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Cohen–Macaulay Approximation and Multiplicity

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INTRODUCTION

Let R be a commutative noetherian Gorenstein local ring with maximal ideal $\mathbf{m}$ and an infinite residue field k. In studying minimal Cohen–Macaulay approximation of a finitely generated R-module C, M. Auslander introduced in [1] the numerical invariant $\delta(C)$ which is defined to be the smallest integer n such that there is an epimorphism $X \coprod R^n \to C$ with X a maximal Cohen–Macaulay module with no free summands. This gives rise to the new numerical invariants $\delta(R/\mathbf{m}^i)$ for $i = 1, 2, ...$, for the ring R. We show there is always an integer n such that $\delta(R/\mathbf{m}^n) \neq 0$ and we define index(R) to be the smallest integer n such that $\delta(R/\mathbf{m}^n) \neq 0$. In general one has that $\text{mult}(R) \geqslant \text{index}(R)$, where mult$(R)$ is the multiplicity of R. Our main objective in this paper is to show that $\text{mult}(R) = \text{index}(R)$ if and only if R is an abstract hypersurface, i.e., the completion $\hat{R}$ of R can be written as $S/(x)$ with S a regular local ring.

Our proof is based on the following general result about hypersurfaces $R = S/(x)$ with $x \neq 0$ in $\mathbf{m}_S^2$. Let $\mathbf{a}$ be an ideal of R and let $\mathbf{A}$ be its preimage in S. Then $\delta(R/\mathbf{a}) = 0$ if and only if $x \in \mathbf{m}_S \mathbf{A}$.

In Section 1 we recall some definitions and results about the theory of minimal Cohen–Macaulay approximations over a Gorenstein local ring, which were initiated by M. Auslander and R.-O. Buchweitz [1, 2]. Section 2 is devoted to studying conditions for an R-module C to have the property $\delta(C) = 0$. In Section 3 we show our main result. We end the paper with a discussion of the structure of the minimal Cohen–Macaulay approximations of finitely generated modules over hypersurfaces and its connection with minimal free resolutions.

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1. CM Approximations and Index of R

Throughout this section we assume that R is a commutative Gorenstein local ring with maximal ideal $\mathfrak{m}$ and residue field k. We denote by $\operatorname{mod} R$ the category of all finitely generated R-modules and by $\mathrm{CM}(R)$ the category of all maximal CM R-modules. For C in $\operatorname{mod} R$, we say C is *stable* if C has no non-zero free summands.

In this section we recall some definitions and results about the theory of minimal Cohen–Macaulay (CM) approximations of a Gorenstein local ring which will be used in the rest of the paper. We then introduce a new numerical invariant $\operatorname{index}(R)$ and show that there is an inequality connecting the index of R with the multiplicity of R and other standard invariants of R.

Let C be in $\operatorname{mod} R$. Then *the stable CM trace* of C is the submodule $\tau(C)$ of C which is generated by the homomorphic images in C of all stable maximal CM modules. Since C is noetherian, it follows that there is a morphism $f: X \to C$ such that X is a stable maximal CM module and $\operatorname{Im} f = \tau(C)$. Therefore we have that $C = \tau(C)$ if and only if C can be *covered* (i.e., there is an epimorphism) by a stable maximal CM module. For any C in $\operatorname{mod} R$, the number $\delta(C)$ is defined to be the minimal number of generators of the factor module $C/\tau(C)$. It is clear that $\delta(C)$ is an invariant of C and $\delta(C) = 0$ if and only if C is a homomorphic image of a stable maximal CM module.

Let C be in $\operatorname{mod} R$. A CM approximation of C is an exact sequence $0 \to A \to B \xrightarrow{f} C \to 0$ with $B \in \mathrm{CM}(R)$ and $\operatorname{pd} A < \infty$. It is called a CM approximation of C because it has the universal mapping property that any morphism $h: X \to C$ with $X \in \mathrm{CM}(R)$ factors through f. A CM approximation of C is called minimal if the morphism $f: B \to C$ is right minimal in the sense that morphism $g: B \to B$ is an isomorphism whenever $f = fg$.

Dually, an exact sequence $0 \to C \xrightarrow{g} A \to B \to 0$ is called a finite projective hull of C if $B \in \mathrm{CM}(R)$ and $\operatorname{pd} A < \infty$. This exact sequence has the universal mapping property that any morphism $h: C \to Y$ with $\operatorname{pd} Y < \infty$ can be extended to A. A finite projective hull of C is called minimal if the morphism $g: C \to A$ is left minimal, i.e., morphism $h: A \to A$ is an isomorphism whenever $g = hg$.

Theorem 1.1 [1, 2]. *Let R be a commutative Gorenstein local ring. Then each C in* $\operatorname{mod} R$ *has a minimal CM approximation and a minimal finite projective hull. They are unique up to isomorphisms.*

Let C be in $\operatorname{mod} R$. Since the minimal CM approximation of C is unique (up to isomorphisms), we usually use the notation $0 \to Y_C \to X_C \xrightarrow{f} C \to 0$ to denote a minimal CM approximation of C. We also denote by $\mu(C)$ the minimal number of generators of C over R.

The following proposition gives a criterion of when a CM approximation is minimal.

PROPOSITION 1.2 [1]. *Let* $0 \to Y \to X \xrightarrow{f} C \to 0$ *be an exact sequence in* mod R *such that* pd $Y < \infty$ *and* $X \in CM(R)$. *Then the following are equivalent*:

(a) *f is right minimal.*

(b) *Given any decomposition* $X = U \coprod F$ *such that U has no free summand and F is a free module, the induced map* $F \to C/f(U)$ *is a projective cover.*

The connections between $\delta(C)$ and the minimal CM approximation and finite projective hull of C are shown in the following proposition.

PROPOSITION 1.3 [1]. *Let C be in* mod R *and let* $0 \to Y_C \to X_C \to C \to 0$ *and* $0 \to C \to Y \to X \to 0$ *be a minimal CM approximation and a finite projective hull of C, respectively. Then the following numbers are the same.*

(a) $\delta(C)$.

(b) *Maximum of the ranks of free summands of* X_C.

(c) $\mu(Y) - \mu(X)$.

In particular we have $\delta(C) = 0$ if and only if $\mu(Y) = \mu(X)$ for any finite projective hull of C. Using this proposition we also have a necessary and sufficient condition for $\delta(C) = 0$ in terms of morphisms $C \to B$ with pd $B < \infty$.

COROLLARY 1.4 [1]. *Let C be in* mod R *and let* $0 \to Y_C \to X_C \to C \to 0$ *be a minimal CM approximation of C. Then the following are equivalent for C.*

(a) $\delta(C) = 0$.

(b) *X_C has no non-zero free summands.*

(c) *Given any morphism* $\varphi: C \to B$ *with* pd $B < \infty$, *we have* Im $\varphi \subset \mathbf{m}B$.

Another useful result about the invariant $\delta(C)$ is the following.

PROPOSITION 1.5 [1]. *Let C be in* mod R. *Then*

(a) pd $C < \infty$ *if and only if X_C is a free module. In particular, if* pd $C < \infty$, *then* $\delta(C) > 0$.

(b) *If* $D \to C \to 0$ *is an epimorphism, then* $\delta(D) \geqslant \delta(C)$.

Now we define the index of R. It is known that a Gorenstein local ring R is a regular local ring if and only if all maximal CM modules over R are free modules. Therefore if R is not a regular local ring, then there exists a non-zero stable maximal CM module over R. It follows that $k = R/\mathbf{m}$ can be covered by a stable maximal CM module, i.e., $\delta(R/\mathbf{m}) = 0$. On the other hand we know that there is an integer n_0 such that $\delta(R/\mathbf{m}^n) \neq 0$ for all $n \geqslant n_0$. For let $\mathbf{a}$ be an ideal of R generated by a regular sequence of R. Then $\operatorname{pd} \mathbf{a} < \infty$ and $\mathbf{a}$ is an $\mathbf{m}$-primary ideal of R. Therefore there exists an integer n_0 such that $\mathbf{m}^{n_0} \subset \mathbf{a}$. Then we have an epimorphism $R/\mathbf{m}^{n_0} \to R/\mathbf{a} \to 0$ and so $\delta(R/\mathbf{m}^{n_0}) \geqslant \delta(R/\mathbf{a}) > 0$ by Proposition 1.5. Also for $n > n_0$, we have a natural epimorphism $R/\mathbf{m}^n \to R/\mathbf{m}^{n_0} \to 0$ which shows that $\delta(R/\mathbf{m}^n) \geqslant \delta(R/\mathbf{m}^{n_0}) > 0$. Therefore the integer

$$\min\{i \mid \delta(R/\mathbf{m}^i) \neq 0\}$$

is well defined which we call the index of R and denote by $\operatorname{index}(R)$. For any Gorenstein local ring R, we have that $\operatorname{index}(R) < \infty$ and $\operatorname{index}(R) = 1$ if and only if R is a regular local ring.

In the rest of this section we assume that R is a Gorenstein local ring having infinite residue field. Later in Section 3 we show that this additional hypothesis can be removed since the minimal CM approximations behave well under faithfully flat local ring extensions.

Let $\operatorname{mult}(R)$ denote the multiplicity of R which is defined in terms of the Hilbert polynomial of R. Since R has an infinite residue field and R is CM, we have that $\operatorname{mult}(R) = \min\{l(R/\mathbf{a})\}$ for ideal $\mathbf{a}$ of R which is generated by a regular sequence of R. We denote by $\operatorname{edim}(R)$ the embedding dimension of R (i.e., the minimal number of generators of $\mathbf{m}$). Then we have the following result.

Proposition 1.6. *Let R be a local Gorenstein ring having infinite residue field. Assume R is not regular. Then*

$$\operatorname{mult}(R) \geqslant \operatorname{index}(R) + (\operatorname{edim} R - \dim R) - 1.$$

In particular, $\operatorname{mult}(R) \geqslant \operatorname{index}(R)$.

Proof. Let $\mathbf{a}$ be an ideal of R generated by a regular sequence of R such that $\operatorname{mult}(R) = l(R/\mathbf{a})$. Let n be the least integer such that $\mathbf{m}^n \subseteq \mathbf{a}$ $(n > 1)$. Then $l(R/\mathbf{a}) = l(R/\mathbf{m}) + l(\mathbf{m}/\mathbf{m}^2 + \mathbf{a}) + \cdots + l(\mathbf{m}^{n-1} + \mathbf{a}/\mathbf{a})$ and by a simple counting argument, we have

$$\begin{aligned} \operatorname{mult}(R) &= l(R/\mathbf{a}) \\ &\geqslant 1 + (\operatorname{edim} R - \dim R) + n - 2. \end{aligned}$$

Since $\mathbf{m}^n \subseteq \mathbf{a}$ and pd $R/\mathbf{a} < \infty$, we know $\text{index}(R) \leqslant n$ and so

$$\text{mult}(R) \geqslant 1 + (\text{edim } R - \dim R) + \text{index}(R) - 2.$$

Hence we obtain the result

$$\text{mult}(R) \geqslant \text{index}(R) + (\text{edim } R - \dim R) - 1.$$

Since R is not regular, we have $\text{edim } R - \dim R \geqslant 1$ and so $\text{mult}(R) \geqslant \text{index}(R)$.

2. A Sufficient Condition for $\delta(C) = 0$

Throughout this section we assume that R is in the form $R = S/(x)$, where S is a local Gorenstein ring with maximal ideal $\mathbf{m}_S$ and $x \in \mathbf{m}_S$ is an S-regular element. For a finitely generated R-module (resp. S-module) C, we denote by $\Omega^i_R(C)$ (resp. $\Omega^i_S(C)$) the ith syzygy of C in the minimal free resolution of C over R (resp. over S). We denote by $\mu(C)$ the minimal number of generators of C as an S-module and by $\text{Ann}_S(C)$ the annihilator of C in S. We write $\bar{C}$ for the reduction C/xC. We set $\delta^i(C) = \delta(\Omega^i_R(C))$ for $i \geqslant 0$ with the convention that $\Omega^0_R(C) = C$.

The purpose of this section is to study conditions for an R-module C to have the property $\delta^i(C) = 0$ for $i \geqslant 0$.

Our first result is a sufficient condition for $\delta^i(C) = 0$ for all $i \geqslant 0$.

Theorem 2.1. *Let C be in* mod R. *If $x \in \mathbf{m}_S \text{Ann}_S(C)$, then $\delta^i(C) = 0$ for all $i \geqslant 0$.*

The proof of Theorem 2.1 requires some preparation. The basic idea of the proof is as follows. Let C be in mod R. Then the syzygies $\Omega^i_R(C)$ are stable maximal CM R-modules for all $i > \dim R$. Therefore if C can be covered by some syzygy $\Omega^i_R(C)$ with $i > \dim R$, then we have $\delta(C) = 0$. Now let C be in mod R and let $0 \to \Omega^1_S(C) \to S^n \to C \to 0$ be a projective cover of C over S. Tensoring it with $S/(x)$ over S, we obtain an exact sequence

$$0 \to \text{Tor}^S_1(C, S/(x)) \to \overline{\Omega^1_S(C)} \to R^n \to C \to 0.$$

Since $C \in \text{mod } R$, we have $\text{Tor}^S_1(C, S/(x)) \simeq C$. That is, we have an exact sequence of R-modules

$$0 \to C \to \overline{\Omega^1_S(C)} \to R^n \to C \to 0$$

from which we obtain the short exact sequence

$$0 \to C \to \overline{\Omega_S^1(C)} \to \Omega_R^1(C) \to 0.$$

Now consider the commutative exact diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Omega_R^2(C) & \longrightarrow & P & \longrightarrow & \Omega_R^1(C) & \longrightarrow & 0 \\ & & \downarrow h & & \downarrow & & \| & & \\ 0 & \longrightarrow & C & \longrightarrow & \overline{\Omega_S^1(C)} & \longrightarrow & \Omega_R^1(C) & \longrightarrow & 0, \end{array}$$

where $P \to \Omega_R^1(C) \to 0$ is a projective cover of $\Omega_R^1(C)$ over R. It follows that h is an epimorphism if and only if $P \to \overline{\Omega_S^1(C)}$ is an epimorphism. It is also easy to see that $P \to \overline{\Omega_S^1(C)}$ is an epimorphism if and only if $\mu(\Omega_S^1(C)) = \mu(\Omega_R^1(C))$. Repeating this procedure, we obtain the following

LEMMA 2.2. *Let C be in* mod R. *If $\mu(\Omega_S^1(\Omega_R^i(C))) = \mu(\Omega_R^{i+1}(C))$ for all $i \geq 0$, then $\delta^i(C) = 0$ for all $i \geq 0$.*

Proof. For any $j \geq 0$, the above argument shows that $\mu(\Omega_S^1(\Omega_R^j(C))) = \mu(\Omega_R^{j+1}(C))$ implies that the jth syzygy $\Omega_R^j(C)$ can be covered by the $(j+2)$th syzygy $\Omega_R^{j+2}(C)$. Since $\mu(\Omega_S^1(\Omega_R^i(C))) = \mu(\Omega_R^{i+1}(C))$ for all $i \geq 0$, using the same argument, we have that $\Omega_R^{j+2}(C)$ can be covered by $\Omega_R^{j+4}(C)$, ... and so on. This shows that for any $j \geq 0$, the jth syzygy $\Omega_R^j(C)$ of C can be covered by $\Omega_R^{j+2i}(C)$ for all $i \geq 0$. Then our desired result follows from the fact that $\Omega_R^{j+2i}(C)$ is a stable maximal CM R-module whenever $j + 2i > \dim R$.

Remark. If C in mod R has the property $\mathrm{pd}_S(C) < \infty$, then the converse of Lemma 2.2 holds for C (Proposition 2.6).

In order to prove Theorem 2.1, we use a construction of Shamash of free resolutions of finitely generated modules over a local ring. We denote by $(\mathscr{S}, d)\colon \cdots \to S_i \xrightarrow{d_i} S_{i-1} \to \cdots \to S_1 \xrightarrow{d_1} S_0 \to C \to 0$ the minimal S-free resolution of C. We also regard $(\mathscr{S}, d)$ as a differential graded module, i.e., $\mathscr{S} = \{S_i\}_{i \in Z}$ is a graded module with $S_i = 0$ for $i < 0$ and $d = \{d_i\}_{i \in Z}$ is an endomorphism of $\mathscr{S}$ of degree -1 ($d_i = 0$ for $i \leq 0$). An endomorphism of $\mathscr{S}$ is a homogeneous homomorphism $\varphi\colon \mathscr{S} \to \mathscr{S}$ such that $d\varphi = \varphi d$. We present Shamash's results in the following proposition.

PROPOSITION 2.3 [5]. *Let $C \in$ mod R. Let $(\mathscr{S}, d)$ be the minimal free resolution of C over S. Then there exists a family of endomorphisms $\{c_n\}_{n \geq 0}$ of $(\mathscr{S}, d)$ (regarded as a differential graded module) such that*

(a) *degree of* $c_n = 2n-1$.

(b) $c_0 = d$.

(c) $c_0 c_1 + c_1 c_0 = x$.

(d) $\sum_{i=0}^{n} c_i c_{n-i} = 0$ *for* $n \geqslant 2$.

In particular, if $x \in \mathbf{m}_S \operatorname{Ann}_S(C)$, *then* $\operatorname{Im} c_n \subset \mathbf{m}_S^n \mathscr{S}$.

As a consequence, we show that $x \in \mathbf{m}_S \operatorname{Ann}_S(C)$ implies that $\mu(\Omega_S^1(C)) = \mu(\Omega_R^1(C))$.

LEMMA 2.4. *Let* $C \in \operatorname{mod} R$. *Then* $\mu(\Omega_S^1(C)) = \mu(\Omega_R^1(C))$ *if and only if there exists a map* $c\colon S_0 \to S_1$ *such that* $d_1 c = f$ *and* $c(S_0) \subset \mathbf{m}_S S_1$, *where* $f\colon S_0 \to S_0$ *is the scalar multiplication by* x.

Proof. Let $\cdots \to S_1 \xrightarrow{d_1} S_0 \to C \to 0$ be the minimal free resolution of C over S. From the above proposition we know that there exists a map $c\colon S_0 \to S_1$ such that $d_1 c = f$. Now consider the following commutative exact diagram

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & S_0 & \longrightarrow & \Omega_S^1(C) & & \\
 & & & & \downarrow{\scriptstyle f} & & \downarrow & & \\
 & & & & S_0 & = & S_0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \Omega_R^1(C) & \longrightarrow & S_0/xS_0 & \longrightarrow & C & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

This diagram gives rise to an exact sequence

$$0 \longrightarrow S_0 \xrightarrow{f} \Omega_S^1(C) \longrightarrow \Omega_R^1(C) \longrightarrow 0.$$

Therefore we have that $\mu(\Omega_S^1(C)) = \mu(\Omega_R^1(C))$ if and only if $\operatorname{Im} f \subset \mathbf{m}_S(C)$. If $c(S_0) \subset \mathbf{m}_S S_1$, then we have $\operatorname{Im} f \subset \mathbf{m}_S \Omega_S^1(C)$ since $\Omega_S^1(C) = \operatorname{Im} d_1$ and $d_1 c = f$. On the other hand, if $\operatorname{Im} f \subset \mathbf{m}_S \Omega_S^1(C)$, we can always choose c such that $c(S_0) \subset \mathbf{m}_S S_1$ and $d_1 c = f$.

We are now ready to prove Theorem 2.1; i.e., for $C \in \operatorname{mod} R$, the condition $x \in \mathbf{m}_S \operatorname{Ann}_S(C)$ implies $\delta^i(C) = 0$ for all $i \geqslant 0$. By Lemma 2.2, it is enough to show that $x \in \mathbf{m}_S \operatorname{Ann}_S(C)$ implies that $\mu(\Omega_S^1(\Omega_R^i(C)) =$

$\mu(\Omega_R^{i+1}(C))$ for all $i \geqslant 0$. By Lemma 2.4 this is equivalent to showing that the condition $x \in \mathbf{m}_S \operatorname{Ann}_S(C)$ implies that the following condition (G) holds for all $i \geqslant 0$.

Condition (G). Let $S^{n_i} \xrightarrow{g_i} S^{m_i} \to \Omega_R^i(C) \to 0$ be a minimal free presentation of $\Omega_R^i(C)$ over S. Then there exists a morphism $\varphi_i \colon S^{m_i} \to S^{n_i}$ such that $\varphi_i(S^{m_i}) \subset \mathbf{m}_S S^{n_i}$ and $g_i \varphi_i = f_i$, where $f_i \colon S^{m_i} \to S^{m_i}$ is the scalar multiplication by x.

We prove (G) by induction on i. The proof involves a construction of the minimal free resolution of $\Omega_R^i(C)$ over S from the minimal free resolution of C over S.

Before we proceed with the proof, we fix some notation. We denote by

$$\mathscr{S} \colon \cdots \longrightarrow S_i \xrightarrow{d_i} S_{i-1} \longrightarrow \cdots \xrightarrow{d_1} S_0 \longrightarrow C \longrightarrow 0$$

the minimal free resolution of C. We regard $\mathscr{S} = \{S_i, d_i\}$ as a differential graded module with $S_i = 0$ for $i < 0$ and $d_i = 0$ for $i \leqslant 0$. We denote by $c_n = \{c_n^k\}_{k \geqslant 0}$ the endomorphism of $\mathscr{S}$ given in Proposition 2.3, where c_n^k denotes the morphism from S_k to S_{k+2n-1}. We also denote by $f = \{f^k\}_{k \geqslant 0}$ the scalar multiplication by x on $\mathscr{S}$. Since $x \in \mathbf{m}_S \operatorname{Ann}_S(C)$, we have $\operatorname{Im} c_n^k \subset \mathbf{m}_S S_{k+2n-1}$ for all k, $n \geqslant 0$ by Proposition 2.3.

Now we show that the condition $x \in \mathbf{m}_S \operatorname{Ann}_S(C)$ implies that Condition (G) is true for all $i \geqslant 0$. For the case $i = 0$, we take $\varphi_0 = c_1^0 \colon S_0 \to S_1$. Then we have $c_1^0(S_0) \subset \mathbf{m}_S S_1$ and $d_1 c_1^0 = f^0$ by Proposition 2.3. So Condition (G) holds for $i = 0$. We suppose Condition (G) is true for some $i \geqslant 0$ and $\Omega_R^i(C)$ has a minimal S-free resolution of the form

$$\cdots \to S_{i+3} \xrightarrow{d_{i+3}} S_{i+2} \xrightarrow{h_i} \coprod_{k \geqslant 0} S_{i+1-2k} \xrightarrow{g_i} \coprod_{k \geqslant 0} S_{i-2k} \to \Omega_R^i(C) \to 0,$$

where g_i is defined as follows:

Assume $v = \sum_{k \geqslant 0} v_{i+1-2k} \in \coprod_{k \geqslant 0} S_{i+1-2k}$. Then

$$g_i(v) := \sum_{k \geqslant 0} \sum_{0 \leqslant j \leqslant k} c_j(v_{i+1-2k}).$$

This definition of g_i is due to Shamash [5]. The morphism h_i is defined by $h_i(v_{i+2}) = d_{i+2}(v_{i+2})$. It is easy to check that the definitions h_i and g_i are satisfied when $i = 0$. We define the morphism $\varphi_i \colon \coprod_{k \geqslant 0} S_{i-2k} \to \coprod_{k \geqslant 0} S_{i+1-2k}$ as follows:

Assume $u = \sum_{k \geqslant 0} u_{i-2k} \in \coprod_{k \geqslant 0} S_{i-2k}$. Then

$$\varphi_i(u) = \sum_{k \geqslant 0} \sum_{0 \leqslant j \leqslant k+1} c_j(u_{i-2k}).$$

Since $x \in \mathbf{m}_S \operatorname{Ann}_S(C)$, we have

$$g_i\left(\coprod_{k\geq 0} S_{i+1-2k}\right)\subset \mathbf{m}_S\left(\coprod_{k\geq 0} S_{i-2k}\right)$$

$$\varphi_i\left(\coprod_{k\geq 0} S_{i-2k}\right)\subset \mathbf{m}_s\left(\coprod_{k\geq 0} S_{i+1-2k}\right).$$

We now show that $g_i\cdot\varphi_i=f_i$. For any $u=\sum_{k\geq 0} u_{i-2k}\in\coprod_{k\geq 0} S_{i-2k}$ we have

$$\begin{aligned} g_i\circ\varphi_i(u) &= g_i\left(\sum_{k\geq 0}\ \sum_{0\leq j\leq k+1} c_j(u_{i-2k})\right)\\ &=\sum_{k\geq 0}\ \sum_{0\leq j\leq k+1}\ \sum_{0\leq l\leq (k+1-j)} c_l c_j(u_{i-2k})\\ &=\sum_{k\geq 0}\ \sum_{0\leq j+l\leq k+1} c_l c_j(u_{i-2k}); \end{aligned}$$

thus

$$\begin{aligned} g_i\circ\varphi_i(u_{i-2k}) &= \sum_{0\leq l+j\leq k+1} c_l c_j(u_{i-2k})\\ &= \sum_{0\leq t\leq k+1}\ \sum_{l+j=t} c_l c_j(u_{i-2k}), \end{aligned}$$

Since $c_0c_0=0$, $c_0c_1+c_1c_0=x$, and $\sum_{o\leq j\leq n} c_jc_{n-j}=0$ for $n>1$ by Proposition 2.3, we get $g_i\circ\ \varphi_i(u_{i-2k})=f_i(u_{i-2k})$ for all $k\geq 1$. So we have $g_i\varphi_i=f_i$.

Now we want to construct the minimal S-free resolution of $\Omega_R^{i+1}(C)$ from the minimal S-free resolution of $\Omega^i_R(C)$ and show that Condition (G) is also satisfied for $i+1$.

For brevity, we write G_i for $\coprod_{k\geq 0} S_{i-2k}$ and F_i for $\coprod_{k\geq 0} S_{i+1-2k}$ and C_i for $\Omega^i_R(C)$. Then the following commutative exact diagram

$$\begin{array}{ccccccccc} & & & & 0 & & 0 & & \\ & & & & \downarrow & & \downarrow & & \\ & & & & G_i & \longrightarrow & \Omega^1_S(C_i) & & \\ & & & & \downarrow{\scriptstyle f_i} & & \downarrow & & \\ & & & & G_i & = & G_i & & \\ & & & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \Omega_R^{i+1}(C) & \longrightarrow & G_i/xG_i & \longrightarrow & C_i & \longrightarrow & 0\\ & & & & \downarrow & & \downarrow & & \\ & & & & 0 & & 0 & & \end{array}$$

gives rise to an exact sequence

$$0 \longrightarrow G_i \xrightarrow{f_i} \Omega_S^1(C_i) \longrightarrow \Omega_R^{i+1}(C) \longrightarrow 0.$$

Since $\mu(\Omega_S^1(C_i)) = \mu(\Omega_R^{i+1}(C))$ by the inductive hypothesis and Lemma 2.4, we have that the composition $F_i \to \Omega_S^1(C_i) \to \Omega_R^{i+1}(C)$ gives a projective cover of $\Omega_R^{i+1}(C)$ over S. Now consider the following commutative exact diagram

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & G_i \coprod \Omega_S^2(C_i) & = & \Omega_S^1(\Omega_R^{i+1}(C)) & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & G_i & \longrightarrow & G_i \coprod F_i & \xrightarrow{\pi} & F_i & \longrightarrow & 0 \\
 & & \| & & \downarrow{\scriptstyle (f_i,\, g_i)} & & \downarrow & & \\
0 & \longrightarrow & G_i & \xrightarrow{f_i} & \Omega_S^1(C_i) & \longrightarrow & \Omega_R^{i+1}(C) & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

From this diagram we obtain the following exact sequence

$$\cdots \xrightarrow{d_{i+4}} S_{i+3} \xrightarrow{h_{i+1}} S_{i+2} \coprod G_i \xrightarrow{\tilde{g}_{i+1}} F_i \coprod G_i \xrightarrow{(f_i,\, g_i)} \Omega_S^1(C_i) \to 0,$$

where $h_{i+1} = [d_{i+3}, 0]$ and $\tilde{g}_{i+1} = \begin{pmatrix} h_i & 0 \\ -\varphi_i & I_{G_i} \end{pmatrix}$. From this exact sequence we obtain the exact sequence

$$\cdots \xrightarrow{d_{i+4}} S_{i+3} \xrightarrow{h_{i+1}} S_{i+2} \coprod G_i \xrightarrow{g_{i+1}} F_i \to \Omega_R^{i+1}(C) \to 0, \qquad (*)$$

where $g_{i+1} = \pi \circ \tilde{g}_{i+1} = \begin{pmatrix} h_i \\ -\varphi_i \end{pmatrix}$. Since all the coefficients of h_{i+1} and g_{i+1} are in $\mathbf{m}_S$, it is easy to see that $(*)$ is a minimal S-free resolution of $\Omega_R^{i+1}(C)$ of the form

$$\cdots \xrightarrow{d_{i+4}} S_{i+3} \xrightarrow{h_{i+1}} \coprod_{k \geqslant 0} S_{i+2-2k} \xrightarrow{g_{i+1}} \coprod_{k \geqslant 0} S_{i+1-2k} \to \Omega_R^{i+1}(C) \to 0,$$

where $g_{i+1} = c_0^{i+2} \coprod \varphi_i$ and $h_{i+1} = d_{i+3}$. It is clear that this minimal S-free resolution of $\Omega_R^{+1}(C)$ has the same form (just change i to $i+1$) as the minimal S-free resolution of $\Omega_R^i(C)$ we started with. We define φ_{i+1}:

$\coprod_{k \geqslant 0} S_{i+1-2k} \to \coprod_{k \geqslant 0} S_{i+2-2k}$ by setting $\varphi_{i+1} = (c_1^{i+1} + c_2^{i-1} + \cdots + c_{k+1}^{i+1-2k} + \cdots) \coprod g_i$, i.e.,

$$\varphi_{i+1}\left(\sum_{k \geqslant 0} u_{i+1-2k}\right) = \sum_{k \geqslant 0} \sum_{0 \leqslant j \leqslant k+1} c_j(u_{i+1-2k})$$

for all $\sum_{k \geqslant 0} u_{i+1-2k} \in \coprod_{k \geqslant 0} S_{i+1-2k}$. Using the same argument as for i, we can check that $g_{i+1}\varphi_{i+1} = f_{i+1}$. Therefore Condition (G) holds for $i+1$. By repeating this procedure, we obtain that (G) holds for all $i \geqslant 0$ and the proof is complete.

Recall that a local ring S is called a (codimension i) deformation of R if there exists a surjective homomorphism $\varphi: S \to R$ such that $\ker \varphi$ is generated by an S-regular sequence (of length i). The deformation is called embedded if $\ker \varphi \subset \mathbf{m}_S^2$. Given a deformation of R, we view every R-module as an S-module via φ. Then Theorem 2.1 can be stated in the following form:

THEOREM 2.5. *Let R be a Gorenstein local ring and $C \in \operatorname{mod} R$. If there is a deformation S of R such that $R = S/(x_1, \ldots, x_r)$ and the induced morphism*

$$(x_1, \ldots, x_r)/\mathbf{m}_S(x_1, \ldots, x_r) \to \operatorname{Ann}_S(C)/\mathbf{m}_S \operatorname{Ann}_S(C)$$

is not a monomorphism, then $\delta^i(C) = 0$ for all $i \geqslant 0$.

Proof. If the above induced map is not a monomorphism, then there is an element $y \in (x_1, \ldots, x_r)$ such that $y \notin \mathbf{m}_S(x_1, \ldots, x_r)$ and $y \in \mathbf{m}_S \operatorname{Ann}_S(C)$. We may assume that $y = x_r$. Then set $S' = S/(x_1, \ldots, x_{r-1})$. We have that S' is also a deformation of R with $R = S'/(\bar{x}_r)$ where $\bar{x}_r$ denotes the image of x_r in S'. Moreover we have $\bar{x}_r \in \mathbf{m}_{S'} \operatorname{Ann}_{S'}(C)$. Therefore by Theorem 2.1, we know $\delta^i(C) = 0$ for all $i \geqslant 0$.

Remark. The converse of Theorem 2.5 is not true, since we know that for any Gorenstein local ring R which is not regular, the residue field $R/\mathbf{m}$ has the property that $\delta^i(R/\mathbf{m}) = 0$ for all $i \geqslant 0$ [1]. Thus if the converse of Theorem 2.5 were true, then every non-regular Gorenstein local ring R would have a non-trivial embedded deformation which we know is not the case. For example, let R be a complete Gorenstein local ring which is not a complete intersection such that $\dim R = \operatorname{edim} R - 3$. Then such R has no non-trivial embedded deformation.

Our next aim is to give a partial converse of Theorem 2.1. Let R be a Gorenstein local ring and $C \in \operatorname{mod} R$. Suppose there is a deformation S of

R such that $R = S/(x)$ and $\mathrm{pd}_S(C) < \infty$. In this case the induced exact sequence

$$0 \to C \to \overline{\Omega_S^1(C)} \to \Omega_R^1(C) \to 0$$

has the property that $\mathrm{pd}_R(\overline{\Omega_S^1(C)}) < \infty$, since $\mathrm{pd}_S(\Omega_S^1(C)) < \infty$ and x is $\Omega_S^1(C)$-regular. Therefore if $\delta(C) = 0$, then we have $\mu(\Omega_S^1(C)) = \mu(\Omega_R^1(C))$ by Corollary 1.3.

To illustrate the above property of the module C, we show that the converse of Lemma 2.2 holds in this case.

PROPOSITION 2.6. *Let R be a Gorenstein local ring and $C \in \mathrm{mod}\, R$. Suppose there is a deformation S of R such that $R = S/(x)$ and $\mathrm{pd}_S(C) < \infty$. Then $\delta^i(C) = 0$ for all $i \geq 0$ if and only if $\mu(\Omega_S^1(\Omega_R^i(C))) = \mu(\Omega_R^{i+1}(C))$ for all $i \geq 0$.*

Proof. We have seen by Lemma 2.2 that $\mu(\Omega_S^1(\Omega_R^i(C))) = \mu(\Omega_R^{i+1}(C))$ for all $i \geq 0$ implies that $\delta^i(C) = 0$ for all $i \geq 0$. Now we show the converse. Let $C \in \mathrm{mod}\, R$. Then we have an exact sequence

$$0 \to C \to \overline{\Omega_S^1(C)} \to \Omega_R^1(C) \to 0.$$

Since $\mathrm{pd}_S(C) < \infty$, we have $\mathrm{pd}_R \overline{\Omega_S^1(C)} < \infty$. Thus $\delta(C) = 0$ implies $\mu(\Omega_S^1(C)) = \mu(\Omega_R^1(C))$ by Corollary 1.3. Since $\mathrm{pd}_S(C) < \infty$ implies $\mathrm{pd}_S(\Omega_R^i(C)) < \infty$ for all $i \geq 0$, replacing C by $\Omega_R^i(C)$ we obtain our desired result.

Remark. In particular, if $R = S/(x)$ is a hypersurface, then for all $C \in \mathrm{mod}\, R$, the hypothesis of Proposition 2.6 is satisfied.

Now let $R = S/(x)$ be a hypersurface, where S is a regular local ring and $x \in \mathbf{m}_S^2$. Then the following result gives a partial converse of Theorem 2.1.

THEOREM 2.7. *Let $R = S/(x)$ be a hypersurface with $x \in \mathbf{m}_S^2$. Let $\mathbf{a}$ be an ideal of R and let $\mathbf{A}$ be its preimage in S. Then $\delta(R/\mathbf{a}) = 0$ if and only if $x \in \mathbf{m}_S \mathbf{A}$. If $\delta(R/\mathbf{a}) = 0$, then $\delta^i(R/\mathbf{a}) = 0$ for all $i \geq 0$.*

Proof. Since $\mathrm{Ann}_S(R/\mathbf{a}) = \mathbf{A}$, Theorem 2.1 asserts that $x \in \mathbf{m}_S \mathbf{A}$ implies $\delta^i(R/\mathbf{a}) = 0$ for all $i \geq 0$. Conversely, if $\delta(R/\mathbf{a}) = 0$, since $\mathrm{pd}_R(\mathbf{A}/x\mathbf{A}) < \infty$ then applying Corollary 1.3 to the exact sequence

$$0 \to R/\mathbf{a} \to \mathbf{A}/x\mathbf{A} \to \mathbf{a} \to 0,$$

we obtain $\mu(\mathbf{A}) = \mu(\mathbf{a})$. To show $x \in \mathbf{m}_S \mathbf{A}$, we consider the following commutative exact diagram

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & S & \longrightarrow & \mathbf{A} & & \\
 & & & & \downarrow{\scriptstyle x} & & \downarrow & & \\
 & & & & S & = & S & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathbf{a} & \longrightarrow & R & \longrightarrow & R/\mathbf{a} & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

This diagram gives rise to an exact sequence

$$0 \longrightarrow S \xrightarrow{x} \mathbf{A} \longrightarrow \mathbf{a} \longrightarrow 0.$$

Since $\mu(\mathbf{A}) = \mu(\mathbf{a})$, we have $x \in \mathbf{m}_S \mathbf{A}$ and this completes the proof.

3. A Characterization of Hypersurfaces

In this section we prove our main result which gives a characterization of hypersurfaces in terms of the relation between mult(R) and index(R). First we show a result on faithfullly flat local ring extensions which enables us to reduce the general case to the case where R is a complete Gorenstein local ring having infinite residue field.

Proposition 3.1. *Let $\varphi\colon R \to S$ be a local homomorphism of rings such that* (i) *R and S are local Gorenstein rings*; (ii) $\dim R = \dim S$; (iii) $\mathbf{m}_R S = \mathbf{m}_S$; (iv) *$S$ is flat over R. Let $C \in \operatorname{mod} R$ and let $0 \to Y \to X \to C \to 0$ be a CM approximation of C over R. Then it is minimal if and only if the exact sequence $0 \to S \otimes_R Y \to S \otimes_R X \to S \otimes_R C \to 0$ is a minimal CM approximation of $S \otimes_R C$ over S.*

Proof. Since $\varphi\colon R \to S$ is a flat extension, we have

$$0 \to S \otimes_R Y \to S \otimes_R X \to S \otimes_R C \to 0$$

is an exact sequence and $\operatorname{pd}_S Y < \infty$. Since $\operatorname{Ext}^i_S(S \otimes_R X, S) \simeq$

$S \otimes_R \mathrm{Ext}^i_R(X, R) = 0$ for $i \geqslant 0$, we have that $S \otimes_R X$ is a maximal CM S-module and so

$$0 \to S \otimes_R Y \to S \otimes_R X \to S \otimes_R C \to 0$$

is a CM approximation of $S \otimes_R C$ over S. We now show it is also minimal.

Suppose $X = U \coprod F$ such that U has no free summands and F is a free R-module. Since U has no free summands over R if and only if the natural inclusion $\mathbf{m}_R \to R$ induces an isomorphism $\mathrm{Hom}_R(U, \mathbf{m}_R) \xrightarrow{\sim} \mathrm{Hom}_R(U, R)$, tensoring with S over R, we obtain an isomorphism $\mathrm{Hom}_S(S \otimes_R U, \mathbf{m}_S) \xrightarrow{\sim} \mathrm{Hom}_S(S \otimes_R U, S)$ since $\mathbf{m}_R S = \mathbf{m}_S$. Therefore $S \otimes_R U$ has no free summands over S. Put $M = \mathrm{coker}(U \to C)$. Then $S \otimes_R M = \mathrm{coker}(S \otimes_R U \to S \otimes_R C)$. Since $\mathbf{m}_R S = \mathbf{m}S$, we have that $F \to M \to 0$ is a projective cover over R if and only if $S \otimes_R F \to S \otimes_R M \to 0$ is a projective cover of $S \otimes_R M$ over S. By Proposition 1.2 this shows that

$$0 \to S \otimes_R Y \to S \otimes_R X \to S \otimes_R C \to 0$$

is a minimal CM approximation of $S \otimes_R C$ over S.

An immediate consequence of Proposition 3.1 is the following.

COROLLARY 3.2. *Hypotheses as in Proposition* 3.1. *Let* $C \in \mathrm{mod}\, R$. *Then* $\delta_R(C) = \delta_S(S \otimes_R C)$.

For a Gorenstein local ring R, let $k = R/\mathbf{m}$ denote the residue field of R. If k is infinite, we set $\tilde{R} = \hat{R}$, the $\mathbf{m}$-adic completion of R. If k is finite, we set $\tilde{R} = (R[z]_{(\mathbf{m}R)[z]})^{\wedge}$, where z is an indeterminate over R. Our main result is the following theorem.

THEOREM 3.3. *The following are equivalent for a Gorenstein local ring R.*

(a) $\mathrm{index}(R) = \mathrm{mult}(R)$.

(b) *$\tilde{R}$ is a hypersurface.*

Proof. It is easy to see that $\tilde{R}$ is a ring extension of R satisfying the hypotheses of Proposition 3.1. Since $\mathbf{m}_R \tilde{R} = \mathbf{m}_{\tilde{R}}$, we have $\tilde{R} \otimes_R R/\mathbf{m}^i_R = \tilde{R}/\mathbf{m}^i_{\tilde{R}}$. Then we have $\mathrm{index}(R) = \mathrm{index}(\tilde{R})$ by Corollary 3.2. Since $\dim R = \dim \tilde{R}$ and $\tilde{R} \otimes_R (\mathbf{m}^i_R/\mathbf{m}^{i+1}_R) = \mathbf{m}^i_{\tilde{R}}/\mathbf{m}^{i+1}_{\tilde{R}}$, we also have $\mathrm{mult}(R) = \mathrm{mult}(\tilde{R})$ and $\mathrm{edim}\, R = \mathrm{edim}\, \tilde{R}$. Therefore we have the following inequality by Proposition 1.6:

$$\mathrm{mult}(\tilde{R}) - \mathrm{index}(\tilde{R}) \geqslant (\mathrm{edim}\, \tilde{R} - \dim \tilde{R}) - 1 \geqslant 0.$$

(a) $\Rightarrow$ (b). By the above comments, we have that $\mathrm{mult}(R) = \mathrm{index}\, R$ implies $\mathrm{mult}(\tilde{R}) = \mathrm{index}(\tilde{R})$. Thus the above formula shows $\mathrm{edim}\, \tilde{R} = \dim \tilde{R} + 1$. By Cohen's structure theorem, we have $\tilde{R} = S/I$ where S is a regular local ring whose dimension is dim $\tilde{R}+1$. Since R is Gorenstein, we have that I is a principal ideal and hence $\tilde{R}$ is a hypersurface.

(b) $\Rightarrow$ (a). If $\tilde{R}=S/(x)$ is a hypersurface, then we have $\operatorname{mult}(\tilde{R})=\max\{i \mid x\in \mathbf{m}_S^i\}$. Theorem 2.7 shows that $\delta(\tilde{R}/\mathbf{m}_{\tilde{R}}^i)=0$ if and only if $x\in m_S^{i+1}$. It follows that $\operatorname{index}(\tilde{R})=\operatorname{mult}(\tilde{R})$. Since $\operatorname{mult}(R)=\operatorname{mult}(\tilde{R})$ and $\operatorname{index}(R)=\operatorname{index}(\tilde{R})$, we obtain our result.

In the rest of this section we consider the structure of CM approximations of modules over a hypersurface. Let $R=S/(x)$ be a hypersurface with S a regular local ring and $x\in\mathbf{m}_S^2$. Our next result is that for any $C\in\operatorname{mod} R$, we have $X_C\simeq\Omega_R^n(C)\coprod F$ for some $n\geqslant\dim R$ and free module F.

LEMMA 3.4. *Let $R=S/(x)$ be a hypersurface and $C\in\operatorname{mod} R$. Then for every $i\geqslant 0$, there exists an exact sequence of the form*

$$0\to Y_i\to\Omega_R^{2i}(C)\coprod F_i\to C\to 0$$

such that F_i is a free R-module and pd $Y_i<\infty$.

Proof. We prove this by induction on i. When $i=1$, consider the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \Omega_R^2(C) & \longrightarrow & P & \longrightarrow & \Omega_R^1(C) & \longrightarrow & 0\\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & C & \longrightarrow & \overline{\Omega_S^1(C)} & \longrightarrow & \Omega_R^1(C) & \longrightarrow & 0,
\end{array}$$

where $P\to\Omega_R^1(C)$ is a projective cover of $\Omega_R^1(C)$ over R. It is easy to see that P is also a part of the projective cover of $\overline{\Omega_S^1(C)}$ over R. Let $F_1\coprod P\to\overline{\Omega_S^1(C)}\to 0$ be a projective cover of $\overline{\Omega_S^1(C)}$ over R. We obtain the following commutative exact diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & Y_1 & = & \overline{\Omega_S^2(C)} & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & \Omega_R^2(C)\coprod F_1 & \longrightarrow & P\coprod F_1 & \longrightarrow & \Omega_R^1(C) & \longrightarrow & 0\\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & C & \longrightarrow & \overline{\Omega_S^1(C)} & \longrightarrow & \Omega_R^1(C) & \longrightarrow & 0\\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

Since S is a regular local ring, we have $\mathrm{pd}_R(\overline{\Omega^i_S(C)}) < \infty$ for all $i \geqslant 1$. Therefore $\mathrm{pd}_R\, Y_1 < \infty$ and the left column gives the result for $i = 1$.

Suppose the Lemma is true for all $j < i$. Then there is an exact sequence

$$0 \to Y_{i-1} \to \Omega_R^{2i-2}(C) \coprod F_{i-1} \to C \to 0$$

with F_{i-1} a free R-module and $\mathrm{pd}_R\, Y_{i-1} < \infty$. Applying the result of the case $i = 1$ to $\Omega_R^{2i-2}(C)$, we obtain an exact sequence of the form

$$0 \to Y \to G \coprod \Omega_R^{2i}(C) \to \Omega_R^{2i-2}(C) \to 0$$

with $\mathrm{pd}_R\, Y < \infty$ and G a free R-module. Then we have the following commutative exact diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & Y & = & Y & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & Y_i & \longrightarrow & F_{i-1} \coprod G \coprod \Omega_R^{2i}(C) & \longrightarrow & C & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & Y_{i-1} & \longrightarrow & F_{i-1} \coprod \Omega_R^{2i-2}(C) & \longrightarrow & C & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

Since $\mathrm{pd}_R\, Y_i < \infty$, the middle row gives our desired exact sequence and finishes the proof.

As an easy consequence of Lemma 3.4, we have the following.

PROPOSITION 3.5. *Let $R = S/(x)$ be a hypersurface and $C \in \mathrm{mod}\, R$. Then $X_C \simeq \Omega_R^n(C) \coprod F$ for some $n \geqslant \dim R$* and free module F.

This also gives us a characterization of hypersurfaces.

PROPOSITION 3.6. *Let R be a Gorenstein local ring. Then the following are equivalent*:

(a) *$\hat{R}$ is a hypersurface.*

(b) *$X_{R/\mathbf{m}} \simeq \Omega_R^n(R/\mathbf{m})$ for some $n \geqslant \dim R$.*

Proof. (a) ⇒ (b). Proposition 3.5 shows that $X_{R/\mathbf{m}} \simeq \Omega^n_R(R/\mathbf{m}) \coprod F$ for some $n \geqslant 0$, where F is a free R-module. Since we assume that R is not regular, we have $\delta(R/\mathbf{m}) = 0$ and so $F = 0$. Since $X_{R/\mathbf{m}} \in \mathrm{CM}(R)$, we have $n \geqslant \dim R$.

(b) ⇒ (a). Since $\mathrm{Ext}^i_R(R/\mathbf{m},\ ) \xrightarrow{\sim} \mathrm{Ext}^i_R(X_{R/\mathbf{m}},\ )$ for all $i > \dim R$, we have $\Omega^{i-1}_R(R/\mathbf{m}) \simeq \Omega^{i-1}_R(X_{R/\mathbf{m}})$ for all $i > \dim R + 1$. Then the fact that $X_{R/\mathbf{m}} = \Omega^n_R(R/\mathbf{m})$ for some $n \geqslant \dim R$ shows that the minimal free resolution of $R/\mathbf{m}$ over R is eventually periodic. In particular the ranks of free modules in the minimal free resolution of $R/\mathbf{m}$ are bounded. J. Herzog has shown in [4] that in this case, $\hat{R}$ is a hypersurface. This completes the proof.

In [3], L. L. Avramov studied the periodic property of a module in terms of its virtual projective dimension. Let R be a complete local ring having infinite residue field. Then the virtual projective dimension of a finitely generated R-module C is defined to be

$$\mathrm{vpd}_R\, C = \min\{\mathrm{pd}_Q\, C \mid Q \text{ is a deformation of } R\}.$$

One of the results on $\mathrm{vpd}_R\, C$ [3, Theorem 4.4] is that if $\mathrm{vpd}_R\, C < \infty$ and the ranks of free modules in the minimal free resolution of C are bounded, then there exists a local ring S such that $R = S/(x)$ for some S-regular element x and $\mathrm{pd}_S(C) < \infty$. Combining this result and Proposition 3.5, we obtain the following.

PROPOSITION 3.7. *Let R be a complete local Gorenstein ring with infinite residue field. Let $C \in \mathrm{mod}\, R$ such that $\mathrm{vpd}_R\, C < \infty$. Then the following are equivalent.*

(a) *$X_C \simeq \Omega^n_R(C) \coprod F$ for some $n \geqslant \dim R$ and free module F.*

(b) *There exists a local ring S such that $R = S/(x)$ for some S-regular element x and $\mathrm{pd}_S(C) < \infty$.*

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