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D-Koszul algebras

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Abstract

In this paper we study d-Koszul algebras which were introduced by Berger. We show that when $d \geqslant 3$, these are classified by the Ext-algebra being generated in degrees 0, 1, and 2. We show the Ext-algebra, after regrading, is a Koszul algebra and present the structure of the Ext-algebra.

MSC: 16S37; 16E30; 16E65; 16W50

1. Introduction

In [2], Roland Berger introduced what he called "generalized" Koszul algebras. He became interested in this class of algebras since Artin-Schelter regular algebras of global dimension 3 which are generated in degree 1 are such algebras. The generalized Koszul algebras are graded algebras $A = K \oplus A_1 \oplus A_2 \oplus \cdots$ which are generated in

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degrees 0 and 1 such that there is a graded projective resolution of K for which the nth projective in the resolution is generated in degree $\delta(n)$ where

$$\delta(n)=\begin{cases}\frac{n}{2}d \text{ if } n \text{ is even,}\\ \frac{n-1}{2}d+1 \text{ if } n \text{ is odd,}\end{cases} \quad \text{for some } d.$$

We generalize this definition to the nonlocal case, i.e., K is replaced by a semisimple K-algebra and we call this class of algebras *d-Koszul algebras*. We give a formal definition in Section 4.

The paper begins with a section on notation. Section 3 provides some general tools to study when the Yoneda product map $\mathrm{Ext}^n_A(A_0,A_0)\otimes \mathrm{Hom}_A(M,A_0)\to \mathrm{Ext}^n(M,A_0)$ is surjective where $A=A_0\oplus A_1\oplus\cdots$ is a graded algebra and M is a graded left A-module. We introduce d-Koszul algebras in Section 4 and prove (Theorem 4.1) that an algebra with generating relations in degree d is d-Koszul if and only if the Ext-algebra $E(A)=\bigoplus_{n\geqslant 0}\mathrm{Ext}^n_A(A_0,A_0)$ is generated in degrees 0, 1, and 2. In Section 5, we define d-Koszul modules and show that other than projective and simple modules, then second syzygy of d-Koszul module (shifted) is a d-Koszul module. In this section, we also show that the first syzygy of $\mathbf{r}M$ is d-Koszul (shifted), where M is a d-Koszul module and $\mathbf{r}$ is the graded Jacobson radical of A.

In sections 6 and 7 we prove that the even Ext-algebra of a d-Koszul algebra is a Koszul algebra and the whole Ext-algebra of a d-Koszul algebra, after a regrading, is also a Koszul algebra (Theorems 6.1 and 7.1). It is also shown that the Ext of a d-Koszul module is a Koszul module over the regraded Ext-algebra. In Section 8, we provide a new proof Berger's result that there is a generalized Koszul complex which is exact if and only if the algebra is a d-Koszul algebra (Theorem 8.3).

Section 9 gives a description of the Ext-algebra of a d-Koszul algebra by generators and relations. In particular, we show how the Ext-algebra can be described using the dual algebra $A^!$. The final section of the paper shows that algebras with relations generated in a fixed degree d of global dimension 2 are d-Koszul. A classification of monomial algebras which are d-Koszul is also given in this final section.

2. Notation

Let K be a commutative ring and $A=A_0\oplus A_1\oplus A_2\oplus\cdots$ be an associative graded K-algebra where the direct sum is as K-modules. Note that $K\to A$ has image contained in the center of A. Assume that A is generated in degrees 0 and 1; that is, $A_i\cdot A_j=A_{i+j}$ for all $0\leqslant i,j<\infty$. Let $\mathrm{Gr}(A)$ denote the category of graded A-modules and degree 0 homomorphisms and $\mathrm{Mod}(A)$ denote the category of left A-modules. We denote by $\mathrm{gr}(A)$ and $\mathrm{mod}(A)$ the full subcategories of $\mathrm{Gr}(A)$ and $\mathrm{Mod}(A)$, respectively, consisting of finitely generated modules. Let $F:\mathrm{Gr}(A)\to\mathrm{Mod}(A)$ denote the forgetful functor and $\mathrm{Gr}_0(A)$ (respectively, $\mathrm{gr}_0(A)$) be the full subcategory of $\mathrm{Gr}(A)$ whose objects are the graded modules (respectively, finitely generated modules) generated in degree 0.

We assume A_0 is a semisimple Artin algebra. The graded Jacobson radical of A, which we denote by $\mathbf{r}_A$, or simply $\mathbf{r}$, when no confusion can arise, is $A_1\oplus A_2\oplus\cdots$.

Since A is generated in degrees 0 and 1, it follows that $\mathbf{r}^i = A_i \oplus A_{i+1} \oplus \cdots$. For the remainder of this paper, we fix a minimal graded projective resolution of A_0,

$$\mathscr{P}^\bullet : \cdots \to P^n \to \cdots \to P^1 \to P^0 \to A_0 \to 0,$$

where A_0 is viewed as a graded A-module generated in degree 0. (Here "minimal graded" means that the image of $P^i \to P^{i-1}$ is contained in $\mathbf{r}P^{i-1}$ and, since A_0 is semisimple, minimal graded resolutions of graded modules exist in $\mathrm{Gr}(A)$.)

Note $E(A) = \coprod_{n\geqslant 0} \mathrm{Ext}^n_A(F(A_0), F(A_0))$ is a graded algebra using the canonical grading; that is, $E(A)_n = \mathrm{Ext}^n_A(F(A_0), F(A_0))$.

In this paper, there will be a number of different gradings occurring and we will be careful about which grading we are using. The grading described above for the Ext-algebra $E(A)$ will be called the *ext-grading* and the degree of an element will be called the *ext-degree*.

If $M = \coprod_i M_i$ is a graded A-module, we denote the nth-shift of M by $M[n]$ where $M[n]$ is the graded A-module $X = \coprod_i X_i$ where $X_i = M_{i-n}$. Thus, if M is generated in degree d, $M[-d]$ is generated in degree 0 and if $M \in \mathrm{Gr}_0(A)$, then $\mathbf{r}^i M[-i] \in \mathrm{Gr}_0(A)$. Let $M = \coprod_i M_i$ and $N = \coprod_i N_i$ be graded A-modules. If M is finitely generated, then we may grade the abelian group $\mathrm{Hom}_A(F(M), F(N))$ by setting $\mathrm{Hom}_A(F(M), F(N))_i = \mathrm{Hom}_{\mathrm{Gr}(A)}(M, N[i])$. This grading will be called the *hom-grading*. More generally, if $\mathscr{Q}^\bullet : \cdots \to Q^n \to \cdots \to Q^1 \to Q^0 \to M \to 0$ is a graded projective resolution of M, we may apply the ith-shift to the resolution $\mathscr{Q}^\bullet$. Thus, we get a graded resolution $\mathscr{Q}^\bullet[i] : \cdots \to Q^1[i] \to Q^0[i] \to M[i] \to 0$. If Q^n is finitely generated, we grade $\mathrm{Ext}^A_n(F(M), F(N))$ as follows. We now assume each Q^n is finitely generated for all $n \geqslant 0$. For each $i \in \mathbf{Z}$, consider the complex induced by applying $\mathrm{Hom}_{\mathrm{Gr}(A)}(-, N[i])$ to $\mathscr{Q}^\bullet$. Define $\mathrm{Ext}^n_A(F(M), F(N))_i = \mathrm{Ext}^n_{\mathrm{Gr}(A)}(M, N[i])$, which is the homology of the complex obtained by applying $\mathrm{Hom}_{\mathrm{Gr}(A)}(-, N[i])$ to $\mathscr{Q}^\bullet$. We call this grading on $\mathrm{Ext}^n(F(M), F(N))$ the *shift-grading*. It is routine to check that the shift-grading grades $\mathrm{Ext}^n_A(F(M), F(N))$ as a K-module.

We have the following useful result. The proof relies on the semisimplicity of N and is left to the reader.

Proposition 2.1. *Let $\mathscr{Q}^\bullet$ be a minimal graded projective resolution of a graded A-module M. Assume that Q^n is finitely generated. Suppose that N is a graded A-module such that* $\mathbf{r}N = (0)$. *Then*

$$\begin{aligned}\mathrm{Ext}^n_A(F(M), F(N))_i &\simeq \mathrm{Hom}_{\mathrm{Gr}(A)}(\Omega^n(M), N[i])\\ &\simeq \mathrm{Hom}_{\mathrm{Gr}(A)}(\Omega^n(M)[-i], N),\end{aligned}$$

where $\Omega^n(M)$ denotes the nth syzygy of M with respect to the resolution $\mathscr{Q}^\bullet$. □

If $\mathscr{J}$ is a subset of the nonnegative integers, then we say a graded module M is *supported in* $\mathscr{J}$ if $M_i = 0$ whenever $i \notin \mathscr{J}$. Note that if M is generated in degree d then M is supported in $\{i \mid i \geqslant d\}$.

To simplify notation, we will omit the functor F in the Ext_A and Hom_A notation.

3. Fundamental results

In this section, we provide results, Proposition 3.5 and its consequence, Proposition 3.6, which are fundamental in what follows. These results, when applicable, provide necessary and sufficient conditions for a Yoneda product map to be surjective.

Lemma 3.1. *Let M be a graded A-module supported in $\{i \mid i \geqslant k\}$ and $\mathcal{Q}^\bullet$ be a minimal graded projective resolution of M. Then, for each $n \geqslant 0$, $(Q^n)_i = 0$, for all $i < k+n$.*

Proof. If $f : Q^0 \to M \to 0$ is a graded projective cover, then $f_k : Q^0_k \to M_k$ is an isomorphism. Hence, $\Omega^1(M)$ is supported in $\{i \mid i \geqslant k+1\}$. The result now follows by induction. □

Recall that $\mathcal{P}^\bullet$ is a fixed minimal graded projective resolution of A_0, where A_0 is viewed as a graded A-module that has support in $\{0\}$.

Lemma 3.2. *Let $M \in \mathrm{Gr}(A)$ supported in $\{j \geqslant 0\}$ with minimal graded projective resolution $\mathcal{Q}^*$. Let $n \geqslant 1$. Assume that P^n is supported in $\{j \mid j \geqslant s\}$. Then Q^n is supported in $\{j \mid j \geqslant s\}$.*

Proof. Let $M = \coprod_{i \geqslant 0} M_i$ and let $M_{\geqslant t}$ denote the submodule $\coprod_{i \geqslant t} M_i$. We have a short exact sequence $0 \to M_{\geqslant s+1} \to M \to M/M_{\geqslant s+1} \to 0$. Note that $M/M_{\geqslant s+1}$ is supported in $\{0, 1, \ldots, s\}$. There is a projective resolution $\mathcal{R}^\bullet$ obtained from the minimal projective resolutions of $M_{\geqslant s+1}$ and $M/M_{\geqslant s+1}$. Since $\mathcal{Q}^\bullet$ is a summand of $\mathcal{R}^\bullet$, if we show the result for $\mathcal{R}^\bullet$, we will be done. Next, note that the projective modules generated in degree $\leqslant s$ in $\mathcal{R}^*$ are obtained only from the projective modules in the projective resolution of $M/M_{\geqslant s+1}$.

Thus, to prove the result, we can assume that M is supported in $\{0, 1, \ldots, s\}$. We proceed by induction on the cardinality of the support of M. If M is supported in exactly one degree, then M is a semisimple module and the result easily follows. Assume the cardinality of the support of M is greater than 1. We can assume $M_0 \neq 0$ since, if not, we can replace M by $M[-i]$ for an appropriate i. Consider $M_{\geqslant 1}$. Since the cardinality of the support of $M_{\geqslant 1}$ is less than the cardinality of the support of M, by induction, the result holds for $M_{\geqslant 1}$. Again we have a short exact sequence $0 \to M_{\geqslant 1} \to M \to M/M_{\geqslant 1} \to 0$. The result holds for the minimal projective resolutions of the two end modules and hence, holds for the projective resolution of M obtained from these two resolutions. Hence the result holds for the minimal projective resolution of M and we are done. □

We have the following immediate consequence.

Corollary 3.3. *Let $M \in \mathrm{Gr}_0(A)$ with minimal graded projective resolution $\mathcal{Q}^\bullet$. Assume that P^n is generated in degree s and Q^n is generated in degree t. Then $t \geqslant s$.*

If M_1, M_2, M_3 are A-modules, we let

$$\mathscr{Y}_{m,n} : \mathrm{Ext}_A^m(M_2, M_3) \bigotimes_K \mathrm{Ext}_A^n(M_1, M_2) \to \mathrm{Ext}_A^{m+n}(M_1, M_3)$$

be the Yoneda product. We will usually write $\mathscr{Y}_{m,n}$ as $\mathscr{Y}$ when no confusion can arise. Furthermore, we will denote the image of $\mathscr{Y}$ in $\mathrm{Ext}_A^{m+n}(M_1, M_3)$ by $\mathrm{Ext}_A^m(M_2, M_3) \cdot \mathrm{Ext}_A^n(M_1, M_2)$.

Lemma 3.4. *In the commutative diagram*

$$\begin{array}{ccc} \mathrm{Ext}_A^n(A_0, A_0) \bigotimes_K \mathrm{Hom}_A(M/\mathbf{r}M, A_0) & \longrightarrow & \mathrm{Ext}_A^n(M/\mathbf{r}M, A_0) \\ \downarrow & & \downarrow \\ \mathrm{Ext}_A^n(A_0, A_0) \bigotimes_K \mathrm{Hom}_A(M, A_0) & \longrightarrow & \mathrm{Ext}_A^n(M, A_0) \end{array}$$

the left vertical map is a bijection and the upper horizontal map is a surjection.

Proof. This is clear since $\mathbf{r}A_0 = 0$ and $M/\mathbf{r}M$ is a summand of a finite sum of copies of A_0. □

Note that if $(\mathscr{Q}^\bullet)$ is a graded projective resolution of a graded module M such that Q^n is finitely generated and is supported in $\mathscr{J}$, then $\mathrm{Ext}_A^n(M, A_0)$, in the shift-grading, is supported in $\mathscr{J}$; that is, $\mathrm{Ext}^n(M, A_0)_i = 0$ for $i \notin \mathscr{J}$. This is a consequence of

$$\begin{aligned} \mathrm{Ext}_A^n(M, A_0) &= \mathrm{Hom}_A(\Omega^n(M), A_0) \\ &= \mathrm{Hom}_A(\Omega^n(M)/\mathbf{r}\Omega^n(M), A_0) = \mathrm{Hom}_A(Q^n/\mathbf{r}Q^n, A_0). \end{aligned}$$

The next proposition is of fundamental importance in that it has many important consequences in what follows.

Proposition 3.5. *Let $M \in \mathrm{Gr}_0(A)$ be finitely generated with minimal graded projective resolution $(\mathscr{Q}^\bullet)$. Assume that P^n is generated in degree s and Q^n is finitely generated. Then Q^n is generated in degree s if and only if $\mathrm{Ext}_A^n(A_0, A_0) \bigotimes_K \mathrm{Hom}_A(M, A_0) \to \mathrm{Ext}_A^n(M, A_0)$ is surjective; that is, $\mathrm{Ext}_A^n(A_0, A_0) \cdot \mathrm{Hom}_A(M, A_0) = \mathrm{Ext}_A^n(M, A_0)$.*

Proof. Consider the graded short exact sequence

$$0 \to \mathbf{r}M \to M \to M/\mathbf{r}M \to 0.$$

Apply the functor $\mathrm{Hom}_A(-, A_0)$, we obtain an exact sequence

$$\mathrm{Ext}_A^n(M/\mathbf{r}M, A_0) \to \mathrm{Ext}_A^n(M, A_0) \to \mathrm{Ext}_A^n(\mathbf{r}M, A_0).$$

By Lemma 3.2 and the remarks before the proposition, since $\mathbf{r}M[-1] \in \mathrm{Gr}_0(A)$, we see that, in the shift-grading, $\mathrm{Ext}_A^n(\mathbf{r}M, A_0)$ is supported in $\{i \mid i \geqslant s+1\}$. In particular,

$\mathrm{Ext}^n(\mathbf{r}M, A_0)_s = 0$. Since $M/\mathbf{r}M$ is semisimple and supported in $\{0\}$, $\mathrm{Ext}_A^n(M/\mathbf{r}M, A_0)$ is supported in $\{s\}$. It follows that $\mathrm{Ext}_A^n(M/\mathbf{r}M, A_0)_s \to \mathrm{Ext}_A^n(M, A_0)_s$ is surjective. Clearly, $\mathrm{Ext}_A^n(A_0, A_0) \bigotimes_K \mathrm{Hom}_A(M, A_0) \to \mathrm{Ext}_A^n(M, A_0)$ factors through $\mathrm{Ext}_A^n(M/\mathbf{r}M, A_0)$.

Now $\mathrm{Ext}_A^n(M/\mathbf{r}M, A_0)$ is supported in $\{s\}$. Next we note that Q^n is generated in degree s if and only if $\mathrm{Ext}_A^n(M, A_0)$ is supported in $\{s\}$. Hence, Q^n is generated in degree s if and only if $\mathrm{Ext}_A^n(M/\mathbf{r}M, A_0) \to \mathrm{Ext}_A^n(M, A_0)$ is surjective. The result now follows from Lemma 3.4. □

We apply the above proposition to get a result that will play an important role in what follows.

Proposition 3.6. *Suppose that P^i is finitely generated with generators in degree d_i, for $i = \alpha, \beta, \alpha + \beta$. Assume that*

$$d_{\alpha+\beta} = d_\alpha + d_\beta.$$

Then the Yoneda map

$$\mathrm{Ext}_A^\alpha(A_0, A_0) \bigotimes_K \mathrm{Ext}_A^\beta(A_0, A_0) \to \mathrm{Ext}_A^{\alpha+\beta}(A_0, A_0)$$

is surjective. Thus,

$$\begin{aligned}\mathrm{Ext}_A^{\alpha+\beta}(A_0, A_0) &= \mathrm{Ext}_A^\alpha(A_0, A_0) \cdot \mathrm{Ext}_A^\beta(A_0 A_0) \\ &= \mathrm{Ext}_A^\beta(A_0, A_0) \cdot \mathrm{Ext}_A^\alpha(A_0, A_0).\end{aligned}$$

Proof. By hypothesis, $\Omega^\beta(A_0)$ is generated in degree d_β. Applying Proposition 3.5 to $M = \Omega^\beta(A_0)[-d_\beta]$, we conclude that $\mathrm{Ext}_A^\alpha(A_0, A_0) \bigotimes_K \mathrm{Hom}_A(\Omega^\beta(A_0), A_0) \to \mathrm{Ext}_A^\alpha(\Omega^\beta A_0, A_0)$ is a surjection. Since $\mathrm{Ext}_A^\beta(A_0, A_0) \simeq \mathrm{Hom}_A(\Omega^\beta(A_0), A_0)$ and $\mathrm{Ext}_A^{\alpha+\beta}(A_0, A_0) \simeq \mathrm{Ext}_A^\alpha(\Omega^\beta(A_0), A_0)$, we have shown the desired surjection. □

We conclude this section with an application of Proposition 3.5 and, although it is not needed in the remainder of the paper, it is of some independent interest. Let $\delta : \{0, 1, 2, \ldots\} \to \{0, 1, 2, \ldots\}$. We say a graded A-module $M \in \mathrm{Gr}_0(A)$ has a δ-*homogeneous resolution* if M has a graded projective resolution $(Q^\bullet, d^\bullet)$ such that Q^n is generated in degree $\delta(n)$ for $n \geqslant 0$. If A_0 has a δ-homogeneous resolution, we say a graded A-module M is *relatively Koszul* if M has a δ-homogeneous resolution.

Theorem 3.7. *Let $A = A_0 + A_1 + A_2 + \cdots$ be a graded K-algebra where K is a commutative ring and $\delta : \{0, 1, 2, \ldots\} \to \{0, 1, 2, \ldots\}$. Assume that A_0 is a semisimple K-algebra with a δ-homogeneous resolution. Then a graded A-module M is relatively Koszul if and only if, as an $E(A)$-module, $\mathrm{Ext}_A^*(M, A_0) = \bigoplus_{n \geqslant 0} \mathrm{Ext}_A^n(M, A_0)$ is generated in degree* 0. □

Of course, if A is either a Koszul algebra or a d-Koszul algebra, then A_0 has a δ-homogeneous resolution for some δ.

4. d-Koszul algebras

Let $A = A_0 + A_1 + A_2 + \cdots$ be a graded K-algebra generated in degrees 0 and 1 where K is a commutative noetherian ring. Assume that A_0 is a finitely generated semisimple K-algebra, A_1 is a finitely generated K-module and that $\mathscr{P}^\bullet$ is a minimal graded A-projective resolution of A_0. We say that A is a *d-Koszul algebra* if, for each $n \geqslant 0$, P^n can be generated in exactly one degree, $\delta(n)$, and

$$\delta(n) = \begin{cases} \frac{n}{2}d & \text{if } n \text{ is even,} \\ \left(\frac{n-1}{2}d\right) + 1 & \text{if } n \text{ is odd.} \end{cases} \tag{1}$$

By our assumption that A is generated in degrees 0 and 1, we see that A is a quotient of the tensor algebra $T_{A_0}(A) = A_0 + A_1 + (A_1 \bigotimes_{A_0} A_1) + (\bigotimes_{A_0}^3 A_1) + \cdots$. If $A = T_{A_0}(A_1)/I$ is a d-Koszul algebra, we see that I is finitely generated and can be generated by elements in $\bigotimes_{A_0}^d A_1$ since P^2 can be generated in degree d. Furthermore, the finiteness assumptions on A_0 and A_1 and that K is noetherian imply that each P^n is finitely generated.

We note that if $d = 2$, then A is a Koszul algebra since A_0 has linear projective resolution [4]. For $d \geqslant 3$, A is not a Koszul algebra. We now fix a representation of A as a quotient of the tensor algebra $T_{A_0}(A_1)$. Let $A = T_{A_0}(A_1)/I$ where $I \subset \sum_{j\geqslant 2} \bigotimes_{A_0}^j A_1$. We now present a characterization of d-Koszul algebras.

Theorem 4.1. *Let $A = T_{A_0}(A_1)/I$ where I can be generated by elements of $\bigotimes_{A_0}^d A_1$ for some $d \geqslant 2$. Then A is a d-Koszul algebra if and only if the* Ext*-algebra $E(A)$ can be generated in degrees* 0, 1, *and* 2 *in the ext-degree grading.*

Proof. First assume that A is a d-Koszul algebra. We proceed by induction on n, to show that $\mathrm{Ext}_A^n(A_0, A_0)$ is generated by $\mathrm{Ext}_{A_0}^i(A_0, A_0)$ for i=0,1,2. For n=0,1,2, the result is trivial. Assume true for $n \leqslant k$ and consider $\mathrm{Ext}_A^{k+1}(A_0, A_0)$. If $k+1 = 2m$ is even, then $\delta(2) + \delta(m-2) = d + ((m-2)/2)\,d = (m/2)\,d = \delta(2m)$. Thus we may apply Proposition 3.6 with $\alpha = 2$ and $\beta = m-2$. We have that $\mathrm{Ext}_A^2(A_0, A_0) \bigotimes_K \mathrm{Ext}_A^{m-2}(A_0, A_0) \to \mathrm{Ext}_A^{k+1}(A_0, A_0)$ is surjective. By induction, $\mathrm{Ext}_A^{m-2}(A_0, A_0)$ is generated by $\mathrm{Ext}^i(A_0, A_0)$ for $i = 0,1,2$ and we conclude that $\mathrm{Ext}_A^{k+1}(A_0, A_0)$ is generated by $\mathrm{Ext}^i(A_0, A_0)$ for $i = 0,1,2$. If $k+1 = 2m+1$, then $\delta(1) + \delta(2m) = 1 + (m/2)d = \delta(2m+1)$. Again we apply Proposition 3.6 with $\alpha = 1$ and $\beta = 2m$ this time to conclude that $\mathrm{Ext}_A^{k+1}(A_0, A_0)$ is generated by $\mathrm{Ext}^i(A_0, A_0)$ for $i = 0,1,2$ and the proof of the only if part is complete.

Now assume that $E(A)$ is generated in degrees 0,1,2 in the ext-degree grading. We begin with case d=2. Then P^1 is generated in degree 1 and P^2 is generated in degree 2. Applying Proposition 3.6 with $\alpha = \beta = 1$, we conclude that $\mathrm{Ext}_A^1(A_0, A_0) \bigotimes_K \mathrm{Ext}_A^1(A_0, A_0) \to \mathrm{Ext}_A^2(A_0, A_0)$ is surjective. Hence, $E(A)$ can be generated in degrees 0,1. It follows by [4] that A is a Koszul algebra and hence a 2-Koszul algebra. Now assume that $d > 2$. First, we note that since P^1 is generated in degree 1, and since $\mathrm{Ext}_A^1(A_0, A_0) \simeq$

$\mathrm{Hom}_A(P^1, A_0)$, every extension in $\mathrm{Ext}^1_A(A_0, A_0)$ is of the form $0 \to A_0[-1] \to E \to A_0 \to 0$ as a short exact sequence of graded modules. If $0 \to A_0[-1] \to E' \to A_0 \to 0$ is another short exact sequence, then we may shift the sequence by -1 to get $0 \to A_0[-2] \to E'[-1] \to A_0[-1] \to 0$. Pasting the sequences together gives $0 \to A_0[-2] \to E'[-1] \to E \to A_0 \to 0$. It follows that the image of $\mathrm{Ext}^1_A(A_0, A_0) \otimes \mathrm{Ext}^1_A(A_0, A_0) \to \mathrm{Ext}^2_A(A_0, A_0)$ lies in $\mathrm{Ext}^2_A(A_0, A_0)_2$. But by hypothesis, $\mathrm{Ext}^2_A(A_0, A_0) = \mathrm{Ext}^2_A(A_0, A_0)_d$ (the shift-grading) and $d > 2$. Thus, $\mathrm{Ext}^1_A(A_0, A_0)^2 = 0$. Now $\mathrm{Ext}^3_A(A_0, A_0)$ is generated by $\mathrm{Ext}^2_A(A_0, A_0)\,\mathrm{Ext}^1_A(A_0, A_0) + \mathrm{Ext}^1_A(A_0, A_0)\,\mathrm{Ext}^2_A(A_0, A_0)$. Since $\mathrm{Ext}^2_A(A_0, A_0) = \mathrm{Ext}^2_A(A_0, A_0)_d$ and $\mathrm{Ext}^1_A(A_0, A_0) = \mathrm{Ext}^1_A(A_0, A_0)_1$, it follows that $\mathrm{Ext}^3_A(A_0, A_0) = \mathrm{Ext}^3_A(A_0, A_0)_{d+1}$. Hence, P^3 must be generated in degree $d+1$. But now we may apply Proposition 3.6 to $\alpha = 1$ and $\beta = 2$, to see that

$$\begin{aligned}\mathrm{Ext}^3_A(A_0, A_0) &= \mathrm{Ext}^2_A(A_0, A_0)\,\mathrm{Ext}^1_A(A_0, A_0)\\ &= \mathrm{Ext}^1_A(A_0, A_0)\,\mathrm{Ext}^2_A(A_0, A_0).\end{aligned} \tag{2}$$

Now consider $\mathrm{Ext}^4_A(A_0, A_0)$. Since $(\mathrm{Ext}^1_A(A_0, A_0))^2 = 0$, by (2), and by our hypothesis, we see that $\mathrm{Ext}^4_A(A_0, A_0) = (\mathrm{Ext}^2_A(A_0, A_0))^2$. As above, we conclude that P^4 is generated in degree $2d$. For $\mathrm{Ext}^5_A(A_0, A_0)$, using that $(\mathrm{Ext}^1_A(A_0, A_0))^2 = 0$ and (2), we conclude that

$$\begin{aligned}\mathrm{Ext}^5_A(A_0, A_0) &= \mathrm{Ext}^1_A(A_0, A_0)\,\mathrm{Ext}^2_A(A_0, A_0)\,\mathrm{Ext}^2_A(A_0, A_0)\\ &= \mathrm{Ext}^2_A(A_0, A_0)\,\mathrm{Ext}^1_A(A_0, A_0)\,\mathrm{Ext}^2_A(A_0, A_0)\\ &= \mathrm{Ext}^2_A(A_0, A_0)\,\mathrm{Ext}^2_A(A_0, A_0)\,\mathrm{Ext}^1_A(A_0, A_0).\end{aligned}$$

It follows that P^5 is generated in degree $2d+1$.

Continuing in this fashion, the result follows. □

From the proof of the above theorem, we get the following important result.

Corollary 4.2. *If A is a d-Koszul algebra, with $d > 2$, then*

$$\mathrm{Ext}^{2m+1}_A(A_0, A_0) \cdot \mathrm{Ext}^{2n+1}_A(A_0, A_0) = (0),$$

for all $n, m \geqslant 0$.

5. *d*-Koszul modules

Throughout this section, $A = A_0 + A_1 + \cdots$ will denote a d-Koszul algebra. We say a left graded A-module M is a *d-Koszul module* if there is a graded A-projective resolution $\cdots \to Q^2 \to Q^1 \to Q^0 \to M \to 0$ such that Q^n is generated in degree $\delta(n)$ where

$$\delta(n) = \begin{cases} \dfrac{n}{2}\,d & \text{if } n \text{ is even,} \\[2ex] \left(\dfrac{n-1}{2}\,d\right) + 1 & \text{if } n \text{ is odd.} \end{cases}$$

If M is a d-Koszul module then $M \in \mathrm{Gr}_0(A)$ since Q^0 is generated in degree 0. Note that if $d = 2$, then a module is d-Koszul if and only if it has a linear projective resolution. Thus, in this case, being a d-Koszul module coincides with being a Koszul module.

We begin by showing that if M is a d-Koszul module, then there are exact sequences of the form

$$0 \to \Omega^t(M) \to \Omega^t(M/\mathbf{r}M) \to \Omega^{t-1}(\mathbf{r}M) \to 0,$$

for $t \geqslant 1$. Assume that M is a d-Koszul module. Then, in degree 0, we have an isomorphism $M_0 \to (M/\mathbf{r}M)_0$ induced from the canonical surjection $M \to M/\mathbf{r}M$. Hence, the graded projective cover $Q^0 \to M$, when composed with $M \to M/\mathbf{r}M$, is also a graded projective cover. From these observations, we see that we have an exact commutative diagram

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \mathbf{r}M & \longrightarrow & M & \longrightarrow & M/\mathbf{r}M & \longrightarrow & 0 \\
 & & & & \uparrow & & \uparrow & & \\
 & & 0 & \longrightarrow & Q^0 & = & Q^0 & \longrightarrow & 0 \\
 & & & & \uparrow & & \uparrow & & \\
 & & 0 & \longrightarrow & \Omega^1_A(M) & \longrightarrow & \Omega^1_A(M/\mathbf{r}M) & \longrightarrow & \mathbf{r}M \longrightarrow 0 \\
 & & & & \uparrow & & \uparrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

By assumption, A is a d-Koszul algebra and it follows that $M/\mathbf{r}M$, being a semisimple $A/\mathbf{r}$-module, is a d-Koszul module. Hence, $\Omega^1_A(M/\mathbf{r}M)$ is generated in degree 1. Hence $\mathbf{r}M$ is generated in degree 1. We now have a short exact sequence $0 \to \Omega^1(M) \to \Omega^1(M/\mathbf{r}M) \to \mathbf{r}M \to 0$ where each module is generated in degree 1. It follows that if $Q^1 \to \Omega^1(M)$ and $L^0 \to \mathbf{r}M$ are graded projective covers then we get an exact commutative diagram

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \Omega^1(M) & \longrightarrow & \Omega^1(M/\mathbf{r}M) & \longrightarrow & \mathbf{r}M & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & Q^1 & \longrightarrow & Q^1 \oplus L^0 & \longrightarrow & L^0 & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \Omega^2(M) & \longrightarrow & \Omega^2(M/\mathbf{r}M) & \longrightarrow & \Omega^1(\mathbf{r}M) & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

Since the top row of the diagram is composed of modules, all generated in degree 1, we see that $Q^1 \oplus L^0 \to \Omega^1(M/\mathbf{r}M)$ is a projective cover. Since M and $M/\mathbf{r}M$ are d-Koszul modules, both $\Omega^2(M)$ and $\Omega^2(M/\mathbf{r}M)$ are generated in degree d. It follows that $\Omega^1(\mathbf{r}M)$ is generated in degree d. Proceeding by induction, we obtain exact sequences

$$0 \to \Omega^t(M) \to \Omega^t(M/\mathbf{r}M) \to \Omega^{t-1}(\mathbf{r}M) \to 0, \tag{3}$$

for $t \geqslant 1$ and that each of the modules is generated in degree $\delta(t)$.

We apply the above result in the following proposition.

Proposition 5.1. *Let A be a d-Koszul algebra and M a left A-module which is d-Koszul. Then, for $n \geqslant 1$, we have exact sequences*

$$0 \to \mathrm{Ext}_A^{n-1}(\mathbf{r}M, A_0) \to \mathrm{Ext}_A^n(M/\mathbf{r}M, A_0) \to \mathrm{Ext}_A^n(M, A_0) \to 0.$$

In the shift-grading, $\mathrm{Ext}_A^{n-1}(\mathbf{r}M, A_0)$, $\mathrm{Ext}_A^n(M/\mathbf{r}M, A_0)$, *and* $\mathrm{Ext}_A^n(M, A_0)$ *have support in degree* $\delta(n)$.

Proof. We have seen that under the hypothesis of the proposition, we have exact sequences

$$0 \to \Omega^n(M) \to \Omega^n(M/\mathbf{r}M) \to \Omega^{n-1}(\mathbf{r}M) \to 0,$$

for $n \geqslant 1$ and that each of the modules is generated in degree $\delta(n)$. It follows that, applying $\mathrm{Hom}_A(-, A_0)$ to sequences, we obtain short exact sequences

$$0 \to \mathrm{Hom}_A(\Omega^{n-1}(\mathbf{r}M), A_0) \to \mathrm{Hom}_A(\Omega^n(M/\mathbf{r}M), A_0)$$

$$\to \mathrm{Hom}_A(\Omega^n(M), A_0) \to 0.$$

The result now follows. □

The next result provides a method for constructing new d-Koszul modules from known ones. For $t \geqslant 0$, we use the formula

$$(*) \quad \delta(t+2) = \delta(t) + d,$$

whose proof we leave to the reader.

Proposition 5.2. *Let A be a d-Koszul algebra and M a d-Koszul module. Then $\Omega^2(M)[-d]$ and $\Omega^1(\mathbf{r}M)[-d]$ are both d-Koszul A-modules.*

Proof. Let M be a d-Koszul module. Then $\Omega^2(M)$ is generated degree $\delta(n+2) - d = \delta(n)$ and we conclude that $\Omega^1(\mathbf{r}M)[-d]$ is a d-Koszul module. This completes the proof. □

We end this section with a result for modules analogous to Theorem 4.1. If M is a left A-module, let $\mathscr{E}(M)$ denote the left $E(A)$-module $\bigoplus_{n\geqslant 0} \mathrm{Ext}^n(M, A_0)$, where the module structure is given by the Yoneda product.

Theorem 5.3. *Let $A = A_0 + A_1 + \cdots$ be a d-Koszul algebra with $d \geqslant 3$ and let M be a graded left A-module. Then M is a d-Koszul module if and only if $\mathscr{E}(M)$ can be generated in degree* 0. *Moreover, if M is a d-Koszul module, then* $\mathrm{Ext}^{2n+1}(A_0, A_0) \cdot \mathrm{Ext}^{2m+1}(M, A_0) = (0)$ *for all $n, m \geqslant 0$.*

Proof. Applying Proposition 3.5, we see that

$$\mathrm{Ext}^n(A_0, A_0) \cdot \mathrm{Hom}_A(M, A_0) = \mathrm{Ext}^n_A(M, A_0)$$

for each n if and only if M is a d-Koszul module. Using this and Corollary 4.2, we conclude that $\mathrm{Ext}^{2n+1}(A_0, A_0) \cdot \mathrm{Ext}^{2m+1}(M, A_0) = (0)$ for all $n, m \geqslant 0$. □

6. The even Ext of modules

In this section, we investigate the properties of the even Ext-algebra $E(A)$ of a d-Koszul algebra. We show that this algebra is a Koszul algebra after a regrading. We also show that the even Ext of a d-Koszul module is a Koszul module over the even Ext algebra. For the remainder of the section, assume that A is a d-Koszul algebra.

Let M be a left d-Koszul A-module. We let

$$E^{\mathrm{ev}}(A) = \bigoplus_{n\geqslant 0} \mathrm{Ext}^{2n}(A_0, A_0)$$

and

$$\mathscr{E}^{\mathrm{ev}}(M) = \bigoplus_{n\geqslant 0} \mathrm{Ext}^{2n}(M, A_0).$$

We grade $E^{\mathrm{ev}}(A)$ by $E^{\mathrm{ev}}(A)_n=\mathrm{Ext}^{2n}(A_0,A_0)$ and view $\mathscr{E}^{\mathrm{ev}}(M)$ as a graded $E^{\mathrm{ev}}(A)$-module where $\mathscr{E}^{\mathrm{ev}}(M)_n=\mathrm{Ext}_A^{2n}(M,A_0)$. We call this the *even-grading*.

The following result is the main result of this section.

Theorem 6.1. *Let $A=A_0+A_1+\cdots$ be a d-Koszul algebra and M a d-Koszul module. Then, in the even-grading, $E^{\mathrm{ev}}(A)$ is a Koszul algebra and $\mathscr{E}^{\mathrm{ev}}(M)$ is a Koszul $E^{\mathrm{ev}}(A)$-module.*

Proof. Let M be a d-Koszul A-module. We begin by showing that $\mathscr{E}^{\mathrm{ev}}(M)$ is generated in degree 0 (in the even-grading). Now $E^{\mathrm{ev}}(A)_n$. $\mathscr{E}(M)_0=\mathrm{Ext}^{2n}(A_0,A_0)\cdot\mathrm{Hom}(M,A_0)$ and $\mathscr{E}(M)_n=\mathrm{Ext}^{2n}(M,A_0)$. By Proposition 3.5 we see that $E^{\mathrm{ev}}(A)_n\cdot\mathscr{E}(M)_0=\mathscr{E}(M)_n$ and we conclude that $\mathscr{E}(M)$ is generated in degree 0.

Next we show that $\mathscr{E}^{\mathrm{ev}}(M)$ has a linear $E^{\mathrm{ev}}(A)$-projective resolution in the even-grading. By Proposition 5.1, we have exact sequences

$$0\to\mathrm{Ext}_A^{n-1}(\mathbf{r}M,A_0)\to\mathrm{Ext}_A^{n}(M/\mathbf{r}M,A_0)\to\mathrm{Ext}_A^{n}(M,A_0)\to 0.$$

In the shift-grading, $\mathrm{Ext}_A^{n-1}(\mathbf{r}M,A_0),\mathrm{Ext}_A^{n}(M/\mathbf{r}M,A_0)$, and $\mathrm{Ext}_A^{n}(M,A_0)$ have support in degree $\delta(n)$.

Now $\mathscr{E}^{\mathrm{ev}}(M/\mathbf{r}M)$ is a projective $E^{\mathrm{ev}}(A)$-module generated in degree 0. Thus, we have a projective cover $\mathscr{E}^{\mathrm{ev}}(M/\mathbf{r}M)\to\mathscr{E}^{\mathrm{ev}}(M)$. The first syzygy $\bigoplus_{n\geqslant 1}\mathrm{Ext}_A^{2n-1}(\mathbf{r}M,A_0)=\bigoplus_{n\geqslant 0}\mathrm{Ext}^{2n-2}(\Omega^1(\mathbf{r}M),A_0)$. We have seen that $\Omega^1(\mathbf{r}M)$ is generated in degree d. By Proposition 5.2, $\Omega^1(\mathbf{r},M)[-d]$ is a d-Koszul A-module. We have just seen that if X is a d-Koszul module then $\mathscr{E}(X)$ is generated in degree 0. Thus $\mathscr{E}^{\mathrm{ev}}(\Omega^1(\mathbf{r}M)[-d])$ is generated in degree 0 (in the even-grading). In the shift-grading $\mathscr{E}^{\mathrm{ev}}(\Omega^1(\mathbf{r}M)[-d])$ is generated in degree d. But degree 1 in the even-grading is degree d in the shift-grading. Hence the first syzygy of $\mathscr{E}^{\mathrm{ev}}(M)$, as an $E^{\mathrm{ev}}(A)$-module, is $\mathscr{E}^{\mathrm{ev}}(\Omega^1(\mathbf{r}M))$ and is generated in degree 1. Furthermore, $\Omega^1(\mathbf{r}M)[-d]$ is a d-Koszul module and we now may repeat the above argument to show that the second syzygy of $\mathscr{E}^{\mathrm{ev}}(M)$ is generated in degree 2 (in the even-grading) and is of the form $\mathscr{E}^{\mathrm{ev}}(N)$ where $N[-2d]$ is a d-Koszul module. Continuing in this fashion, we see that $\mathscr{E}^{\mathrm{ev}}(M)$ has a linear $E^{\mathrm{ev}}(A)$-projective resolution in the even-grading.

Since $E^{\mathrm{ev}}(A)=\mathscr{E}^{\mathrm{ev}}(A_0)$, we see that $E^{\mathrm{ev}}(A)$ has a linear graded projective resolution (in the even-grading). Hence $E^{\mathrm{ev}}(A)$ is a Koszul algebra. This competes the proof. □

If A is a d-Koszul algebra, and M is a d-Koszul A-module let $\mathscr{E}^{\mathrm{odd}}(M)=\bigoplus_{n\geqslant 0}\mathrm{Ext}^{2n+1}(M,A_0)$. We view this as a graded object by $\mathscr{E}^{\mathrm{odd}}(M)_n=\mathrm{Ext}^{2n+1}(M,A_0)$. We note that

$$\begin{aligned}\mathrm{Ext}^{2m}(A_0,A_0)\cdot\mathrm{Ext}^{2n+1}(M,A_0)&=\mathrm{Ext}^{2m}(A_0,A_0)\cdot\mathrm{Hom}(\Omega^{2n+1}(M),A_0)\\&=\mathrm{Ext}^{2m}(\Omega^{2n+1}(M),A_0)=\mathrm{Ext}^{2(n+m)+1}(M,A_0),\end{aligned}$$

where the second equality follows from Proposition 3.6. Thus, $\mathscr{E}^{\mathrm{odd}}(M)$ is a graded $E^{\mathrm{ev}}(A)$-module and is generated in degree 0 (in the even-grading). If M is a d-Koszul module, it is open whether or not $\mathscr{E}^{\mathrm{odd}}(M)$ is a Koszul module.

7. Ext of *d*-Koszul algebras and modules

In this section we show that the Ext-algebra of a d-Koszul algebra is a Koszul algebra after regrading. Furthermore, we show that the Ext of a d-Koszul module is a Koszul module over the Ext-algebra after regrading. If A is a Koszul algebra, then these results are well-known [5]. Throughout this section, let A be a d-Koszul algebra where $d \geqslant 3$. We begin by describing the new grading.

We let $\hat{E}(A)$ be the Ext-algebra $E(A)$ graded as follows. $\hat{E}(A)_0 = \mathrm{Ext}^0_A(A_0, A_0)$, $\hat{E}(A)_1 = \mathrm{Ext}^1_A(A_0, A_0) \oplus \mathrm{Ext}^2_A(A_0, A_0)$, $\hat{E}(A)_2 = \mathrm{Ext}^3(A_0, A_0) \oplus \mathrm{Ext}^4(A_0, A_0)$. In general, if $n \geqslant 1$,

$$\hat{E}(A)_n = \mathrm{Ext}^{2n-1}_A(A_0, A_0) \oplus \mathrm{Ext}^{2n}_A(A_0, A_0).$$

This is a well-defined grading by Corollary 4.2.

If M is a d-Koszul A-module, we define $\hat{\mathscr{E}}(M)$ to be $\mathscr{E}(M)$ with grading given as follows. If $n \geqslant 0$,

$$\hat{\mathscr{E}}(M)_n = \mathrm{Ext}^{2n-1}_A(M, A_0) \oplus \mathrm{Ext}^{2n}_A(M, A_0),$$

where $\mathrm{Ext}^{-1}(M, A_0) = (0)$. By Theorem 5.3 we see that $\hat{\mathscr{E}}(M)$ is a graded $\hat{E}(A)$-module. We will call this grading the *hat-grading* and the degree is call the *hat-degree*. We now state and prove the main result of this section.

Theorem 7.1. *Let $A = A_0 + A_1 + \cdots$ be a d-Koszul algebra with $d \geqslant 3$. Let M be a d-Koszul A-module. Then $\hat{E}(A)$ is a Koszul algebra and $\hat{\mathscr{E}}(M)$ is a Koszul $\hat{E}(A)$-module.*

Proof. We begin by showing that (graded) semisimple $\hat{E}(A)$-modules have linear presentations. First consider $0 \to L \to \hat{E}(A) \to \hat{E}(A)_0 \to 0$. By Theorem 4.1, we see that L, which is $\coprod_{n \geqslant 1} \mathrm{Ext}^n_A(A_0, A_0)$, is generated in hat-degree 1. If W is a graded semisimple module with support in hat-degree 0, then W is a summand of a direct sum of copies of $\hat{E}(A)_0$. It follows that W has a linear presentations as an $\hat{E}(A)$-module.

Next we show that $\hat{\mathscr{E}}(M)$ has a linear presentation. By Theorem 5.3 $\hat{\mathscr{E}}(M)$ is generated in degree 0 as a $\hat{E}(A)$-module. Applying Proposition 5.1, we see that we have a short exact sequence of $\hat{E}(A)$-modules

$$0 \to N \to \hat{\mathscr{E}}(M/\mathbf{r}M) \to \hat{\mathscr{E}}(M) \to 0,$$

where $N = \coprod_{n \geqslant 0} \mathrm{Ext}^n_A(\mathbf{r}M, A_0)$ with $\mathrm{Ext}^n(\mathbf{r}M, A_0)$ in ext-degree $n-1$. In hat-degree, $N_n = \mathrm{Ext}^{n-1}(\mathbf{r}M, A_0) + \mathrm{Ext}^n(\mathbf{r}M, A_0)$. Thus, $N' = \coprod_{n \geqslant 0} \mathrm{Ext}^n(\Omega^1(\mathbf{r}M)), A_0)$ is $E(A)$-submodule of N. We have the following short exact sequence:

$$0 \to N' \to N \to \mathrm{Hom}_A(\mathbf{r}M, A_0) \to 0. \tag{4}$$

Grading N' by $N'_1 = \mathrm{Ext}^0(\Omega^1(\mathbf{r}M), A_0)$ and, for $n \geqslant 2$,

$$N'_n = \mathrm{Ext}^{2n-3}(\Omega^1(\mathbf{r}M), A_0) + \mathrm{Ext}^{2n-2}(\Omega^1(\mathbf{r}M), A_0)$$

and grading $\mathrm{Hom}_A(\mathbf{r}M, A_0)$ to have support in degree 1, we see that (4) is a short exact sequence of hat-graded $\hat{E}(A)$-modules. But $\hat{\mathscr{E}}(\Omega^1(\mathbf{r}M)[-d])[1] = N'$ where the shift

$-d$ is as graded A-modules and the shift 1 is in the hat-grading. But $\Omega^1(\mathbf{r}M)[-d])$ is a d-Koszul A-module by Proposition 5.2. Thus, N' is an $\hat{E}(A)$-module generated in degree 1. We conclude that N is generated in degree 1 and hence $\hat{\mathscr{E}}(M)$ has a linear presentation.

Consider (4) again. We have seen that this is an exact sequence of $\hat{E}(A)$-modules all generated in degree 1. Furthermore, $\mathrm{Hom}_A(\mathbf{r}M, A_0)$ is a semisimple $\hat{E}(A)$-module and N is of the form $\hat{\mathscr{E}}(M')[1]$ for a d-Koszul module M'. But both $\mathrm{Hom}_A(\mathbf{r}M, A_0)$ and $\hat{\mathscr{E}}(M')$ have linear presentations by our work above. We will show that a semisimple $\hat{E}(A)$-module supported in degree 0 has first syzygy U such that there is a short exact sequence of graded $\hat{E}(A)$-modules all generated in hat-degree 1,

$$0 \to \hat{\mathscr{E}}(M') \to U \to V \to 0,$$

where V is semisimple and M' is d-Koszul. From this, by a standard induction argument, conclude that $\hat{\mathscr{E}}(M)$ has a linear $\hat{E}(A)$-projective resolution.

We need to show that a semisimple $\hat{E}(A)$-module has first syzygy U such that there is a short exact sequence of graded $\hat{E}(A)$-modules, generated in hat-degree 1

$$0 \to \hat{\mathscr{E}}(M') \to U \to V \to 0,$$

where V is semisimple and M' is d-Koszul. Let W be a semisimple $\hat{E}(A)$-module supported in degree 0. There is a semisimple A-module S such that $\mathrm{Hom}_A(S, A_0) = W$. Consider the short exact sequence

$$0 \to \coprod_{n \geqslant 1} \mathrm{Ext}_A^n(\mathbf{r} \bigotimes_{A_0} S, A_0) \to \hat{\mathscr{E}}(A \bigotimes_{A_0} S) \to W \to 0.$$

Then, in a similar fashion to our investigation of N above, we get a short exact sequence of graded $\hat{E}(A)$-modules

$$0 \to \hat{\mathscr{E}}(\Omega^1(\mathbf{r} \bigotimes_{A_0} S, A_0)[-d])[1] \to \hat{\mathscr{E}}(A \bigotimes_{A_0} S) \to \mathrm{Hom}_A(S, A_0) \to 0.$$

It follows that $\hat{\mathscr{E}}(\Omega^1(\mathbf{r} \otimes_{A_0} S)[-d])[1]$ has hat-degree 1, $\Omega^1(\mathbf{r} \otimes_{A_0} S)[-d] = \Omega^2(S)[-d]$ is a d-Koszul module and $\mathrm{Hom}_A(\mathbf{r} \otimes_{A_0} S, A_0)$ is a semisimple $\hat{E}(A)$-module supported in hat-degree 1. This is the desired result.

Finally, to show that $\hat{E}(A)$ is a Koszul algebra, we note that $\hat{E}(A)_0 = \hat{\mathscr{E}}(A)$ and hence $\hat{E}(A)_0$ has a linear projective resolution in the hat-grading. It follows that $\hat{E}(A)$ is a Koszul algebra and the proof is complete. □

8. Generalized Koszul complexes

As mentioned in the introduction, Berger introduced d-Koszul algebras and generalized Koszul complexes. We briefly summarize some of his results in [2] and provide a new proof of one of his main results. We also extend the definitions to the nonlocal case.

We begin with some notation and conventions. Recall that we have $A = T_{A_0}(A_1)/I$ where I is generated by elements of degree d. Let $R = I \cap (\bigotimes_{A_0}^d A_1)$. Note that R is an A_0-A_0-submodule of $\bigotimes_{A_0}^d A_1$. We now assume K is a field and that A_0 is not only semisimple, but, as a ring, A_0 is $K \times K \times \cdots \times K$. Let $T = T_{A_0}(A_1)$ and if $x \in T$, let $\bar{x}$ denote $\pi(x)$ where $\pi : T \to A$ is the canonical surjection. In this case, T is isomorphic to a path algebra $K\Gamma$ for some quiver Γ. Let $\{v_1, \ldots, v_n\}$ be the arrows of Γ. Then the v_i's are a full set of orthogonal idempotents. We say a nonzero element $x \in T$ is *left uniform* if there exists a vertex v_i such that $x = v_i x$. If x is left uniform, we let $o(x) = v_i$ if $x = v_i x$.

We define the *generalized Koszul complex of* R as follows. Let $H_0 = A_0$, $H_1 = A_1$, and, for $n \geqslant d$,

$$H_n = \bigcap_{i+j+d=n} \left(\bigotimes_{A_0}^{i} A_1 \right) \bigotimes_{A_0} R \bigotimes_{A_0} \left(\bigotimes_{A_0}^{j} A_1 \right).$$

As usual, we let

$$\delta(n) = \begin{cases} \dfrac{n}{2} d & \text{if } n \text{ is even,} \\ \left(\dfrac{n-1}{2} d \right) + 1 & \text{if } n \text{ is odd.} \end{cases}$$

We define $Q^n = A \bigotimes_{A_0} H_{\delta(n)}$ and note that Q^n is a projective left A-module for $n \geqslant 0$.

We wish to define maps $d^n : Q^n \to Q^{n-1}$ for $n \geqslant 1$. For this we need the following lemma which relates to the condition (ec) in Berger's work. To simplify notation, we will denote $\bigotimes_{A_0}^i A_1$ as simply A_1^i and write $\bigotimes_{A_0}$ as simply $\otimes$.

Lemma 8.1. *Keeping the above notation, if A is d-Koszul then, for $2 \leqslant i < d$,*

$$(R \otimes A_1^i) \cap (A_1^i \otimes R) \subseteq A_1^{i-1} \otimes R \otimes A_1.$$

Proof. To prove this result, we use the results in [6]. Since we are considering left modules in this paper, we switch their notation from right modules to left modules. It is shown in [6] that, for each $n \geqslant 0$, there are subsets of left uniform elements of T, $\{g_i^n\}_{i \in U_n}, \{g{*}_i^n\}_{i \in V_n}$, and $\{h_{i,j}^{n,n-1}\}_{i \in U_n,\ j \in U_{n-1}}$ such that

(i) $(\bigoplus_{i \in U_{n-1}} Tg_i^{n-1}) \cap (\bigoplus_{i \in U_{n-2}} Ig_i^{n-2}) = (\bigoplus_{i \in U_n} Tg_i^n) \oplus (\bigoplus_{i \in V_n} Tg{*}_i^n)$.
(ii) For $i \in U_n$, $g_i^n \notin \bigoplus_{i \in U_{n-1}} Ig_i^{n-1}$.
(iii) For $i \in V_n$, $g{*}_i^n \in \bigoplus_{i \in U_{n-1}} Ig_i^{n-1}$.
(iv) For $i \in U_n$, $g_i^n = \sum_{j \in U_{n-1}} h_{i,j}^{n,n-1} g_j^{n-1}$.
(v) Setting $L^n = \bigoplus_{i \in U_n} A\overline{o(g_i^n)}$, then $\{L^n, d^n\}$ is a minimal A-projective resolution of A_0 where

$$d^n(\overline{\mathrm{o}(g_i^n)}) = \sum_{j \in U_{n-1}} \overline{h_{i,j}^{n,n-1}}\, \overline{\mathrm{o}(g_j^{n-1})}.$$

Since A is a graded d-Koszul algebra, we may assume that the g_s^n's are homogeneous left uniform elements of degree $\delta(n)$. Furthermore, $A_1 = \bigoplus_{s \in U_1} A_0 g_s^1$ and $R = \bigoplus_{s \in U_2} A_0 g_s^2$. Note that the g_s^1 are just the arrows of Q and are of degree 1.

The element g_s^2 are homogeneous of degree d and the elements g_s^3 are homogeneous of degree $d+1$.

Now, from the above, we see that

$$(A_1^i \otimes R) = \bigoplus_{s \in U_2} A_1^i \otimes A_0 g_s^2 = (\bigoplus_{s \in U_2} T g_s^2)_{d+i},$$

where the last subscript denotes the elements of $(\bigoplus_{s \in U_2} T g_s^2)$ of degree $d+i$.

We see that $R \otimes A_1^i \subseteq \bigoplus_{s \in U_1} I g_s^1$ since $i \geqslant 2$ and $R \subset I$. We also have that $A_1^i \otimes R \subset \bigoplus_{s \in U_2} T g_s^2$. Hence

$$(R \otimes A_1^i) \cap (A_1^i \otimes R) \subset (\bigoplus_{s \in U_2} T g_s^2) \cap (\bigoplus_{s \in U_1} I g_s^1).$$

By (i) above,

$$(R \otimes A_1^i) \cap (A_1^i \otimes R) \subset (\bigoplus_{s \in U_3} T g_s^3) \oplus (\bigoplus_{s \in V_3} T g{*}_s^3).$$

The left-hand side are elements of degree $d+i$. Noting that each $g{*}_s^3 \in \bigoplus_t I g_t^2$ and I is generated in degree d, we see that the $g{*}_s^3$ are degree at least $2d$. But, i is assumed to be less that d. Hence we conclude that

$$(R \otimes A_1^i) \cap (A_1^i \otimes R) \subset (\bigoplus_{s \in U_3} T g_s^3). \tag{5}$$

Now each g_s^3 is degree $d+1$ since A is d-Koszul. By (iv) above, $g_s^3 \in \bigoplus_t I g_t^1$. It follows by degree that each $g_s^3 = \sum_{t,u} c_{t,u} g_t^2 g_u^1$ where $c_{t,u} \in K$. Hence $g_s^3 \in R \otimes A_1$. Thus, from (5) and degree, that

$$(R \otimes A_1^i) \cap (A_1^i \otimes R) \subset A_1^{i-1} \otimes R \otimes A_1.$$

The proof is complete. □

The following result is an immediate consequence of the above lemma and we leave the proof to the reader.

Corollary 8.2. *Keeping the notation of this section, for* $n \geqslant 0$,

$$\begin{aligned} H_{dn+1} = {} & (R \otimes A_1^{(n-1)d+1}) \cap (A_1 \otimes R \otimes A_1^{(n-1)d}) \\ & \cap (A_1^d \otimes R \otimes A_1^{(n-2)d+1}) \cap (A_1^{d+1} \otimes R \otimes A_1^{(n-1)d}) \\ & \vdots \\ & \cap (A_1^{(n-1)d} \otimes R \otimes A_1) \cap (A_1^{d+1} \otimes R), \end{aligned}$$

and

$$H_{dn} = (R \otimes A_1^{(n-1)d})$$
$$\cap (A_1^{d-1} \otimes R \otimes A_1^{(n-2)d+1}) \cap A_1^d \otimes (R \otimes A_1^{(n-2)d})$$
$$\cap (A_1^d \otimes R \otimes A_1^{(n-2)d+1}) \cap (A_1^{d+1} \otimes R \otimes A_1^{(n-1)d})$$
$$\vdots$$
$$\cap (A_1^{(n-1)d-1} \otimes R \otimes A_1) \cap (A_1^{(n-1)d} \otimes R).$$

Using the result of the corollary we now define $d^m : Q^m \to Q^{m-1}$. Recall that $Q^m = A \otimes H_{\delta(m)}$. From the definition and that $R \subset A_1^d$, we note that $H_{\delta(m)} \subset A_1^{\delta(m)}$. We write elements of $H_{\delta(m)}$ as $x_1 \otimes \cdots \otimes x_{\delta(m)}$ where the x_i are in A_1. If $m = 2n$, define

$$d^m(a \otimes x_1 \otimes \cdots \otimes x_{dn}) = ax_1x_2 \cdots x_{d-1} \otimes x_d \otimes \cdots \otimes x_{dn}.$$

If $m = 2n+1$, define

$$d^m(a \otimes x_1 \otimes \cdots \otimes x_{nd+1}) = ax_1 \otimes x_2 \otimes \cdots \otimes x_{nd+1}.$$

It is writing $H_{\delta(n)}$ in the form of the corollary that shows that the maps are well-defined. We now can state one of Berger's main results.

Theorem 8.3 (Berger [2, Theorem 2.1]). *Let $A = K\Gamma/I$ where I is an ideal generated in degree d. The following statements are equivalent.*

(i) *A is a d-Koszul algebra.*
(ii) *$\{Q^n, d^n\}$ is a minimal A-projective resolution of A_0.*

Proof. Note that by construction, Q^n is generated in degree $\delta(n)$. Hence, if $\{Q^n, d^n\}$ is a minimal A-projective resolution of A_0, then A is a d-Koszul algebra.

Now suppose that A is a d-Koszul algebra. As pointed out, the maps d^n are well-defined. The proof of the exactness of d^n is similar to the usual proof found in [1] and we only give a brief sketch. Because of the definition of H_m as an intersection, we note that if $a \otimes x_1 \otimes \cdots \otimes x_{\delta(m)} \in H_m$ then $x_1x_2 \cdots x_d \in R$. From this, it is immediate that $d^{m-1}d^m = 0$ for $m \geqslant 2$. It is immediate that $Q^1 \xrightarrow{d^1} Q^0 \to A_0 \to 0$ is exact.

We now show exactness at P^{2n}, $n \geqslant 1$. By the definitions, it is not hard to show that $d^{2n+1}(P^{2n+1})$ is generated in degree $nd+1$ in P^{2n}. Similarly, it is not hard to see that if $z = \sum_i a_i \otimes x_{1,i} \otimes \cdots \otimes x_{n,i}$ is in the kernel of d^{2n}, then each $a_i \in A_1 \oplus A_2 \oplus \cdots$. Thus, after rewriting, we may write $z = \sum_i b_i a_i \otimes x_{1,i} \otimes \cdots \otimes x_{n,i}$ where $b_i \in A$ and $a_i \in A_1$. We may assume that each a_i is left uniform. Considering each degree, we see that $\sum_i a_i \otimes x_{1,i} \otimes \cdots \otimes x_{n,i}$ is in the kernel of d^{2n}. But then $z' = \sum_i o(a_i) \otimes a_i \otimes x_{1,i} \otimes \cdots \otimes x_{n,i}$ is in Q^{2n+1} and $d^{2n+1}(\sum_i \bar{o}(a_i) \otimes a_i \otimes x_{1,i} \otimes \cdots \otimes x_{n,i}) = z'$. From this we conclude that the image of d^{2n+1} equals the kernel of d^{2n}.

Exactness at P^{2n+1} is similar and we omit the proof. $\square$

We note that Berger also showed that A is d-Koszul if and only if $A^{\rm op}$, the opposite algebra is d-Koszul. He also studied the $A^{\rm e}$-projective resolution of A, where $A^{\rm e} = A \bigotimes_K A^{\rm op}$ and related this to the Hochschild homology of A.

9. Description of the Ext-algebra

In this section we provide a description of the Ext-algebra $E(A)$ when A is a d-Koszul algebra with $d > 2$.

We begin with well-known preliminaries. Recall that since $A_0 = \prod_{i=1}^n K$, indecomposable A_0-A_0-bimodules are 1-dimensional over K and of the form $A_0 e_i \bigotimes_K e_j A_0$, where $e_i = (0,\dots,0,1,0,\dots,0)$ with 1 occurring in the ith component. Furthermore, $A_0^{\rm op} = A_0$ since A_0 is a commutative ring. Since $A_0 \bigotimes_K A_0$ is a semisimple ring, it follows that every A_0-A_0-bimodule is a direct sum of copies of the 1-dimensional simple modules $A_0 e_i \bigotimes_K e_j A_0$, $1 \leqslant i,j \leqslant n$.

Let V be a finitely generated A_0-A_0-bimodule. If W is an A_0-A_0-submodule of V, let $W^* = \operatorname{Hom}_{A_0}(W, A_0)$ where the Hom is as left A_0-modules. The right A_0-module structure on W gives W^* a left A_0-module structure. The right A_0-module structure on A_0 gives W^* an A_0-A_0-bimodule structure. Note that * is a duality on A_0-A_0-bimodules and that if V is a finitely generated bimodule, then V^{**} is naturally isomorphic to V as bimodules. Let $W^\perp = \{f \in V^* \mid f(W) = 0\}$. We see that $W^\perp$ is an A_0-A_0-bimodule if W is.

We have the following facts, assuming all modules are finitely generated A_0-A_0-bimodules, which can be proved by adjusting the usual proofs for vector spaces:

(i) If U and W are submodules of V, then $(U+W)^\perp = U^\perp \cap W^\perp$.
(ii) If U is a submodule of V, then, for each i,j, $1 \leqslant i,j \leqslant n$, $\dim e_i U^\perp e_j = \dim e_i V e_j - \dim e_i U e_j$.
(iii) If we identify U^{**} with U, $(U^\perp)^\perp = U$.
(iv) If U is a submodule of V and W is a finitely generated bimodule, then $(U \bigotimes_{A_0} W)^\perp = U^\perp \bigotimes_{A_0} W^*$.
(v) If U is a submodule of V, and W and Z are finitely generated bimodules, then $(W \bigotimes_{A_0} U \bigotimes_{A_0} Z)^\perp = (W^* \bigotimes_{A_0} U^\perp \bigotimes_{A_0} Z)$.
(vi) $(U^* \bigotimes_{A_0} V^*) \simeq (U \bigotimes_{A_0} V)^*$.

We identify A_0 and A_0^*. There is a natural isomorphism between $(A_1^i)^* = (\overset{i}{\otimes} A_1)^*$ and $\overset{i}{\otimes} A_1^* = (A_1^*)^i$, which we view as an identification. Let $R^\perp = \{f \in (A_1^*)^d \mid f(x) = 0 \text{ for all } x \in R\}$. Let T^* be the tensor algebra $T_{A_0}(A_1^*) = A_0 \oplus A_1^* \oplus (A_1^*)^2 \oplus \cdots$. The *dual algebra* of A is defined to be $A^! = T^*/\langle R^\perp \rangle$.

We see that $A^!$ is a graded algebra since $R^\perp$ is contained in $(A_1^*)^d$. Thus $A^! = A_0^! \oplus A_1^! \oplus A_2^! \oplus \cdots$. Let $B = B_0 \oplus B_1 \oplus B_2 \oplus \cdots$ where $B_n = A^!_{\delta(n)}$ as vector spaces. In the case $d = 2$, we recall that $\delta(n) = n$ and then $B_n = A^!$ as graded algebras, see, for example, [1]. If $d > 2$, we define multiplication as follows: let $x \in B_n$ and $y \in B_m$, then $x \cdot y = 0$ if both m,n are odd, and as xy where multiplication is in $A^!$ if at least one m or n

is even. It is easy to check that B is a graded K-algebra generated in degrees 0, 1, and 2.

Our goal is to prove the following result.

Theorem 9.1. *If A is a d-Koszul algebra and $d \geqslant 2$ then $E(A)$ is isomorphic to B as graded algebras. In particular,* $\mathrm{Ext}^n_A(A_0, A_0)$ *is isomorphic to* $A^!_{\delta(n)}$.

Before starting the proof, we review the notations of the previous section and assume that A is a d-Koszul algebra with $d > 2$. Then $(Q^\bullet, d^\bullet)$ is a minimal graded A-projective resolution of A_0. Recall that $Q^n = A \otimes H_{\delta(n)}$, $H_0 = A_0$, $H_1 = A_1$ and, for $n \geqslant d$, $H_n = \bigcap_{i+d+j=n} A_1^i \otimes R \otimes A_1^j$ where $A_1^i = \bigotimes^i_{A_0} A_1$. Now $H_n \subset A_1^n$ and we write elements of H_n as $\sum x_1 \otimes \cdots \otimes x_n$. The maps $d^n : Q^n \to Q^{n-1}$ are given by

$$d_n\left(\sum a \otimes x_1 \otimes \cdots \otimes x_{\delta(n)}\right) = \begin{cases} \sum ax_1 \otimes x_2 \otimes \cdots \otimes x_{\delta(n)}, & \text{if } n \text{ is odd} \\ \sum ax_1 \cdots x_{d-1} \otimes x_d \otimes \cdots \otimes x_{\delta(n)}, & \text{if } n \text{ is even.} \end{cases}$$

Since A_0 is semisimple, for $n \geqslant 0$, there is a natural A_0-A_0-bimodule isomorphism

$$\mathrm{Ext}^n_A(A_0, A_0) \simeq \mathrm{Hom}_A(Q^n, A_0) = \mathrm{Hom}_A(A \bigotimes_{A_0} H_n, A_0),$$

which we view as an identification. When we "multiply" homomorphisms, we will mean the Yoneda product of the elements viewed as elements in the Exts under this identification.

Proposition 9.2. *Let $f_n \in \mathrm{Hom}_A(Q^n, A_0)$ and $f_m \in \mathrm{Hom}_A(Q^m, A_0)$. Then, $f_n \cdot f_m = 0$ if both m and n are odd. If at least one of m or n is even, then*

$$(f_n f_m)\left(\sum a \otimes x_1 \otimes \cdots x_{\delta(n)+\delta(m)}\right) = f_n\left(\sum a \otimes x_1 \otimes \cdots x_{\delta(n)-1} \otimes x_{\delta(n)} f_m(1 \otimes x_{\delta(n)+1} \otimes \cdots \otimes x_{\delta(n)+\delta(m)})\right).$$

Proof. Since A is a graded algebra, we have two ring homomorphisms $i : A_0 \to A$ and $\pi : A \to A_0$ where the first is the inclusion and the second is the canonical surjection. Also recall that if at least one of m or n is even, then $\delta(n+m) = \delta(n) + \delta(m)$. If both m and n are odd, then all Yoneda products of elements are 0 by Corollary 4.2. Assume that either m or n is even. Consider $f_m : Q^m \to A_0$. We lift f_m to $f_m^0 : Q^m \to A$ by $f_m^0 = i \circ f_m$. We continue lifting f_m as follows. Suppose we have $f_m^{i-1} : Q^{m+i-1} \to Q^{i-1}$.

We want to find $f_m^i : Q^{m+i} \to Q^i$ such that the following diagram commutes:

$$\begin{array}{ccc} Q^{m+i} & \xrightarrow{d^{m+i}} & Q^{m+i-1} \\ \Big\downarrow{\scriptstyle f_m^i} & & \Big\downarrow{\scriptstyle f_m^{i-1}} \\ Q^i & \xrightarrow{d^i} & Q^{i-1}. \end{array}$$

Define

$$f_m^i\left(\sum a \otimes x_1 \otimes \cdots \otimes x_{\delta(m+i)}\right)$$
$$= \sum a \otimes x_1 \cdots x_{d-2} \otimes x_{d-1} \cdots \otimes x_{\delta(i)} f_m^0(1 \otimes x_{\delta(i)+1} \otimes x_{\delta(i+m)}).$$

The reader can verify that $f_m^{i-1} d^{i+m} = d^i f_m^i$.

The Yoneda product $f_n f_m$ is given by $f_n \circ f_m^n$. The proposition now follows. □

We prove Theorem 9.1.

Proof. Now consider B_n. By definition, $B_n = A^!_{\delta(n)}$ and $A^! = T_{A_0}(A_1^*)/\langle R^\perp \rangle$. Hence,

$$B_n = \frac{(A_1^*)^{\delta(n)}}{\sum_{0 \leqslant i \leqslant \delta(n)} (A_1^*)^{\delta(n)-d-i} \otimes R^\perp \otimes (A_1^*)^i}.$$

Dualizing and using our remarks above, we obtain

$$B_n^* \simeq \sum ((A_1^*)^{\delta(n)-i-d} \otimes R^\perp \otimes (A_1^*)^i)^\perp = \bigcap (A_1^{\delta(n)-i-d} \otimes R \otimes A_1^i) \simeq H_{\delta(n)}.$$

Next, consider the natural isomorphisms

$$\begin{aligned} \mathrm{Ext}_A^n(A_0, A_0) &\simeq \mathrm{Hom}_A(A \otimes H_{\delta(n)}, A_0) \\ &\simeq \mathrm{Hom}_{A_0}(H_{\delta(n)}, \mathrm{Hom}_A(A, A_0)) \\ &\simeq \mathrm{Hom}_{A_0}((A^!_{\delta(n)})^*, \mathrm{Hom}_A(A_0, A_0)) \simeq A^!_{\delta(n)} = B_n. \end{aligned}$$

Let $\psi : B_n \to \mathrm{Hom}_A(A \otimes H_{\delta(n)}, A_0)$ given above. A careful analysis of these isomorphisms shows that if $\bar{f} = \overline{f_1 \otimes \cdots \otimes f_{\delta(n)}} \in B_n$ then $\psi(\bar{f}) \in \mathrm{hom}_A(A \otimes H_{\delta(n)}, A_0)$ is given by

$$\psi(\bar{f})\left(\sum a \otimes x_1 \otimes \cdots \otimes x_{\delta(n)}\right) = \sum f_1(x_1) \cdots f_{\delta(n)}(x_{\delta(n)}) \cdot \bar{a}.$$

Applying Proposition 9.2, we see that if $\bar{f} \in B_n$, $\bar{g} \in B_m$ and either n or m is even, then

$$\psi(\bar{f} \cdot \bar{g}) = \psi(\bar{f}) \cdot \psi(\bar{g}),$$

where the product on the right-hand side is given by the Yoneda product of the elements as in 9.2. This completes the proof. □

10. Concluding results

We end the paper with some examples of d-Koszul algebras with $d > 2$.

Proposition 10.1. *Let A_0 be a semisimple ring, A_1 a finitely generated A_0-A_0-bimodule, and $T = T_{A_0}(A_1)$ the tensor algebra. Suppose that $A = T/I$ where I is an ideal in T with generators in $\bigotimes_{A_0}^{d} A_1$. If the global dimension of A is 2, then A is d-Koszul.*

Proof. Since I can be generated by homogeneous elements, A has a grading induced by $A_n = \bigotimes^{n} A_1/(I \cap \bigotimes^{n} A_1)$. If $0 \to Q^2 \to Q^1 \to Q^0 \to A_0 \to 0$ is a minimal graded A-projective resolution of A_0 viewed as a graded module with support in degree 0, it is clear that $Q_0 = A$, generated in degree 0 and Q_1 is generated in degree 1 since the kernel of $A \to A_0$ is $A_1 \oplus A_2 \oplus \cdots$. Since T is hereditary, we see that Q_1 is isomorphic to $J/IJ = A \bigotimes_T J$ where $J = A_1 \oplus A_2 \oplus \cdots$. The kernel $Q_1 \to J/I$ is I/IJ, which is generated in degree d and we are done. □

For the remainder of this section, we restrict our attention to quotients of path algebras. More precisely, let Γ be a quiver, which is just a finite directed multigraph with loops. Let K be a field and we denote the path algebra by $K\Gamma$. It is well-known that $K\Gamma$ is isomorphic to a tensor algebra $T_{A_0}(A_1)$ where, for some n, $A_0 = \prod_{i=1}^{n} K$ and A_1 is a finitely generated A_0-A_0-bimodule. If ρ is a finite set of paths in Γ and I is the ideal generated by ρ, we say the quotient algebra, $A = K\Gamma/I$ is a *monomial algebra*. We give a characterization of monomial d-Koszul algebras. Berger gives such a classification in the local case [2]. We need I to be generated in degree d so we assume ρ is a set of paths of length d. We say ρ is *d-covering* if whenever $pq, qr \in \rho$ with q of length at least 1, then every subpath of pqr of length d is in ρ.

Theorem 10.2. *Let $A = K\Gamma/I$ where I is an ideal generated by a set ρ of paths of length d, $d \geqslant 2$. Then A is a d-Koszul algebra if and only if ρ is d-covering.*

Proof. In [3], the authors give a construction of a minimal graded projective resolution of A_0 of the monomial algebra A. The degrees of generators the projectives in P^n correspond to the lengths of the admissible n-sequences. It is a straightforward combinatorial check that if a path p is an admissible n-sequence (see [3] for a definition) then length of p is $\delta(n)$. From this we conclude that the nth projective in the minimal projective resolution is generated in degree $\delta(n)$ and hence A is a d-Koszul algebra. □

Corollary 10.3. *Let $A = K\Gamma/I$ where I is the ideal generated by all paths of length d, for some $d \geqslant 2$. Then A is a d-Koszul algebra.*

Proof. If ρ is the set of paths of length d, then ρ is clearly d-covering. □

Corollary 10.4. *Let $A = K\Gamma/I$ where I is an ideal generated by some paths of length d, for some $d \geqslant 2$. Suppose the longest path in Γ has length $d+1$. Then A is d-Koszul.*

Proof. Let ρ be a set of path of length d. Since the longest path in Γ is of length $d+1$, it is immediate that ρ is d-covering. □

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