(\tau), \tag{4.4}$$
$$r_Q\bigl(p^2\bigr)=r_Q(1)\bigl(1+\chi(p)p+p^2\bigr)\quad\Longleftrightarrow\quad \Theta_Q(\tau)|T_{p^2,2,\chi}=\bigl(1+\chi(p)p+p^2\bigr)\Theta_Q(\tau). \tag{4.5}$$

First, suppose that p satisfies (4.2). For any integer $n\geqslant 1$ which is prime to p, we obtain from the definition (4.1) and (4.4) that

$$r_Q\bigl(p^2\bigr)+\chi(p)pr_Q(1)=\bigl(1+\chi(p)p\bigr)r_Q(p)\quad \text{by comparing the coefficients of } q^p.$$

Thus we derive by (4.2) that

$$r_Q\bigl(p^2\bigr)=r_Q(1)\bigl(1+\chi(p)p+p^2\bigr),$$

which becomes the condition (4.3).

So we may assume that p satisfies (4.3). Now, for $n\geqslant 1$ prime to p, we achieve from (4.1) and (4.5) that

$$\begin{aligned} r_Q\bigl(p^2n\bigr)&=\bigl(1+\chi(p)p+p^2\bigr)r_Q(n)\quad \text{by comparing the coefficients of } q^n\\ &=r_Q\bigl(p^2\bigr)r_Q(n)/r_Q(1)\quad \text{by (4.3)}.\end{aligned}$$

This completes the proof. □

Remark 4.4.

(i) If $p\neq 13$ is a prime satisfying (4.2), then $\chi(p)$ should be 1 because $r_Q(p)\geqslant 0$.
(ii) By using the explicit Fourier expansion of $\Theta_Q(\tau)$ given in Example 3.4 we can find small primes satisfying the condition (4.2) or (4.3). For example, $p=3$, 17, 23, 29, 43, 53, 61, 79, 101, 103, 107, 113, 127, 131, 139, 157, 173, 179, 181, 191, 199, 233, 251, 247, 263, 269, 277, 283, 311, 313, 337, 347 satisfy (4.2). And, $p=2$, 5, 7, 11 satisfy (4.3).
(iii) We predict that every prime $p\neq 13$ satisfies (4.3).
(iv) If a prime p satisfies (4.2) or (4.3), then one can easily find a formula for $r_Q(p^n)$ for $n\geqslant 1$. For example, $p=3$ satisfies (4.2). It follows from (4.4) that

$$\Theta_Q(\tau)|T_{3,2,\chi}=\bigl(1+\chi(3)3\bigr)\Theta_Q(\tau)=4\Theta_Q(\tau).$$

Comparing the coefficients of the term $q^{3^{n-1}}$ $(n\geqslant 2)$ on both sides we have

$$r_Q(3^n) + 3r_Q(3^{n-2}) = 4r_Q(3^{n-1}),$$

which can be rewritten as

$$r_Q(3^n) - r_Q(3^{n-1}) = 3\big(r_Q(3^{n-1}) - r_Q(3^{n-2})\big).$$

Hence we conclude that

$$r_Q(3^n) = r_Q(3^1) + \big(r_Q(3^1) - r_Q(3^0)\big)\sum_{j=1}^{n-1} 3^j = 6\big(3^{n+1} - 1\big) \quad \text{for } n \geqslant 2.$$

Observe that this formula is also true for $n = 0$ and 1.

(v) Let A be any 4×4 positive definite symmetric matrix over $\mathbb{Z}$ with even diagonal entries. Suppose further that $\det(A)A^{-1}$ has even diagonal entries. Then, one can apply Proposition 4.3 to the coefficients of the theta function associated with A provided that the space $M_2(\Gamma_0(\det(A)), (\frac{\det(A)}{\cdot}))$ is of dimension 2. Unfortunately, however, there does not seem to be general results on the construction of a basis of the space $M_2(\Gamma_1(\det(A)))$ (by means of products of Klein forms).

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