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Dynamics of homoclinic tangles in periodically perturbed second-order equations

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ABSTRACT

We obtain a comprehensive description on the overall geometrical and dynamical structures of homoclinic tangles in periodically perturbed second-order ordinary differential equations with dissipation. Let μ be the size of perturbation and Λ_μ be the *entire* homoclinic tangle. We prove in particular that (i) for infinitely many disjoint open sets of μ, Λ_μ contains *nothing else* but a horseshoe of infinitely many branches; (ii) for infinitely many disjoint open sets of μ, Λ_μ contains *nothing else* but one sink and one horseshoe of infinitely many branches; and (iii) there are positive measure sets of μ so that Λ_μ admits strange attractors with Sinai–Ruelle–Bowen measure. We also use the equation

$$\frac{d^2q}{dt^2} + \left(\lambda - \gamma q^2\right)\frac{dq}{dt} - q + q^2 = \mu q^2 \sin \omega t$$

to illustrate how to apply our theory to the analysis and to the numerical simulations of a given equation.

1. Introduction

The chaos theory has been built around the following two themes:

(1) to develop mathematical theories that help us to understand and to describe the complicated dynamics of non-linear systems;
(2) to apply these theories to the analysis of given differential equations coming from the real world.

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Concerning theme (1), there have been spectacular developments and the vast literature on this subject dates back to Poincaré [26] and Birkhoff [4]. Here we list a few theories that are closely related to the studies of this paper: The theory of Smale's horseshoes and homoclinic tangles [38,39,1,22]; the Newhouse theory on homoclinic tangency [25,29,16,17,20]; the theory of SRB measures [37,6,31, 46], and the theory of Hénon-like and rank one attractors [5,24,7,43,44]. There have also been a large body of work from Shilnikov's school [40,41,3,32–36] and Hale's school [10,9] concerning chaos and homoclinic bifurcations in ordinary differential equations.

The study of homoclinic tangles formed by periodically perturbed homoclinic solutions goes back to Poincaré [26] and Birkhoff [4]. The construction of horseshoes of two branches in homoclinic tangles is due to Smale [38,39]. After Smale's work, there has been an extensive literature on periodically perturbed second-order differential equations arising in applications. The main objective of these studies is to prove the existence of horseshoes in differential equations through Smale's construction (see, for example, [15,18,27,28] and the references therein). For this purpose it suffices to explicitly compute the Melnikov function and to verify that this function has a simple zero.

The purpose of this paper is to systematically develop a way of applying, rigorously, the dynamics theories on non-uniformly hyperbolic maps to the analysis of homoclinic tangles in time-periodic differential equations coming from applications. It resembles the previous development of the Melnikov method for the rigorous proof of the existence of Smale's horseshoe in periodically and almost-periodically perturbed equations [21,18,27,23,19]. Though the majority of our results are new as far as theme (1) is concerned, the overall purpose of this paper falls into theme (2). It is an attempt to bridge the gap between the theory on maps and the analysis of homoclinic tangles in given ordinary differential equations.

We start with an autonomous second-order ordinary differential equation that contains a *non-resonant, dissipative* saddle fixed point with a homoclinic solution. This autonomous equation is then subjected to time-periodic perturbations. The homoclinic solution becomes a homoclinic loop for the time-T map of the unperturbed equation and this homoclinic loop can be broken into two intersecting curves by a small perturbation, creating a homoclinic tangle. In this case there exists a *maximal* invariant set in the neighborhood of the unperturbed loop, which we denote as Λ.

Λ is the object of study and the *ultimate* goal is to completely understand the geometric and dynamic structure of the *entire* homoclinic tangle Λ. Smale [38,39] constructed a horseshoe of two branches, which is an invariant subset of Λ. Conley and Moser constructed a horseshoe of infinitely many branches, which is also an invariant subset of Λ nearby a transversal homoclinic point [22]. If inside of Λ there is a point of transversal non-degenerate homoclinic tangency, then there could be Newhouse sinks or SRB measures supported on an invariant subset of Λ nearby the point of tangency.

In this paper, we derive a return map in the extended phase space that catches dynamical activities of *all* solutions staying in the neighborhood of the unperturbed homoclinic loop in the extended phase space (see the Main Theorem in Section 2), and through the maps derived, we give a comprehensive description on the overall geometrical and dynamical structures of the associated homoclinic tangles (see Sections 4.1 and 4.3). In particular, we prove that, let μ be the size of perturbation,

(i) for infinitely many disjoint open sets of μ, Λ_μ contains *nothing else* but a horseshoe of infinitely many branches (Theorem 1 in Section 4.2);
(ii) for infinitely many disjoint open sets of μ, Λ_μ contains *nothing else* but one sink and one horseshoe of infinitely many branches (Theorem 2 in Section 4.2); and
(iii) we also prove, rigorously, that there are infinitely many values of μ, for which the unstable manifold of a periodic saddle in Smale's horseshoe and the stable manifold of the same saddle form non-degenerate, transversal homoclinic tangency in Λ_μ (Theorem 3 in Section 4.2).

For the parameters of (i) and (ii), we now understand completely the structure of the *entire* homoclinic tangle Λ. As for (iii): it is the first time that the Newhouse theory and the theory of Sinai–Ruelle–Bowen measures are rigorously applied to the analysis of homoclinic tangles in given differential equations (see Corollary 4.1).

Our theory is then applied to the analysis of a periodically perturbed equation. We start with a 2D autonomous equation

$$\frac{d^2q}{dt^2} - q + q^2 = 0. \tag{1.1}$$

We add a term of dissipation and a term of non-linear damping to (1.1) to form a new autonomous equation

$$\frac{d^2q}{dt^2} + \left(\lambda - \gamma q^2\right)\frac{dq}{dt} - q + q^2 = 0 \tag{1.2}$$

where $\lambda, \gamma > 0$ are parameters. Denote $p = \frac{dq}{dt}$. $(q, p) = (0, 0)$ is now a dissipative saddle. Let $\lambda = \lambda_0 > 0$. Then for λ_0 sufficiently small there exists a $\gamma_0 > 0$ uniquely determined by λ_0, so that $(q, p) = (0, 0)$ admits a homoclinic saddle connection. To Eq. (1.2) we now add a time-periodic forcing to obtain a new equation

$$\frac{d^2q}{dt^2} + \left(\lambda_0 - \gamma_0 q^2\right)\frac{dq}{dt} - q + q^2 = \mu q^2 \sin\omega t \tag{1.3}$$

where μ is a parameter representing the magnitude of the forcing and ω is the forcing frequency. We prove that our descriptions on the overall dynamics of the homoclinic tangles, in particular, the (i)–(iii) above, apply to Eq. (1.3) (Sections 5.1 and 5.2). We also use Eq. (1.3) to illustrate how to systematically explore and to explain, by using the theory of this paper, the various dynamical scenarios observed in numerical simulations of a periodically perturbed homoclinic solution (Sections 5.3–5.5).

The idea of using the return maps induced by the solutions around the unperturbed homoclinic loop in extended phase space goes back to Afraimovich and Shilnikov [3], see also [2]. The same constructions are used in the studies of homoclinic bifurcations in autonomously perturbed equations [33,34,36,14,8] including the studies of Shilnikov's attractor [35,13]. Similar maps are also used in the study of the Arnold diffusion in Hamiltonian equations (see [30] and the long list of references therein).

2. Return map in extended phase space

2.1. Equations of study

Let $(x, y) \in \mathbb{R}^2$ be the phase variables and t be the time. We start with an autonomous differential equation

$$\frac{dx}{dt} = -\alpha x + f(x, y), \qquad \frac{dy}{dt} = \beta y + g(x, y) \tag{2.1}$$

where $f(x, y)$, $g(x, y)$ are defined on an open domain $\mathbf{V} \subset \mathbb{R}^2$ satisfying $f(0, 0) = g(0, 0) = \partial_x f(0, 0) = \partial_y f(0, 0) = \partial_x g(0, 0) = \partial_y g(0, 0) = 0$. We assume that $f(x, y)$, $g(x, y)$ are C^r on $\mathbf{V}$ for some $r \geqslant 3$ and are real-analytic at $(x, y) = (0, 0)$. We also assume that $(0, 0)$ is a *non-resonant, dissipative* saddle point. To be more precise we assume

(H1) (i) there exist $d_1, d_2 > 0$ so that for all $n, m \in \mathbb{Z}^+$,

$$|n\alpha - m\beta| > d_1\left(|n| + |m|\right)^{-d_2};$$

(ii) $0 < \beta < \alpha$.

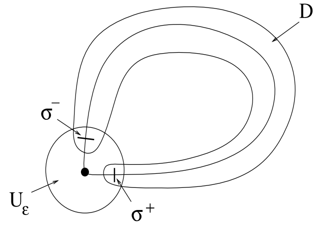

Fig. 1. U_ε, D and $\sigma^{\pm}$.

(H1)(i) is a Diophantine non-resonance condition on α and β. (H1)(ii) claims that the saddle point $(0,0)$ is dissipative. Let us also assume that the positive x-side of the local stable manifold and the positive y-side of the local unstable manifold of $(0,0)$ are included as part of a homoclinic solution, which we denote as

$$\ell = \{\ell(t) = (a(t), b(t)) \in \mathbf{V},\ t \in \mathbb{R}\}.$$

Let $P(x,y,t), Q(x,y,t) : \mathbf{V} \times \mathbb{R} \to \mathbb{R}$ be C^r functions for some $r \geqslant 3$. We assume that $P(x,y,t) = P(x,y,t+T)$, $Q(x,y,t) = Q(x,y,t+T)$ for a constant $T > 0$. To the right of Eq. (2.1) we add forcing terms to form a non-autonomous equation

$$\frac{dx}{dt} = -\alpha x + f(x,y) + \mu P(x,y,t),$$
$$\frac{dy}{dt} = \beta y + g(x,y) + \mu Q(x,y,t). \tag{2.2}$$

We also assume that $P(x,y,t)$, $Q(x,y,t)$ are real-analytic, high order terms at $(x,y)=(0,0)$. This is to say that $P(0,0,t) = Q(0,0,t) = 0$, $\partial_x P(0,0,t) = \partial_y P(0,0,t) = \partial_x Q(0,0,t) = \partial_y Q(0,0,t) = 0$. We regard α, β, $f(x,y), g(x,y)$, $P(x,y,t)$, $Q(x,y,t)$ as fixed. μ is the forcing parameter.

2.2. The return map

We study the solutions of Eq. (2.2) in the surroundings of the homoclinic loop ℓ in the original phase space (x,y), which we divide into a small neighborhood U_ε of $(0,0)$ and a small neighborhood D around ℓ out of $U_{\frac{1}{4}\varepsilon}$. See Fig. 1. Let $\sigma^{\pm} \in U_\varepsilon \cap D$ be the two line segments depicted in Fig. 1, both perpendicular to the homoclinic solution. We use an angular variable $\theta \in S^1$ to represent the time. In the extended phase space (x,y,θ) we denote

$$\mathcal{U}_\varepsilon = U_\varepsilon \times S^1, \qquad \mathbf{D} = D \times S^1$$

and let

$$\Sigma^{\pm} = \sigma^{\pm} \times S^1.$$

Let $\mathcal{N} : \Sigma^+ \to \Sigma^-$ be the maps induced by the solutions on $\mathcal{U}_\varepsilon$ and $\mathcal{M} : \Sigma^- \to \Sigma^+$ be the maps induced by the solutions on $\mathbf{D}$. See Fig. 2. We follow the steps of [42] in deriving the return maps. We first compute $\mathcal{M}$ and $\mathcal{N}$ separately, then compose $\mathcal{N}$ and $\mathcal{M}$ to obtain an explicit formula for the return map $\mathcal{R} = \mathcal{N} \circ \mathcal{M} : \Sigma^- \to \Sigma^-$.

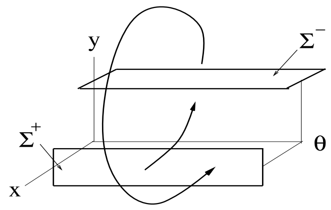

Fig. 2. $\mathcal{N}$ and $\mathcal{M}$.

2.3. The Main Theorem

From this point on all functions are regarded as functions in phase variables, time t and the parameter μ. Let X, Y be such that

$$x = X + \mathbb{P}(X, Y) + \mu\tilde{\mathbb{P}}(X, Y, t; \mu),$$
$$y = Y + \mathbb{Q}(X, Y) + \mu\tilde{\mathbb{Q}}(X, Y, t; \mu) \tag{2.3}$$

where $\mathbb{P}, \mathbb{Q}, \tilde{\mathbb{P}}, \tilde{\mathbb{Q}}$ as functions of X and Y are real-analytic on $|(X, Y)| < 2\varepsilon$, and the values of these functions and their first derivatives with respect to X and Y at $(X, Y) = (0, 0)$ are all zero. As is explicitly indicated in (2.3), $\mathbb{P}$ and $\mathbb{Q}$ are independent of t and μ. We also assume that

$$\tilde{\mathbb{P}}(X, Y, t + T; \mu) = \tilde{\mathbb{P}}(X, Y, t; \mu), \qquad \tilde{\mathbb{Q}}(X, Y, t + T; \mu) = \tilde{\mathbb{Q}}(X, Y, t; \mu)$$

are periodic of period T in t and they are also real-analytic with respect to t and μ for all $t \in \mathbb{R}$ and $|\mu| < \mu_0$ for some $\mu_0 > 0$ sufficiently small. (2.3) defines a non-autonomous, near identity coordinate transformation from x, y to (X, Y). First we have

Proposition 2.1. *Assume that α and β satisfy the Diophantine non-resonance condition* (H1)(i)*. Then there exists a small neighborhood U_ε of* $(0, 0)$*, the size of which is completely determined by Eq.* (2.1) *and d_1, d_2 in* (H1)(i)*, such that there exists an analytic coordinate transformation in the form of* (2.3) *that transforms Eq.* (2.2) *into*

$$\frac{dX}{dt} = -\alpha X, \qquad \frac{dY}{dt} = \beta Y.$$

Moreover, the C^r-norms of $\mathbb{P}, \mathbb{Q}, \tilde{\mathbb{P}}, \tilde{\mathbb{Q}}$ as functions of X, Y, t, μ are all uniformly bounded from above by a constant K that is independent of both ε and μ on $(X, Y) \in U_\varepsilon$, $t \in \mathbb{R}$ and $|\mu| < \mu_0$.

Proof. This is a standard linearization result. See for instance [11] for a proof. $\square$

Let $\omega = 2\pi T^{-1}$ be the forcing frequency and $\theta = \omega t$ be the extended phase variable for t. In the extended phase space (X, Y, θ) we define $\Sigma^{\pm}$ by letting

$$\Sigma^- = \left\{(X, Y, \theta)\colon\ Y = \varepsilon,\ |X| < \mu,\ \theta \in S^1\right\}$$

and

$$\Sigma^+ = \left\{(X, Y, \theta)\colon\ X = \varepsilon,\ |Y| < K_1(\varepsilon)\mu,\ \theta \in S^1\right\}$$

where $K_1(\varepsilon) > 1$ is a constant depending on ε.

Two small scales. $\mu \ll \varepsilon \ll 1$ represents two small scales of different magnitude. 2ε represents the size of a small neighborhood of $(x, y) = (0, 0)$, where the linearization above is valid. Let L^+, $-L^-$ be the respective times at which the homoclinic solution $\ell(t)$ enters U_ε in the positive and the negative time directions, respectively. L^+ and L^- are related, both determined completely by ε and $\ell(t)$. The parameter $\mu \ll \varepsilon$ controls the magnitude of the time-dependent perturbation.

Notations. 1. (*Generic constant*) Quantities that are independent of the phase variables, time and μ are regarded as constants and K is used to denote a generic constant, the precise value of which is allowed to change from line to line. We will also make distinctions between constants that depend on ε and those that do not by making such dependencies explicit. A constant that depends on ε is written as $K(\varepsilon)$. A constant written as K is independent of ε.

2. (*The $\mathcal{O}$-terms*) The intended formula for the return maps would inevitably contain terms that are explicit and terms that are implicit. Implicit terms are usually "error" terms, and the usefulness of a derived formula would depend completely on how well the error terms are controlled. In the rest of this paper $r \geqslant 3$ is reserved for an integer arbitrarily fixed, and we aim at C^r-control on all error terms. We adopt the following conventions indicating controls on magnitude. For a given constant, we write $\mathcal{O}(1)$, $\mathcal{O}(\varepsilon)$ or $\mathcal{O}(\mu)$ to indicate that the magnitude of this constant is bounded by K, $K\varepsilon$ or $K(\varepsilon)\mu$, respectively. For a function of a set V of variables, we write $\mathcal{O}_V(1), \mathcal{O}_V(\varepsilon)$ or $\mathcal{O}_V(\mu)$ to indicate that the C^r-norm of the function on a specified domain is bounded by $K, K\varepsilon$ or $K(\varepsilon)\mu$, respectively. We chose to specify the domain in the surrounding text rather than explicitly involving it in this notation. For example, $\mathcal{O}_{Z,\theta}(\mu)$ represents a function of Z, θ, the C^r-norm of which is bounded by $K(\varepsilon)\mu$ on a domain specified in the proceeding text.

3. (*The new parameter $p = \ln \mu^{-1}$*) We also need to estimate derivatives with respect to parameter but simply regarding μ as the parameter would lead to trouble, for taking derivative with respect to μ would create new non-perturbational terms in variational equations. To deal with this problem we let $p = \ln \mu^{-1}$ and regard p, not μ, as a bottom-line parameter. In other words, we regard μ as a shorthand for e^{-p}, and all functions in μ as functions in p. Observe that $\mu \in (0, \mu_0)$ corresponds to $p \in (\ln \mu_0^{-1}, +\infty)$. By regarding a function $F(\mu)$ of μ as a function of p, we have

$$\partial_p F(\mu) = -\mu \partial_\mu F(\mu),$$

and this allows us to keep copies of μ that would be otherwise lost.

We introduce a function, which we call the Melnikov function for Eq. (2.2), as follows: Let

$$\big(u(t), v(t)\big) = \left|\frac{d}{dt}\ell(t)\right|^{-1} \frac{d}{dt}\ell(t)$$

be the unit tangent vector of ℓ at $\ell(t)$, and let

$$\begin{aligned} E(t) = {} & v^2(t)\big(-\alpha + \partial_x f\big(a(t), b(t)\big)\big) + u^2(t)\big(\beta + \partial_y g\big(a(t), b(t)\big)\big) \\ & - u(t)v(t)\big(\partial_y f\big(a(t), b(t)\big) + \partial_x g\big(a(t), b(t)\big)\big). \end{aligned} \tag{2.4}$$

The quantity $E(t)$ measures the rate of expansion of the solutions of Eq. (2.1) in the direction normal to ℓ at $\ell(t)$. The *Melnikov function* $\mathcal{W}(\theta)$ for Eq. (2.2) is defined as

$$\mathcal{W}(\theta) = \int_{-\infty}^{\infty} \big(v(s)P\big(a(s), b(s), s + \omega^{-1}\theta\big) - u(s)Q\big(a(s), b(s), s + \omega^{-1}\theta\big)\big)e^{-\int_0^s E(\tau)\,d\tau}\,ds. \tag{2.5}$$

Observe that $E(t) \to \beta$ as $t \to +\infty$ and $E(t) \to -\alpha$ as $t \to -\infty$. $\mathcal{W}(\theta)$ is well defined and it measures the distances between the stable manifold W^s and the unstable manifold W^u of the solution $(x, y) = (0, 0)$ in the extended phase space.

Let

$$\mathbb{X} = \mu^{-1} X, \qquad \mathbb{Y} = \mu^{-1} Y.$$

Recall that $L^+, -L^-$ are the respective times at which the homoclinic solution $\ell(t)$ enters U_ε in the positive and the negative time directions. We obtain $\mathcal{W}_L(\theta)$ by replacing the integral bounds $+\infty, -\infty$ in $\mathcal{W}(\theta)$ with $L^+, -L^-$ respectively. This is to say that

$$\mathcal{W}_L(\theta) = \int_{-L^-}^{L^+} \big(v(s) P\big(a(s), b(s), s + \omega^{-1}\theta\big) - u(s) Q\left(a(s), b(s), s + \omega^{-1}\theta\right)\big) e^{-\int_0^s E(\tau)\, d\tau}\, ds. \tag{2.6}$$

We also denote

$$P_L = e^{\int_{-L^-}^{L^+} E(s)\, ds}, \qquad P_L^+ = e^{\int_0^{L^+} E(s)\, ds}. \tag{2.7}$$

Main Theorem *(Formula for $\mathcal{R}$). Let $(\theta, \mathbb{X}) \in \Sigma^-$, and $(\theta_1, \mathbb{X}_1) = \mathcal{R}(\theta, \mathbb{X})$. We have*

$$\begin{aligned} \theta_1 &= \theta + \mathbf{a} - \frac{\omega}{\beta} \ln \mathbb{F}(\theta, \mathbb{X}, \mu) + \mathcal{O}_{\theta, \mathbb{X}, p}(\mu), \\ \mathbb{X}_1 &= \mathbf{b}\big[\mathbb{F}(\theta, \mathbb{X}, \mu)\big]^{\frac{\alpha}{\beta}} \end{aligned} \tag{2.8}$$

where

$$\begin{aligned} \mathbf{a} &= \frac{\omega}{\beta} \ln \mu^{-1} + \omega\big(L^+ + L^-\big) + \frac{\omega}{\beta} \ln\big(\varepsilon\big(1 + \mathcal{O}(\varepsilon)\big) P_L^+\big), \\ \mathbf{b} &= \big(\mu\varepsilon^{-1}\big)^{\frac{\alpha}{\beta} - 1} \big[\big(1 + \mathcal{O}(\varepsilon)\big) P_L^+\big]^{\frac{\alpha}{\beta}} \end{aligned} \tag{2.9}$$

are constants and

$$\mathbb{F}(\theta, \mathbb{X}, \mu) = \mathcal{W}(\theta) + \mathbf{k}\mathbb{X} + \mathbb{E}(\theta, \mu) + \mathcal{O}_{\theta, \mathbb{X}, p}(\mu) \tag{2.10}$$

is a function of $(\theta, \mathbb{X}, \mu)$ where $\mathcal{W}(\theta)$ is as in (2.5)*,*

$$\mathbf{k} = P_L\big(P_L^+\big)^{-1}\big(1 + \mathcal{O}(\varepsilon)\big) \tag{2.11}$$

and

$$\mathbb{E}(\theta, \mu) = \big(P_L^+\big)^{-1} (1 + P_L) \mathcal{O}_{\theta, p}(1) + \mathcal{W}_L(\theta) - \mathcal{W}(\theta). \tag{2.12}$$

Remarks. As we will see momentarily (Lemma 3.1), it holds by definition that

$$P_L \sim \varepsilon^{\frac{\alpha}{\beta} - \frac{\beta}{\alpha}} \ll 1, \qquad P_L^+ \sim \varepsilon^{-\frac{\beta}{\alpha}} \gg 1.$$

We have the following for $\mathbf{a}, \mathbf{b}, \mathbf{k}$ and $\mathbb{E}(\theta, \mu)$:

(i) $\mathbf{a} \approx \frac{\omega}{\beta} \ln \mu^{-1}$. $\mathbf{a} \to +\infty$ as $\mu \to 0$. We also observe that $\mathbf{a} \sim \ln \mu^{-1}$. So concerning the parameter for $\mathcal{R}$, using $\mathbf{a}$ is as good as using p in the Main Theorem. For $\mu \in (0, \mu_0)$, $\mathbf{a} \in (\mathbf{a}_0, +\infty)$ for some $\mathbf{a}_0$ corresponding to μ_0.

(ii) $\mathbf{b} \sim \mu^{\frac{\alpha}{\beta}-1}$. $\mathbf{b} \to 0$ as $\mu \to 0$ by (H1)(ii). Following [43] and [44], we call the 1D family

$$f(\theta) = \theta + \mathbf{a} - \frac{\omega}{\beta} \ln\bigl(\mathcal{W}(\theta) + \mathbb{E}(\theta, 0)\bigr)$$

the 1D *singular limit* for the 2D return map $\mathcal{R}$.

(iii) $\mathbf{k} \sim \varepsilon^{\frac{\alpha}{\beta}}$. Since $\mathbf{k} \gg \mu$, the first derivative of $\mathbb{F}(\theta, \mathbb{X}, \mu)$ with respect to $\mathbb{X}$ is approximately $\mathbf{k}$ and the unfolding from $f(\theta)$ to $\mathcal{R}$ in $\mathbb{X}$-direction is determined mainly by the linear term $\mathbf{k}\mathbb{X}$.

(iv) $\mathbb{E}(\theta, \mu) \sim \varepsilon^{\frac{\beta}{\alpha}} \mathcal{O}_{\theta,p}(1)$. When ε is sufficiently small, $\mathbb{E}(\theta, \mu)$ is a C^r-small perturbation to $\mathcal{W}(\theta)$.

3. Proof of the Main Theorem

3.1. A standard form around homoclinic loop

In this subsection we derive a standard form for Eq. (2.2) around the homoclinic loop of Eq. (2.1) outside of $U_{\frac{1}{2}\varepsilon}$.

Let us regard t in $\ell(t) = (a(t), b(t))$ not as time, but as a parameter that parameterizes the curve ℓ in (x, y)-space. We replace t by s to write the homoclinic loop as $\ell(s) = (a(s), b(s))$. We have

$$\frac{da(s)}{ds} = -\alpha a(s) + f\bigl(a(s), b(s)\bigr), \qquad \frac{db(s)}{ds} = \beta b(s) + g\bigl(a(s), b(s)\bigr). \tag{3.1}$$

By definition,

$$\begin{aligned} u(s) &= \frac{-\alpha a(s) + f(a(s), b(s))}{\sqrt{(-\alpha a(s) + f(a(s), b(s)))^2 + (\beta b(s) + g(a(s), b(s)))^2}}, \\ v(s) &= \frac{\beta b(s) + g(a(s), b(s))}{\sqrt{(-\alpha a(s) + f(a(s), b(s)))^2 + (\beta b(s) + g(a(s), b(s)))^2}}. \end{aligned} \tag{3.2}$$

Let

$$\mathbf{e}(s) = \bigl(v(s), -u(s)\bigr).$$

We now introduce new variables (s, z) such that

$$(x, y) = \ell(s) + z\mathbf{e}(s).$$

This is to say that

$$x = x(s, z) := a(s) + v(s)z, \qquad y = y(s, z) := b(s) - u(s)z. \tag{3.3}$$

We derive the equations for (2.2) in the new variables (s, z) defined through (3.3). Differentiating (3.3) we obtain

$$\begin{aligned} \frac{dx}{dt} &= \bigl(-\alpha a(s) + f\bigl(a(s), b(s)\bigr) + v'(s)z\bigr)\frac{ds}{dt} + v(s)\frac{dz}{dt}, \\ \frac{dy}{dt} &= \bigl(\beta b(s) + g\bigl(a(s), b(s)\bigr) - u'(s)z\bigr)\frac{ds}{dt} - u(s)\frac{dz}{dt} \end{aligned} \tag{3.4}$$

where $u'(s) = \frac{du(s)}{ds}$, $v'(s) = \frac{dv(s)}{ds}$. Let us denote

$$\begin{aligned}
F(s,z) &= -\alpha\big(a(s)+zv(s)\big)+f\big(a(s)+zv(s),b(s)-zu(s)\big),\\
G(s,z) &= \beta\big(b(s)-zu(s)\big)+g\big(a(s)+zv(s),b(s)-zu(s)\big),\\
P(s,z,t) &= P\big(a(s)+zv(s),b(s)-zu(s),t\big),\\
Q(s,z,t) &= Q\big(a(s)+zv(s),b(s)-zu(s),t\big).
\end{aligned}$$

By using Eq. (2.2), we obtain from Eq. (3.4) the new equations for s, z as

$$\begin{aligned}
\frac{ds}{dt} &= \frac{u(s)F(s,z)+v(s)G(s,z)+\mu(u(s)P(s,z,t)+v(s)Q(s,z,t))}{\sqrt{F(s,0)^2+G(s,0)^2}+z(u(s)v'(s)-v(s)u'(s))},\\
\frac{dz}{dt} &= v(s)F(s,z)-u(s)G(s,z)+\mu\big(v(s)P(s,z,t)-u(s)Q(s,z,t)\big).
\end{aligned}$$

We re-write these equations as

$$\begin{aligned}
\frac{ds}{dt} &= 1+zw_1(s,z,t)+\frac{\mu(u(s)P(s,z,t)+v(s)Q(s,z,t))}{\sqrt{F(s,0)^2+G(s,0)^2}},\\
\frac{dz}{dt} &= E(s)z+z^2w_2(s,z)+\mu\big(v(s)P(s,z,t)-u(s)Q(s,z,t)\big)
\end{aligned}\tag{3.5}$$

where

$$\begin{aligned}
E(s) = {} & v^2(s)\big(-\alpha+\partial_x f\big(a(s),b(s)\big)\big)+u^2(s)\big(\beta+\partial_y g\big(a(s),b(s)\big)\big)\\
& -u(s)v(s)\big(\partial_y f\big(a(s),b(s)\big)+\partial_x g\big(a(s),b(s)\big)\big).
\end{aligned}$$

Also in the rest of this section we let $K_0(\varepsilon)$ be a given constant independent of μ and regard Eq. (3.5) as being defined on

$$\big\{s\in\big[-2L^-,2L^+\big],\ t\in\mathbb{R},\ |z|<K_0(\varepsilon)\mu\big\}.$$

The C^r-norms of $w_1(s,z,t)$ and $w_2(s,z)$ are bounded above by a constant $K(\varepsilon)$.

Finally we re-scale the variable z by letting

$$Z=\mu^{-1}z. \tag{3.6}$$

We arrive at the following equations

$$\begin{aligned}
\frac{ds}{dt} &= 1+\mu\tilde{w}_1(s,Z,t),\\
\frac{dZ}{dt} &= E(s)Z+\mu\tilde{w}_2(s,Z,t)+\big(v(s)P(s,0,t)-u(s)Q(s,0,t)\big)
\end{aligned}\tag{3.7}$$

where (s,Z,t) is defined on

$$\mathbf{D}=\big\{(s,Z,t)\colon s\in\big[-2L^-,2L^+\big],\ |Z|\leqslant K_0(\varepsilon),\ t\in\mathbb{R}\big\}.$$

Here we assume that μ is sufficiently small so that

$$\mu\ll\min_{s\in[-2L^-,2L^+]}\big(F(s,0)^2+G(s,0)^2\big).$$

Again, the C^r-norms of the functions $\tilde{w}_1$, $\tilde{w}_2$ are uniformly bounded by a constant $K(\varepsilon)$ on **D**. Eq. (3.7) is the one we need. Note that

$$P(s,0,t) = P\big(a(s), b(s), t\big), \qquad Q(s,0,t) = Q\big(a(s), b(s), t\big).$$

3.2. *Poincaré sections* $\Sigma^{\pm}$

Recall that $\Sigma^{\pm}$ are defined by letting

$$\Sigma^{-} = \big\{(X,Y,\theta)\colon\ Y = \varepsilon,\ |X| < \mu,\ \theta \in S^1\big\}$$

and

$$\Sigma^{+} = \big\{(X,Y,\theta)\colon\ X = \varepsilon,\ |Y| < K_1(\varepsilon)\mu,\ \theta \in S^1\big\}$$

where (X,Y) is as in (2.3). $K_1(\varepsilon)$ will be precisely defined momentarily.

Let $q \in \Sigma^+$ or Σ^-. We can also use (s, Z, θ)-coordinate to represent q, for which the defining equations for $\Sigma^{\pm}$ are not as direct. To compute the return maps, we need to first attend two issues that are technical in nature. First, we need to derive the defining equations for $\Sigma^{\pm}$ for (s, Z, θ). Second, we need to be able to change coordinates from (X, Y, θ) to (s, Z, θ) and vice versa on $\Sigma^{\pm}$. In what follows

$$\mathbb{X} = \mu^{-1}X, \qquad \mathbb{Y} = \mu^{-1}Y.$$

Proposition 3.1. *Coordinate conversions on* $\Sigma^{\pm}$ *are as follows*:

(a) *On* Σ^+, (i) $s = L^+ + \mathcal{O}_{Z,\theta,p}(\mu)$, (ii) $\mathbb{Y} = (1 + \mathcal{O}(\varepsilon))Z + \mathcal{O}_{\theta,p}(1) + \mathcal{O}_{Z,\theta,p}(\mu)$.
(b) *On* Σ^-, (i) $s = -L^- + \mathcal{O}_{Z,\theta,p}(\mu)$, (ii) $Z = (1 + \mathcal{O}(\varepsilon))\mathbb{X} + \mathcal{O}_{\theta,p}(1) + \mathcal{O}_{\mathbb{X},\theta,p}(\mu)$.

The proof of Proposition 3.1 is postponed to Section 3.5.

3.3. *The map* $\mathcal{M} : \Sigma^- \to \Sigma^+$

First we prove

Lemma 3.1.

$$P_L \sim \varepsilon^{\frac{\alpha}{\beta} - \frac{\beta}{\alpha}} \ll 1, \qquad P_L^+ \sim \varepsilon^{-\frac{\beta}{\alpha}} \gg 1.$$

Proof. By the definition of $L^{\pm}$ we have

$$\varepsilon \sim e^{-\alpha L^+} \sim e^{-\beta L^-}.$$

We also have

$$P_L \sim e^{\beta L^+ - \alpha L^-}, \qquad P_L^+ \sim e^{\beta L^+}.$$

Lemma 3.1 follows directly from these estimates. □

Recall that $\theta = \omega t$. For $q = (s^-, Z, \theta_0) \in \Sigma^-$, the value of s^- is uniquely determined by that of (Z, θ_0) through Proposition 3.1(b)(i). So we can use (Z, θ_0) for q. Let $(s(t), Z(t))$ be the solution of

Eq. (3.7) initiated from (s^-, Z) at $t_0 := \omega^{-1}\theta_0$, and $\hat{t}$ be the time $(s(\hat{t}), Z(\hat{t}))$ hits Σ^+. In what follows we write

$$s^+ = s(\hat{t}), \qquad \hat{Z} = Z(\hat{t}).$$

Proposition 3.2. *Denote* $(\hat{Z}, \hat{\theta}) = \mathcal{M}(Z, \theta_0)$. *We have*

$$\begin{aligned} \hat{Z} &= P_L^+ \mathcal{W}_L\big(\theta_0 + \omega L^-\big) + P_L Z + \mathcal{O}_{Z,\theta_0,p}(\mu), \\ \hat{\theta} &= \theta_0 + \omega\big(L^+ + L^-\big) + \mathcal{O}_{Z,\theta_0,p}(\mu). \end{aligned} \tag{3.8}$$

Proof. We re-write Eq. (3.7) as

$$\begin{aligned} \frac{dZ}{ds} &= E(s)Z + \big(v(s)P\big(a(s), b(s), t\big) - u(s)Q\big(a(s), b(s), t\big)\big) + \mathcal{O}_{s,Z,t,p}(\mu), \\ \frac{dt}{ds} &= 1 + \mathcal{O}_{s,Z,t,p}(\mu) \end{aligned} \tag{3.9}$$

on $\mathbf{D} \times (0, \mu_0)$ where

$$\mathbf{D} = \big\{(s, Z, t)\colon s \in \big[-2L^-, 2L^+\big],\ |Z| < K_1(\varepsilon),\ t \in \mathbb{R}\big\}.$$

From the second item of (3.9) we obtain

$$t = t_0 + s + L^- + \mathcal{O}_{s,Z,t_0,p}(\mu),$$

from which the claim on $\hat{\theta}$ follows. Substituting it into the first item of (3.9) we obtain

$$\frac{dZ}{ds} = E(s)Z + \big(v(s)P\big(a(s), b(s), t_0 + s + L^-\big) - u(s)Q\big(a(s), b(s), t_0 + s + L^-\big)\big) + \mathcal{O}_{s,Z,t_0,p}(\mu),$$

from which it follows that

$$\hat{Z} = P_L\big(Z + \Phi_L(t_0)\big) + \mathcal{O}_{Z,t_0,p}(\mu)$$

where P_L is as in (2.7) and

$$\Phi_L(t) = \int_{-L^-}^{L^+} \big(v(s)P\big(a(s), b(s), t + s + L^-\big) - u(s)Q\big(a(s), b(s), t + s + L^-\big)\big) \cdot e^{-\int_{-L^-}^{s} E(\tau)\,d\tau}\, ds. \tag{3.10}$$

We also observe that

$$P_L\Phi_L(t) = P_L^+ \cdot \mathcal{W}_L\big(\theta + \omega L^-\big).$$

This proves the line for $\hat{Z}$.

Let

$$K_1(\varepsilon) = \max_{t\in\mathbb{R},\ s\in[-2L^-, 2L^+]} P_s \cdot \big(2 + \big|\Phi_s(t)\big|\big) \tag{3.11}$$

where P_s and Φ_s are obtained by replacing L^+ with s in P_L and Φ_L respectively. $K_1(\varepsilon)$ is the one we use for $\mathbf{D}$ and Σ^+. Solutions of (3.9) initiated on Σ^- will stay inside of $\mathbf{D}$ before hitting Σ^+. $\square$

3.4. Proof of the Main Theorem

First we compute $\mathcal{N}: \Sigma^+ \to \Sigma^-$. For $(\mathbb{X}, \mathbb{Y}, \theta) \in \Sigma^+$ we have $\mathbb{X} = \varepsilon\mu^{-1}$. Similarly, for $(\mathbb{X}, \mathbb{Y}, \theta) \in \Sigma^-$ we have $\mathbb{Y} = \varepsilon\mu^{-1}$. Denote a point on Σ^+ by using $(\mathbb{Y}, \theta)$ and a point on Σ^- by using $(\mathbb{X}, \theta)$. For $(\mathbb{Y}, \theta) \in \Sigma^+$, let

$$(\tilde{\mathbb{X}}, \tilde{\theta}) = \mathcal{N}(\mathbb{Y}, \theta).$$

Proposition 3.3. *We have for* $(\mathbb{Y}, \theta) \in \Sigma^+$,

$$\begin{aligned}
\tilde{\mathbb{X}} &= \left(\mu\varepsilon^{-1}\right)^{\frac{\alpha}{\beta}-1}\mathbb{Y}^{\frac{\alpha}{\beta}}, \\
\tilde{\theta} &= \theta + \frac{\omega}{\beta}\ln\left(\varepsilon\mu^{-1}\right) - \frac{\omega}{\beta}\ln\mathbb{Y}.
\end{aligned} \tag{3.12}$$

Proof. Let $\mathbb{T}$ be the time it takes for the solution of the linear equation of Proposition 2.1 from $(\varepsilon, Y, \theta) \in \Sigma^+$ to get to $(\tilde{X}, \varepsilon, \tilde{\theta}) \in \Sigma^-$. We have

$$\tilde{X} = \varepsilon e^{-\alpha\mathbb{T}}, \qquad \varepsilon = Ye^{\beta\mathbb{T}}, \qquad \tilde{\theta} = \theta + \omega\mathbb{T},$$

from which (3.12) follows. $\square$

We are now ready to compute the return map $\mathcal{R} = \mathcal{N} \circ \mathcal{M} : \Sigma^- \to \Sigma^-$. We use $(\mathbb{X}, \theta)$ to represent a point on Σ^- and denote $(\tilde{\mathbb{X}}, \tilde{\theta}) = \mathcal{R}(\mathbb{X}, \theta)$. By using Propositions 3.2 and 3.1(b)(ii), we have

$$\begin{aligned}
\hat{Z} &= P_L\left(1 + \mathcal{O}(\varepsilon)\right)\mathbb{X} + P_L^+\mathcal{W}_L\left(\theta + \omega L^-\right) + P_L\mathcal{O}_{\theta,p}(1) + \mathcal{O}_{\mathbb{X},\theta,p}(\mu), \\
\hat{\theta} &= \theta + \omega\left(L^+ + L^-\right) + \mathcal{O}_{\mathbb{X},\theta,p}(\mu).
\end{aligned}$$

Let $\hat{\mathbb{Y}}$ be the $\mathbb{Y}$-coordinate for $(\hat{Z}, \hat{\theta})$, we have from Proposition 3.1(a)(ii),

$$\hat{\mathbb{Y}} = \left(1 + \mathcal{O}(\varepsilon)\right)P_L^+\mathbb{F}(\theta, \mathbb{X}, \mu) \tag{3.13}$$

where

$$\begin{aligned}
\mathbb{F}(\theta, \mathbb{X}, \mu) = {} & \mathcal{W}_L\left(\theta + \omega L^-\right) + P_L\left(P_L^+\right)^{-1}\left(1 + \mathcal{O}(\varepsilon)\right)\mathbb{X} \\
& + \left(P_L^+\right)^{-1}(1 + P_L)\mathcal{O}_{\theta,p}(1) + \mathcal{O}_{\mathbb{X},\theta,p}(\mu).
\end{aligned} \tag{3.14}$$

We then obtain (2.8) by using (3.12). Note that we also shifted the angular variable θ by $-\omega L^-$ to change $\mathcal{W}(\theta + \omega L^-)$ to $\mathcal{W}(\theta)$ for $\mathbb{F}(\theta, \mathbb{X}, \mu)$ in (2.10).

3.5. Proof of Proposition 3.1

In this subsection we prove Proposition 3.1. We start with the defining equations for Σ^+ in (s, Z, θ).

Lemma 3.2. *We have for* $(s, Z, \theta) \in \Sigma^+$,

$$s = L^+ + \mathcal{O}_{Z,\theta,p}(\mu).$$

Proof. We have on Σ^+,

$$\begin{aligned} a(s)+v(s)z &= \varepsilon+\mathbb{P}(\varepsilon,Y)+\mu\tilde{\mathbb{P}}\big(\varepsilon,Y,\omega^{-1}\theta\big),\\ b(s)-u(s)z &= Y+\mathbb{Q}(\varepsilon,Y)+\mu\tilde{\mathbb{Q}}\big(\varepsilon,Y,\omega^{-1}\theta\big). \end{aligned} \tag{3.15}$$

By the definition

$$\begin{aligned} a\big(L^+\big) &= \varepsilon+\mathbb{P}(\varepsilon,0),\\ b\big(L^+\big) &= \mathbb{Q}(\varepsilon,0). \end{aligned} \tag{3.16}$$

Let

$$\begin{aligned} W_1 &= a(s)-a\big(L^+\big)+v(s)z-\mu\tilde{\mathbb{P}}\big(\varepsilon,0,\omega^{-1}\theta\big),\\ W_2 &= b(s)-b\big(L^+\big)-u(s)z-\mu\tilde{\mathbb{Q}}\big(\varepsilon,0,\omega^{-1}\theta\big). \end{aligned} \tag{3.17}$$

We have from (3.15) and (3.16),

$$\begin{aligned} W_1 &= \mathbb{P}(\varepsilon,Y)-\mathbb{P}(\varepsilon,0)+\mu\big(\tilde{\mathbb{P}}\big(\varepsilon,Y,\omega^{-1}\theta\big)-\tilde{\mathbb{P}}\big(\varepsilon,0,\omega^{-1}\theta\big)\big),\\ W_2 &= Y+\mathbb{Q}(\varepsilon,Y)-\mathbb{Q}(\varepsilon,0)+\mu\big(\tilde{\mathbb{Q}}\big(\varepsilon,Y,\omega^{-1}\theta\big)-\tilde{\mathbb{Q}}\big(\varepsilon,0,\omega^{-1}\theta\big)\big) \end{aligned}$$

which we re-write as

$$\begin{aligned} W_1 &= \big(\mathcal{O}(\varepsilon)+\mu\mathcal{O}_{\theta,p}(1)\big)Y+\mathcal{O}_{Y,\theta,p}(1)Y^2,\\ W_2 &= \big(1+\mathcal{O}(\varepsilon)+\mu\mathcal{O}_{\theta,p}(1)\big)Y+\mathcal{O}_{Y,\theta,p}(1)Y^2. \end{aligned} \tag{3.18}$$

We first obtain

$$Y=\big(1+\mathcal{O}(\varepsilon)+\mu\mathcal{O}_{\theta,p}(1)\big)W_2+\mathcal{O}_{W_2,\theta,p}(1)W_2^2 \tag{3.19}$$

by inverting the second line in (3.18). We then substitute it into the first line in (3.18) to obtain

$$\begin{aligned} W_1 &= \big(\mathcal{O}(\varepsilon)+\mu\mathcal{O}_{\theta,p}(1)\big)\big(\big(1+\mathcal{O}(\varepsilon)+\mu\mathcal{O}_{\theta,p}(1)\big)W_2+\mathcal{O}_{W_2,\theta,p}(1)W_2^2\big)\\ &\quad+\mathcal{O}_{Y,\theta,p}(1)\big(\big(1+\mathcal{O}(\varepsilon)+\mu\mathcal{O}_{\theta,p}(1)\big)W_2+\mathcal{O}_{W_2,\theta,p}(1)W_2^2\big)^2\\ &= \big(\mathcal{O}(\varepsilon)+\mu\mathcal{O}_{\theta,p}(1)\big)W_2+\mathcal{O}_{W_2,\theta,p}(1)W_2^2. \end{aligned}$$

Consequently,

$$F(s,Z,\theta):=W_1-\big(\mathcal{O}(\varepsilon)+\mu\mathcal{O}_{\theta,p}(1)\big)W_2+\mathcal{O}_{W_2,\theta,p}(1)W_2^2=0 \tag{3.20}$$

where W_1, W_2 as functions of s, Z, θ are defined by (3.17). To re-write W_1 and W_2, we let

$$\xi=s-L^+ \tag{3.21}$$

and expand $a(s)$ in terms of ξ as

$$a(s)=a\big(L^{+}\big)+a'\big(L^{+}\big)\xi+\sum_{i=2}^{\infty}a_i\big(L^{+}\big)\xi^{i}.$$

Expansions for $b(s)$, $u(s)$, and $v(s)$ are similar. We have

$$\begin{aligned}W_1 &= a'\big(L^{+}\big)\xi+\sum_{i=2}^{\infty}a_i\big(L^{+}\big)\xi^{i}+v\big(L^{+}\big)z+\left(v'\big(L^{+}\big)\xi+\sum_{i=2}^{\infty}v_i\big(L^{+}\big)\xi^{i}\right)z\\ &\quad-\mu\tilde{\mathbb{P}}\big(\varepsilon,0,\omega^{-1}\theta\big),\\ W_2 &= b'\big(L^{+}\big)\xi+\sum_{i=2}^{\infty}b_i\big(L^{+}\big)\xi^{i}-u\big(L^{+}\big)z-\left(u'\big(L^{+}\big)\xi+\sum_{i=2}^{\infty}u_i\big(L^{+}\big)\xi^{i}\right)z\\ &\quad-\mu\tilde{\mathbb{Q}}\big(\varepsilon,0,\omega^{-1}\theta\big).\end{aligned}\tag{3.22}$$

We now put (3.22) for W_1 and W_2 back into Eq. (3.20) and replace z by μZ. We obtain

$$\big(a'\big(L^{+}\big)-\mathcal{O}(\varepsilon)b'\big(L^{+}\big)+h(\theta,\xi)\xi\big)\xi=\mathcal{O}_{Z,\theta,p}(\mu)$$

where the C^r-norm of $h(t,\xi)$ is bounded from above by $K(\varepsilon)$. Also note that $a'(L^{+})\approx-\alpha\varepsilon$, $b'(L^{+})=\mathcal{O}(\varepsilon^2)$. We finally obtain

$$s=L^{+}+\mathcal{O}_{Z,\theta,p}(\mu)$$

by solving ξ. This completes the proof of Lemma 3.2. □

Lemma 3.2 is not precise enough. We need the following refinement.

Lemma 3.3. *We have on Σ^{+},*

$$s-L^{+}=-\frac{v(L^{+})+\mathcal{O}(\varepsilon)u(L^{+})}{a'(L^{+})-\mathcal{O}(\varepsilon)b'(L^{+})}z+\frac{\mu}{a'(L^{+})-\mathcal{O}(\varepsilon)b'(L^{+})}\mathcal{O}_{\theta,p}(1)+\mathcal{O}_{Z,\theta,p}\big(\mu^2\big).$$

Proof. It suffices for us to drop all terms that are $\mathcal{O}_{Z,\theta,p}(\mu^2)$ in Eq. (3.20) to solve for ξ. From Lemma 3.2 we conclude that all terms in ξ,z of degree higher than one are $\mathcal{O}_{Z,\theta,p}(\mu^2)$. With these terms all dropped, (3.20) becomes

$$\big(a'\big(L^{+}\big)-\mathcal{O}(\varepsilon)b'\big(L^{+}\big)\big)\xi+\big(v\big(L^{+}\big)+\mathcal{O}(\varepsilon)u\big(L^{+}\big)\big)z=\mu\mathcal{O}_{\theta,p}(1),\tag{3.23}$$

from which the estimates of Lemma 3.3 on Σ^{+} follow. □

Recall that $\mathbb{X}=\mu^{-1}X$, $\mathbb{Y}=\mu^{-1}Y$.

Lemma 3.4. *On Σ^{+} we have*

$$\mathbb{Y}=\big(1+\mathcal{O}(\varepsilon)\big)Z+\mathcal{O}_{\theta,p}(1)+\mathcal{O}_{Z,\theta,p}(\mu).$$

Proof. We have

$$
\begin{aligned}
Y &= \big(1+\mathcal{O}(\varepsilon)\big)\big(b'\big(L^+\big)\xi - u\big(L^+\big)z - \mu\tilde{\mathbb{Q}}(\varepsilon,0,t)\big) + \mathcal{O}_{Z,t}\big(\mu^2\big) \\
&= \big(1+\mathcal{O}(\varepsilon)\big)\bigg(-\bigg(u\big(L^+\big) + b'\big(L^+\big)\frac{v(L^+)+\mathcal{O}(\varepsilon)u(L^+)}{a'(L^+)-\mathcal{O}(\varepsilon)b'(L^+)}\bigg)z \\
&\quad + \frac{\mu b'(L^+)}{a'(L^+)-\mathcal{O}(\varepsilon)b'(L^+)}\mathcal{O}_{\theta,p}(1) - \mu\tilde{\mathbb{Q}}\big(\varepsilon,0,\omega^{-1}\theta\big)\bigg) + \mathcal{O}_{Z,\theta,p}\big(\mu^2\big) \\
&= \big(1+\mathcal{O}(\varepsilon)\big)z + \mu\mathcal{O}_{\theta,p}(1) + \mathcal{O}_{Z,\theta,p}\big(\mu^2\big)
\end{aligned}
\tag{3.24}
$$

where the first equality follows from using (3.19), (3.22) and Lemma 3.2, and the second equality from using Lemma 3.3. To obtain the third equality we use $u(L^+) = -1+\mathcal{O}(\varepsilon)$, $a'(L^+) \approx -\alpha\varepsilon$, $b'(L^+) = \mathcal{O}(\varepsilon^2)$. □

Lemma 3.2 is Proposition 3.1(a)(i) and Lemma 3.4 is Proposition 3.1(a)(ii). Proposition 3.1(b) follows from parallel computations.

4. Dynamics of homoclinic tangles

In Sections 4.1 and 4.2 we let

$$
P(x,y,t) = A(x,y)\sin\omega t, \qquad Q(x,y) = B(x,y)\sin\omega t
\tag{4.1}
$$

in Eq. (2.2) where $A(x,y)$, $B(x,y)$ are high order terms at $(x,y)=(0,0)$. We have

$$
\mathcal{W}(\theta) = J_s\cos\theta + J_c\sin\theta
\tag{4.2}
$$

where

$$
\begin{aligned}
J_s &= \int_{-\infty}^{+\infty} \big(v(s)A(s) - u(s)B(s)\big)\sin(\omega s)e^{-\int_0^s E(\tau)\,d\tau}\,ds, \\
J_c &= \int_{-\infty}^{+\infty} \big(v(s)A(s) - u(s)B(s)\big)\cos(\omega s)e^{-\int_0^s E(\tau)\,d\tau}\,ds,
\end{aligned}
\tag{4.3}
$$

and $A(s) = A(a(s), b(s), s)$, $B(s) = B(a(s), b(s), s)$. For the rest of this section we assume

(H2) $J_c^2 + J_s^2 \neq 0$.

4.1. A comprehensive description

First observe that the return map $\mathcal{R}$ is only defined on the part of Σ^- satisfying

$$
\mathbb{F}(\theta,\mathbb{X},\mu) > 0.
\tag{4.4}
$$

This part of Σ^- follows the homoclinic loop of the unperturbed equation to hit Σ^+ on the upper side of the local stable manifold of $(X,Y)=(0,0)$ where they return to Σ^-. The rest of Σ^- satisfies

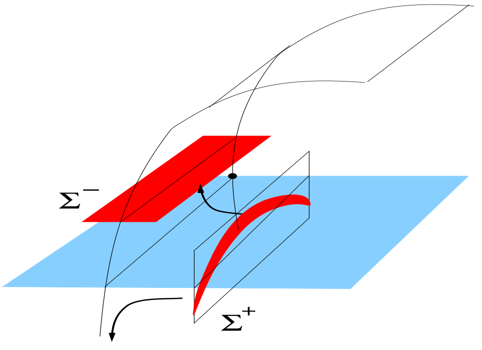

Fig. 3. Partial returns to Σ^-.

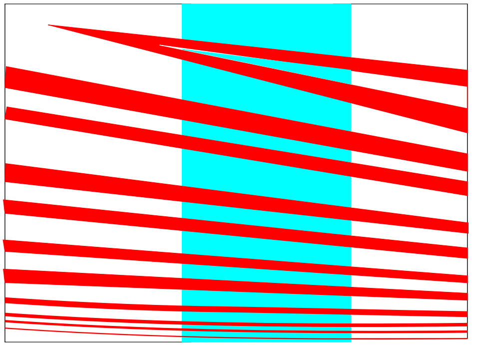

Fig. 4. The geometry of $\mathcal{R}$.

$$\mathbb{F}(\theta, \mathbb{X}, \mu) \leqslant 0. \tag{4.5}$$

From there the solutions hit the lower side of the local stable manifold of $(X, Y) = (0, 0)$ where they sneak out of $\mathcal{U}_\varepsilon \cup \mathbf{D}$. See Fig. 3.

From this point on we call the θ-direction in Σ^- the horizontal direction and the $\mathbb{X}$-direction the vertical direction. With $\mathcal{W}(\theta)$ as in (4.2),

$$\mathbb{F}(\theta, \mathbb{X}, \mu) = 0$$

defines two curves of slope $\sim \varepsilon^{-\frac{\beta}{\alpha}}$. These two curves divide Σ^- into two vertical strips, which we denote as V and U respectively. Let us assume that $\mathcal{R}$ is defined on V, where (4.4) holds. By using the Main Theorem for $\mathcal{R}$, we know that $\mathcal{R}$ compresses V in the vertical direction and stretches it in the horizontal direction, making the image infinitely long towards both ends. $\mathcal{R}$ then folds it and wraps it around Σ^- infinitely many times. See Fig. 4.

Denote the return map derived in the Main Theorem as $\mathcal{R}_\mu$ and let[1]

$$\Omega_\mu = \{(\theta, \mathbb{X}) \in V\colon \mathcal{R}^n_\mu(\theta, \mathbb{X}) \in V\ \forall n \geqslant 0\}, \qquad \Lambda_\mu = \bigcap_{n \geqslant 0} \mathcal{R}^n_\mu(\Omega_\mu).$$

[1] We caution that V and U depend also on μ. So to be completely rigorous we ought to write V_μ and U_μ instead of V and U. However, for $\mathcal{R}_\mu$ derived from the periodically perturbed equations, V and U vary only slightly as μ varies, and we could practically think of them as being independent of μ.

Then Ω_μ represents *all* solutions that stay close to the unperturbed homoclinic loop in forward times; Λ_μ is the set Ω_μ is attracted to, representing *all* solutions that stay close to the unperturbed homoclinic loop in both the forward and the backward times. The geometrical and dynamical structures of the *entire* homoclinic tangle are manifested in those of Ω_μ and Λ_μ.

From Fig. 4 for $\mathcal{R}$, it is obvious that Λ_μ contains a horseshoe of infinitely many symbols as a subset for all μ. We will refer to this horseshoe as *the* Smale horseshoe. The horseshoes constructed previously by Smale and others near homoclinic intersections all came from this one horseshoe of infinitely many branches.

The structures of Ω_μ and Λ_μ depend sensitively on the location of the folded part of $\mathcal{R}_\mu(V)$. If this part is deep inside of U, then the entire homoclinic tangle is reduced to one horseshoe of infinitely many symbols. If it is located inside of V, then the homoclinic tangles are likely to have attracting periodic solutions or sinks and observable chaos associated with non-degenerate transversal homoclinic tangency. According to the formula for $\mathcal{R}_\mu$ obtained in the Main Theorem, $\mathcal{R}_\mu(V)$ moves horizontally towards $\theta = +\infty$ as $\mu \to 0$. With a roughly constant speed with respect to $p = \ln \mu^{-1}$, it crosses V and U infinitely many times along the way. As the folded part of $\mathcal{R}_\mu$ traverses V, we encounter complicated dynamics caused by our allowing the folded images of the unstable manifold of the Smale horseshoe to come back to form tangential intersections to the stable manifold of the same horseshoe. In this case the dynamics of the associated homoclinic tangles depends sensitively on μ and it is not possible to obtain a bifurcation diagram for $\mathcal{R}_\mu$. However, we know from (2.8) that, in the limit, there is a dynamics pattern that repeats periodically with respect to $\mathbf{a}$ of period 2π, for $\mathbf{a} \to \infty$ is a parameter appearing additively in the angular component of the map $\mathcal{R}_\mu$ in the Main Theorem. In terms of μ, this is a “multiplicative period” of size $\approx e^{\beta T}$. This is to say that, let $[\mu_1, \mu_2]$ be a base interval for the indicated periodicity, then $\mu_2 \approx \mu_1 e^{\beta T}$.

4.2. The dynamics of homoclinic tangles

In Eq. (2.2) we let $P(x, y, t)$ and $Q(x, y, t)$ be as in (4.1) and assume that (H1) and (H2) hold. Let $\mathcal{R}_\mu$ be the return map derived in the Main Theorem. First we have

Theorem 1 *(Tangle as one full shift of countably many symbols). Assume that $\omega\beta^{-1} > 100$, then there exists a sequence of μ that approaches to zero, which we denote as*

$$1 \gg \mu_1^{(r)} > \mu_1^{(l)} > \cdots > \mu_n^{(r)} > \mu_n^{(l)} > \cdots \to 0$$

such that for all $\mu \in [\mu_n^{(l)}, \mu_n^{(r)}]$, $\mathcal{R}_\mu$ on

$$\Lambda = \{(\theta, \mathbb{X}) \in V\colon\ \mathcal{R}_\mu^i(\theta, \mathbb{X}) \in V,\ \forall i \in \mathbb{Z}\}$$

conjugates to a full shift of countably many symbols.

For the parameters of Theorem 1, the *entire* homoclinic tangle consists of one single horseshoe of infinitely many symbols.

We caution that, though the horseshoe of Theorem 1 represents all solutions of the perturbed equation that stay forever inside of a small neighborhood of the homoclinic loop ℓ, solutions sneaking out through U might find a way to come back to Σ^-, creating more complicated structures. One particular mechanism for such a come back is for the unperturbed equation to have two homoclinic loops. In this case, part of U would come back to Σ^- following the other homoclinic loop. On the other hand, it is easy to obtain examples for which the solutions sneaking out of U would never come back. One such example is presented in Section 5.

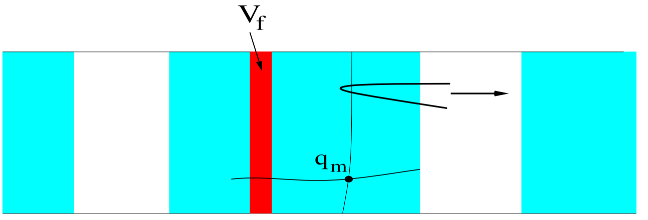

Fig. 5. Transversal homoclinic tangency.

We also have

Theorem 2 *(Tangle as the union of one sink and one horseshoe). Let the assumptions be identical to that of Theorem* 1. *Then there exists an open set of μ inside each of the intervals $[\mu_n^{(r)}, \mu_{n+1}^{(l)}]$, such that the corresponding homoclinic tangle consists of one periodic sink and one horseshoe of countably many branches.*

We again remark that, for the parameters of Theorem 2, the entire homoclinic tangle consists of one attractive periodic solution and one horseshoe. The periodic sink we obtained here is not Newhouse sink associated to homoclinic tangency.

Our next theorem is about the existence of non-degenerate transversal homoclinic tangency.

Theorem 3 *(Homoclinic tangency). Let the assumptions be identical to that of Theorem* 1. *Then for every $n > 0$ given, there exists $\hat{\mu} \in [\mu_n^{(r)}, \mu_{n+1}^{(l)}]$, the corresponding value for $\mathbf{a}$ we denote as $\hat{\mathbf{a}}$, such that*:

(i) *$\mathcal{R}_{\hat{\mathbf{a}}}$ has a saddle fixed point, which we denote as $q(\hat{\mathbf{a}})$, so that $W^u(q(\hat{\mathbf{a}})) \cap W^s(q(\hat{\mathbf{a}}))$ contains a point of non-degenerate tangency.*

(ii) *Let $q(\mathbf{a})$ be the continuous extension of $q(\hat{\mathbf{a}})$ for $\mathbf{a}$ sufficiently close to $\hat{\mathbf{a}}$. Then as $\mathbf{a}$ passes through $\hat{\mathbf{a}}$, $W^u(q(\mathbf{a}))$ crosses $W^s(q(\mathbf{a}))$ at the tangential intersection point of* (i) *with a relative speed $> \frac{1}{2}$ with respect to $\mathbf{a}$ in θ-direction.*

Outline of proof. Let V_f be the vertical strip in Σ^- so that $\mathcal{R}(V_f)$ is the turning part of the image $\mathcal{R}(V)$. Our plan of proof is as follows. We know that $\mathcal{R}_{\mathbf{a}}$ induces a horseshoe of infinitely many symbols in V, creating many saddle fixed points. Pick one and denote it as q. We prove that q is continuously extended over the μ-interval $[\mu_n^{(r)}, \mu_{n+1}^{(l)}]$, which we denote as $q(\mathbf{a})$. Let $W^u(q(\mathbf{a}))$ be the unstable and $W^s(q(\mathbf{a}))$ be the stable manifold of $q(\mathbf{a})$. We prove that $W^u(q(\mathbf{a}))$ traverses V_f in horizontal direction and it has a horizontal segment inside of V_f, which we denote as $\ell^u(\mathbf{a})$. We also prove that $W^s(q(\mathbf{a}))$ has a vertical segment fully extended in V, which we denote as $\ell^s(\mathbf{a})$. We observe that $\mathcal{R}_{\mathbf{a}}(\ell^u(\mathbf{a}))$ has a sharp quadratic turn, and as μ varies from $\mu_n^{(r)}$ to $\mu_{n+1}^{(l)}$, it moves from one side of V to the other, transversally crossing $\ell^s(\mathbf{a})$. See Fig. 5.

Detailed proof for Theorem 3 is long and includes some tedious computations. A complete proof is included in Appendices B and C. □

The following is a direct consequence of Theorem 3.[2]

Corollary 4.1. *Let the assumptions be identical to that of Theorem* 1. *Then inside of every parameter interval $[\mu_n^{(r)}, \mu_{n+1}^{(l)}]$, there is a set of parameters of positive Lebesgue measure, such that the homoclinic tangle associated with these parameters admits strange attractors with SRB measures.*

Proof. This follows from Theorem 3 applying [24] and [7], both are based on [5], to $\mathcal{R}_{\mathbf{a}}$. □

Detailed proofs for Theorems 1–3 are included in Appendices A–C.

[2] We thank Marcelo Viana for assuring us that, with Theorem 3, [24] directly applies.

4.3. Homoclinic tangles for general perturbation

In this subsection we consider the geometric and dynamic structure of homoclinic tangles for general forcing functions. Let $P(x, y, t)$ and $Q(x, y, t)$ be as in Section 2.1, and $\mathcal{W}(\theta)$ be as in (2.5). We assume that $P(x, y, t)$ and $Q(x, y, t)$ are such that the graph of $\mathbb{X} = \mathcal{W}(\theta)$ is transversal to the θ-axis. We also assume that $\mathcal{W}(\theta)$ is a Morse function. Let

$$M = \max_{\theta \in S^1} \mathcal{W}(\theta), \qquad m = \min_{\theta \in S^1} \mathcal{W}(\theta).$$

We divide into the following three cases: (a) $M < 0$; (b) $m > 0$; and (c) $m < 0 < M$. Among the three, (a) is a trivial case for which $\Lambda = \emptyset$. For (b) the return map $\mathcal{R}$ in the Main Theorem is completely well defined on Σ^- and it is a family of rank one maps studied previously by Wang and Young in [43–45]. The entire theory on rank one attractors is applicable and we obtain a variety of dynamical scenarios from a globally attracting invariant tori to strange attractors with SRB measures. For more details see [42].

(c) is the case for homoclinic tangles. By the assumption that the graph of $\mathbb{X} = \mathcal{W}(\theta)$ is transversal to the θ-axis, it follows that

$$\mathbb{F}(\theta, \mathbb{X}, \mu) = 0$$

divides Σ^- into a collection of vertical strips, which we denote as $V_1, U_1, \ldots, V_{n-1}, U_{n-1}$, $V_n = V_1$. The sign of $\mathbb{F}$ alternates on V_i and U_i. The return map $\mathcal{R}$ is well defined on all V_i but not on U_i. The actions of $\mathcal{R}$ on each of the vertical strips V_i are similar to the ones on V described previously in Section 4.1: First it compresses V_i in the vertical direction and stretches it in the horizontal direction. It then folds it, but probably more than one time. The number of turns of this image is determined by the number of critical points of $\mathcal{W}(\theta)$ on the corresponding θ-interval for V_i. This image is then put back into Σ^-, wrapping around Σ^- infinitely many times in θ-direction. Finally, the images of all V_i are bound together, moving towards $\theta = \infty$ in a roughly constant speed with respect to $\ln \mu^{-1}$ as $\mu \to 0$. The rich possibility on the number of folds for each vertical strip, and the possibilities on the different locations of all these folded parts create a rich array of complicated structures where the three dynamical scenarios of horseshoes, sinks, and SRB measures represented in Theorems 1–3 co-exist. In general, these are all that are out there for these homoclinic tangles.

5. Analysis of a periodically forced second-order equation

5.1. An integrable equation

We start with the second-order equation

$$\frac{d^2q}{dt^2} - q + q^2 = 0. \tag{5.1}$$

Observe that the non-linear term in Eq. (5.1) is q^2, not q^3 as in the Duffing equation [12]. Though our theory also applies to periodically perturbed Duffing equation, (5.1) is a better choice for our purpose of comparing the three theorems of Section 4.2 to the results of numerical simulations of Sections 5.3–5.5. Duffing's equation has two homoclinic loops, and the mixture of two homoclinic tangles for the perturbed equation would make the comparison between the theory and the numerical integration less direct. A separate paper on periodically perturbed Duffing's equation, where the dynamics of strange attractors formed by the mixture of two homoclinic tangles are studied, is currently under preparation and will appear elsewhere.

Let

$$H(q, p) = \frac{1}{2}p^2 + \frac{1}{3}q^3 - \frac{1}{2}q^2$$

where $(q, p) \in \mathbb{R}^2$. We re-write Eq. (5.1) as the following integrable Hamiltonian equations

$$\frac{dp}{dt} = -\frac{\partial H}{\partial q} = q - q^2, \qquad \frac{dq}{dt} = \frac{\partial H}{\partial p} = p. \tag{5.2}$$

$(q, p) = (0, 0)$ is a saddle point with a homoclinic solution, which we denote as $(q, p) = (a(t), b(t))$. From a direct computation we obtain

$$a(t) = \frac{6e^{-t}}{(1+e^{-t})^2}, \qquad b(t) = \frac{6e^{-t}(e^{-t}-1)}{(1+e^{-t})^3}. \tag{5.3}$$

5.2. Periodically perturbed equation

We add a term of dissipation and a term of non-linear damping to (5.1) to form a new autonomous equation

$$\frac{d^2q}{dt^2} + \left(\lambda - \gamma q^2\right)\frac{dq}{dt} - q + q^2 = 0 \tag{5.4}$$

where $\lambda, \gamma > 0$ are parameters. Denote $p = \frac{dq}{dt}$. $(q, p) = (0, 0)$ is now a dissipative saddle.

Proposition 5.1. *There exists $\varepsilon_0 > 0$ sufficiently small, such that for $\lambda \in [0, \varepsilon_0)$, there exists a γ_λ, $|\gamma_\lambda| \leqslant K\lambda$ such that for $\gamma = \gamma_\lambda$, Eq. (5.4) has a homoclinic solution for $(q, p) = (0, 0)$.*

Proof. Re-write Eq. (5.4) as

$$\frac{d^2q}{dt^2} - q + q^2 = -\lambda\left(1 - \gamma\lambda^{-1}q^2\right)\frac{dq}{dt}.$$

Let $\tau := \gamma\lambda^{-1}$. λ is the magnitude of an autonomous perturbation and the Melnikov function in this case is a constant. As a matter of fact,

$$\mathcal{W} = \int_{-\infty}^{\infty} \left(1 - \tau a^2(s)\right) b^2(s)\, ds.$$

We obtain a unique τ for a homoclinic solution provided that

$$\int_{-\infty}^{\infty} a^2(s) b^2(s)\, ds \neq 0$$

which is clearly the case here. $\square$

Let $0 < \lambda_0 < \varepsilon_0$ be fixed and $\gamma_0 = \gamma_{\lambda_0}$ be as in Proposition 5.1. $(q, p) = (0, 0)$ is a dissipative saddle and the eigenvalues are $-\alpha$ and β where

$$\alpha = \frac{1}{2}\left(\sqrt{\lambda_0^2 + 4} + \lambda_0\right), \qquad \beta = \frac{1}{2}\left(\sqrt{\lambda_0^2 + 4} - \lambda_0\right).$$

The set of λ_0 satisfying (H1)(i) has full measure in $(0, \varepsilon_0)$, and (H1)(ii) holds because $-\alpha + \beta = -\lambda_0 < 0$.

Let us now fix a $\lambda_0 \in (0, \varepsilon_0)$ so that (H1)(i) holds. Let the homoclinic solution be denoted as $(a_\lambda(t), b_\lambda(t))$ and the unit tangent vector be denoted as $(u_\lambda(t), v_\lambda(t))$, and so on. We add a periodic perturbation to Eq. (5.4) to obtain a new equation

$$\frac{d^2q}{dt^2} + \left(\lambda_0 - \gamma_0 q^2\right)\frac{dq}{dt} - q + q^2 = \mu q^2 \sin \omega t \tag{5.5}$$

where μ is a parameter representing the magnitude of the forcing and ω is the forcing frequency. In order to prove that Theorems 1–3 of Section 4.2 apply to Eq. (5.5), it suffices for us to verify (H2). That is

$$J_c^2 + J_s^2 \neq 0.$$

Let $J_\lambda(\omega) = J_c + i J_s$. We could make a direct estimate by first computing J_λ at $\lambda = 0$, then arguing that J_λ is close to J_0, or we take advantage of the fact that, as a function of ω, J_λ is the Fourier transform of the function

$$R_\lambda(s) = \left(v_\lambda(s) A_\lambda(s) - u_\lambda(s) B_\lambda(s)\right) e^{-\int_0^s E_\lambda(\tau)\, d\tau}.$$

Since $R_\lambda(s)$ decays exponentially as a function of s, the Fourier transform $J_\lambda(\omega)$ is analytic in a strip containing the real ω-axis by the Paley–Wiener theorem. It follows that $J_\lambda(\omega) = 0$ for at most a discrete set of values of ω unless $R_\lambda(s)$ is identically zero, which is obviously not the case here.

We have rigorously proved that the theory developed in this paper, in particular, Theorems 1–3, apply to Eq. (5.5).

5.3. Numerical simulations

In Sections 5.3–5.5 we use Eq. (5.5) to illustrate how to systematically explore and to explain, by using the theory of this paper, the various dynamical scenarios observed in numerical simulations of a periodically perturbed homoclinic solution (Sections 5.3–5.5). We adopt throughout the point of view that (1) a dynamical object is *observable* in phase space only if it affords an attractive basin that is of positive Lebesgue measure in the extended phase space and (2) observable objects are the ones expected to show up in numerical simulations. If the equations are with parameters, then there is also an issue of observability in the parameter space. A dynamical scenario is observable *in parameter space* if it is supported by a positive measure set of parameters.

The conclusions we wish to verify through numerical simulations are as follows:

(I) (Observable objects) There are three main observable dynamical scenarios in parameter space:

(a) (*Transient tangle*) there exist open sets of parameters so that the entire homoclinic tangle is a uniformly hyperbolic horseshoe of infinitely many symbols (Theorem 1);
(b) (*Non-chaotic tangle*) there exist open sets of parameters so that the homoclinic tangle admits periodic sinks (Theorems 2 and 3); and
(c) (*Chaotic tangle*) there exist positive measure sets of parameters so that the homoclinic tangle admits Sinai–Ruelle–Bowen measure (Corollary 4.1).

(II) (Multiplicative periodicity) Let T be the period of the forcing function in time, β be the positive eigenvalue of the perturbed saddle, and μ be the magnitude of the forcing. There is a well-defined pattern for the occurrence of the three dynamical scenarios of (I); and this pattern materializes over every μ-interval (μ_1, μ_2) satisfying $\mu_2/\mu_1 \approx e^{\beta T}$, repeating infinitely many times as $\mu \to 0$.

A horseshoe of (I)(a) has an attractive basin of measure zero therefore is *not* observable in phase space. Consequently, corresponding to (I)(a), we would expect in numerical simulations a scenario in which no solution will stay in the surroundings of the unperturbed homoclinic solution for long. The sinks of (I)(b) and the Sinai–Ruelle–Bowen measures of (I)(c), on the other hand, are both observable in phase space.

We proceed as follows:

Step 1. We arbitrarily pick a λ_0, say for example, $\lambda_0 = 0.5$. We then numerically integrate Eq. (5.4) to find the corresponding value of γ_0 from Proposition 5.1. Set $\lambda = \lambda_0$, $\gamma = \gamma_0$ in (5.5).

Step 2. We let μ, the small parameter representing the magnitude of the forcing, vary in between 10^{-3} and 10^{-6}.

Step 3. We take $(q_0, p_0) = (0.01, 0)$ as our initial phase position. The dynamics of homoclinic tangles, according to the Main Theorem, are best revealed by the various behavior of solutions of different initial times. We set $\omega = 2\pi$ in Eq. (5.5). With (q_0, p_0) being fixed throughout, we run the initial time over the interval $[0, 1)$. We then compare the simulation results to the predictions (I) and (II). We remark that using a different initial phase position (q_0, p_0) does not change the qualitative nature of our simulation results provided that it is inside of, and reasonably close to the homoclinic loop of Eq. (5.1).

From this point on we let $\lambda_0 = 0.5$. Note that though Proposition 5.1 requires that λ_0 is small, numerical simulation indicates that homoclinic solutions for Eq. (5.4) exist for much larger λ_0. For $\lambda_0 = 0.5$, the corresponding value of $\gamma_0 \approx 0.577028548901$. Let $(q_{\mu,t_0}(t), p_{\mu,t_0}(t))$ be the solution of Eq. (5.5) satisfying $(q_{\mu,t_0}(t_0), p_{\mu,t_0}(t_0)) = (q_0, p_0)$. We numerically integrate Eq. (5.5) to compute $(q_{\mu,t_0}(t), p_{\mu,t_0}(t))$. We then plot the following:

(a) $(q_{\mu,t_0}(n), p_{\mu,t_0}(n))$ on the (q, p)-plane.
(b) $(n, q_{\mu,t_0}(n))$. We save the redundancy of plotting $(n, p_{\mu,t_0}(n))$.
(c) The Fourier spectrum of $(n, q_{\mu,t_0}(n))$.

It is expected that either a solution satisfies $q > 0$ for all t, in which we obtain a plot of an observable object in the homoclinic tangle, or a solution reaches the negative side of q. When the latter occur, we know that the solution has moved out of the homoclinic tangle and our computation will be terminated, often at a time much earlier than the targeted ending time. There will be no plots for these solutions.

5.4. Simulation results on Section 5.3(I)

Our simulation results fall *always* into one of the three types:

(a) (*Transient tangle*): One instance of this type is for $\mu = 8 \times 10^{-7}$. We run the initial time t_0 over $[0, 1)$, skipping by 0.001 along the way. This amounts to numerically computing one thousand solutions of different initial time. All solutions reach the negative side of q rather quickly, indicating that they all pass through the homoclinic tangle. In this instance we have a homoclinic tangle without any observable objects. We hit a transient tangle of (I)(a).

(b) (*Non-chaotic tangle*): One instance of this type is for $\mu = 3.26 \times 10^{-6}$. Again, we run the initial time t_0 over $[0, 1)$, but this time we hit some initial time t_0, for which the solutions stay on the positive q side for all t. The plots for $t_0 = 0.15$ are illustrated in Fig. 6. Fig. 6(a) is for $(q_{\mu,t_0}(n), p_{\mu,t_0}(n))$; Fig. 6(b) is for $(n, q_{\mu,t_0}(n))$; and Fig. 6(c) is for the Fourier spectrum for $(n, q_{\mu,t_0}(n))$. This solution catches a sink.

(c) (*Chaotic tangle*): One instance of this type is for $\mu = 7.25 \times 10^{-7}$. For the solution plotted in Fig. 7, $t_0 = 0.25$. The observable object illustrated in Fig. 7 has a dominating periodic behavior and a random deviation that is somewhat secondary. These are the plots of a Hénon-like attractor associated to homoclinic tangency of the stable and unstable manifolds of a saddle of high period from the Smale horseshoe.

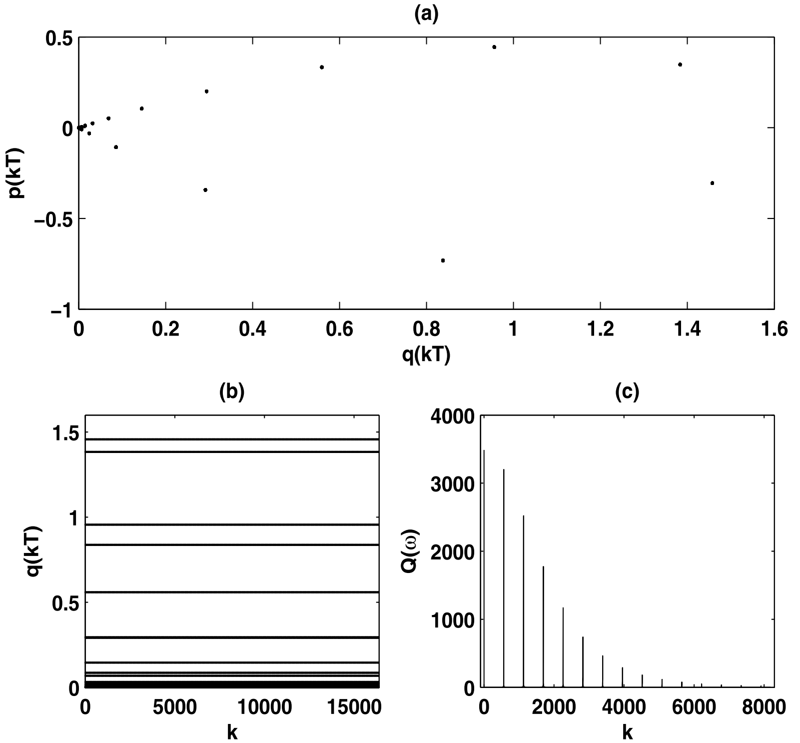

Fig. 6. Sinks inside of homoclinic tangle, $\mu = 3.26 \times 10^{-6}$, $t_0 = 0.15$.

5.5. Simulation results on Section 5.3(II)

The explicit formula for the multiplicative ratio of μ serves more than an item for verification. It helps in anticipating the future simulation results. For the case considered here, $T = 1$, $\beta = +0.78077641$. Therefore the theoretical ratio for μ is $P \approx e^{\beta T} \approx 2.18316664$.

Simulation results are presented in Table 1. *All* our simulations have resulted in one of the three cases listed above. We record a *transient tangle* in Table 1 if, by running t_0 over $[0, 1)$ with a skip of 0.001 each step along the way, the one thousand solutions integrated all reach the negative side for q, passing the homoclinic tangle without hitting anything observable. We record a *non-chaotic tangle* if there are values of initial time t_0, for which the solutions stay on the positive side of q for all t; furthermore, all these solutions return plots similar to Fig. 6. We record a *chaotic tangle* if the plots of some of these solutions are like Fig. 7. The μ values listed in Table 1 are where the plots change from one scenarios to another. The actual multiplicative ratios, computed by using the consecutive μ values for the same dynamics scenario, are listed in the third column of Table 1. Again, the theoretical ratio is ≈ 2.18316664.

Acknowledgments

We would like to thank V. Afraimovich for leading us to his previous work with Shilnikov, in particular, [3]. We would also like to thank Kening Lu for numerous conversations that have helped shaping the outlook of this research project.

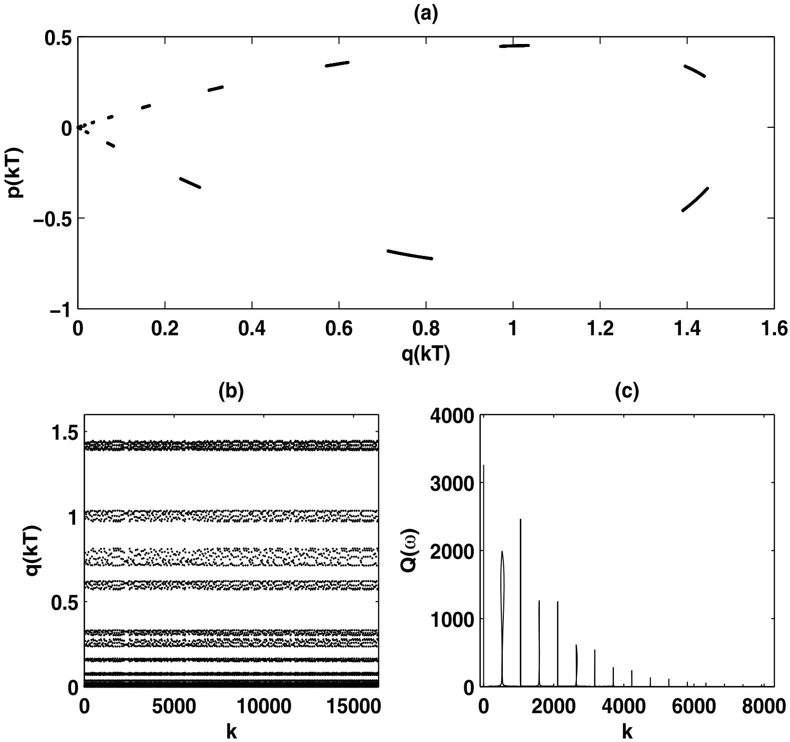

Fig. 7. Chaotic tangle, $\mu = 7.25 \times 10^{-7}$, $t_0 = 0.25$.

Appendix A. Proof of Theorems 1 and 2

From this point on we use z for $\mathbb{X}$ and write the annulus Σ^- as $\mathcal{A}$. We also use $\mathbf{a}$ in place of $p = \ln \mu^{-1}$ (see remark (i) at the end of Section 2.3). Let

$$J = \sqrt{J_c^2 + J_s^2},$$

$J \neq 0$ by (H2). The return map $(\theta_1, z_1) = \mathcal{R}(\theta, z)$ of the Main Theorem is written as

$$\begin{aligned} \theta_1 &= \theta + \mathbf{a} - \frac{\omega}{\beta} \ln \mathbb{F}(\theta, z, \mu) + \mathcal{O}_{\theta,z,\mathbf{a}}(\mu), \\ z_1 &= \mathbf{b}\big[\mathbb{F}(\theta, z, \mu)\big]^{\frac{\alpha}{\beta}} \end{aligned} \tag{A.1}$$

where

$$\mathbb{F}(\theta, z, \mu) = \mathbf{k}z + \sin\theta + \mathbb{E}(\theta, \mu) + \mathcal{O}_{\theta,z,\mathbf{a}}(\mu). \tag{A.2}$$

Note that $\mathbf{a}$, $\mathbf{b}$, $\mathbf{k}$, $\mathbb{E}$ and $\mathbb{F}$ here are slightly different from those in the Main Theorem because we now take a factor of J out of $\mathbb{F}$. The variable θ is shifted one more time to take away the initial phase θ_0 caused by writing $J_c \sin\theta + J_s \cos\theta$ as $J \sin(\theta + \theta_0)$ where $\theta_0 = \tan^{-1} J_s J_c^{-1}$. On the other hand, it remains true that

Table 1
Multiplicative ratio for μ.

$\lambda = 0.5$ $\gamma = 0.577028548901$ $\beta = 0.78077641$ Theoretical multiplicity $= e^{\beta T} = 2.1831$		
μ	Behavior type	Actual ratio
1.575×10^{-3}	Transient tangle (I)(a)	2.1141
7.952×10^{-4}	Chaotic tangle (I)(c)	2.1498
7.677×10^{-4}	Non-chaotic tangle (I)(b)	2.1492
7.450×10^{-4}	Transient tangle (I)(a)	2.1495
3.699×10^{-4}	Chaotic tangle (I)(c)	2.1670
3.572×10^{-4}	Non-chaotic tangle (I)(b)	2.1820
3.466×10^{-4}	Transient tangle (I)(a)	2.1676
1.707×10^{-4}	Chaotic tangle (I)(c)	2.1754
1.637×10^{-4}	Non-chaotic tangle (I)(b)	2.1745
1.599×10^{-4}	Transient tangle (I)(a)	2.1749
7.847×10^{-5}	Chaotic tangle (I)(c)	2.1797
7.528×10^{-5}	Non-chaotic tangle (I)(b)	2.1795
7.352×10^{-5}	Transient tangle (I)(a)	2.1797
3.600×10^{-5}	Chaotic tangle (I)(c)	2.1818
3.454×10^{-5}	Non-chaotic tangle (I)(b)	2.1751
3.373×10^{-5}	Transient tangle (I)(a)	2.1818
1.650×10^{-5}	Chaotic tangle (I)(c)	2.1825
1.588×10^{-5}	Non-chaotic tangle (I)(b)	2.1753
1.546×10^{-5}	Transient tangle (I)(a)	–
7.560×10^{-6}	Chaotic tangle (I)(c)	–
7.300×10^{-6}	Non-chaotic tangle (I)(b)	–

$$\mathbf{a} \approx \frac{\omega}{\beta} \ln \mu^{-1}, \qquad \mathbf{b} \sim \mu^{\frac{\alpha}{\beta}-1}, \qquad \mathbf{k} \sim \varepsilon^{\frac{\alpha}{\beta}}, \qquad \mathbb{E} \sim \varepsilon^{\frac{\beta}{\alpha}} \mathcal{O}_{\theta,\mathbf{a}}(1). \tag{A.3}$$

For $q = (\theta, z) \in \mathcal{A}$, let $\mathbf{v} = (u, v)$ be a tangent vector of $\mathcal{A}$ at q and let $s(\mathbf{v}) = vu^{-1}$. $s(\mathbf{v})$ is the slope of $\mathbf{v}$. We say that $\mathbf{v}$ is *horizontal* if $|s(\mathbf{v})| < \frac{1}{100}$ and $\mathbf{v}$ is *vertical* if $|s(\mathbf{v})| > 100$. A curve in $\mathcal{A}$ is a *horizontal curve* if all its tangent vectors are horizontal and it is a *vertical curve* if all its tangent vectors are vertical. A vertical curve is *fully extended* if it reaches both boundaries of $\mathcal{A}$ in z-direction. A region in $\mathcal{A}$ that is bounded by two non-intersecting, fully extended vertical curves is a *vertical strip*. For a given vertical strip V, *a horizontal strip* in V is a region bounded by two non-intersecting horizontal curves traversing V in θ-direction.

Observe that

$$\mathbb{F}(\theta, z, \mu) = \mathbf{k}z + \sin\theta + \mathbb{E}(\theta, \mu) + \mathcal{O}_{\theta,z,\mathbf{a}}(\mu) = 0 \tag{A.4}$$

defines two fully extended vertical curves that divide $\mathcal{A}$ into two vertical strips, which we denote as V and U. Let $\mathbb{F} > 0$ on V and $\mathbb{F} < 0$ on U. $\mathcal{R}_{\mathbf{a}}$ is well defined on V but not on U. U is the window through which the solutions of Eq. (2.2) sneak out.

Let

$$\Omega_{\mathbf{a}} = \{(\theta, z) \in V\colon\ \mathcal{R}^n_{\mathbf{a}}(\theta, z) \in V,\ \forall n \geqslant 0\}, \qquad \Lambda_{\mathbf{a}} = \bigcap_{n \geqslant 0} \mathcal{R}^n_{\mathbf{a}}(\Omega_{\mathbf{a}}). \tag{A.5}$$

$\Omega_{\mathbf{a}}$ represents all solutions of Eq. (2.2) that stay close to ℓ in forward times; $\Lambda_{\mathbf{a}}$ is the set $\Omega_{\mathbf{a}}$ is attracted to, representing all solutions that stay close to ℓ in both the forward and the backward times. $\Omega_{\mathbf{a}}$ and $\Lambda_{\mathbf{a}}$ together represent the homoclinic tangles, the structure of which we now unravel through $\mathcal{R}_{\mathbf{a}}$.

For a fixed $z \in [-1, 1]$, let

$$I_z = \left\{\theta \in \left(-\frac{1}{2}\pi, \frac{3}{2}\pi\right]: (\theta, z) \in V\right\}.$$

I_z is an interval in $(-\frac{1}{2}\pi, \frac{3}{2}\pi)$, which we denote as $(\theta_l(z), \theta_r(z))$. Let $h_z = \{(\theta, z): \theta \in I_z\}$. $\mathcal{R}_{\mathbf{a}}(h_z)$ is a 1D curve in $\mathcal{A}$ parameterized in θ, which we denote as $(z_1(\theta), \theta_1(\theta))$. By definition

$$\theta_1(\theta) = \theta + \mathbf{a} - \frac{\omega}{\beta} \ln \mathbb{F}(\theta, z, \mu) + \mathcal{O}_{\theta,z,\mathbf{a}}(\mu). \tag{A.6}$$

We have

Lemma A.1. *Assume that $\omega\beta^{-1} > 100$.*

(a) $\lim_{\theta\to\theta_r(z)^-}(\theta_1, z_1) = \lim_{\theta\to\theta_l(z)^+}(\theta_1, z_1) = (+\infty, 0)$.
(b) *For every fixed $z \in [-1, 1]$, there exists a unique value of θ, which we denote as $\theta_c(z)$, such that*

$$\frac{d\theta_1}{d\theta}\big(\theta_c(z)\big) = 0.$$

(c) *Let*

$$V_f = \bigcup_{z\in[-1,1]} \left\{(\theta, z) \in V: \left|\frac{d\theta_1}{d\theta}\right| < 1.5\right\}. \tag{A.7}$$

Then V_f is a vertical strip, the horizontal size of which is $< 10\omega^{-1}\beta$.

Proof. Observe, from (A.4), that $\theta_l \in (-\frac{1}{4}\pi, \frac{1}{4}\pi)$ where $\cos\theta > 0$, and $\theta_r \in (\frac{3}{4}\pi, \frac{5}{4}\pi)$ where $\cos\theta < 0$. (a) follows directly from the fact that, as $\theta \to \theta_l^+, \theta_r^-$, $\mathbb{F} \to 0$. To prove (b) we first observe that, because $\mathbb{F} \to 0$ as $\theta \to \theta_r^-$,

$$\left|\sin\theta_r^-\right| < K\varepsilon^{\beta\alpha^{-1}} \ll 1,$$

and it follows that

$$\frac{\partial\mathbb{F}}{\partial\theta}\big(\theta_r(z)^-, z\big) \approx \cos\theta_r < -1/2.$$

Consequently,

$$\lim_{\theta\to\theta_r^-}\frac{d\theta_1}{d\theta} = \lim_{\theta\to\theta_r^-}\left(1 - \omega\beta^{-1}\frac{1}{\mathbb{F}}\frac{\partial\mathbb{F}}{\partial\theta}\right) + \mathcal{O}_{\theta,z,\mathbf{a}}(\mu) = +\infty.$$

Similarly, we have

$$\lim_{\theta\to\theta_l^+}\frac{d\theta_1}{d\theta} = \lim_{\theta\to\theta_l^+}\left(1 - \omega\beta^{-1}\frac{1}{\mathbb{F}}\frac{\partial\mathbb{F}}{\partial\theta}\right) + \mathcal{O}_{\theta,z,\mathbf{a}}(\mu) = -\infty.$$

Therefore there exists at least one $\theta_c(z)$ satisfying $\frac{d\theta_1}{d\theta} = 0$. For the uniqueness we observe that

$$\frac{d^2\theta_1}{d\theta^2} = -\frac{\omega\beta^{-1}}{\mathbb{F}^2}\left(\frac{\partial^2\mathbb{F}}{\partial\theta^2}\mathbb{F} - \left(\frac{\partial\mathbb{F}}{\partial\theta}\right)^2\right) + \mathcal{O}_{\theta,z,\mathbf{a}}(\mu) \approx \frac{\omega\beta^{-1}}{\mathbb{F}^2} > 0$$

for all θ.

To prove (c) we observe that the boundary of V_f is defined by

$$\left|1 - \omega\beta^{-1}\frac{1}{\mathbb{F}}\frac{\partial\mathbb{F}}{\partial\theta}\right| \approx 1.5,$$

from which we obtain

$$|\cos\theta| \leqslant \frac{5}{2}\omega^{-1}\beta + K\varepsilon^{\beta\alpha^{-1}}.$$

(c) follows directly from this estimate. $\square$

Proof of Theorem 1. For different values of μ, the corresponding vertical curves in $\mathcal{A}$ defined by (A.4) are $\mathcal{O}(\mu)$ close. So V and U are almost stationary as $\mathbf{a}$ varies from $\mathbf{a}_0$ to $+\infty$. On the other hand, it follows from (2.8) that, by varying $\mathbf{a}$ from $\mathbf{a}_0$ to $+\infty$, we move $\mathcal{R}_{\mathbf{a}}(V)$ horizontally towards $\theta = +\infty$. Denote $\mathcal{R} = \mathcal{R}_{\mathbf{a}}$ and let V_f be the vertical strip defined through (A.7). The horizontal size of $\mathcal{R}(V_f)$ is smaller than $20\beta\omega^{-1}$ from Lemma A.1 assuming $\omega\beta^{-1} > 100$, which is in turn smaller than the horizontal size of U. Therefore $\mathcal{R}(V_f)$ traverses $\mathcal{A}$ infinitely many times in the horizontal direction as we vary $\mathbf{a}$ from $\mathbf{a}_0$ to $+\infty$ and there are infinitely many sub-intervals of $\mathbf{a}$, such that $\mathcal{R}(V_f) \subset U$. For these parameter values $\mathcal{R}(V) \cap V$ consists of countably many horizontal strips in V (see Fig. 4 in Section 4.1), to each of which we assign a positive integer according naturally to the order in which these strips are stacked in the downward z-direction.

For $q \in \mathcal{A}$, let $\mathbf{v}$ be a tangent vector at q. Let $\mathcal{C}_h(q)$ be the collection of all $\mathbf{v}$ satisfying $|s(\mathbf{v})| < \frac{1}{100}$, and $\mathcal{C}_v(q)$ be the collection of all $\mathbf{v}$ satisfying $|s(\mathbf{v})| > 100$. To prove that Λ conjugates to a full shift of all positive integers, it suffices to verify that we have, assuming $\mathcal{R}(V_f) \subset U$,

(i) $D\mathcal{R}(\mathcal{C}_h(q)) \subset \mathcal{C}_h(\mathcal{R}(q))$ on $\mathcal{R}^{-1}(\mathcal{R}(V) \cap V)$, and
(ii) $D\mathcal{R}^{-1}(\mathcal{C}_v(q)) \subset \mathcal{C}_v(\mathcal{R}(q))$ on $\mathcal{R}(V) \cap V$.

To prove (i) we first compute $D\mathcal{R}$ by using (A.1). Let $(\theta_1, z_1) = \mathcal{R}(\theta, z)$, we have

$$D\mathcal{R} = \begin{pmatrix} -\omega\beta^{-1}\frac{1}{\mathbb{F}}\frac{\partial\mathbb{F}}{\partial\theta} + \mathcal{O}_{\theta,z,\mathbf{a}}(\mu) & \omega\beta^{-1}\frac{1}{\mathbb{F}}\frac{\partial\mathbb{F}}{\partial z} + \mathcal{O}_{\theta,z,\mathbf{a}}(\mu) \\ \alpha\beta^{-1}\mathbf{b}\mathbb{F}^{\alpha\beta^{-1}-1}\frac{\partial\mathbb{F}}{\partial\theta} & \alpha\beta^{-1}\mathbf{b}\mathbb{F}^{\alpha\beta^{-1}-1}\frac{\partial\mathbb{F}}{\partial z} \end{pmatrix} \tag{A.8}$$

where $\mathbb{F} = \mathbb{F}(\theta, z, \mu)$ is as in (A.2) and

$$\frac{\partial\mathbb{F}}{\partial\theta} = \cos\theta + \varepsilon^{\beta\alpha^{-1}}\mathcal{O}_{\theta,\mathbf{a}}(1) + \mathcal{O}_{\theta,z,\mathbf{a}}(\mu),$$
$$\frac{\partial\mathbb{F}}{\partial z} = \mathbf{k} + \mathcal{O}_{\theta,z,\mathbf{a}}(\mu).$$

Let $\mathbf{v}$ be such that $|s(\mathbf{v})| < \frac{1}{100}$, we have from (A.8)

$$\left|s\big(D\mathcal{R}(\mathbf{v})\big)\right| = \left|\frac{\alpha\beta^{-1}\mathbf{b}\mathbb{F}^{\alpha\beta^{-1}-1}\frac{\partial\mathbb{F}}{\partial\theta} + \alpha\beta^{-1}\mathbf{b}\mathbb{F}^{\alpha\beta^{-1}-1}\frac{\partial\mathbb{F}}{\partial z}s(\mathbf{v})}{(1 - \omega\beta^{-1}\frac{1}{\mathbb{F}}\frac{\partial\mathbb{F}}{\partial\theta}) + \omega\beta^{-1}\frac{1}{\mathbb{F}}\frac{\partial\mathbb{F}}{\partial z}s(\mathbf{v}) + (1 + s(\mathbf{v}))\mathcal{O}_{\theta,z,\mathbf{a}}(\mu)}\right|. \tag{A.9}$$

We have two cases to consider.

Case 1: $\mathbb{F} \geqslant \sqrt{\mathbf{k}}$. In this case we have

$$\omega\beta^{-1}\frac{1}{\mathbb{F}}\frac{\partial\mathbb{F}}{\partial z} < \omega\beta^{-1}\sqrt{\mathbf{k}} \ll 1.$$

From $(\theta, z) \in \mathcal{R}^{-1}(\mathcal{R}(V) \cap V)$ and $\mathcal{R}(V_f) \subset U$, it follows that $(\theta, z) \notin V_f$ therefore

$$\left|\frac{\partial\theta_1}{\partial\theta}\right| \approx \left|1 - \omega\beta^{-1}\frac{1}{\mathbb{F}}\frac{\partial\mathbb{F}}{\partial\theta}\right| > 1.5.$$

These two estimates together imply that the denominator for $|s(D\mathcal{R}(\mathbf{v}))|$ in (A.9) is > 1, and it follows that $|s(D\mathcal{R}(\mathbf{v}))| < \frac{1}{100}$.

Case 2: $\mathbb{F} < \sqrt{\mathbf{k}}$. In this case

$$|\sin\theta| < K\varepsilon^{\frac{\beta}{\alpha}} + \sqrt{\mathbf{k}},$$

from which we have

$$|\cos\theta| > 1/2. \tag{A.10}$$

It then follows that the denominator for $|s(D\mathcal{R}(\mathbf{v}))|$ in (A.9) is $> \frac{\omega\beta^{-1}}{2\sqrt{\mathbf{k}}}$, which implies $|s(D\mathcal{R}(\mathbf{v}))| < \frac{1}{100}$. This finishes our proof for (i).

To prove (ii) we let $\mathbf{v}$ be such that $|s(\mathbf{v})| > 100$. From (A.8),

$$D\mathcal{R}^{-1} = \frac{1}{\alpha\beta^{-1}\mathbf{b}\mathbb{F}^{\alpha\beta^{-1}-1}\frac{\partial\mathbb{F}}{\partial z}}\begin{pmatrix} \alpha\beta^{-1}\mathbf{b}\mathbb{F}^{\alpha\beta^{-1}-1}\frac{\partial\mathbb{F}}{\partial z} & -\omega\beta^{-1}\frac{1}{\mathbb{F}}\frac{\partial\mathbb{F}}{\partial z} \\ -\alpha\beta^{-1}\mathbf{b}\mathbb{F}^{\alpha\beta^{-1}-1}\frac{\partial\mathbb{F}}{\partial\theta} & 1 - \omega\beta^{-1}\frac{1}{\mathbb{F}}\frac{\partial\mathbb{F}}{\partial\theta} \end{pmatrix}. \tag{A.11}$$

Note that in obtaining (A.12) we dropped the $\mathcal{O}_{\theta,z,p}(\mu)$ terms in (A.8) to avoid tedious writing, which is not consequential for the outcome of the rest of the estimates that follow. We have

$$\left|s\left(D\mathcal{R}^{-1}(\mathbf{v})\right)\right| = \left|\frac{-\alpha\beta^{-1}\mathbf{b}\mathbb{F}^{\alpha\beta^{-1}-1}\frac{\partial\mathbb{F}}{\partial\theta}s^{-1}(\mathbf{v}) + (1 - \omega\beta^{-1}\frac{1}{\mathbb{F}}\frac{\partial\mathbb{F}}{\partial\theta})}{\alpha\beta^{-1}\mathbf{b}\mathbb{F}^{\alpha\beta^{-1}-1}\frac{\partial\mathbb{F}}{\partial z}s^{-1}(\mathbf{v}) - \omega\beta^{-1}\frac{1}{\mathbb{F}}\frac{\partial\mathbb{F}}{\partial z}}\right|.$$

We again divide into the cases of $\mathbb{F} > \sqrt{k}$ and $\mathbb{F} < \sqrt{k}$. For the case of $\mathbb{F} > \sqrt{k}$, the magnitude of the denominator $\ll 1$ and that of the numerator is > 1 again because

$$\left|1 - \omega\beta^{-1}\frac{1}{\mathbb{F}}\frac{\partial\mathbb{F}}{\partial\theta}\right| > 1.5$$

from the assumption that $(\theta, z) \notin V_f$. For the case of $\mathbb{F} < \sqrt{\mathbf{k}}$, we re-write $|s(D\mathcal{R}^{-1}(\mathbf{v}))|$ as

$$\left|s\left(D\mathcal{R}^{-1}(\mathbf{v})\right)\right| = \left|\frac{-\alpha\beta^{-1}\mathbf{b}\mathbb{F}^{\alpha\beta^{-1}}\frac{\partial\mathbb{F}}{\partial\theta}s^{-1}(\mathbf{v}) + (\mathbb{F} - \omega\beta^{-1}\frac{\partial\mathbb{F}}{\partial\theta})}{\alpha\beta^{-1}\mathbf{b}\mathbb{F}^{\alpha\beta^{-1}}\frac{\partial\mathbb{F}}{\partial z}s^{-1}(\mathbf{v}) - \omega\beta^{-1}\frac{\partial\mathbb{F}}{\partial z}}\right|.$$

The denominator is again $\ll \omega\beta^{-1}$ and the dominating term in the numerator is

$$\left|\omega\beta^{-1}\frac{\partial\mathbb{F}}{\partial\theta}\right| > \frac{1}{2}\omega\beta^{-1}.$$

The last estimate is from

$$\left|\frac{\partial \mathbb{F}}{\partial \theta}\right| \approx |\cos\theta| > 1/2$$

again by (A.10). This proves (ii). □

We refer the reader to Chapter III.1 of [22] for a detailed discussion on horseshoes of infinitely many symbols.

Proof of Theorem 2. Let $\theta_c(z)$ be as in Lemma A.1(b). To make the dependency on μ explicit we write it as $\theta_c(z,\mu)$. Let $\mathbf{a}_n$ be the value of $\mathbf{a}$ at $\mu = \mu_n^{(r)}$ and $[\mathbf{a}_n] = \mathbf{a}_n - \mathbf{a}_n \mod (2\pi)$. Observe that there exists a $\hat{\mu} \in [\mu_n^{(r)}, \mu_{n+1}^{(l)}]$ so that $\theta_1(\theta_c) = \theta_c + [\mathbf{a}_n]$ where $\theta_c = \theta_c(0,\hat{\mu})$. This is because when μ traverses $[\mu_n^{(r)}, \mu_{n+1}^{(l)}]$, $\theta_1(\theta_c)$ traverses the interval $(\theta_l + [\mathbf{a}_n], \theta_r + [\mathbf{a}_n])$. Let $\hat{\mathbf{a}}$ be the value of $\mathbf{a}$ for $\hat{\mu}$. To solve for a fixed point we let

$$\begin{aligned} \theta + [\mathbf{a}_n] &= \theta + \hat{\mathbf{a}} - \omega\beta^{-1}\ln\mathbb{F} + \mathcal{O}_{\theta,z,\mathbf{a}}(\hat{\mu}), \\ z &= \mathbf{b}\mathbb{F}^{\alpha\beta^{-1}} \end{aligned} \tag{A.12}$$

to obtain

$$\begin{aligned} \mathbb{F} &= e^{\omega^{-1}\beta(\hat{\mathbf{a}}-[\mathbf{a}_n])+\mathcal{O}_{\theta,z,\mathbf{a}}(\hat{\mu})}, \\ z &= \mathbf{b}e^{\omega^{-1}\alpha(\hat{\mathbf{a}}-[\mathbf{a}_n])+\mathcal{O}_{\theta,z,\mathbf{a}}(\hat{\mu})}. \end{aligned} \tag{A.13}$$

From the first line we have

$$\sin\theta + \mathbb{E}(\theta,\hat{\mu}) + \mathcal{O}_{\theta,z,\mathbf{a}}(\hat{\mu}) = e^{\omega^{-1}\beta(\hat{\mathbf{a}}-[\mathbf{a}_n])+\mathcal{O}_{\theta,z,\mathbf{a}}(\hat{\mu})}. \tag{A.14}$$

To solve (A.14) for θ, first we observe that $\theta_c = \theta_c(0,\hat{\mu})$ is a solution of (A.14) for $z = 0$. We then observe that

$$|\cos\theta_c| > K^{-1}.$$

This estimate follows from the fact that θ_c is defined by

$$1 - \omega\beta^{-1}\frac{1}{\mathbb{F}}\frac{\partial\mathbb{F}}{\partial\theta} + \mathcal{O}_{\theta,\mathbf{a}}(\hat{\mu}) = 0$$

and $\mathbb{F} = e^{\omega^{-1}\beta(\hat{\mathbf{a}}-[\mathbf{a}_n])+\mathcal{O}_{\theta,z,\mathbf{a}}(\hat{\mu})}$ from (A.13). Applying the inverse value theorem to (A.14) we obtain a solution $\hat{\theta}$ satisfying

$$|\hat{\theta} - \theta_c| < K\hat{\mu}.$$

In summary we have obtained a fixed point $(\hat{\theta}, \hat{z})$ satisfying

$$\hat{\theta} \approx \theta_c; \qquad \hat{z} \approx \mathbf{b}e^{\omega^{-1}\alpha(\hat{\mathbf{a}}-[\mathbf{a}_n])}.$$

To prove that $(\hat{\theta}, \hat{z})$ is an attracting fixed point, we compute the eigenvalues. The eigen-equation for $D\mathcal{R}$ is

$$\lambda^2 - \mathrm{Tr}(D\mathcal{R})\lambda + \det(D\mathcal{R}) = 0.$$

From (A.8) we have

$$\begin{aligned} \mathrm{Tr}(D\mathcal{R}) &\approx \frac{\partial \theta_1}{\partial \theta} + \alpha\beta^{-1}\mathbf{b}\mathbb{F}^{\alpha\beta^{-1}-1}\frac{\partial \mathbb{F}}{\partial z} \ll 1, \\ \det(D\mathcal{R}) &\approx \alpha\beta^{-1}\mathbf{b}\mathbb{F}^{\alpha\beta^{-1}-1}\frac{\partial \mathbb{F}}{\partial z} \ll 1 \end{aligned} \tag{A.15}$$

where for the first inequality we use

$$\frac{\partial \theta_1}{\partial \theta} < K\big(|\hat{\theta} - \theta_c| + |\hat{z}|\big)$$

at $(\hat{\theta}, \hat{z})$ with

$$K = \max_{\theta \in (\theta_c, \hat{\theta}),\ z \in [0, \hat{z}]} \left(\left| \frac{\partial^2 \theta_1}{\partial \theta^2} \right| + \left| \frac{\partial^2 \theta_1}{\partial \theta \partial z} \right| + 1 \right).$$

Note that, on the domain the maximum is taken, $\mathbb{F} > \frac{1}{2}$. The rest of (A.15) is obvious. It follows from (A.15) that both eigenvalues of $D\mathcal{R}$ are close to 0.

Let $q_c = (\hat{\theta}, \hat{z})$ be the stable fixed point obtained above and let

$$V_f = \big\{(\theta, z) \in \mathcal{A},\ |\theta - \hat{\theta}| < 1.5\omega^{-1}\beta\big\}.$$

To prove that the homoclinic tangle Λ contains only q_c and a horseshoe of infinitely many symbols, it suffices to prove that

$$q_c = \bigcap_{n=0}^{\infty} \mathcal{R}^n(V_f). \tag{A.16}$$

Assuming (A.16), the horseshoe inside of Λ is constructed by first adding V_f to U, then repeating the proof for Theorem 1.

Observe that $\mathcal{R}$ on V_f is a 2D map unfolded from the 1D singular limit

$$f(\theta) = \theta + \mathbf{a} + \omega\beta^{-1}\ln\big(\sin\theta + E(\theta)\big)$$

where $E(\theta) = \varepsilon^{\beta\alpha^{-1}}\mathcal{O}_\theta(1)$ is also a perturbation to $\sin\theta$. Denote

$$F(\theta) = \theta + \omega\beta^{-1}\ln\sin\theta,$$

and let θ_c be the critical point of $F(\theta)$. Also denote

$$I_\tau = \big\{\theta\colon\ |\theta - \theta_c| < \tau\omega^{-1}\beta\big\}.$$

To prove (A.16) it suffices to prove that

$$F(I_{1.5}) \subset I_{1.4}. \tag{A.17}$$

(A.17) follows directly from an elementary computation. □

Appendix B. Proof of Theorem 3

In this appendix we prove Theorem 3.[3] Let $\mathbf{a}_n$ be the value of $\mathbf{a}$ at $\mu = \mu_n^{(r)}$ and $[\mathbf{a}_n] = \mathbf{a}_n - \mathbf{a}_n \bmod (2\pi)$. Let $\mathbf{a}(\mu)$ be the value of $\mathbf{a}$ at $\mu \in [\mu_n^{(r)}, \mu_{n+1}^{(l)}]$. We divide the proof of this theorem into the following steps.

Step 1. Solving for hyperbolic fixed points. For $\mu \in [\mu_n^{(r)}, \mu_{n+1}^{(l)}]$, let m be an integer $\geqslant 3\omega\beta^{-1}$ and $q_m(\mathbf{a}) = (\theta_m, z_m)$ be the solution of the equations

$$\begin{aligned} \theta + [\mathbf{a}_n] + 2\pi m &= \theta + \mathbf{a}(\mu) - \frac{\omega}{\beta} \ln \mathbb{F}(\theta, z, \mu) + \mathcal{O}_{\theta,z,\mathbf{a}}(\mu), \\ z &= \mathbf{b}\big[\mathbb{F}(\theta, z, \mu)\big]^{\frac{\alpha}{\beta}}. \end{aligned} \tag{B.1}$$

θ_m is determined by

$$\mathbb{F}(\theta_m, z_m, \mu) = e^{\omega^{-1}\beta(\mathbf{a}(\mu) - [\mathbf{a}_n] - 2\pi m) + \mathcal{O}_{\theta,z,\mathbf{a}}(\mu)}, \tag{B.2}$$

and

$$z_m = \mathbf{b} e^{\omega^{-1}\alpha(\mathbf{a}(\mu) - [\mathbf{a}_n] - 2\pi m) + \mathcal{O}_{\theta,z,\mathbf{a}}(\mu)}. \tag{B.3}$$

Let us note again that, in all estimates that follow, the $\mathcal{O}_{\theta,z,\mathbf{a}}(\mu)$ terms in (B.1)–(B.3) are inconsequential. We will drop them to avoid tedious writings.

Claim B.1. *$q_m(\mathbf{a}) = (\theta_m, z_m)$ is a saddle fixed point.*

Proof. Recall that

$$\begin{aligned} \mathrm{Tr}(D\mathcal{R}) &= \frac{\partial \theta_1}{\partial \theta} + \alpha\beta^{-1}\mathbf{b}\mathbb{F}^{\alpha\beta^{-1}-1}\frac{\partial \mathbb{F}}{\partial z}, \\ \det(D\mathcal{R}) &= \alpha\beta^{-1}\mathbf{b}\mathbb{F}^{\alpha\beta^{-1}-1}\frac{\partial \mathbb{F}}{\partial z}. \end{aligned}$$

Observe that, from (B.2) and the assumption that $m \geqslant 3\omega\beta^{-1}$, $\mathbb{F}(\theta_m, z_m, \mu) < \frac{1}{100}$. It follows that $|\cos\theta_m| > 2/3$, and

$$\left|\frac{\partial \theta_1}{\partial \theta}\right| > 51.$$

This implies

$$\big|\mathrm{Tr}(D\mathcal{R})\big| > 50.$$

[3] Minus Claim B.6(b), which we prove in Appendix C.

Observe that we also have $\det(D\mathcal{R}) \ll 1$. Therefore we have two eigenvalues, one is close to 0 and the other is with magnitude > 1. $\square$

We also have

Claim B.2. *For $m \geqslant 3\omega\beta^{-1}$,*

$$\left|\frac{d\theta_m}{d\mathbf{a}}\right| < \frac{1}{100}, \qquad \left|\frac{dz_m}{d\mathbf{a}}\right| < K\mathbf{b}.$$

Proof. Estimate for $\frac{dz_m}{d\mathbf{a}}$ follows directly from (B.3). To estimate $\frac{d\theta_m}{d\mathbf{a}}$ we take the derivative with respect to $\mathbf{a}$ on both sides of (B.2) and use $\mathbb{F} < \frac{1}{100}$ to obtain $|\cos\theta_m| > 2/3$. $\square$

Step 2. The stable and the unstable manifolds for q_m. In the rest of this proof we let m be the smallest integer $> 3\omega\beta^{-1}$. Denote $q(\mathbf{a}) = q_m(\mathbf{a})$. Let

$$\hat{V} = \{(\theta, z) \in V,\ \mathbb{F} > \mathbb{F}(q_{100m}(\mathbf{a}), \mu)\}.$$

We obtain $\hat{V}$ from V by taking away two thin vertical strips at the vertical boundaries of V. Observe that, by definition, $q(\mathbf{a}) \in \hat{V}$. We make ε sufficiently small so that (1) the distance from $q(\mathbf{a})$ to the vertical boundary of $\hat{V}$ is $\gg \mathbf{k}$; and (2) $K_m := \mathbb{F}(q_{100m}) \gg \mathbf{k}$.

Denote the stable and the unstable manifolds of $q = q(\mathbf{a})$ as $W^s(q)$ and $W^u(q)$ respectively. The local stable and the local unstable manifolds are denoted as $W^s_{loc}(q)$ and $W^u_{loc}(q)$. Let $\ell^u(q)$ be the connected branch of $W^u(q)$ in $\hat{V} \setminus V_f$ that contains $W^u_{loc}(q)$, and $\ell^u_1(q) = \mathcal{R}_{\mathbf{a}}(\ell^u(q))$. $\ell^u_1(q)$ is a horizontal curve traversing V_f in θ-direction: it is straightforward to verify that (1) $\ell^u(q)$ is a horizontal curve, (2) the image of $\ell^u(q)$ is also a horizontal curve, and (3) the length of that image is at least double the length of $\ell^u(q)$ so it traverses V_f. Let $z = w^u(\theta)$ be such that $(\theta, w^u(\theta)) \in \ell^u_1(q)$.

Claim B.3. *We have on $\ell^u_1(q)$,*

(a) $|\frac{dw^u}{d\theta}| < \mathbf{b}^{\frac{1}{2}}$;
(b) $|\frac{d^2w^u}{d\theta^2}| < \mathbf{b}^{\frac{1}{2}}$.

Proof. Denote $\mathcal{R} = \mathcal{R}_{\mathbf{a}}$. For $(\theta, z) \in \ell^u(q)$, let $(\theta_1, z_1) = \mathcal{R}(\theta, z)$. We have from (A.8)

$$\left|\frac{dw^u(\theta_1)}{d\theta_1}\right| = \left|\frac{\alpha\beta^{-1}\mathbf{b}\mathbb{F}^{\alpha\beta^{-1}-1}\frac{\partial\mathbb{F}}{\partial\theta} + \alpha\beta^{-1}\mathbf{b}\mathbb{F}^{\alpha\beta^{-1}-1}\frac{\partial\mathbb{F}}{\partial z}\frac{dw^u(\theta)}{d\theta}}{(1 - \omega\beta^{-1}\frac{1}{\mathbb{F}}\frac{\partial\mathbb{F}}{\partial\theta}) + \omega\beta^{-1}\frac{1}{\mathbb{F}}\frac{\partial\mathbb{F}}{\partial z}\frac{dw^u(\theta)}{d\theta}}\right|. \tag{B.4}$$

(a) holds because the magnitude of the denominator in (B.4) is > 1 for $(\theta, z) \in \hat{V} \setminus V_f$. Remember that since $\ell^u(q)$ is horizontal we have $|\frac{dw^u(\theta)}{d\theta}| < \frac{1}{100}$. To prove (b) we take the derivative one more time to obtain

$$\frac{d^2w^u(\theta_1)}{d^2\theta_1} = \frac{\frac{d}{d\theta}(\frac{dw^u(\theta_1)}{d\theta_1})}{\frac{d\theta_1}{d\theta}}. \tag{B.5}$$

Observe that $\frac{d\theta_1}{d\theta}$ is the denominator in (B.4), the magnitude of which is > 1. Let

$$M = \max_{(\theta,z)\in\ell^u_1}\left|\frac{d^2w^u(\theta)}{d\theta^2}\right|.$$

We have from (B.4) and (B.5),

$$\left|\frac{d^2 z_1}{d^2\theta_1}\right| < K_1\mathbf{b} + K_2\mathbf{b}M.$$

So

$$M < K_1\mathbf{b} + K_2\mathbf{b}M,$$

and $M < K\mathbf{b} < \mathbf{b}^{\frac{1}{2}}$. $\square$

Let $\ell^s_{-1}(q)$ be the segment of $W^s(q)$ in $\hat{V}$ that contains $W^s_{loc}(q)$, and $\ell^s(q) = \mathcal{R}(\ell^s_{-1}(q))$. $\ell^s_{-1}(q)$ is a fully extended vertical curve in $\hat{V}$, which we represent by a function $\theta = w^s(z)$.

Claim B.4. *We have on $\ell^s_{-1}(q)$,*

(a) $|\frac{dw^s(z)}{dz}| < \mathbf{k}^{\frac{1}{2}}$; *and*
(b) $|\frac{d^2 w^s(z)}{dz^2}| < \mathbf{b}^{\frac{1}{2}}$.

Proof. Let $(\theta, z) \in \ell^s(q)$ and denote $(\theta_{-1}, z_{-1}) = \mathcal{R}^{-1}(\theta, z)$. We have from (A.12),

$$\begin{pmatrix} \frac{d\theta_{-1}}{dz} \\ \frac{dz_{-1}}{dz} \end{pmatrix} = \frac{1}{\alpha\beta^{-1}\mathbf{b}\mathbb{F}^{\alpha\beta^{-1}-1}\frac{\partial\mathbb{F}}{\partial z}} \begin{pmatrix} \alpha\beta^{-1}\mathbf{b}\mathbb{F}^{\alpha\beta^{-1}-1}\frac{\partial\mathbb{F}}{\partial z} & -\omega\beta^{-1}\frac{1}{\mathbb{F}}\frac{\partial\mathbb{F}}{\partial z} \\ -\alpha\beta^{-1}\mathbf{b}\mathbb{F}^{\alpha\beta^{-1}-1}\frac{\partial\mathbb{F}}{\partial\theta} & 1-\omega\beta^{-1}\frac{1}{\mathbb{F}}\frac{\partial\mathbb{F}}{\partial\theta} \end{pmatrix} \begin{pmatrix} \frac{dw^s(z)}{dz} \\ 1 \end{pmatrix}, \tag{B.6}$$

and it follows that

$$\left|\frac{dw^s(z_{-1})}{dz_{-1}}\right| = \left|\frac{\alpha\beta^{-1}\mathbf{b}\mathbb{F}^{\alpha\beta^{-1}-1}\frac{\partial\mathbb{F}}{\partial z}\frac{dw^s(z)}{dz} - \omega\beta^{-1}\frac{1}{\mathbb{F}}\frac{\partial\mathbb{F}}{\partial z}}{-\alpha\beta^{-1}\mathbf{b}\mathbb{F}^{\alpha\beta^{-1}-1}\frac{\partial\mathbb{F}}{\partial\theta}\frac{dw^s(z)}{dz} + (1-\omega\beta^{-1}\frac{1}{\mathbb{F}}\frac{\partial\mathbb{F}}{\partial\theta})}\right|. \tag{B.7}$$

Let

$$M_1 = \max_{(\theta,z)\in\ell^s_{-1}(q)}\left|\frac{dw^s(z)}{dz}\right|.$$

We have from (B.7),

$$\left|\frac{dw^s(z_{-1})}{dz_{-1}}\right| < K_1\mathbf{k} + K_2 bM_1$$

because $\mathbb{F} > K_m$ and

$$\left(1-\omega\beta^{-1}\frac{1}{\mathbb{F}}\frac{\partial\mathbb{F}}{\partial\theta}\right) > 2.$$

It then follows that $M_1 < K\mathbf{k} < \mathbf{k}^{\frac{1}{2}}$.

To prove (b) we write

$$\frac{d^2 w^s(z_{-1})}{dz_{-1}^2} = \frac{\frac{d}{dz}(\frac{dw^s(z_{-1})}{dz_{-1}})}{\frac{dz_{-1}}{dz}} \tag{B.8}$$

where $\frac{dw^s(z_{-1})}{dz_{-1}}$ is as in (B.7) and

$$\frac{dz_{-1}}{dz} = \frac{1}{\alpha\beta^{-1}\mathbf{b}\mathbb{F}^{\alpha\beta^{-1}-1}\frac{\partial\mathbb{F}}{\partial z}}\left(-\alpha\beta^{-1}\mathbf{b}\mathbb{F}^{\alpha\beta^{-1}-1}\frac{\partial\mathbb{F}}{\partial\theta}\frac{dw^s(z)}{dz} + 1 - \omega\beta^{-1}\frac{1}{\mathbb{F}}\frac{\partial\mathbb{F}}{\partial\theta}\right). \tag{B.9}$$

Let

$$M_2 = \max_{(\theta,z)\in\ell^s_{-1}(q)}\left|\frac{d^2w^s(z)}{dz^2}\right|.$$

We have from (B.7), (B.8) and (B.9) that

$$\left|\frac{d^2w^s(z_{-1})}{dz_{-1}^2}\right| < K\mathbf{b}(K_1\mathbf{b}M_2 + K_2),$$

from which we obtain $M_2 < K\mathbf{b} < \mathbf{b}^{\frac{1}{2}}$. $\square$

Step 3. Non-degenerate, transversal tangency. Let $\ell^u_{\mathbf{a}}$ be a connected segment of $\ell^u_1(q)\cap V_f$, and $\ell^s_{\mathbf{a}}$ be the vertical curve $\ell^s_{-1}(q)$ where $\ell^u_1(q)$, $\ell^s_{-1}(q)$ are as in Step 2. We use $z = w^u(\theta)$ to represent $\ell^u_{\mathbf{a}}$ and $\theta = w^s(z)$ to represent $\ell^s_{\mathbf{a}}$. $\mathcal{R}_{\mathbf{a}}(\ell^u_{\mathbf{a}})$ traverses $\hat{V}$ in horizontal direction as μ runs through $[\mu_n^{(r)}, \mu_{n+1}^{(l)}]$. Consequently there exists $\hat{\mu}\in[\mu_n^{(r)}, \mu_{n+1}^{(l)}]$, the corresponding value for $\mathbf{a}$ we denote as $\hat{\mathbf{a}}$, so that $\ell^s_{\hat{\mathbf{a}}}$ and $\mathcal{R}_{\hat{\mathbf{a}}}(\ell^u_{\hat{\mathbf{a}}})$ intersect tangentially at a point we denote as $\tilde{q} = (\tilde{\theta}, \tilde{z})$. Let $(\theta_0, z_0)\in\ell^u_{\hat{\mathbf{a}}}$ be such that $(\tilde{\theta}, \tilde{z}) = \mathcal{R}_{\hat{\mathbf{a}}}(\theta_0, z_0)$. Our next claim implies that the tangential intersection of $\ell^s_{\hat{\mathbf{a}}}$ and $\mathcal{R}_{\hat{\mathbf{a}}}(\ell^u_{\hat{\mathbf{a}}})$ at $\tilde{q}$ is not degenerate.

Claim B.5. *For $(\theta, z)\in\ell^u_{\hat{\mathbf{a}}}$, let $(\theta_1, z_1) = \mathcal{R}_{\hat{\mathbf{a}}}(\theta, z)$. Then at $(\theta, z) = (\theta_0, z_0)$, we have*

$$\left|\frac{d^2\theta_1}{dz_1^2}\right| \gg 1.$$

Proof. From (A.8) we have

$$\frac{d\theta_1}{dz_1} = \frac{(1-\omega\beta^{-1}\frac{1}{\mathbb{F}}\frac{\partial\mathbb{F}}{\partial\theta}) + \omega\beta^{-1}\frac{1}{\mathbb{F}}\frac{\partial\mathbb{F}}{\partial z}\frac{dw^u(\theta)}{d\theta}}{\alpha\beta^{-1}\mathbf{b}\mathbb{F}^{\alpha\beta^{-1}-1}\frac{\partial\mathbb{F}}{\partial\theta} + \alpha\beta^{-1}\mathbf{b}\mathbb{F}^{\alpha\beta^{-1}-1}\frac{\partial\mathbb{F}}{\partial z}\frac{dw^u(\theta)}{d\theta}}. \tag{B.10}$$

At the point of tangential intersection, we have $|\frac{d\theta_1}{dz_1}| < \mathbf{k}^{\frac{1}{2}}$, which is not possible unless

$$\left|1 - \omega\beta^{-1}\frac{1}{\mathbb{F}}\frac{\partial\mathbb{F}}{\partial\theta}\right| < \mathbf{b}^{\frac{1}{4}} \tag{B.11}$$

from (B.10). This is because $\frac{dw^u(\theta)}{d\theta} < \mathbf{b}^{\frac{1}{2}}$ from Claim B.3(a). The effect of $\mathbf{b}$ in the denominator cannot be possibly balanced if (B.11) is false.

For the estimate on the second derivative we start from

$$\frac{d^2\theta_1}{dz_1^2} = \frac{\frac{d}{d\theta}(\frac{d\theta_1}{dz_1})}{\frac{dz_1}{d\theta}} \tag{B.12}$$

where $\frac{dz_1}{d\theta}$ is the denominator in (B.10). To compute $\frac{d}{d\theta}(\frac{d\theta_1}{dz_1})$, we take the derivative of the function on the right-hand side of (B.10) with respect to θ. Applying the quotient rule we obtain a fraction, the bottom of which has a factor $\mathbf{b}^2$. On the top, we have a collection of finitely many terms, each of which is $< K\mathbf{b}^{1+\frac{1}{4}}$ in magnitude except one in the form of

$$\left(\alpha\beta^{-1}\mathbf{b}\mathbb{F}^{\alpha\beta^{-1}-1}\frac{\partial\mathbb{F}}{\partial\theta}\right)\frac{d}{d\theta}\left(1-\omega\beta^{-1}\frac{1}{\mathbb{F}}\frac{\partial\mathbb{F}}{\partial\theta}\right). \tag{B.13}$$

Remember that we have $\mathbb{F} > K_m$ on $\hat{V}$, and $\frac{d\mathbb{F}}{d\theta} > K^{-1}$ from (B.11). We also have

$$\left|\frac{d}{d\theta}\left(1-\omega\beta^{-1}\frac{1}{\mathbb{F}}\frac{\partial\mathbb{F}}{\partial\theta}\right)\right| \approx \frac{\omega\beta^{-1}}{\mathbb{F}^2} > 1.$$

Therefore, (B.13) is the dominating term on top and we obtain

$$\left|\frac{d^2\theta_1}{dz_1^2}\right| > K\mathbf{b}^{-2}$$

at $\tilde{q}$. □

To finish our proof of Theorem 3, we also need to prove that, as $\mathbf{a}$ varies, $\ell^s_{\mathbf{a}}$ and $\mathcal{R}_{\mathbf{a}}(\ell^u_{\mathbf{a}})$ move with different speeds at the point of tangency. To make the dependency on parameter $\mathbf{a}$ explicit, we write $w^u = w^u(\theta, \mathbf{a})$, $w^s = w^s(z, \mathbf{a})$. Claim B.3 applies to $w^u(\theta, \mathbf{a})$ and Claim B.4 applies to $w^s(z, \mathbf{a})$.

Claim B.6. *Let* $(\theta_1(\theta_0, \mathbf{a}), z_1(\theta_0, \mathbf{a})) = \mathcal{R}_{\mathbf{a}}(\theta_0, w^u_{\mathbf{a}}(\theta_0, \mathbf{a}))$. *Then at* $\mathbf{a} = \hat{\mathbf{a}}$ *we have*

(a) $|\frac{\partial}{\partial\mathbf{a}}\theta_1(\theta_0, \mathbf{a})| > \frac{2}{3}$; *and*
(b) $|\frac{\partial}{\partial\mathbf{a}}w^s(\tilde{z}, \mathbf{a})| < \frac{1}{25}$.

Recall that $\tilde{q} = (\tilde{\theta}, \tilde{z})$ *is the point of tangential intersection and* (θ_0, z_0) *is such that* $\mathcal{R}_{\hat{\mathbf{a}}}(\theta_0, z_0) = \tilde{q}$.

Proof. In this proof we use $\partial_z, \partial_\theta$ and $\partial_{\mathbf{a}}$ to denote the partial derivative with respect to z, θ and $\mathbf{a}$ respectively.

To prove (a) we let $q = (\theta_m, z_m)$ be the saddle fixed point and $\ell^u(q)$ and $\ell^u_1(q) = \mathcal{R}_{\mathbf{a}}(\ell^u(q))$ be as in Claim B.3. For $(\theta, z) \in \ell^u(q)$ and $(\theta_0, z_0) = \mathcal{R}_{\mathbf{a}}(\theta, z)$, we have from (2.8),

$$\begin{aligned} \theta_0 &= f(\theta, z, \mathbf{a}) = \theta + \mathbf{a} - \frac{\omega}{\beta}\ln\mathbb{F}(\theta, z, \mu), \\ z_0 &= g(\theta, z, \mathbf{a}) = \mathbf{b}\big[\mathbb{F}(\theta, z, \mu)\big]^{\frac{\alpha}{\beta}}, \end{aligned} \tag{B.14}$$

and in (B.14), $z_0 = w^u(\theta_0, \mathbf{a})$, $z = w^u(\theta, \mathbf{a})$ because both (θ_0, z_0) and (θ, z) are on $\ell^u_1(q)$. We first invert the first equality in (B.14), obtaining $\theta = \theta(\theta_0, \mathbf{a})$; then we put it into the second equality in (B.14) to obtain $z_0 = w^u(\theta_0, \mathbf{a})$. To estimate $\partial_{\mathbf{a}}w^u(\theta_0, \mathbf{a})$, we first let

$$M_{\mathbf{a}} = \max_{(\theta,z)\in\ell^u_1(q)}\big|\partial_{\mathbf{a}}w^u(\theta, \mathbf{a})\big|$$

and obtain from the first equality in (B.14),

$$\big|\partial_{\mathbf{a}}\theta(\theta_0, \mathbf{a})\big| = \left|\frac{\partial_{\mathbf{a}}f + \partial_z f\cdot\partial_{\mathbf{a}}w^u}{\partial_\theta f + \partial_z f\cdot\partial_\theta w^u}\right| < K_1 + K_2M_{\mathbf{a}}$$

because $|\partial_\theta f| > 1$ for $(\theta, z) \in \hat{V} \setminus V_f$ and $|\partial_\theta w^u| < \mathbf{b}^{\frac{1}{2}}$ from Claim B.3(a). From the second equality in (B.14) we have

$$\begin{aligned}\left|\partial_{\mathbf{a}} w^u(\theta_0, \mathbf{a})\right| &= \left|\partial_\theta g \cdot \partial_{\mathbf{a}}\theta + \partial_z g \cdot \left(\partial_\theta w^u \cdot \partial_{\mathbf{a}}\theta + \partial_{\mathbf{a}} w^u\right) + \partial_{\mathbf{a}} g\right| \\ &\leqslant K_3\mathbf{b}(K_1 + K_2 M_{\mathbf{a}}) + K_4\mathbf{b},\end{aligned}$$

from which it follows that

$$M_{\mathbf{a}} < \mathbf{b}^{\frac{1}{2}}. \tag{B.15}$$

(a) now follows by taking $\partial_{\mathbf{a}}$ on

$$\theta_1(\theta_0, \mathbf{a}) = \theta_0 + \mathbf{a} - \omega\beta^{-1} \ln \mathbb{F}\big(\theta_0, w^u(\theta_0, \mathbf{a}), \mu\big)$$

using (B.15).

Proof of (b) is more sophisticated than that of (a). We need to study the stable manifold through the field of most contracted directions, a method originally introduced in [5] and fully developed in [43] and [44]. A detailed proof is included in Appendix C. □

With Claim B.6 we know that, as $\mathbf{a}$ varies passing $\hat{\mathbf{a}}$, $\mathcal{R}_{\mathbf{a}}(\ell^u_{\mathbf{a}})$ crosses $\ell^s_{\mathbf{a}}$ transversally. This finishes our proof of Theorem 3 owing that of Claim B.6(b).

Appendix C. Proof of Claim B.6(b)

In order to produce the desired estimates in Claim B.6(b), we need more precise controls on the stable manifold of the saddle fixed point q_m. The main idea of our proof, that is, to approximate the stable manifold by using the integral curves of the vector field defined by the most contracted directions of the Jacobi matrix, was originated from [5], and was fully developed in [43] and [44]. Here we only need a specific version of the contents developed in the beginning part of Section 3 in [44].

C.1. Most contracted directions

In what follows $u_1 \wedge u_2$ is the wedge product and $\langle u_1, u_2\rangle$ is the inner product for $u_1, u_2 \in \mathbb{R}^2$.

Let M be a 2×2 matrix and assume $M \neq cO$ where O is orthogonal and $c \in \mathbb{R}$. Then there is a unit vector e, uniquely defined up to a sign, that represents *the most contracted direction* of M, i.e. $|Me| \leqslant |Mu|$ for all unit vectors u. From standard linear algebra, we know $f = e^\perp$ is the most expanded direction, meaning $|Me^\perp| \geqslant |Mu|$ for all unit vectors u, and $Me \perp Me^\perp$. The numbers $|Me|$ and $|Me^\perp|$ are the *singular values* of M.

Let $u \perp v$ be two unit vectors in $\mathbb{R}^2$. The following formulas are the results of elementary computations. First, we write down the squares of the singular values of M:

$$|Me|^2 = \frac{1}{2}\big(B - \sqrt{B^2 - 4C}\big) := \lambda, \qquad |Mf|^2 = \frac{1}{2}\big(B + \sqrt{B^2 - 4C}\big) \tag{C.1}$$

where

$$B = |Mu|^2 + |Mv|^2, \qquad C = |Mu \wedge Mv|^2. \tag{C.2}$$

We write $e = \alpha_0 u + \beta_0 v$, and solve for $|Me| = \sqrt{\lambda}$ subject to $\alpha_0^2 + \beta_0^2 = 1$. There are two solutions (a vector and its negative): either $e = \pm v$, or the solution with a positive u-component is given by

$$e = \frac{1}{Z}(\alpha u + \beta v) \tag{C.3}$$

with

$$\alpha = |Mv|^2 - \lambda, \qquad \beta = -\langle Mv, Mu\rangle \tag{C.4}$$

and

$$Z = \sqrt{\alpha^2 + \beta^2}. \tag{C.5}$$

From this we deduce that a solution for f is

$$f = \frac{1}{Z}(-\beta u + \alpha v). \tag{C.6}$$

C.2. Stability of most contracted directions

In what follows we let $q_i = \mathcal{R}^i_{\mathbf{a}}(q_0)$, $M_i = D\mathcal{R}_{\mathbf{a}}(q_{i-1})$;

$$M_i = \begin{pmatrix} A_i & B_i \\ C_i & D_i \end{pmatrix} = \begin{pmatrix} 1 - \omega\beta^{-1}\frac{1}{\mathbb{F}}\frac{\partial\mathbb{F}}{\partial\theta} & \omega\beta^{-1}\frac{1}{\mathbb{F}}\frac{\partial\mathbb{F}}{\partial z} \\ \alpha\beta^{-1}\mathbf{b}\mathbb{F}^{\alpha\beta^{-1}-1}\frac{\partial\mathbb{F}}{\partial\theta} & \alpha\beta^{-1}\mathbf{b}\mathbb{F}^{\alpha\beta^{-1}-1}\frac{\partial\mathbb{F}}{\partial z} \end{pmatrix}.$$

We have for $q_{i-1} \in \hat{V} \setminus V_f$,

$$2 < |A_i| < K, \qquad |B_i| < K, \qquad |C_i|, |D_i| < K\mathbf{b}. \tag{C.7}$$

Let $M^{(n)} = D\mathcal{R}^n_{\mathbf{a}}(q_0)$. $M^{(n)} = M_n \cdot M_{n-1} \cdots M_1$. Let the most contracted direction for $M^{(n)}$ be e_n and the most expanded direction be f_n. Denote the values of α, β and Z in (C.4) and (C.5) for $M^{(n)}$ as α_n, β_n and Z_n. Observe that, assuming $q_i \in \hat{V} \setminus V_f$, $i < n$,

$$\left|M^{(n)} f_n\right| > 1. \tag{C.8}$$

We have

Lemma C.1. *Let q_0 be such that $q_0, \ldots, q_n \in \hat{V} \setminus V_f$. Then for all $1 \leqslant i \leqslant n$,*

(a) $|e_{i+1} - e_i| < (K\mathbf{b})^i$, $|M^{(i)}e_n| < (K\mathbf{b})^i$;
(b) $|\partial_{\mathbf{a}}(e_{i+1} - e_i)| < (K\mathbf{b})^i$, $|\partial_{\mathbf{a}} M^{(i)}e_n| < (K\mathbf{b})^i$.

Proof. Let $\Delta_i := |M^{(i)}u \wedge M^{(i)}v|$. We have

$$\Delta_i = \left|\det\left(M^{(i)}\right)\right| < (K\mathbf{b})^i. \tag{C.9}$$

It then follows from $|M^{(i)}e_i||M^{(i)}f_i| = \Delta_i$ and (C.8),

$$\left|M^{(i)}e_i\right| < (K\mathbf{b})^i. \tag{C.10}$$

We substitute $u = e_i, v = f_i$ and $M = M^{(i+1)}$ into (C.3) for e_{i+1} and (C.5) for f_{i+1}. By using (C.1) for $M^{(i+1)} f_{i+1}$, we have

$$\left|M^{(i+1)} f_i\right| = \left|M^{(i+1)} f_{i+1}\right| \pm \mathcal{O}\big((K\mathbf{b})^i\big). \tag{C.11}$$

From (C.2), (C.9) and (C.10), we also have

$$Z_{i+1} \approx |\alpha_{i+1}| \approx \left|M^{(i+1)} f_i\right|^2. \tag{C.12}$$

We now prove Lemma C.1(a). Using $u = e_i$ and $v = f_i$, we have, from (C.3),

$$e_{i+1} - e_i = \frac{1}{Z_{i+1}}\left(\frac{-\beta_{i+1}^2}{\alpha_{i+1} + Z_{i+1}} e_i + \beta_{i+1} f_i\right). \tag{C.13}$$

To estimate $|e_{i+1} - e_i|$, we need to obtain a suitable upper bound for $|\beta_{i+1}|$ and lower bounds for $|\alpha_{i+1}|$ and Z_{i+1}. We have from (C.4), (C.10) and (C.12),

$$|\beta_{i+1}| \leqslant \left|M^{(i+1)} e_i\right|\left|M^{(i+1)} f_i\right| < K\mathbf{b}^i \sqrt{Z_{i+1}} \tag{C.14}$$

and $|\alpha_{i+1}| \approx Z_{i+1}$. These estimates together with $Z_{i+1} > 1$ tell us

$$|e_{i+1} - e_i| \approx \frac{|\beta_{i+1}|}{Z_{i+1}} < (K\mathbf{b})^i.$$

The second assertion follows easily from

$$\left|M^{(i)} e_n\right| \leqslant \left|M^{(i)}(e_n - e_{n-1})\right| + \cdots + \left|M^{(i)}(e_{i+1} - e_i)\right| + \left|M^{(i)} e_i\right| < (K\mathbf{b})^i.$$

This finishes our proof for Lemma C.1(a).

To prove Lemma C.1(b) we start with

Sublemma C.1. $|\partial_{\mathbf{a}} e_1|, |\partial_{\mathbf{a}} f_1| < K_1$ *for some* K_1.

Proof. Let $u = (0,1)^T$, $v = (1,0)^T$ and use (C.3) for e_1 and (C.6) for f_1. We have $Z_1 > \alpha \geqslant |M_1 v|^2 - K\mathbf{b} > 1$. Differentiating (C.3) and (C.6) gives the desired result. $\square$

In the rest of this proof, $\partial = \partial_{\mathbf{a}}$. Our plan of proof for Lemma C.1(b) is as follows: For $k = 1, 2, \ldots,$ we assume for all $i \leqslant k$:

$(*)$ $|\partial e_i|, |\partial f_i| < 2K_1$ where K_1 is as in Sublemma C.1,

and prove for all $i \leqslant k$:

(A) $|\partial(M^{(i)} f_i)| < K^i$, $|\partial(M^{(i)} e_i)| < (K\mathbf{b})^i$;
(B) $|\partial(e_{i+1} - e_i)|$, $|\partial(f_{i+1} - f_i)| < (K\mathbf{b})^i$.

Observe that for $i = 1$, $(*)$ is given by Sublemma C.1. It is easy to see that (B) above implies $(*)$ with $i = k+1$, namely $|\partial f_{k+1}| \leqslant |\partial(f_{k+1} - f_k)| + \cdots + |\partial(f_2 - f_1)| + |\partial f_1|$. From (B), we have $|\partial(f_{i+1} - f_i)| < (K\mathbf{b})^i$, and from Sublemma C.1, we have $|\partial f_1| < K_1$. Hence $|\partial f_{k+1}| < K\mathbf{b} + K_1$, which, for $\mathbf{b}$ sufficiently small, is $< 2K_1$. The computation for e_{k+1} is identical.

Proof that $(*) \Rightarrow$ (A). First we prove the estimate for $\partial(M^{(i)} f_i)$. Writing

$$\partial\big(M^{(i)} f_i\big) = \sum_{j=1}^{i} M_i \cdots (\partial M_j) \cdots M_1 f_i + M^{(i)} \partial f_i,$$

we obtain easily

$$\left|\partial\left(M^{(i)}f_i\right)\right| \leqslant \sum_{j=1}^{i}\left|M_i\cdots(\partial M_j)\cdots M_1 f_i\right| + \left\|M^{(i)}\right\||\partial f_i| \leqslant iK^i + K^i(2K_1).$$

This estimate is used to estimate $\partial(M^{(i)}e_i)$. Write $\partial(M^{(i)}e_i) = (I) + (II)$ where (I) is its component in the direction of $M^{(i)}f_i$ and (II) is its component orthogonal to $M^{(i)}f_i$. Recall that $\partial\langle M^{(i)}e_i, M^{(i)}f_i\rangle = 0$. We have

$$\left|(I)\right| = \left|\left\langle\partial\left(M^{(i)}e_i\right), \frac{M^{(i)}f_i}{|M^{(i)}f_i|}\right\rangle\right| = \frac{1}{|M^{(i)}f_i|}\left|\left\langle M^{(i)}e_i, \partial\left(M^{(i)}f_i\right)\right\rangle\right| < (K\mathbf{b})^i K^i;$$

$$\left|(II)\right|\left|M^{(i)}f_i\right| = \left|\partial\left(M^{(i)}e_i\right)\wedge M^{(i)}f_i\right| \leqslant \left|\partial\left(M^{(i)}e_i \wedge M^{(i)}f_i\right)\right| + \left|M^{(i)}e_i \wedge \partial\left(M^{(i)}f_i\right)\right|.$$

The first term in the last line is $< (K\mathbf{b})^i$, noting that we have established $|\partial e_i|, |\partial f_i| < 2K_1$; the second term is $< (K\mathbf{b})^i \cdot K^i$. This completes the proof of (A). □

To prove (B), we first compute some quantities associated with the next iterate. Substitute $u = e_i$, $v = f_i$, $M = M^{(i+1)}$ in (C.1)–(C.6). The following is a straightforward computation.

Sublemma C.2. *Assume* $(*)$ *and* (A)*. Then for all* $i \leqslant k$:

(a) $|\partial\lambda_{i+1}| < (K\mathbf{b})^{2(i+1)}$;
(b) $|\partial\beta_{i+1}| < (K\mathbf{b})^i\sqrt{Z_{i+1}}$;
(c) $|\partial\alpha_{i+1}|, |\partial Z_{i+1}| < K^i\sqrt{Z_{i+1}}$.

Proof that $(*)$, (A) ⇒ (B). We work with e_i; the computation for f_i is similar. From (23) we have $\partial(e_{i+1} - e_i) = (III) + (IV) + (V)$ where

$$\left|(III)\right| = \left|\frac{1}{Z_{i+1}}(e_{i+1} - e_i)\partial Z_{i+1}\right| < \frac{K^i\sqrt{Z_{i+1}}}{Z_{i+1}} \cdot (K\mathbf{b})^i < (K\mathbf{b})^i;$$

$$\left|(IV)\right| = \left|\frac{1}{Z_{i+1}}\partial(\beta_{i+1}f_i)\right| < \frac{1}{Z_{i+1}}\left(|\partial\beta_{i+1}| + |\beta_{i+1}||\partial f_i|\right) < (K\mathbf{b})^i;$$

$$\left|(V)\right| = \left|\frac{1}{Z_{i+1}}\partial\left(\frac{\beta_{i+1}^2}{\alpha_{i+1} + Z_{i+1}}e_i\right)\right| \ll (K\mathbf{b})^i.$$

To estimate (III), we have used Sublemma C.2(c) and part (a) of Lemma C.1. To estimate (IV), we have used Sublemma C.2(b), $(*)$ and $|\beta_{i+1}| < (\frac{Kb}{\kappa})^i$. The estimate for (V) is easy. □

This completes the proof of Lemma C.1(b). □

We also need to control the speed of change for the most contracted directions in $\hat{V} \setminus V_f$. Let $q_0(s, \mathbf{a})$ be a curve in $\hat{V} \setminus V_f$ parameterized by a parameter s and assume that

$$\left\|q_0(s, \mathbf{a})\right\|_{C^2} < K.$$

Let $M^{(n)}(s) = D\mathcal{R}_{\mathbf{a}}^n(q_0(s, \mathbf{a}))$, and $e_n(s)$ be the most contracted direction for $M^{(n)}(s)$.

Lemma C.2. *Let q_0 be such that $q_0, \ldots, q_n \in \hat{V} \setminus V_f$. Then for all $1 \leqslant i \leqslant n$,*

(a) $|\partial_s(e_{i+1}(s) - e_i(s))| < (K\mathbf{b})^i$, $|\partial_s M^{(i)}(s)e_n(s)| < (K\mathbf{b})^i$; *and*
(b) $|\partial_s\partial_{\mathbf{a}}(e_{i+1}(s) - e_i(s))| < (K\mathbf{b})^i$, $|\partial_s\partial_{\mathbf{a}} M^{(i)}(s)e_n(s)| < (K\mathbf{b})^i$.

Proof. The proof for Lemma C.2(a) is identical to that of Lemma C.1(b). It suffices to regard all ∂ as ∂_s instead of $\partial_{\mathbf{a}}$. The estimate for the second derivatives is proved by a similar argument. Here we skip the details. $\square$

C.3. Temporary stable curves and the stable manifold

In the rest of this proof we let $\eta = \mathbf{b}^{\frac{1}{10}}$ and denote $\mathcal{A}_\eta = \{(\theta, z) \in \mathcal{A}\colon\ |z| < \eta\}$. We view e_n as a vector field, defined where it makes sense, and let $\gamma_n(s)$ be the integral curve to e_n with $\gamma_n(0) = q_0$.

Lemma C.3. *Let $q_0 = q_m$ be the saddle fixed point of Theorem 3, and $\gamma_n(s)$ be the integral curve to e_n satisfying $\gamma_n(0) = q_0$ in $\mathcal{A}_\eta$. Then, for all $n > 0$,*

(a) $|\mathcal{R}^i_{\mathbf{a}}(q) - \mathcal{R}^i_{\mathbf{a}}(q_0)| < (Kb)^i|s|$ *for all $q = \gamma_n(s)$ and all $i \leqslant n$;*
(b) *$\gamma_n(s)$ is a fully extended vertical curve in $(\hat{V} \setminus V_f) \cap \mathcal{A}_\eta$;*
(c) $|\gamma_{n+1}(s) - \gamma_n(s)|$, $|\partial_{\mathbf{a}}\gamma_{n+1}(s) - \partial_{\mathbf{a}}\gamma_n(s)| < \mathbf{b}^{\frac{n}{10}}$.

Proof. Lemma C.3(a) follows directly from $|M^{(i)}e_n| < (K\mathbf{b})^i$ for all $i \leqslant n$ (Lemma C.1(a)). Denote $\mathcal{R} = \mathcal{R}_{\mathbf{a}}$. Let B_0 be the ball of radius 2η centered at q_0. e_1 is well defined on B_0, and substituting $u = (0, 1)^T$, $v = (1, 0)^T$ into (C.3) we obtain $s(e_1) > K\mathbf{b}^{-1}$. Let $\gamma_1 = \gamma_1(s)$ be the integral curve to e_1 defined for $s \in (-2\eta, 2\eta)$ with $\gamma_1(0) = q_0$.

To construct γ_2, let B_1 be the η^2-neighborhood of γ_1. For $\xi \in B_1$, let ξ' be a point in γ_1 with $|\xi - \xi'| < \eta^2$. Then $|\mathcal{R}(\xi) - \mathcal{R}(q_0)| \leqslant |\mathcal{R}(\xi) - \mathcal{R}(\xi')| + |\mathcal{R}(\xi') - \mathcal{R}(q_0)| \leqslant K\eta^2 + K\mathbf{b}\eta < K\eta^2$. This ensures that e_2 is defined on all of B_1. Let γ_2 be the integral curve to e_2 with $\gamma_2(0) = q_0$. We verify that γ_2 is defined on $(-2\eta, 2\eta)$ and runs alongside γ_1. More precisely, let $t \in [0, 1]$ and

$$q(t, s) = \gamma_1(s) + t\big(\gamma_2(s) - \gamma_1(s)\big).$$

We have

$$\begin{aligned}\left|\frac{d}{ds}\big(\gamma_2(s) - \gamma_1(s)\big)\right| &\leqslant \big|e_2\big(\gamma_2(s)\big) - e_1\big(\gamma_2(s)\big)\big| + \big|e_1\big(\gamma_2(s)\big) - e_1\big(\gamma_1(s)\big)\big| \\ &\leqslant |e_2 - e_1| + |\partial_t e_1|\big|\gamma_2(s) - \gamma_1(s)\big| \\ &\leqslant K\mathbf{b} + K\big|\gamma_2(s) - \gamma_1(s)\big|.\end{aligned}$$

Here we use $|\partial_t e_1| < K$. By Gronwall's inequality, $|\gamma_2(s) - \gamma_1(s)| \leqslant K\mathbf{b}|s|e^{K|s|}$, which is $\ll \eta^2$ for $|s| < 2\eta$. This ensures that γ_2 remains in B_1 and hence is well defined for all $s \in (-2\eta, 2\eta)$.

In general, we inductively construct γ_i by letting B_{i-1} be the η^i-neighborhood of γ_{i-1} in S. Then for all $\xi \in B_{i-1}$, $|\mathcal{R}^j(\xi) - \mathcal{R}^j(q_0)| = |\mathcal{R}^j(\xi) - q_0| < K\eta^j$ for $k < i$. Thus e_i is well defined. Integrating and arguing as above, we obtain γ_i with $|\gamma_i(s) - \gamma_{i-1}(s)| < (K\mathbf{b})^{i-1}|s| \ll \eta^i$ for all s with $|s| < 2\eta$.

To estimate the derivative with respect to $\mathbf{a}$, we let

$$q(t, s) = \gamma_n(s) + t\big(\gamma_{n+1}(s) - \gamma_n(s)\big).$$

We have

$$\begin{aligned}\left|\frac{d}{ds}\partial_{\mathbf{a}}\big(\gamma_{n+1}(s)-\gamma_n(s)\big)\right| &\leqslant \big|\partial_{\mathbf{a}}\big(e_{n+1}\big(\gamma_{n+1}(s)\big)-e_n\big(\gamma_{n+1}(s)\big)\big)\big| \\ &\quad + \big|\partial_{\mathbf{a}}\big(e_n\big(\gamma_{n+1}(s)\big)-e_n\big(\gamma_n(s)\big)\big)\big| \\ &\leqslant |e_{n+1}-e_n| + |\partial_t\partial_{\mathbf{a}}e_n|\big|\gamma_{n+1}(s)-\gamma_n(s)\big| \leqslant K\eta^n.\end{aligned}$$

From this the second item of Lemma C.3(b) follows. □

C.4. The proof of Claim B.6(b)

We are ready to prove Claim B.6(b). First we note that at the point of tangency, $|z| \ll \mathbf{b}^{\frac{1}{10}}$ so it suffices for us to consider $\mathcal{A}_\eta$ in place of $\mathcal{A}$ with $\eta = \mathbf{b}^{\frac{1}{10}}$.

From Lemma C.3(c), we know that $\gamma_n \to \gamma_\infty$ uniformly as $n \to \infty$, and from Lemma C.3(a) we know that γ_∞ is the stable manifold of $q_0 = q_m$, which we write as

$$\theta = \theta_\infty(s, \mathbf{a}), \qquad z = z_\infty(s, \mathbf{a}). \tag{C.15}$$

We also have

$$\big|\partial_{\mathbf{a}}\theta_\infty(s, \mathbf{a}) - \partial_{\mathbf{a}}\theta_1(s, \mathbf{a})\big|, \big|\partial_s\theta_\infty(s, \mathbf{a}) - \partial_s\theta_1(s, \mathbf{a})\big| < 2\mathbf{b}^{\frac{1}{10}} \tag{C.16}$$

from Lemma C.1(b) and Lemma C.2(a).

Write γ_∞ using $\theta = w^s(z, \mathbf{a})$. We first solve $s = s_\infty(z, \mathbf{a})$ from the second item in (C.15), then substitute it to the first item in (C.15) to obtain

$$w^s(z, \mathbf{a}) = \theta_\infty\big(s_\infty(z, \mathbf{a}), a\big).$$

Differentiating on both sides, we obtain

$$\begin{aligned}\partial_{\mathbf{a}} w^s(z, \mathbf{a}) &= \partial_s\theta_\infty(s, \mathbf{a})\partial_{\mathbf{a}} s_\infty(z, \mathbf{a}) + \partial_{\mathbf{a}}\theta_\infty(s, \mathbf{a}) \\ &= -\partial_s\theta_\infty(s, \mathbf{a})\frac{\partial_{\mathbf{a}} z_\infty(s, \mathbf{a})}{\partial_s z_\infty(s, \mathbf{a})} + \partial_{\mathbf{a}}\theta_\infty(s, \mathbf{a}) \\ &= \partial_{\mathbf{a}} w_1^s(z, \mathbf{a}) + \mathcal{O}\big(\mathbf{b}^{\frac{1}{10}}\big)\end{aligned}$$

where $\theta = w_1^s(z, \mathbf{a})$ is the equation for γ_1. To obtain the last estimate we use (C.16) and the fact that $\partial_s z_1(s, \mathbf{a}) > \frac{1}{2}$.

To prove Claim B.6(b), it now suffices for us to confirm that

$$\partial_{\mathbf{a}} w_1^s(z, \mathbf{a}) < \frac{1}{50}.$$

To verify this estimate we observe

$$\left|\partial_{\mathbf{a}}\frac{d}{dz} w_1^s(z, \mathbf{a})\right| = \big|\partial_{\mathbf{a}} s^{-1}(e_1)\big| < K\mathbf{b} \tag{C.17}$$

where $s(e_1)$ is the slope for e_1. This follows from a direct computation using (C.3). From (C.17),

$$\big|\partial_{\mathbf{a}} w_1^s(z, \mathbf{a})\big| < \big|\partial_{\mathbf{a}} w_1^s(z_m, \mathbf{a})\big| + K\mathbf{b}\eta$$

where $\partial_{\mathbf{a}} w_1^s(z_m, \mathbf{a})$ is the value of $\partial_{\mathbf{a}} w_1^s(z, \mathbf{a})$ at $q_0 = q_m$. We now use Claim B.2 for $\partial_{\mathbf{a}} w_1^s(z_m, \mathbf{a})$.

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