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# Boundary behaviour of the unique solution to a singular Dirichlet problem with a convection term ☆

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**Abstract**

By Karamata regular variation theory and constructing comparison functions, we derive that the boundary behaviour of the unique solution to a singular Dirichlet problem $-\Delta u = b(x)g(u) + \lambda|\nabla u|^q$, $u > 0$, $x \in \Omega$, $u|_{\partial\Omega} = 0$, which is independent of $\lambda|\nabla u_\lambda|^q$, where $\Omega$ is a bounded domain with smooth boundary in $\mathbb{R}^N$, $\lambda \in \mathbb{R}$, $q \in (0, 2]$, $\lim_{s\to 0^+} g(s) = +\infty$, and $b$ is non-negative on $\Omega$, which may be vanishing on the boundary.



*Keywords:* Semilinear elliptic equations; Dirichlet problem; Singularity; Convection term; Weight; Karamata regular variation theory; Unique solution; Boundary behaviour

## 1. Introduction and the main results

The purpose of this paper is to investigate the boundary behaviour of the unique classical solution to the following model problem

$$-\Delta u = b(x)g(u) + \lambda|\nabla u|^q, \quad u > 0,\ x \in \Omega, \qquad u|_{\partial\Omega} = 0, \tag{1.1}$$

where $\Omega$ is a bounded domain with smooth boundary in $\mathbb{R}^N$ $(N \geqslant 1)$, $\lambda \in \mathbb{R}$, $q \in (0, 2]$, $g$ satisfies

$(\mathrm{g}_1)$ $g \in C^1((0,\infty),(0,\infty))$, $g'(s) < 0$ for all $s > 0$, $\lim_{s\to 0^+} g(s) = +\infty$;

and $b$ satisfies

$(\mathrm{b}_1)$ $b \in C^\alpha(\overline{\Omega})$ for some $\alpha \in (0, 1)$, is non-negative in $\Omega$ and positive near the boundary $\partial\Omega$.

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The problem arises in the study of non-Newtonian fluids, boundary layer phenomena for viscous fluids, chemical heterogeneous catalysts, as well as in the theory of heat conduction in electrical materials (see [4,7,10,17,21]).

For $\lambda = 0$, i.e., problem (1.1) reads the following one

$$-\Delta u = b(x)g(u), \quad u > 0, \ x \in \Omega, \qquad u|_{\partial\Omega} = 0. \tag{1.2}$$

Problem (1.2) was discussed in a number of works; see, for instance, [3,4,7,8,13–15,20–24,26,27].

For $b \equiv 1$ on $\Omega$: when $g$ satisfies $(g_1)$, Fulks and Maybee [7], Stuart [21], Crandall, Rabinowitz and Tartar [4] showed that problem (1.2) has a unique solution $u \in C^{2+\alpha}(\Omega) \cap C(\overline{\Omega})$. Moreover, Theorems 2.2 and 2.5 in [4] showed that if $\varphi_1 \in C[0,a] \cap C^2(0,a]$ is the local solution to the problem

$$-\varphi_1''(s) = g\big(\varphi_1(s)\big), \quad \varphi_1(s) > 0, \ 0 < s < a, \qquad \varphi_1(0) = 0, \tag{1.3}$$

then there exist positive constants $C_1$ and $C_2$ such that

(I) $C_1\varphi_1(d(x)) \leqslant u(x) \leqslant C_2\varphi_1(d(x))$ near $\partial\Omega$, where $d(x) = \operatorname{dist}(x, \partial\Omega)$.

In particular, when $g(u) = u^{-\gamma}$, $\gamma > 1$, $u$ has the property:

$(\mathrm{I}_1)$ $C_1[d(x)]^{2/(1+\gamma)} \leqslant u(x) \leqslant C_2[d(x)]^{2/(1+\gamma)}$ near $\partial\Omega$.

In [15], by constructing a pair of global subsolution and supersolution, Lazer and McKenna showed that $(\mathrm{I}_1)$ continues to hold on $\overline{\Omega}$. Then $u \in H_0^1(\Omega)$ if and only if $\gamma < 3$. This is a basic character to problem (1.2) for $g(u) = u^{-\gamma}$ with $\gamma > 0$.

When $\lambda = \pm 1$, $0 < q < 2$, $b(x) \equiv 1$ on $\Omega$ and the function $g : (0,\infty) \to (0,\infty)$ is locally Lipschitz continuous and decreasing, Giarrusso and Porru [11] showed that if $g$ satisfies the following conditions:

$(g_2)$ $\int_0^1 g(s)\,ds = \infty$, $\int_1^\infty g(s)\,ds < \infty$,
$(g_3)$ let $G_1(t) = \int_t^\infty g(s)\,ds$, $t > 0$; there exist positive constants $\delta$ and $M$ with $M > 1$ such that $G_1(t) < MG_1(2t)$, $\forall t \in (0,\delta)$,

then the unique solution $u$ to problem (1.1) has the properties:

$(\mathrm{II}_1)$ $|u(x) - \varphi_2(d(x))| < C_0 d(x)$, $\forall x \in \Omega$ for $0 < q \leqslant 1$;
$(\mathrm{II}_2)$ $|u(x) - \varphi_2(d(x))| < C_0 d(x)[G_1(\varphi_2(d(x)))]^{(q-1)/2}$, $\forall x \in \Omega$ for $1 < q < 2$, where $C_0$ is a suitable positive constant and $\varphi_2 \in C[0,\infty) \cap C^2(0,\infty)$ is uniquely determined by

$$\int_0^{\varphi_2(t)} \frac{ds}{\sqrt{2G_1(s)}} = t, \quad t > 0. \tag{1.4}$$

These imply that

$$\lim_{d(x)\to 0} \frac{u(x)}{\varphi_2(d(x))} = 1. \tag{1.5}$$

In particular, if $g(u) = u^{-\gamma}$, $\gamma > 1$, then $\varphi_2(s) = cs^{2/(1+\gamma)}$, $c = [\frac{(1+\gamma)^2}{2(\gamma-1)}]^{1/(1+\gamma)}$.

For other works, see [5,6,9,10,25,28,30] and the references therein.

Our approach relies on Karamata regular variation functions, which was first introduced and established by Karamata in 1930 and is a basic tool in stochastic process, see [16,18,19], and has been applied to study the boundary behaviour of solutions to boundary blow-up elliptic problems (see [1,2,29]) and singular nonlinear Dirichlet problems (see [24,26–28,30]).

**Definition 1.1.** A positive measurable function $f$ defined on $[a,\infty)$, for some $a>0$, is called *regularly varying at infinity* with index $\rho$, written $f\in RV_\rho$, if for each $\xi>0$ and some $\rho\in\mathbb{R}$,

$$\lim_{s\to\infty}\frac{f(\xi s)}{f(s)}=\xi^\rho. \tag{1.6}$$

In particular, when $\rho=0$, $f$ is called *slowly varying at infinity*.

Clearly, if $g\in RV_\rho$, then $L(s):=f(s)/s^\rho$ is slowly varying at infinity.

Some basic examples of slowly varying functions at infinity are:

(i) every measurable function on $[a,\infty)$ which has a positive limit at infinity;
(ii) $(\ln s)^q$ and $(\ln(\ln s))^q$, $q\in\mathbb{R}$;
(iii) $e^{(\ln s)^p}$, $0<p<1$.

We also see that a positive measurable function $g$ defined on $(0,a)$ for some $a>0$, is *regularly varying at zero* with index $\sigma$ (write $g\in RVZ_\sigma$) if $t\to g(1/t)$ belongs to $RV_{-\sigma}$.

**Proposition 1.1** *(Uniform convergence theorem). If $f\in RV_\rho$, then* (1.6) *holds uniformly for $\xi\in[a,b]$ with $0<a<b$.*

**Proposition 1.2** *(Representation theorem). A function $L$ is slowly varying at infinity if and only if it may be written in the form*

$$L(s)=\phi(s)\exp\left(\int_a^s\frac{y(\tau)}{\tau}\,d\tau\right),\quad s\geqslant a, \tag{1.7}$$

*for some $a>0$, where the functions $\phi$ and $y$ are measurable and for $s\to+\infty$, $y(s)\to0$ and $\phi(s)\to c_0$, with $c_0>0$.*

We call that

$$\hat{L}(s)=c_0\exp\left(\int_a^s\frac{y(\tau)}{\tau}\,d\tau\right),\quad s\geqslant a, \tag{1.8}$$

is *normalised* slowly varying at infinity and

$$f(s)=c_0s^\rho\hat{L}(s),\quad s\geqslant a, \tag{1.9}$$

is *normalised* regularly varying at infinity with index $\rho$ (write $f\in NRV_\rho$).

Similarly, $g$ is called *normalised* regularly varying at zero with index $\sigma$, written $g\in NRVZ_\sigma$ if $t\to g(1/t)$ belongs to $NRV_{-\sigma}$.

A function $f\in RV_\rho$ belongs to $NRV_\rho$ if and only if

$$f\in C^1[b,\infty)\quad\text{for some } b>0\quad\text{and}\quad\lim_{s\to\infty}\frac{sf'(s)}{f(s)}=\rho. \tag{1.10}$$

In this paper, by Karamata regular variation theory and constructing comparison functions, we derive the boundary behaviour to the unique solution to problem (1.1) for the weight $b$ which may be vanishing on the boundary.

Our main results are the following.

**Theorem 1.1.** *Let $\lambda\in\mathbb{R}$, $q\in(0,2]$, $b$ satisfy* ($b_1$)*, $g$ satisfy* ($g_1$) *and $g\in NRVZ_{-\gamma}$ with $\gamma>1$. Suppose that there exist a positive non-decreasing $C^1$-function $k\in NRVZ_{\sigma/2}$ with $\sigma\in[0,\gamma-1)$ and a positive constant $b_0$ such that*

($b_2$) $\lim_{d(x)\to0}\frac{b(x)}{k^2(d(x))}=b_0$,

*then the unique solution $u_\lambda\in C(\overline{\Omega})\cap C^2(\Omega)$ to problem* (1.1) *satisfies*

$$\lim_{d(x)\to0}\frac{u_\lambda(x)}{\varphi_1(K(d(x)))}=\xi_0, \tag{1.11}$$

*where* ${\xi_0}^{-\gamma-1} = \frac{2(\gamma-\sigma-1)}{b_0(2+\sigma)(\gamma-1)}$ *and* $\varphi_1 \in C[0,a] \cap C^2(0,a]$ *is the solution to problem* (1.3)*, i.e.,*

$$\int_0^{\varphi_1(t)} \frac{ds}{\sqrt{2G_2(s)}} = t, \quad t \in [0,a] \text{ for small } a > 0, \tag{1.12}$$

$$K(t) = \int_0^t k(s)\,ds, \quad t \in [0,a]; \qquad G_2(t) = \int_t^b g(s)\,ds, \quad t \in (0,b],\ b > 0. \tag{1.13}$$

*Moreover,* $\varphi_1 \in NRVZ_{2/(1+\gamma)}$ *and there exists* $y_2 \in C(0,a]$ *with* $\lim_{s\to 0^+} y_2(s) = 0$ *such that* $\varphi_1(t) = t^{2/(1+\gamma)} e^{\int_t^a \frac{y_2(s)}{s}\,ds}$, $t \in (0,a]$.

**Corollary 1.1.** *When* $g(u) = u^{-\gamma}$ *with* $\gamma > 1$ *in Theorem* 1.1*,* $u_\lambda$ *satisfies*

$$\lim_{d(x)\to 0} \frac{u_\lambda(x)}{[K(d(x))]^{2/(1+\gamma)}} = \left[\frac{b_0(2+\sigma)(1+\gamma)^2}{4(\gamma-\sigma-1)}\right]^{1/(1+\gamma)}. \tag{1.14}$$

**Remark 1.1.** By (1.12), we see that the asymptotic behaviour of $u_\lambda$ is independent of $\lambda|\nabla u_\lambda|^q$.

**Remark 1.2.** By ($g_1$) and the proof of the maximum [12, Theorems 10.1 and 10.2], we see that problem (1.1) has at most one solution in $C^2(\Omega) \cap C(\overline{\Omega})$. For the existence of solutions to problem (1.1), see [28].

**Remark 1.3.** Some examples of the functions which satisfy the conditions in Theorem 1.1 are:

(1) $g(u) = u^{-\gamma}$, where $\gamma > 1$;
(2) $g(u) = u^{-\gamma} \arctan(u^{-1})$, where $\gamma > 1$;
(3) $g(u) = u^{-\gamma_1}(\ln(1+u))^{-\gamma_2}$, where $\gamma_1 > 0$, $\gamma_2 > 0$ and $\gamma_1 + \gamma_2 > 1$;
(4) $g(u) = u^{-\gamma_1}(e^u - 1)^{-\gamma_2}$, where $\gamma_1 > 0$, $\gamma_2 > 0$ and $\gamma_1 + \gamma_2 > 1$;
(5) $g(u) = u^{-\gamma}(-\ln u)^p$, where $\gamma > 1$, $p \in \mathbb{R}$, $0 < u < a < 1$;
(6) $g(u) = u^{-\gamma} e^{(-\ln u)^p}$, where $\gamma > 1$, $p \in (0,1)$, $0 < u < a < 1$;
(7) $g(u) = u^{-\gamma}(\ln(-\ln u))^p$, where $\gamma > 1$, $p \in \mathbb{R}$, $0 < u < a < 1$.

**Remark 1.4.** The key of the paper is the estimation of $|\nabla u|^q$ which is very different from that one in [28] and [30] where the weight $b$ is singular on the boundary.

The outline of this paper is as follows. In Section 2 we recall some basic the properties to Karamata regular variation theory. The proof of Theorem 1.1 is given in Section 3.

## 2. Some basic definitions and the properties to Karamata regular variation theory

We recall some basic properties to Karamata regular variation theory (see [16,18,19]).

**Proposition 2.1.** *If functions* $L, L_1$ *are slowly varying at infinity, then*:

(i) $L^\sigma$ *for every* $\sigma \in \mathbb{R}$, $c_1 L + c_2 L_1$ ($c_1 \geqslant 0$, $c_2 \geqslant 0$ *with* $c_1 + c_2 > 0$), $L \circ L_1$ (*if* $L_1(t) \to +\infty$ *as* $t \to +\infty$) *are also slowly varying at infinity.*
(ii) *For every* $\theta > 0$ *and* $t \to +\infty$,

$$t^\theta L(t) \to +\infty, \quad t^{-\theta} L(t) \to 0.$$

(iii) *For* $\rho \in \mathbb{R}$ *and* $t \to +\infty$, $\ln(L(t))/\ln t \to 0$ *and* $\ln(t^\rho L(t))/\ln t \to \rho$.

**Proposition 2.2** *(Asymptotic behaviour). If a function $H$ is slowly varying at zero, then for $a > 0$ and $t \to 0^+$:*

(i) $\int_0^t s^\beta H(s)\,ds \cong (\beta+1)^{-1} t^{1+\beta} H(t)$, *for* $\beta > -1$;
(ii) $\int_t^a s^\beta H(s)\,ds \cong (-\beta-1)^{-1} t^{1+\beta} H(t)$, *for* $\beta < -1$.

Let $\Psi$ be non-decreasing on $\mathbb{R}$, we define (as in [18]) the inverse of $\Psi$ by

$$\Psi^{\leftarrow}(t) = \inf\{s:\ \Psi(s) \geqslant t\}. \tag{2.1}$$

**Proposition 2.3.** *(See [18, Proposition 0.8].) The following hold*:

(i) *if* $f_1 \in RV_{\rho_1}$, $f_2 \in RV_{\rho_2}$, *then* $f_1 \cdot f_2 \in RV_{\rho_1+\rho_2}$;
(ii) *if* $f_1 \in RV_{\rho_1}$, $f_2 \in RV_{\rho_2}$, *with* $\lim_{t\to+\infty} f_2(t) = +\infty$, *then* $f_1 \circ f_2 \in RV_{\rho_1\rho_2}$;
(iii) *if $f$ is non-decreasing on* $(a,\infty)$, $f \in RV_\rho$ *with* $\rho > 0$, *then* $f^{\leftarrow} \in RV_{\rho^{-1}}$.

By the above propositions, we can directly obtain the following results.

**Corollary 2.1.** *If $g$ satisfies* $(\mathrm{g}_1)$ *and* $g \in NRVZ_{-\gamma}$ *with* $\gamma > 1$, *then*:

(i) $g(t) = t^{-\gamma}\exp(\int_t^a \frac{y(s)}{s}\,ds)$, $0 < t < a$, $y \in C(0,a]$, $\lim_{s\to 0^+} y(s) = 0$;
(ii) $\lim_{t\to 0^+} g(t) = +\infty = \lim_{t\to 0^+} G_2(t)$; $\lim_{t\to 0^+} \frac{G_2(t)}{g(t)} = 0 = \lim_{t\to 0^+} \frac{\sqrt{G_2(t)}}{g(t)}$;
(iii) $\lim_{t\to 0^+} \frac{G_2(t)}{tg(t)} = \frac{1}{\gamma-1}$; $\lim_{t\to 0^+} \frac{tg'(t)}{g(t)} = -\gamma$.

**Corollary 2.2.** *$k$ in Theorem* 1.1 *has the following properties*:

(i) $k(t) = t^{\sigma/2}\exp(\int_t^a \frac{y_1(s)}{s}\,ds)$ *for* $t \in (0,a)$, $y_1 \in C(0,a]$, $\lim_{s\to 0^+} y_1(s) = 0$;
(ii) $\lim_{t\to 0^+} \frac{K(t)}{k(t)} = 0$; $\lim_{t\to 0^+} \frac{tk'(t)}{k(t)} = \frac{\sigma}{2}$; $\lim_{t\to 0^+} \frac{K(t)}{tk(t)} = \frac{2}{2+\sigma}$;
(iii) $\lim_{t\to 0^+} \frac{k'(t)K(t)}{k^2(t)} = \lim_{t\to 0^+} \frac{tk'(t)}{k(t)} \lim_{t\to 0^+} \frac{K(t)}{tk(t)} = \frac{\sigma}{2+\sigma}$.

## 3. The exact asymptotic behaviour

First we give some preliminary considerations.

**Lemma 3.1.** *Under the assumption in Theorem* 1.1:

(i) $\varphi_1 \in NRVZ_{2/(1+\gamma)}$;
(ii) $(g \circ \varphi_1 \circ K)^{q-1} \cdot K^q \cdot k^{q-2} \in RVZ_\beta$ *with* $\beta = \frac{(2-q)\gamma + q(\sigma+1) - \sigma}{1+\gamma}$.

**Proof.** (i) Let $f_1(t) = \int_0^t \frac{ds}{\sqrt{2G_2(s)}}$, $\forall t \in (0,a)$. By l'Hospital's rule and Proposition 2.3(iii), we can easily see that $f_1 \in RVZ_{(1+\gamma)/2}$ and $\varphi_1 = f_1^{-1} \in RVZ_{2/(1+\gamma)}$. Moreover, we see by (1.10), the following Lemma 3.2(i) and Proposition 2.2(i) that $\varphi_1' \in NRVZ_{-(\gamma-1)/(\gamma+1)}$ and $\lim_{t\to 0^+} \frac{t\varphi_1'(t)}{\varphi_1(t)} = \frac{2}{\gamma+1}$. Thus $\varphi_1 \in RVZ_{2/(1+\gamma)}$.

(ii) follows by (i) and Proposition 2.3. □

**Lemma 3.2.** *Let $g$, $k$ and $\varphi_1$ be as in Theorem* 1.1, *then*:

(i) $\lim_{t\to 0^+} \frac{\varphi_1'(t)}{t\varphi_1''(t)} = -\frac{\gamma+1}{\gamma-1}$;
(ii) $\lim_{t\to 0^+} \frac{(\varphi_1'(t))^q}{\varphi_1''(t)} = 0$, $q \in (0,2]$;
(iii) $\lim_{t\to 0^+} \frac{k^q(t)(\varphi_1'(K(t)))^q}{k^2(t)\varphi_1''(K(t))} = 0$, $q \in (0,2]$.

**Proof.** We see by (1.12) and a direct calculation that

$$\varphi_1'(t)=\sqrt{2G_2\big(\varphi(t)\big)},\quad -\varphi_1''(t)=g\big(\varphi_1(t)\big),\quad 0<t<a.$$

(i) It follows by Corollary 2.1 and l'Hospital's rule that

$$\lim_{t\to 0^+}\frac{\varphi_1'(t)}{t\varphi_1''(t)}=-\lim_{t\to 0^+}\frac{\sqrt{2G_2(\varphi(t))}}{tg(\varphi_1(t))}=-\lim_{u\to 0^+}\frac{\sqrt{2G_2(u)}/g(u)}{\int_0^u\frac{ds}{\sqrt{2G_2(s)}}}=-\left(1+2\lim_{u\to 0^+}\frac{g'(u)G_2(u)}{g^2(u)}\right)$$
$$=-\left(1+2\lim_{u\to 0^+}\frac{ug'(u)}{g(u)}\lim_{u\to 0^+}\frac{G_2(u)}{ug(u)}\right)=-\frac{\gamma+1}{\gamma-1}.$$

(ii) It follows by Corollary 2.1 that

$$\lim_{t\to 0^+}\frac{(\varphi_1'(t))^2}{\varphi_1''(t)}=-2\lim_{u\to 0^+}\frac{G(u)}{g(u)}=0.$$

Since $\lim_{t\to 0^+}\varphi_1'(t)=+\infty$, we have

$$\lim_{t\to 0^+}\frac{(\varphi_1'(t))^q}{\varphi_1''(t)}=\lim_{t\to 0^+}\frac{(\varphi_1'(t))^2}{\varphi_1''(t)}\lim_{t\to 0^+}\big(\varphi_1'(t)\big)^{q-2}=0\quad\text{for } 0<q<2.$$

(iii) When $q=2$, (ii) implies (iii). For $q\in(0,2)$, since $\gamma>1$ and $\sigma\in[0,\gamma-1)$, we see that $q(1+\sigma)>\sigma$ for $q\in[1,2)$ and $(2-q)\gamma>\gamma>\sigma$ for $q\in(0,1)$. Thus $(2-q)\gamma+q(1+\sigma)-\sigma>0$ for $q\in(0,2)$. Since $\beta>0$, we see by Lemma 3.1(ii) and Proposition 2.1(ii) that

$$\lim_{t\to 0^+}\big(g\big(\varphi_1\big(K(t)\big)\big)\big)^{q-1}K^q(t)k^{q-2}(t)=\lim_{t\to 0^+}t^\beta H(t)=0,\tag{3.1}$$

where $H$ is slowly varying at zero.

It follows that

$$\lim_{t\to 0^+}\frac{k^q(t)(\varphi_1'(K(t)))^q}{k^2(t)\varphi_1''(K(t))}=\lim_{t\to 0^+}\left(\frac{\varphi_1'(K(t))}{-K(t)\varphi_1''(K(t))}\right)^q\lim_{t\to 0^+}\big(-\varphi_1''\big(K(t)\big)\big)^{q-1}K^q(t)k^{q-2}(t)$$
$$=\left(\frac{\gamma+1}{\gamma-1}\right)^q\lim_{t\to 0^+}\big(g\big(\varphi_1\big(K(t)\big)\big)\big)^{q-1}K^q(t)k^{q-2}(t)$$
$$=0.$$

The proof is finished. □

**Proof of Theorem 1.1.** Let $\xi_0^{-(1+\gamma)}=\tau_0/b_0$, where

$$\tau_0=\frac{2(\gamma-\sigma-1)}{(2+\sigma)(\gamma-1)}>0,\qquad 1-\tau_0=\frac{2(\gamma+1)}{(2+\sigma)(\gamma-1)}>0.$$

Fix $\varepsilon\in(0,\tau_0/4)$ and let

$$\xi_{1\varepsilon}=\left(\frac{b_0}{\tau_0-2\epsilon}\right)^{1/(1+\gamma)},\qquad \xi_{2\varepsilon}=\left(\frac{b_0}{\tau_0+2\epsilon}\right)^{1/(1+\gamma)}.$$

It follows that

$$\left(\frac{2b_0}{3\tau_0}\right)^{1/(1+\gamma)}=C_1<\xi_{2\varepsilon}<\xi_0<\xi_{1\varepsilon}<C_2=\left(\frac{2b_0}{\tau_0}\right)^{1/(1+\gamma)}.$$

Since $\partial\Omega\in C^2$, there exists a constant $\delta\in(0,\delta_0/2)$ which only depends on $\Omega$ such that

(i) $d(x)\in C^2(\overline{\Omega}_\delta)$ and $|\nabla d|\equiv 1$ on $\Omega_\delta=\{x\in\Omega\colon d(x)<\delta\}$.

By $(b_1)$, $(b_2)$, Corollary 2.2 and Lemma 3.2, we see that corresponding to $\varepsilon$, there is $\delta_\varepsilon\in(0,\delta)$ sufficiently small such that:

(ii) for $i = 1, 2$,

$$\left|\frac{k'(d(x))K(d(x))}{k^2(d(x))}\frac{\varphi_1'(s)}{s\varphi_1''(s)} - (\tau_0 - 1) + \frac{K(d(x))}{k(d(x))}\frac{\varphi_1'(s)}{s\varphi_1''(s)}\Delta d(x) + \frac{\lambda\xi_{i\varepsilon}^{q-1}k^q(d(x))}{k^2(d(x))}\frac{(\varphi_1'(K(d(x))))^q}{\varphi_1''(K(d(x)))}\right| < \varepsilon,$$
$$\forall (x, s) \in \Omega_{\delta_\varepsilon} \times (0, \delta_\varepsilon);$$

(iii) $$\frac{\xi_{2\varepsilon}k^2(d(x))g(\varphi_1(K(d(x))))}{g(\xi_{2\varepsilon}\varphi_1(K(d(x))))}(\tau_0 + \varepsilon) < b(x) < \frac{\xi_{1\varepsilon}k^2(d(x))g(\varphi_1(K(d(x))))}{g(\xi_{1\varepsilon}\varphi_1(K(d(x))))}(\tau_0 - \varepsilon), \quad x \in \Omega_{\delta_\varepsilon}.$$

Let $\bar{u}_\varepsilon = \xi_{1\varepsilon}\varphi_1(K(d(x)))$, $\underline{u}_\varepsilon = \xi_{2\varepsilon}\varphi_1(K(d(x)))$, $x \in \Omega_{\delta_\varepsilon}$. We see that for $x \in \Omega_{\delta_\varepsilon}$

$$\begin{aligned}
&\Delta\bar{u}_\varepsilon(x) + b(x)g\big(\bar{u}_\varepsilon(x)\big) + \lambda\big|\nabla\bar{u}_\varepsilon(x)\big|^q \\
&\quad = \xi_{1\varepsilon}\varphi_1''\big(K\big(d(x)\big)\big)k^2\big(d(x)\big) + \xi_{1\varepsilon}\varphi_1'\big(K\big(d(x)\big)\big)k'\big(d(x)\big) + \xi_{1\varepsilon}\varphi_1'\big(K\big(d(x)\big)\big)k\big(d(x)\big)\Delta d(x) \\
&\qquad + b(x)g\big(\xi_1\varphi_1\big(K\big(d(x)\big)\big)\big) + \lambda\xi_{1\varepsilon}^q\big(\varphi_1'\big(K\big(d(x)\big)\big)\big)^q k^q\big(d(x)\big) \\
&\quad = \xi_{1\varepsilon}g\big(\varphi_1\big(K\big(d(x)\big)\big)\big)k^2\big(d(x)\big)\left[\frac{b(x)g(\xi_{1\varepsilon}\varphi_1(K(d(x))))}{\xi_{1\varepsilon}k^2(d(x))g(\varphi_1(K(d(x))))} - \tau_0\right. \\
&\qquad - \left(\frac{k'(d(x))K(d(x))}{k^2(d(x))}\frac{\varphi_1'(K(d(x)))}{K(d(x))\varphi_1''(K(d(x)))} - (\tau_0 - 1)\right) \\
&\qquad \left. - \frac{K(d(x))}{k(d(x))}\frac{\varphi_1'(K(d(x)))}{K(d(x))\varphi_1''(K(d(x)))}\Delta d(x) - \frac{\lambda\xi_{1\varepsilon}^{q-1}k^q(d(x))}{k^2(d(x))}\frac{(\varphi_1'(K(d(x))))^q}{\varphi_1''(K(d(x)))}\right] \\
&\quad \leqslant 0
\end{aligned}$$

and

$$\begin{aligned}
&\Delta\underline{u}_\varepsilon(x) + b(x)g\big(\underline{u}_\varepsilon(x)\big) + \lambda\big|\nabla\underline{u}_\varepsilon(x)\big|^q \\
&\quad = \xi_{2\varepsilon}\varphi_1''\big(K\big(d(x)\big)\big)k^2\big(d(x)\big) + \xi_{2\varepsilon}\varphi_1'\big(K\big(d(x)\big)\big)k'\big(d(x)\big) + \xi_{2\varepsilon}\varphi_1'\big(K\big(d(x)\big)\big)k\big(d(x)\big)\Delta d(x) \\
&\qquad + b(x)g\big(\xi_{2\varepsilon}\varphi_1\big(K\big(d(x)\big)\big)\big) + \lambda\xi_{2\varepsilon}^q k^q\big(d(x)\big)\big(\varphi_1'\big(K\big(d(x)\big)\big)\big)^q \\
&\quad = \xi_{2\varepsilon}g\big(\varphi_1\big(K\big(d(x)\big)\big)\big)k^2\big(d(x)\big)\left[\frac{b(x)g(\xi_{2\varepsilon}\varphi_1(K(d(x))))}{\xi_{2\varepsilon}k^2(d(x))g(\varphi_1(K(d(x))))} - \tau_0\right. \\
&\qquad - \left(\frac{k'(d(x))K(d(x))}{k^2(d(x))}\frac{\varphi_1'(K(d(x)))}{K(d(x))\varphi_1''(K(d(x)))} - (\tau_0 - 1)\right) \\
&\qquad \left. - \frac{K(d(x))}{k(d(x))}\frac{\varphi_1'(K(d(x))}{K(d(x))\varphi_1''(K(d(x)))}\Delta d(x) - \frac{\lambda\xi_{2\varepsilon}^{q-1}k^q(d(x))}{k^2(d(x))}\frac{(\varphi_1'(K(d(x))))^q}{\varphi_1''(K(d(x)))}\right] \\
&\quad \geqslant 0.
\end{aligned}$$

Let $u_\lambda \in C(\overline{\Omega}) \cap C^{2+\alpha}(\Omega)$ be the unique solution to problem (1.1). We assert $\underline{u}_\varepsilon(x) \leqslant u_\lambda(x) \leqslant \bar{u}_\varepsilon(x)$, $\forall x \in \Omega_{\delta_\varepsilon}$. In fact, denote $\Omega_{\delta_\varepsilon} = \Omega_{\delta+} \cup \Omega_{\delta-}$, where $\Omega_{\delta+} = \{x \in \Omega_{\delta_\varepsilon}\colon u_\lambda(x) \geqslant \underline{u}_\varepsilon(x)\}$ and $\Omega_{\delta-} = \{x \in \Omega_{\delta_\varepsilon}\colon u_\lambda(x) < \underline{u}_\varepsilon(x)\}$. We need to show $\Omega_{\delta-} = \emptyset$. Assume the contrary, we see that there exists $x_0 \in \Omega_{\delta-}$ (note that $\underline{u}_\varepsilon(x) = u_\lambda(x)$, $\forall x \in \partial\Omega_{\delta-}$) such that

$$0 < \underline{u}_\varepsilon(x_0) - u_\lambda(x_0) = \max_{x \in \overline{\Omega}_{\delta-}}\big(\underline{u}_\varepsilon(x) - u_\lambda(x)\big)$$

and

$$\nabla\underline{u}_\varepsilon(x_0) = \nabla u_\lambda(x_0), \qquad \Delta(\underline{u}_\varepsilon - u_\lambda)(x_0) \leqslant 0.$$

On the other hand, we see by $(b_1)$ and $(g_1)$ that

$$-\Delta(u_\lambda - \underline{u}_\varepsilon)(x_0) = b(x_0)\big(g\big(\underline{u}_\varepsilon(x_0)\big) - g\big(u_\lambda(x_0)\big)\big) < 0,$$

which is a contradiction. Hence $\Omega_{\delta-} = \emptyset$, i.e., $u(x) \geqslant \underline{u}_\varepsilon(x)$ in $\Omega_\delta$. In the same way, we can see that $u_\lambda(x) \leqslant \bar{u}_\varepsilon(x)$, $\forall x \in \Omega_\delta$. It follows that

$$\xi_{2\varepsilon} \leqslant \lim_{d(x)\to 0} \inf \frac{u_\lambda(x)}{\varphi_1(K(d(x)))} \leqslant \lim_{d(x)\to 0} \sup \frac{u_\lambda(x)}{\varphi_1(K(d(x)))} \leqslant \xi_{1\varepsilon}.$$

Thus let $\varepsilon \to 0$, we see that

$$\lim_{d(x)\to 0} \frac{u_\lambda(x)}{\varphi_1(K(d(x)))} = \xi_0.$$

The last part of the proof follows from Lemma 3.1(i). □

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