Boundary behaviour of the unique solution to a singular Dirichlet problem with a convection term✩

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Abstract

By Karamata regular variation theory and constructing comparison functions, we derive that the boundary behaviour of the unique solution to a singular Dirichlet problem

$$-\Delta u = b(x)g(u) + \lambda|\nabla u|^q, \quad u > 0, \quad x \in \Omega, \quad u|_{\partial\Omega} = 0,$$

which is independent of $\lambda|\nabla u|^q$, where $\Omega$ is a bounded domain with smooth boundary in $\mathbb{R}^N$, $\lambda \in \mathbb{R}$, $q \in (0, 2]$, $\lim_{s \to 0^+} g(s) = +\infty$, and $b$ is non-negative on $\Omega$, which may be vanishing on the boundary.

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1. Introduction and the main results

The purpose of this paper is to investigate the boundary behaviour of the unique classical solution to the following model problem

$$-\Delta u = b(x)g(u) + \lambda|\nabla u|^q, \quad u > 0, \quad x \in \Omega, \quad u|_{\partial\Omega} = 0,$$ (1.1)

where $\Omega$ is a bounded domain with smooth boundary in $\mathbb{R}^N$ ($N \geq 1$), $\lambda \in \mathbb{R}$, $q \in (0, 2]$, $g$ satisfies

(g1) $g \in C^1((0, \infty), (0, \infty))$, $g'(s) < 0$ for all $s > 0$, $\lim_{s \to 0^+} g(s) = +\infty$;

and $b$ satisfies

(b1) $b \in C^\alpha(\overline{\Omega})$ for some $\alpha \in (0, 1)$, is non-negative in $\Omega$ and positive near the boundary $\partial\Omega$.

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The problem arises in the study of non-Newtonian fluids, boundary layer phenomena for viscous fluids, chemical heterogeneous catalysts, as well as in the theory of heat conduction in electrical materials (see [4,7,10,17,21]).

For \( \lambda = 0 \), i.e., problem (1.1) reads the following one
\[
- \Delta u = b(x)g(u), \quad u > 0, \quad x \in \Omega, \quad u|_{\partial \Omega} = 0.
\]

Problem (1.2) was discussed in a number of works; see, for instance, [3,4,7,8,13–15,20–24,26,27].

For \( b \equiv 1 \) on \( \Omega \): when \( g \) satisfies (g1), Fulks and Maybee [7], Stuart [21], Crandall, Rabinowitz and Tartar [4] showed that problem (1.2) has a unique solution \( u \in C^{2+\epsilon}(\Omega) \cap C(\overline{\Omega}) \). Moreover, Theorems 2.2 and 2.5 in [4] showed that if \( \varphi_1 \in C[0, a] \cap C^2(0, a) \) is the local solution to the problem
\[
- \varphi''_1(s) = g(\varphi_1(s)), \quad \varphi_1(s) > 0, \quad 0 < s < a, \quad \varphi_1(0) = 0,
\]
then there exist positive constants \( C_1 \) and \( C_2 \) such that

(I) \( C_1 \varphi_1(d(x)) \leq u(x) \leq C_2 \varphi_1(d(x)) \) near \( \partial \Omega \), where \( d(x) = \text{dist}(x, \partial \Omega) \).

In particular, when \( g(u) = u^{-\gamma}, \gamma > 1, u \) has the property:

(I1) \( C_1[d(x)]^{2/(1+\gamma)} \leq u(x) \leq C_2[d(x)]^{2/(1+\gamma)} \) near \( \partial \Omega \).

In [15], by constructing a pair of global subsolution and supersolution, Lazer and McKenna showed that (I1) continues to hold on \( \overline{\Omega} \). Then \( u \in H^1_0(\Omega) \) if and only if \( \gamma < 3 \). This is a basic character to problem (1.2) for \( g(u) = u^{-\gamma} \) with \( \gamma > 0 \).

When \( \lambda = \pm 1, 0 < q < 2, b(x) \equiv 1 \) on \( \Omega \) and the function \( g : (0, \infty) \rightarrow (0, \infty) \) is locally Lipschitz continuous and decreasing, Giarrusso and Porru [11] showed that if \( g \) satisfies the following conditions:

\( g_2 \) \( \int_0^1 g(s) \, ds = \infty, \int_1^\infty g(s) \, ds < \infty \),
\( g_3 \) let \( G_1(t) = \int_t^\infty g(s) \, ds, \ t > 0 \); there exist positive constants \( \delta \) and \( M \) with \( M > 1 \) such that \( G_1(t) < MG_1(2t), \ \forall t \in (0, \delta) \),

then the unique solution \( u \) to problem (1.1) has the properties:

(P1) \( |u(x) - \varphi_2(d(x))| < C_0 d(x), \ \forall x \in \Omega \) for \( 0 < q \leq 1 \);
(P2) \( |u(x) - \varphi_2(d(x))| < C_0 d(x)G_1(\varphi_2(d(x)))^{(q-1)/2}, \ \forall x \in \Omega \) for \( 1 < q < 2 \), where \( C_0 \) is a suitable positive constant and \( \varphi_2 \in C[0, \infty) \cap C^2(0, \infty) \) is uniquely determined by
\[
\int_0^t \frac{ds}{\sqrt{2G_1(s)}} = t, \quad t > 0.
\]

These imply that
\[
\lim_{d(x) \to 0} \frac{u(x)}{\varphi_2(d(x))} = 1.
\]

In particular, if \( g(u) = u^{-\gamma}, \gamma > 1, \) then \( \varphi_2(s) = cs^{2/(1+\gamma)}, \ c = \left[ \frac{1+\gamma}{2(1-\gamma)} \right]^{1/(1+\gamma)} \).

For other works, see [5,6,9,10,25,28,30] and the references therein.

Our approach relies on Karamata regular variation functions, which was first introduced and established by Karamata in 1930 and is a basic tool in stochastic process, see [16,18,19], and has been applied to study the boundary behaviour of solutions to boundary blow-up elliptic problems (see [1,2,29]) and singular nonlinear Dirichlet problems (see [24,26–28,30]).
Definition 1.1. A positive measurable function $f$ defined on $[a, \infty)$, for some $a > 0$, is called \textit{regularly varying at infinity} with index $\rho$, written $f \in RV_{\rho}$, if for each $\xi > 0$ and some $\rho \in \mathbb{R}$,

$$
\lim_{s \to \infty} \frac{f(\xi s)}{f(s)} = \xi^{\rho}.
$$

(1.6)

In particular, when $\rho = 0$, $f$ is called \textit{slowly varying at infinity}.

Clearly, if $g \in RV_{\rho}$, then $L(s) := f(s)/s^\rho$ is slowly varying at infinity.

Some basic examples of slowly varying functions at infinity are:

(i) every measurable function on $[a, \infty)$ which has a positive limit at infinity;

(ii) $(\ln s)^q$ and $(\ln(\ln s))^q$, $q \in \mathbb{R}$;

(iii) $e(\ln s)^p$, $0 < p < 1$.

We also see that a positive measurable function $g$ defined on $(0, a)$ for some $a > 0$, is \textit{regularly varying at zero} with index $\sigma$ (write $g \in RV_{\sigma}$) if $t \to g(1/t)$ belongs to $RV_{-\sigma}$.

Proposition 1.1 \textbf{(Uniform convergence theorem).} If $f \in RV_{\rho}$, then (1.6) holds uniformly for $\xi \in [a, b]$ with $0 < a < b$.

Proposition 1.2 \textbf{(Representation theorem).} A function $L$ is slowly varying at infinity if and only if it may be written in the form

$$
L(s) = \phi(s) \exp\left(\int_a^s \frac{y(\tau)}{\tau} d\tau\right), \quad s \geq a,
$$

(1.7)

for some $a > 0$, where the functions $\phi$ and $y$ are measurable and for $s \to +\infty$, $y(s) \to 0$ and $\phi(s) \to c_0$, with $c_0 > 0$.

We call that

$$
\hat{L}(s) = c_0 \exp\left(\int_a^s \frac{y(\tau)}{\tau} d\tau\right), \quad s \geq a,
$$

(1.8)

is \textit{normalized} slowly varying at infinity and

$$
f(s) = c_0 s^{\rho} \hat{L}(s), \quad s \geq a,
$$

(1.9)

is \textit{normalized} regularly varying at infinity with index $\rho$ (write $f \in NRV_{\rho}$).

Similarly, $g$ is called \textit{normalized} regularly varying at zero with index $\sigma$, written $g \in NRV_{\sigma}$ if $t \to g(1/t)$ belongs to $NRV_{-\sigma}$.

A function $f \in RV_{\rho}$ belongs to $NRV_{\rho}$ if and only if

$$
f \in C^1[b, \infty) \quad \text{for some } b > 0 \quad \text{and} \quad \lim_{s \to \infty} \frac{s f'(s)}{f(s)} = \rho.
$$

(1.10)

In this paper, by Karamata regular variation theory and constructing comparison functions, we derive the boundary behaviour to the unique solution to problem (1.1) for the weight $b$ which may be vanishing on the boundary.

Our main results are the following.

Theorem 1.1. \textbf{Let} $\lambda \in \mathbb{R}$, $q \in [0, 2]$, $b$ satisfy (b$_1$), $g$ satisfy (g$_1$) and $g \in NRV_{-\rho}$ with $\rho > 1$. \textbf{Suppose that there exist a positive non-decreasing} $C^1$-function $k \in NRV_{\sigma/2}$ \textbf{with} $\sigma \in [0, \gamma - 1)$ and \textbf{a positive constant} $b_0$ \textbf{such that}

(b$_2$) \hspace{1cm} $\lim_{d(x) \to 0} \frac{b_0}{k''(d(x))} = b_0$,

\textbf{then the unique solution} $u_\lambda \in C(\overline{\Omega}) \cap C^2(\Omega)$ \textbf{to problem (1.1) satisfies}

$$
\lim_{d(x) \to 0} \frac{u_\lambda(x)}{\varphi_1(K(d(x)))} = \xi_0.
$$

(1.11)
where \( \xi_0^{-\gamma} = 2(y-\sigma-1) \) and \( \varphi_1 \in C[0,a] \cap C^2(0,a) \) is the solution to problem (1.3), i.e.,

\[
\int_0^{\varphi_1(t)} \frac{ds}{\sqrt{2G_2(s)}} = t, \quad t \in [0,a] \text{ for small } a > 0,
\]

(1.12)

\[
K(t) = \int_0^t k(s) \, ds, \quad t \in [0,a]; \quad G_2(t) = \int_t^b g(s) \, ds, \quad t \in (0,b), \ b > 0.
\]

(1.13)

Moreover, \( \varphi_1 \in NRVZ_{2/(1+\gamma)} \) and there exists \( \gamma_2 \in C(0,a) \) with \( \lim_{s \to 0^+} \gamma_2(s) = 0 \) such that \( \varphi_1(t) = t^{2/(1+\gamma)} e^{\sigma y(t)} \), \( t \in (0,a] \).

**Corollary 1.1.** When \( g(u) = u^{-\gamma} \) with \( \gamma > 1 \) in Theorem 1.1, \( u_\lambda \) satisfies

\[
\lim_{d(x) \to 0} \frac{u_\lambda(x)}{[K(d(x))]^{2/(1+\gamma)}} = \left[ \frac{b_0(2+\sigma)(1+\gamma)2^{\gamma/(1+\gamma)}}{4(\gamma-\sigma-1)} \right]^{1/(1+\gamma)}.
\]

(1.14)

**Remark 1.1.** By (1.12), we see that the asymptotic behaviour of \( u_\lambda \) is independent of \( \lambda |\nabla u_\lambda|^\gamma \).

**Remark 1.2.** By \( g_1 \) and the proof of the maximum [12, Theorems 10.1 and 10.2], we see that problem (1.1) has at most one solution in \( C(I) \cap C(I) \). For the existence of solutions to problem (1.1), see [28].

**Remark 1.3.** Some examples of the functions which satisfy the conditions in Theorem 1.1 are:

1. \( g(u) = u^{-\gamma} \), where \( \gamma > 1 \);
2. \( g(u) = u^{-\gamma} \arctan(u^{-1}) \), where \( \gamma > 1 \);
3. \( g(u) = u^{-\gamma_1}(\ln(1+u))^{-\gamma_2} \), where \( \gamma_1 > 0, \gamma_2 > 0 \) and \( \gamma_1 + \gamma_2 > 1 \);
4. \( g(u) = u^{-\gamma_1}(e^u - 1)^{-\gamma_2} \), where \( \gamma_1 > 0, \gamma_2 > 0 \) and \( \gamma_1 + \gamma_2 > 1 \);
5. \( g(u) = u^{-\gamma}(-\ln u)^p \), where \( \gamma > 1, p \in \mathbb{R}, 0 < u < a < 1 \);
6. \( g(u) = u^{-\gamma} e^{(\ln u)^p} \), where \( \gamma > 1, p \in (0,1), 0 < u < a < 1 \);
7. \( g(u) = u^{-\gamma} (\ln(-\ln u))^p \), where \( \gamma > 1, p \in \mathbb{R}, 0 < u < a < 1 \).

**Remark 1.4.** The key of the paper is the estimation of \( |\nabla u|^\gamma \) which is very different from that one in [28] and [30] where the weight \( b \) is singular on the boundary.

The outline of this paper is as follows. In Section 2 we recall some basic the properties to Karamata regular variation theory. The proof of Theorem 1.1 is given in Section 3.

2. Some basic definitions and the properties to Karamata regular variation theory

We recall some basic properties to Karamata regular variation theory (see [16,18,19]).

**Proposition 2.1.** If functions \( L, L_1 \) are slowly varying at infinity, then:

1. \( L_\sigma \) for every \( \sigma \in \mathbb{R} \), \( c_1 L + c_2 L_1 \) (\( c_1 \geq 0, c_2 \geq 0 \) with \( c_1 + c_2 > 0 \)), \( L \circ L_1 \) (if \( L_1(t) \to +\infty \) as \( t \to +\infty \)) are also slowly varying at infinity.
2. For every \( \theta > 0 \) and \( t \to +\infty \),

\[
t^n L(t) \to +\infty, \quad t^{-n} L(t) \to 0.
\]

3. For \( \rho \in \mathbb{R} \) and \( t \to +\infty \),\n
\[
\ln(L(t))/\ln t \to 0 \text{ and } \ln(t^n L(t))/\ln t \to \rho.
\]
Proposition 2.2 (Asymptotic behaviour). If a function $H$ is slowly varying at zero, then for $a > 0$ and $t \to 0^+$:

(i) $\int_0^t s^\beta H(s) \, ds \cong (\beta + 1)^{-1} t^{1+\beta} H(t)$, for $\beta > -1$;

(ii) $\int_t^\infty s^\beta H(s) \, ds \cong (-\beta - 1)^{-1} t^{1+\beta} H(t)$, for $\beta < -1$.

Let $\Psi$ be non-decreasing on $\mathbb{R}$, we define (as in [18]) the inverse of $\Psi$ by

$$\Psi^{-1}(t) = \inf\{s : \Psi(s) \geq t\}. \quad (2.1)$$

Proposition 2.3. (See [18, Proposition 0.8].) The following hold:

(i) if $f_1 \in RV_{\rho_1}$, $f_2 \in RV_{\rho_2}$, then $f_1 \cdot f_2 \in RV_{\rho_1+\rho_2}$;

(ii) if $f_1 \in RV_{\rho_1}$, $f_2 \in RV_{\rho_2}$, with $\lim_{t \to +\infty} f_2(t) = +\infty$, then $f_1 \circ f_2 \in RV_{\rho_1\rho_2}$;

(iii) if $f$ is non-decreasing on $(a, \infty)$, $f \in RV_\rho$ with $\rho > 0$, then $f^{-1} \in RV_{\rho^{-1}}$.

By the above propositions, we can directly obtain the following results.

Corollary 2.1. If $g$ satisfies (g1) and $g \in NRVZ_\gamma$ with $\gamma > 1$, then:

(i) $g(t) = t^{-\gamma} \exp\left(\int_1^t s^{\gamma/(\gamma-1)} \, ds\right)$, $0 < t < a$, $y \in C(0, a]$, $\lim_{s \to 0^+} y(s) = 0$;

(ii) $\lim_{t \to 0^+} g(t) = +\infty = \lim_{t \to 0^+} G_2(t)$; $\lim_{t \to 0^+} \frac{G_2(t)}{g(t)} = 0 = \lim_{t \to 0^+} \frac{\sqrt{G_2(t)}}{g(t)}$;

(iii) $\lim_{t \to 0^+} \frac{G_2(t)}{g(t)} = 1 - \frac{1}{\gamma}$; $\lim_{t \to 0^+} \frac{t g'(t)}{g(t)} = -\gamma$.

Corollary 2.2. $k$ in Theorem 1.1 has the following properties:

(i) $k(t) = t^{\sigma/2} \exp\left(\int_1^t s^{\sigma/(\sigma-1)} \, ds\right)$ for $t \in (0, a)$, $y_1 \in C(0, a]$, $\lim_{s \to 0^+} y_1(s) = 0$;

(ii) $\lim_{t \to 0^+} \frac{k(t)}{\sqrt{2\sigma(t)}} = 0$; $\lim_{t \to 0^+} \frac{i k'(t)}{k(t)} = \frac{2}{\sigma+\sigma}$;

(iii) $\lim_{t \to 0^+} \frac{k'(t)(K(t))}{k^2(t)} = \lim_{t \to 0^+} \frac{i k(t)}{k^2(t)} \lim_{t \to 0^+} \frac{K(t)}{i k(t)} = 1 - \frac{\sigma}{\sigma+\sigma}$.

3. The exact asymptotic behaviour

First we give some preliminary considerations.

Lemma 3.1. Under the assumption in Theorem 1.1:

(i) $\varphi_1 \in NRVZ_{2/(1+\gamma)}$;

(ii) $(g \circ \varphi_1 \circ K)^{q-1} \cdot K^q \cdot k^q \in RVZ_\beta$ with $\beta = \frac{(2-q)\gamma+q(\sigma+1)-\sigma}{1+\gamma}$.

Proof. (i) Let $f_1(t) = \int_0^t \frac{ds}{\sqrt{2\sigma(t)}}$, $\forall t \in (0, a)$. By l’Hospital’s rule and Proposition 2.3(iii), we can easily see that $f_1 \in RVZ_{(1+\gamma)/2}$ and $\varphi_1 = f_1^{-1} \in RVZ_{2/(1+\gamma)}$. Moreover, we see by (1.10), the following Lemma 3.2(i) and Proposition 2.2(ii) that $\varphi_1 \in NRVZ_{(\gamma-1)/(\gamma+1)}$ and $\lim_{t \to 0^+} \frac{g(t)}{\varphi_1(t)} = \frac{2}{\gamma+1}$. Thus $\varphi_1 \in RVZ_{2/(1+\gamma)}$.

(ii) follows by (i) and Proposition 2.3. \qed

Lemma 3.2. Let $g$, $k$ and $\varphi_1$ be as in Theorem 1.1, then:

(i) $\lim_{t \to 0^+} \frac{\varphi_1(t)}{\varphi_1'(t)} = \frac{\gamma+1}{\gamma-1}$;

(ii) $\lim_{t \to 0^+} \frac{(\varphi_1(t))^q}{\varphi_1'(t)} = 0$, $q \in (0, 2]$;

(iii) $\lim_{t \to 0^+} \frac{k^q(\varphi_1'(K(t)))}{k^{(q+1)/2}(\varphi_1'(K(t)))} = 0$, $q \in (0, 2]$. 
Proof. We see by (1.12) and a direct calculation that
\[ \phi_1'(t) = \sqrt{2G_2(\phi(t))}, \quad -\phi_1''(t) = g(\phi_1(t)), \quad 0 < t < a. \]

(i) It follows by Corollary 2.1 and l’Hospital’s rule that
\[ \lim_{t \to 0^+} \phi_1'(t) = -\lim_{t \to 0^+} \frac{\sqrt{2G_2(\phi(t))}}{tg(\phi_1(t))} = -\lim_{u \to 0^+} \frac{\sqrt{2G_2(u)/g(u)}}{\int_0^u \frac{ds}{\sqrt{2G_2(s)}}} = -\left(1 + 2 \lim_{u \to 0^+} \frac{g'(u)G_2(u)}{g^2(u)}\right) = -\left(1 + 2 \lim_{u \to 0^+} \frac{ug'(u)}{g(u)} \lim_{u \to 0^+} G_2(u)\right) = -\frac{\gamma + 1}{\gamma - 1} . \]

(ii) It follows by Corollary 2.1 that
\[ \lim_{t \to 0^+} \frac{(\phi_1'(t))^2}{\phi_1''(t)} = -2 \lim_{u \to 0^+} \frac{G(u)}{g(u)} = 0. \]

Since \( \lim_{t \to 0^+} \phi_1'(t) = +\infty \), we have
\[ \lim_{t \to 0^+} \frac{(\phi_1'(t))^2}{\phi_1''(t)} = \lim_{t \to 0^+} \frac{(\phi_1'(t))^2}{\phi_1''(t)} \lim_{t \to 0^+} (\phi_1'(t))^{q-2} = 0 \quad \text{for} \quad 0 < q < 2. \]

(iii) When \( q = 2 \), (ii) implies (iii). For \( q \in (0, 2) \), since \( \gamma > 1 \) and \( \sigma \in [0, \gamma - 1] \), we see that \( q(1 + \sigma) > \sigma \) for \( q \in [1, 2] \) and \( (2 - q)\gamma > \gamma > \sigma \) for \( q \in (0, 1) \). Thus \( (2 - q)\gamma + q(1 + \sigma) - \sigma > 0 \) for \( q \in (0, 2) \). Since \( \beta > 0 \), we see by Lemma 3.1(ii) and Proposition 2.1(ii) that
\[ \lim_{t \to 0^+} (g(\phi_1(K(t))))^q - 1 K^q(t) k^{q-2}(t) = \lim_{t \to 0^+} t^\beta H(t) = 0, \quad \text{(3.1)} \]
where \( H \) is slowly varying at zero.

It follows that
\[ \lim_{t \to 0^+} k^q(t) (\phi_1'(K(t)))^q = \lim_{t \to 0^+} \left( \frac{\phi_1'(K(t))}{K(t) \phi_1''(K(t))} \right)^q \lim_{t \to 0^+} (\phi_1''(K(t)))^{q-1} K^q(t) k^{q-2}(t) = \frac{\gamma + 1}{\gamma - 1} \lim_{t \to 0^+} (g(\phi_1(K(t))))^q - 1 K^q(t) k^{q-2}(t) = 0. \]

The proof is finished. \( \square \)

Proof of Theorem 1.1. Let \( \xi_0^{-(1+\gamma)} = \tau_0/b_0 \), where
\[ \tau_0 = \frac{2(\gamma - \sigma - 1)}{(2 + \sigma)(\gamma - 1)} > 0, \quad 1 - \tau_0 = \frac{2(\gamma + 1)}{(2 + \sigma)(\gamma - 1)} > 0. \]

Fix \( \epsilon \in (0, \tau_0/4) \) and let
\[ \xi_1 = \left( \frac{b_0}{\tau_0 - 2\epsilon} \right)^{1/(1+\gamma)}, \quad \xi_2 = \left( \frac{b_0}{\tau_0 + 2\epsilon} \right)^{1/(1+\gamma)}. \]

It follows that
\[ \left( \frac{2b_0}{\tau_0} \right)^{1/(1+\gamma)} = C_1 < \xi_2 < \xi_0 < \xi_1 < C_2 = \left( \frac{2b_0}{\tau_0} \right)^{1/(1+\gamma)}. \]

Since \( \partial \Omega \in C^2 \), there exists a constant \( \delta \in (0, \delta_0/2) \) which only depends on \( \Omega \) such that

(i) \( d(x) \in C^2(\overline{\Omega}_\delta) \) and \( |\nabla d| \equiv 1 \) on \( \partial \Omega = \{ x \in \Omega : d(x) < \delta \} \).

By (b1), (b2), Corollary 2.2 and Lemma 3.2, we see that corresponding to \( \epsilon \), there is \( \delta_\epsilon \in (0, \delta) \) sufficiently small such that:
(ii) for \( i = 1, 2, \)
\[
\left| \frac{k'_i(d(x))K_i(d(x))}{k^{2_i}(d(x))} \psi'_i(s) - \frac{k^{2_i}(d(x))}{s \psi''_i(s)} \Delta d(x) \right| < \varepsilon,
\]
\[\forall (x, s) \in \Omega_{\delta_i} \times (0, \delta_e) ;
\]
(iii) \[
\frac{\xi_{2_i}k^{2_i}(d(x))g((K_i(d(x))))}{g(\xi_{2_i} \psi_1(K_i(d(x))))} < b(x) < \frac{\xi_{1_i}k^{2_i}(d(x))g((K_i(d(x))))}{g(\xi_{1_i} \psi_1(K_i(d(x))))}, \quad \forall (x, s) \in \Omega_{\delta_i}.
\]
Let \( \bar{u}_e = \xi_{1_i} \psi_1(K_i(d(x))), \) \( u_e = \xi_{2_i} \psi_1(K_i(d(x))), \) \( x \in \Omega_{\delta_i} \). We see that for \( x \in \Omega_{\delta_i} \)
\[
\begin{align*}
\Delta \bar{u}_e(x) + b(x)g(\bar{u}_e(x)) + \lambda \left| \nabla \bar{u}_e(x) \right|^q & = \xi_{1_i} \psi_1'(K_i(d(x)))k^{2_i}(d(x)) + \xi_{1_i} \psi_1'(K_i(d(x)))k'(d(x)) \Delta d(x) \\
& + b(x)g(\xi_{1_i} \psi_1(K_i(d(x)))) + \lambda \xi_{1_i}^q \psi_1'(K_i(d(x)))^q k^q(d(x)) \\
& = \xi_{1_i} g(\psi_1(K_i(d(x))))k^{2_i}(d(x)) \left[ \frac{b(x)g(\xi_{1_i} \psi_1(K_i(d(x))))}{\xi_{1_i}k^{2_i}(d(x))g(\psi_1(K_i(d(x))))} - \tau_0 \right. \\
& \left. - \frac{K(d(x))}{k^{2_i}(d(x))} \frac{\psi_1'(K_i(d(x)))}{\psi_1''(K_i(d(x)))} \Delta d(x) - \frac{\lambda \xi_{1_i}^q - 1}{k^q(d(x))} \psi_1'(K_i(d(x))) \right] \\
& \leq 0
\end{align*}
\]
and
\[
\begin{align*}
\Delta u_e(x) + b(x)g(u_e(x)) + \lambda \left| \nabla u_e(x) \right|^q & = \xi_{2_i} \psi_1''(K_i(d(x)))k^{2_i}(d(x)) + \xi_{2_i} \psi_1'(K_i(d(x)))k'(d(x)) \Delta d(x) \\
& + b(x)g(\xi_{2_i} \psi_1(K_i(d(x)))) + \lambda \xi_{2_i}^q \psi_1'(K_i(d(x)))^q k^q(d(x)) \\
& = \xi_{2_i} g(\psi_1(K_i(d(x))))k^{2_i}(d(x)) \left[ \frac{b(x)g(\xi_{2_i} \psi_1(K_i(d(x))))}{\xi_{2_i}k^{2_i}(d(x))g(\psi_1(K_i(d(x))))} - \tau_0 \right. \\
& \left. - \frac{K(d(x))}{k^{2_i}(d(x))} \frac{\psi_1'(K_i(d(x)))}{\psi_1''(K_i(d(x)))} \Delta d(x) - \frac{\lambda \xi_{2_i}^q - 1}{k^q(d(x))} \psi_1'(K_i(d(x))) \right] \\
& \geq 0.
\end{align*}
\]
Let \( u_\lambda \in C(\overline{\Omega}) \cap C^{2+q}(\Omega) \) be the unique solution to problem (1.1). We assert \( u_\varepsilon(x) \leq u_\lambda(x) \leq \bar{u}_e(x), \) \( \forall x \in \Omega_{\delta_i}. \) In fact, denote \( \Omega_{\delta_+} = \Omega_{\delta_+} \cup \Omega_{\delta_-}, \) where \( \Omega_{\delta_+} = \{ x \in \Omega_{\delta_i} : u_\lambda(x) \geq u_\varepsilon(x) \} \) and \( \Omega_{\delta_-} = \{ x \in \Omega_{\delta_i} : u_\lambda(x) < u_\varepsilon(x) \}. \) We need to show \( \Omega_{\delta_-} = \emptyset. \) Assume the contrary, we see that there exists \( x_0 \in \Omega_{\delta_-} \) (note that \( u_\varepsilon(x) = u_\lambda(x), \) \( \forall x \in \partial \Omega_{\delta_-} \)) such that
\[
0 < u_\varepsilon(x_0) - u_\lambda(x_0) = \max_{x \in \Omega_{\delta_-}} (u_\varepsilon(x) - u_\lambda(x))
\]
and
\[
\nabla u_\varepsilon(x_0) = \nabla u_\lambda(x_0), \quad \Delta (u_\varepsilon - u_\lambda)(x_0) \leq 0.
\]
On the other hand, we see by (b1) and (g1) that
\[
- \Delta (u_\lambda - u_\varepsilon)(x_0) = b(x_0)(g(u_\varepsilon(x_0)) - g(u_\lambda(x_0))) < 0,
\]
which is a contradiction. Hence $\Omega_{\delta} = \emptyset$, i.e., $u(x) \geq u_{\delta}(x)$ in $\Omega_{\delta}$. In the same way, we can see that $u_{\gamma}(x) \leq \bar{u}_{\gamma}(x)$, $\forall x \in \Omega_{\delta}$. It follows that
\[
\xi_{2\varepsilon} \leq \liminf_{d(x) \to 0} \frac{u_{\gamma}(x)}{\varphi_{1}(K(d(x)))} \leq \limsup_{d(x) \to 0} \frac{u_{\delta}(x)}{\varphi_{1}(K(d(x)))} \leq \xi_{1\varepsilon}.
\]
Thus let $\varepsilon \to 0$, we see that
\[
\lim_{d(x) \to 0} \frac{u_{\gamma}(x)}{\varphi_{1}(K(d(x)))} = \xi_{0}.
\]

The last part of the proof follows from Lemma 3.1(i). $\square$

References


