Cores for Feller semigroups with an invariant measure

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Abstract

Let \( A = \sum_{i,j=1}^{N} a_{ij}(x) D_{ij} + \sum_{i=1}^{N} b_{i}(x) D_{i} \) be an elliptic differential operator with unbounded coefficients on \( \mathbb{R}^{N} \) and assume that the associated Feller semigroup \((T(t))_{t \geq 0}\) has an invariant measure \( \mu \). Then \((T(t))_{t \geq 0}\) extends to a strongly continuous semigroup \((T_{p}(t))_{t \geq 0}\) on \( L^{p}(\mu) = L^{p}(\mathbb{R}^{N}, \mu) \) for every \( 1 \leq p < \infty \). We prove that, under mild conditions on the coefficients of \( A \), the space of test functions \( C_{c}^{\infty}(\mathbb{R}^{N}) \) is a core for the generator \((A_{p}, D_{p})\) of \((T_{p}(t))_{t \geq 0}\) in \( L^{p}(\mu) \) for \( 1 \leq p < \infty \).

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Second order elliptic operators with unbounded coefficients on \( \mathbb{R}^{N} \) of type

\[
A = \sum_{i,j=1}^{N} a_{ij}(x) D_{ij} + \sum_{i=1}^{N} b_{i}(x) D_{i} \tag{0.1}
\]

have received a lot of attention in the last years, mainly because of their connections with the study of Markov processes. In particular, the existence of an invariant measure \( \mu \) for the...
associated Feller semigroup \((T(t))_{t \geq 0}\) is a relevant problem for the study of the asymptotic behaviour of the solutions of the parabolic equation \(u_t = Au\) (see, e.g., [9,18]). When an invariant measure exists, the semigroup extends to a strongly continuous semigroup on \(L^p(\mathbb{R}^N, \mu)\) for every \(1 \leq p < \infty\). In general there is not an explicit representation of the domain of the generator \((A_p, D_p)\) of \((T(t))_{t \geq 0}\) in \(L^p(\mathbb{R}^N, \mu)\). For this reason it is interesting to determine “nice” cores for \((A_p, D_p)\). This problem is closely related with uniqueness of semigroups generated by diffusion operators (see [10] and the references quoted therein).

In this paper we provide a criterion, based on the existence of a Lyapunov function, in order that the space of test functions is a core for \((A_p, D_p)\) and we present several examples.

The article is organized as follows. In the first section we state the main theorem (Theorem 1.1), after collecting some known results about elliptic differential operators on \(\mathbb{R}^N\). The section also includes several examples. The second section is splitted into two subsections which deal with results of generation in the space \(C_0(\mathbb{R}^N)\) of continuous functions on \(\mathbb{R}^N\) vanishing at infinity and in weighted spaces. These results are needed to prove Theorem 1.1 but they can be of independent interest. The key result is a criterion in order that the space of test functions is a core for an elliptic differential operator on \(C_0(\mathbb{R}^N)\). The proof of this criterion is based on a result of local regularity for distributional solutions of elliptic differential equations and some localization techniques. Finally, Section 3 is devoted to the proof of the main result.

**Notation.** For \(x \in \mathbb{R}^N\), let \(|x|\) denote its euclidean norm and let \(B_\rho := \{ x \in \mathbb{R}^N \mid |x| < \rho \}\) for \(\rho > 0\). For every \(m \in \mathbb{N}_0\), \(C^m(\mathbb{R}^N)\) stands for the space of \(m\)-times continuously differentiable real-valued functions on \(\mathbb{R}^N\), \(C^m_p(\mathbb{R}^N)\) for its subspace of \(m\)-times differentiable real-valued functions on \(\mathbb{R}^N\) that are continuous and bounded with their derivatives up to the order \(m\), and \(C^m_c(\mathbb{R}^N)\) for the space of \(m\)-times continuously differentiable functions with compact support. Finally, we write \(C^\infty_c(\mathbb{R}^N)\) for the space of all infinitely differentiable real-valued functions on \(\mathbb{R}^N\) having compact support. Let \(C_0(\mathbb{R}^N)\) be the space of real-valued continuous functions on \(\mathbb{R}^N\) vanishing at infinity.

For \(k \geq 0\) and \(0 < \alpha < 1\), \(C^{k+\alpha}_{\text{loc}}(\mathbb{R}^N)\) denotes the space of \(k\)-times differentiable real-valued functions on \(\mathbb{R}^N\), with the derivatives of order \(k\) Hölder continuous with exponent \(\alpha\) in every compact subset of \(\mathbb{R}^N\). If \(p \geq 1\) and \(k \in \mathbb{N}\), \(W^{k,p}_{\text{loc}}(\mathbb{R}^N)\) is the space of all real-valued measurable functions \(u\) on \(\mathbb{R}^N\) such that, for every \(\phi \in C^\infty_c(\mathbb{R}^N)\), \(\phi u\) belongs to the Sobolev space \(W^{k,p}(\mathbb{R}^N)\) of measurable functions on \(\mathbb{R}^N\) with \(p\)-summable weak derivatives up to order \(k\).

**1. The main results**

Let \(A\) be the second order elliptic partial differential operator defined by

\[
Au(x) := \sum_{i,j=1}^{N} a_{ij}(x) D_{ij} u(x) + \sum_{i=1}^{N} b_{i}(x) D_{i} u(x), \quad x \in \mathbb{R}^N, \tag{1.1}
\]
where, for all \( i, j = 1, \ldots, N \), \( a_{ij}, b_i \in C^\alpha_{\text{loc}}(\mathbb{R}^N) \) for some \( 0 < \alpha < 1 \), \( a_{ij} = a_{ji} \), and the ellipticity condition

\[
\sum_{i,j=1}^N a_{ij}(x)\xi_i \xi_j \geq \nu(x)|\xi|^2, \quad x, \xi \in \mathbb{R}^N,
\]

holds with \( \inf_K \nu > 0 \) for every compact \( K \subset \mathbb{R}^N \).

In the sequel, we will set \( b(x) := (b_1(x), \ldots, b_N(x)) \) and \( a(x) := (a_{ij}(x))_{i,j=1}^N \) for every \( x \in \mathbb{R}^N \).

We now recall some results on the differential operator \( A \) given in (1.1) from the survey paper [14] (for more details see also the references therein).

In [14] the authors presented the construction of a semigroup \((T(t))_{t \geq 0}\) of positive contractions on \( C_b(\mathbb{R}^N) \) which gives, for positive \( f \in C_b(\mathbb{R}^N) \), the minimal solution among all positive solutions of the parabolic problem

\[
\begin{aligned}
D_t u(t, x) &= A u(t, x), \quad t > 0, \ x \in \mathbb{R}^N, \\
 u(0, x) &= f(x), \quad x \in \mathbb{R}^N.
\end{aligned}
\]

The semigroup \((T(t))_{t \geq 0}\) is irreducible, that is \( T(t)f(x) > 0 \) for every \( t > 0, x \in \mathbb{R}^N \) whenever \( f \geq 0, f \neq 0 \). Moreover \((T(t))_{t \geq 0}\) has the strong Feller property, i.e., \( T(t)f \in C_b(\mathbb{R}^N) \) for every bounded Borel function \( f \) (see [14]). We point out that \((T(t))_{t \geq 0}\) is not strongly continuous in \( C_b(\mathbb{R}^N) \), but \( T(t)f \to f \) as \( t \to 0 \), uniformly on compact subsets of \( \mathbb{R}^N \). This is a typical behaviour for semigroups associated with elliptic operators with unbounded coefficients.

On the other hand, the generator of \((T(t))_{t \geq 0}\) can be defined through the Laplace transform of the semigroup, following, e.g., the approach of [12] (see also [6,17]). Indeed, for \( \lambda > 0, f \in C_b(\mathbb{R}^N) \), the bounded operators on \( C_b(\mathbb{R}^N) \) defined by

\[
R(\lambda) f(x) = \int_0^\infty e^{-\lambda t} T(t) f(x) \, dt, \quad x \in \mathbb{R}^N,
\]

satisfy the resolvent identity and every operator \( R(\lambda) \) is injective. Thus there exists a unique operator \((\widehat{A}, \widehat{D})\) such that \( R(\lambda) = R(\lambda, \widehat{A}) \). The operator \( \widehat{A} \) is called the generator of \((T(t))_{t \geq 0}\). It holds the following direct description of \((\widehat{A}, \widehat{D})\) (whose proof can be found in [12] or in [17]):

\[
\widehat{A} u := \tau_c - \lim_{t \to 0^+} \frac{T(t)u - u}{t},
\]

\[
\widehat{D} := \left\{ u \in C_b(\mathbb{R}^N) \mid \sup_{t \in [0,1]} \left\| T(t)u - u \right\|_\infty < +\infty, \ \exists \tau_c - \lim_{t \to 0^+} \frac{T(t)u - u}{t} \in C_b(\mathbb{R}^N) \right\},
\]

(1.2)
where \( \tau_c \) denotes the topology of uniform convergence on compact subsets of \( \mathbb{R}^N \). In general \( \hat{D} \) is not \( \| \cdot \|_\infty \)-dense in \( C_b(\mathbb{R}^N) \), but for every \( f \in C_b(\mathbb{R}^N) \) there exists a \( \| \cdot \|_\infty \)-bounded sequence \( (f_n)_n \) in \( \hat{D} \) such that \( f_n \xrightarrow{\tau_c} f \). Moreover, \( (\hat{A}, \hat{D}) \) is closed with respect to this type of convergence (see, e.g., [12]).

The connection between \( A \) and \( \hat{A} \) is the following. By [14, Proposition 3.5, Sections 4, 5], it holds that \( D_d(A) \subseteq \hat{D} \subseteq D_{\text{max}}(A) \) and \( \hat{A}u = Au \) for every \( u \in \hat{D} \), where \( D_{\text{max}}(A) \) is the maximal domain of the operator \( A \) in \( C_b(\mathbb{R}^N) \), that is

\[
D_{\text{max}}(A) := \left\{ u \in \bigcap_{1 < p < \infty} W^{2,p}_{\text{loc}}(\mathbb{R}^N) \cap C_b(\mathbb{R}^N) \mid Au \in C_b(\mathbb{R}^N) \right\},
\]

while \( D_d(A) := D_{\text{max}}(A) \cap C_0(\mathbb{R}^N) \) is the Dirichlet domain of \( A \). In [14, Theorems 3.7 and 3.12] sufficient conditions are given in order that \( \hat{D} = D_{\text{max}}(A) \) or \( \hat{D} = D_d(A) \).

By [14, Proposition 4.3 and Corollary 5.4] the maximal subspace of \( C_b(\mathbb{R}^N) \) of strong continuity for the semigroup \( (T(t))_{t \geq 0} \) is given by

\[
X = \left\{ f \in C_b(\mathbb{R}^N) : \| T(t) f - f \|_\infty \to 0 \text{ as } t \to 0 \right\}.
\]

In particular, \( X \) is \( T(t) \)-invariant, contains \( C_0(\mathbb{R}^N) \) and coincides with the closure of \( \hat{D} \) in \( C_b(\mathbb{R}^N) \). We stress that, in general, \( (T(t))_{t \geq 0} \) does not preserve \( C_0(\mathbb{R}^N) \).

A probability measure \( \mu \) defined on the Borel subsets of \( \mathbb{R}^N \) is called an invariant measure for the semigroup \( (T(t))_{t \geq 0} \) if for every \( f \in C_b(\mathbb{R}^N) \) and \( t \geq 0 \),

\[
\int_{\mathbb{R}^N} T(t)f \, d\mu = \int_{\mathbb{R}^N} f \, d\mu.
\]

An immediate consequence of the existence of an invariant measure is that \( T(t)1 = 1 \) for every \( t \geq 0 \) and hence \( \lambda - A \) is injective on \( D_{\text{max}}(A) \) and the generator of \( (T(t))_{t \geq 0} \) is \( (A, D_{\text{max}}(A)) \) (see [14, Propositions 5.1 and 5.9]). Since \( (T(t))_{t \geq 0} \) is irreducible and has the strong Feller property, in the case that an invariant measure exists, by [9, Theorem 4.2.1] it is unique and is also absolutely continuous with respect to the Lebesgue measure. Moreover, it holds that a probability measure \( \mu \) is invariant for \( (T(t))_{t \geq 0} \) if and only if

\[
\int_{\mathbb{R}^N} A\varphi \, d\mu = 0
\]

for every \( \varphi \in D_{\text{max}}(A) \).

For more information about invariant measures we refer to [9,18].

We recall that, if \( (T(t))_{t \geq 0} \) has an invariant measure \( \mu \), then \( (T(t))_{t \geq 0} \) extends to a strongly continuous positive semigroup of contractions \( (T_p(t))_{t \geq 0} \) on \( L^p(\mu) = L^p(\mathbb{R}^N, d\mu) \) for every \( 1 \leq p < +\infty \). Its generator \( (A_p, D_p) \) is an extension of \( (A, D_{\text{max}}(A)) \) as one can deduce from the description of \( D_{\text{max}}(A) \) given in (1.2), by using
dominated convergence. Clearly, $D_{\text{max}}(A)$ is invariant with respect to $(T_p(t))_{t \geq 0}$ and then it is a core for $(A_p, D_p)$ in $L^p(\mu)$ for every $1 \leq p < +\infty$.

We are now able to state the main result, whose proof relies heavily on the results of Sections 2.1 and 2.2 and will be given after them.

**Theorem 1.1.** Assume that there exists a strictly positive function $V \in C^2(\mathbb{R}^N)$ such that $\lim_{|x| \to +\infty} V(x) = +\infty$, $AV \leq \lambda V$ for some $\lambda > 0$, and $(a \nabla V) \cdot \nabla V \leq KV^2$ for some $K > 0$. Assume also that there exists an invariant measure $\mu$ on $\mathbb{R}^N$ such that $V \in L^p(\mu)$ for some $1 \leq p < +\infty$. Then $C^\infty_c(\mathbb{R}^N)$ is a core for $(A_q, D_q)$ in $L^q(\mu)$ for all $1 \leq q \leq p$.

**Remark 1.1.** Theorem 1.1 and the examples given below should be compared with the results about regular diffusion operators on $\mathbb{R}^N$ in [10, Chapter 2, c]), where growth conditions on the coefficients of the differential operators are given in order that $C^\infty_c(\mathbb{R}^N)$ is a core for $(A_p, D_p)$ with $p > 1$. However, the application of Eberle’s conditions seems to require the explicit representation of the invariant measure, while our assumptions can be verified also when the invariant measure is not explicitly represented since several conditions in order that $V \in L^p(\mu)$ are already known (see, e.g., the proof of Proposition 1.1 and the references quoted therein).

**Example 1.1.** Let us consider the Ornstein–Uhlenbeck operator

$$A = \sum_{i,j=1}^N q_{ij} D_{ij} + \sum_{i,j=1}^N b_{ij} x_j D_i, \quad x \in \mathbb{R}^N,$$

where $Q = (q_{ij})$ is a real, symmetric and nonnegative matrix and $B = (b_{ij})$ is a nonzero real matrix. The associated Feller semigroup $(T(t))_{t \geq 0}$ has the following explicit representation due to Kolmogorov:

$$(T(t)f)(x) = \frac{1}{(4\pi t)^{N/2}(\det Q_t)^{1/2}} \int_{\mathbb{R}^N} e^{-\langle Q_t^{-1}y, y \rangle / 4} f(e^t B x - y) \, dy,$$

where $Q_t = \int_0^t e^{sB} Q e^{sB^*} \, ds$ and $B^*$ denotes the adjoint matrix of $B$. We refer to [7] for more details.

Assume that $\det Q_t > 0$ and $\sigma(B) \subset \mathbb{C}^-$ (where $\sigma(B)$ denotes the spectrum of $B$). Under these conditions, in [8, Section 11.2.3] it was proved that $(T(t))_{t \geq 0}$ has an invariant measure $\mu$ given by $d\mu(x) = \rho(x) \, dx$, where

$$\rho(x) = \frac{1}{(4\pi)^{N/2}(\det Q_{\infty})^{1/2}} e^{-\langle Q_{\infty}^{-1}x, x \rangle / 4}$$

and $Q_{\infty} = \int_0^\infty e^{sB} Q e^{sB^*} \, ds$.

Taking $V(x) = 1 + |x|^2$ for every $x \in \mathbb{R}^N$, it is easily seen that the assumptions of Theorem 1.1 are satisfied with $V \in L^p(\mu)$ for every $1 \leq p < \infty$. Then we can conclude that $C^\infty_c(\mathbb{R}^N)$ is a core for $(A_p, D_p)$ in $L^p(\mu)$ for every $1 \leq p < \infty$. 

In the following proposition, we present useful conditions in order to apply our main result in other cases.

**Proposition 1.1.** Assume that, for all \(i, j = 1, \ldots, N\), \(a_{ij} \in C^1(\mathbb{R}^N)\) and \(b_i\) is locally Lipschitz continuous on \(\mathbb{R}^N\), and that there exists a strictly positive function \(\alpha \in C^1([0, +\infty[)\) satisfying the following properties:

1. \((a(x)x) \cdot x \leq \alpha(|x|)|x|^2\) for all \(x \in \mathbb{R}^N\),
2. \(\frac{1}{\sqrt{\alpha}} \notin L^1([0, +\infty[)\),
3. \[
\limsup_{|x| \to +\infty} \left\{ p - \frac{1}{2} \frac{\alpha'(|x|)}{\alpha^{3/2}(|x|)} \frac{(a(x)x) \cdot x}{|x|^2} + \frac{1}{\sqrt{\alpha(|x|)}} \left( \sum_{i=1}^{N} a_{ii}(x) + \sum_{i=1}^{N} b_i(x)x_i \right) \right\} < 0
\]
   for some \(p \geq 1\).

Then \((A, D_{\text{max}}(A))\) generates a semigroup \((T(t))_{t \geq 0}\) of positive contractions on \(C_b(\mathbb{R}^N)\), \((T(t))_{t \geq 0}\) has a unique invariant measure \(\mu\) and \(C_c^\infty(\mathbb{R}^N)\) is a core for \((A_p, D_p)\) in \(L^p(\mu)\).

**Proof.** Set

\[
V(x) = \exp \left( \int_0^{|x|} \frac{1}{\sqrt{\alpha(r)}} \, dr \right) \quad \text{for all } x \in \mathbb{R}^N.
\]

Then \(V \in C^2(\mathbb{R}^N)\) and \(\lim_{|x| \to +\infty} V(x) = +\infty\) by (2). Moreover, for every \(1 \leq q \leq p\) and \(x \in \mathbb{R}^N\),

\[
D_i(V^q)(x) = q \frac{V^q(x)}{\sqrt{\alpha(|x|)}} \frac{x_i}{|x|}
\]

\[
D_{ij}(V^q)(x) = q^2 \frac{x_i x_j}{|x|^2 \alpha(|x|)} V^q(x) - q \frac{V^q(x)}{2} \frac{\alpha'(|x|)}{\alpha^{3/2}(|x|)} \frac{x_i x_j}{|x|^2} + q \frac{V^q(x)}{\sqrt{\alpha(|x|)}} \left( \delta_{ij} \frac{x_i x_j}{|x|} - \frac{x_i x_j}{|x|^3} \right).
\]

Therefore, for every \(1 \leq q \leq p\) and \(x \in \mathbb{R}^N\), \((a(x) \nabla V^q(x)) \cdot \nabla V^q(x) \leq q^2 V^{2q}(x)\) and

\[
AV^q(x) = q V^q(x) \frac{(a(x)x) \cdot x}{|x|^2} \left[ \frac{q}{\alpha(|x|)} - \frac{1}{2} \frac{\alpha'(|x|)}{\alpha^{3/2}(|x|)} \right] + \frac{q V^q(x)}{\sqrt{\alpha(|x|)}} \sum_{i=1}^{N} \frac{a_{ii}(x)}{|x|} + \frac{q V^q(x)}{\sqrt{\alpha(|x|)}} \sum_{i=1}^{N} \frac{b_i(x)x_i}{|x|}
\]

\[
\leq q V^q(x) \left[ q - \frac{1}{2} \frac{\alpha'(|x|)}{\alpha^{3/2}(|x|)} \frac{(a(x)x) \cdot x}{|x|^2} \right]
\]
\[
+ \frac{1}{\sqrt{\alpha(|x|)|x|}} \left( \sum_{i=1}^{N} a_{ii}(x) + \sum_{i=1}^{N} b_{i}(x)x_{i} \right).
\]

By (3) it follows that \( \lim_{|x| \to +\infty} AV^q(x) = -\infty \); hence, by [14, Theorem 3.7 and Proposition 5.1] \((A, D_{\text{max}}(A))\) generates a semigroup \((T(t))_{t \geq 0}\) of positive contractions on \(C_b(\mathbb{R}^N)\) and by [11, Chapter III, Theorem 5.1] (see also [3,4]) \((T(t))_{t \geq 0}\) has a unique invariant measure \(\mu\) such that \(AV^q \in L^1(\mu)\) for all \(1 \leq q \leq p\). Moreover, \(V^q(x) \leq C_q|AV^q(x)|\) for large \(|x|\), thereby implying that \(V \in L^p(\mu)\). By Theorem 1.1, \(C^\infty_c(\mathbb{R}^N)\) is a core for \((A_p, D_p)\) in \(L^p(\mu)\).

**Remark 1.2.** Condition (2) in the previous proposition can be somehow justified in view of the following considerations. Assume that \(V(x) = V(|x|)\) for all \(x \in \mathbb{R}^N\), i.e., \(V\) is radial, and \((a \nabla V) \cdot \nabla V \leq KV^2\) for some \(K > 0\). Put \(0 < \lambda(r) := \inf_{|x|=r} \lambda(x)\) for all \(r \geq 0\), where \(\lambda(x)\) is the minimal eigenvalue of the matrix \(a(x)\); then

\[
\lambda(r) \left| \nabla V(x) \right|^2 \leq (a(x) \nabla V(x)) \cdot \nabla V(x) \leq KV^2(x)
\]

for all \(|x| = r\) and \(r > 0\). Consequently, it holds that

\[
\frac{|\nabla V(x)|^2}{V^2(x)} \leq \frac{K}{\lambda(r)} \quad \text{or equivalently} \quad \frac{|\nabla V(x)|}{V(x)} \leq \frac{\sqrt{K}}{\sqrt{\lambda(r)}}
\]

for all \(|x| = r\) and \(r > 0\); therefore \(V(r) \leq e^{\int_{0}^{r} (\sqrt{K}/\sqrt{\lambda(s)}) ds}\) for large \(r\). Thus \(\lim_{|x| \to +\infty} V(x) = +\infty\) only if \(\lambda^{-1/2} \notin L^1([0, +\infty[)\).

**Remark 1.3.** If \(\alpha\) is increasing, then condition (3) is verified if

\[
\limsup_{|x| \to +\infty} \left[ p + \frac{1}{\sqrt{\alpha(|x|)|x|}} \left( \sum_{i=1}^{N} a_{ii}(x) + \sum_{i=1}^{N} b_{i}(x)x_{i} \right) \right] < 0
\]

for some \(p \geq 1\). On the other hand, if \(\alpha\) is decreasing, then, by taking (1) into account, condition (3) is satisfied if

\[
\limsup_{|x| \to +\infty} \left[ p - \frac{1}{2} \frac{\alpha'(|x|)}{\sqrt{\alpha(|x|)|x|}} + \frac{1}{\sqrt{\alpha(|x|)|x|}} \left( \sum_{i=1}^{N} a_{ii}(x) + \sum_{i=1}^{N} b_{i}(x)x_{i} \right) \right] < 0
\]

for some \(p \geq 1\).

As an immediate consequence, in the case of a diffusion part with bounded coefficients we get:
Corollary 1.1. Assume that, for all \( i, j = 1, \ldots, N \), \( a_{ij} \in C_b(\mathbb{R}^N) \cap C^1(\mathbb{R}^N) \) and \( b_i \) is locally Lipschitz continuous on \( \mathbb{R}^N \), and that for some \( p \geq 1 \)
\[
\limsup_{|x| \to +\infty} \frac{\sum_{i=1}^{N} b_i(x) x_i}{|x|} < -N^2 p \sqrt{\|a\|_{\infty}}
\]
where \( \|a\|_{\infty} = \max_{i,j=1,\ldots,N} \|a_{ij}\|_{\infty} \). Then \((A, D_{\max}(A))\) generates a semigroup \((T(t))_{t \geq 0}\) of positive contractions on \( C_b(\mathbb{R}^N) \), \((T(t))_{t \geq 0}\) has a unique invariant measure \( \mu \) and \( C^\infty_c(\mathbb{R}^N) \) is a core for \((A_p, D_p)\) in \( L^p(\mu) \).

We now collect further examples which can be discussed using the above results.

Example 1.2. Let us consider operators \( A \) as in (1.1) verifying the condition
\[
\sum_{i,j=1}^{N} a_{ij}(x)x_i x_j \leq C(1 + |x|^2)^{\beta} |x|^2
\]
for large \(|x|\) and for some \( C > 0 \). Set \( \alpha(r) = C(1 + r^2)^{\beta} \) for all \( r \in [0, +\infty[ \), where \( \beta \in \mathbb{R} \). Then \( \alpha \in C^1([0, +\infty[) \) is a positive function such that \( \frac{1}{\sqrt{\alpha}} \notin L^1([0, +\infty[) \) if \( \beta < 1 \) and \( \alpha'(r) = 2C\beta r(1 + r^2)^{\beta-1} \) for all \( r \in [0, +\infty[ \). If \( 0 \leq \beta \leq 1 \) and
\[
\limsup_{|x| \to +\infty} \left[ p + \frac{1}{\sqrt{C}|x|(1 + |x|^2)^{\beta/2}} \left( \sum_{i=1}^{N} a_{ii}(x) + \sum_{i=1}^{N} b_i(x) x_i \right) \right] < 0,
\]
then by Proposition 1.1 and Remark 1.3, \((A, D_{\max}(A))\) generates a semigroup \((T(t))_{t \geq 0}\) of positive contractions on \( C_b(\mathbb{R}^N) \), \((T(t))_{t \geq 0}\) has a unique invariant measure \( \mu \) and \( C^\infty_c(\mathbb{R}^N) \) is a core for \((A_p, D_p)\) in \( L^p(\mu) \).

If \( \beta < 0 \), then \( \alpha \) is decreasing and \( \lim_{|x| \to +\infty} \frac{\alpha'(|x|)}{\sqrt{\alpha(|x|)}} = 0 \). In this case, if
\[
\limsup_{|x| \to +\infty} \left[ p + \frac{1}{\sqrt{C}|x|(1 + |x|^2)^{\beta/2}} \left( \sum_{i=1}^{N} a_{ii}(x) + \sum_{i=1}^{N} b_i(x) x_i \right) \right] < 0,
\]
then by Proposition 1.1 \((A, D_{\max}(A))\) generates a semigroup \((T(t))_{t \geq 0}\) of positive contractions on \( C_b(\mathbb{R}^N) \), \((T(t))_{t \geq 0}\) has a unique invariant measure \( \mu \) and \( C^\infty_c(\mathbb{R}^N) \) is a core for \((A_p, D_p)\) in \( L^p(\mu) \).

Example 1.3. Let us consider operators \( A \) as in (1.1) verifying the condition
\[
\sum_{i,j=1}^{N} a_{ij}(x)x_i x_j \leq C|x|^4 \log^2 |x|
\]
for large \(|x|\) and for some \( C > 0 \).
Set \( \alpha(r) = Cr^2 \log^2 r \) for all \( r \geq r_0 > 1 \). Then \( \alpha \in C^1([r_0, +\infty[) \) is an increasing positive function such that \( \frac{1}{\sqrt{\alpha}} \notin L^1([r_0, +\infty[) \). If

\[
\limsup_{|x| \to +\infty} \left[ p + \frac{1}{\sqrt{C}|x|^2 \log |x|} \left( \sum_{i=1}^N a_{ii}(x) + \sum_{i=1}^N b_i(x)x_i \right) \right] < 0
\]

for some \( p \geq 1 \), then by Proposition 1.1 \((A, D_{\max}(A))\) generates a semigroup \((T(t))_{t \geq 0}\) of positive contractions on \( C_b(\mathbb{R}^N) \), \((T(t))_{t \geq 0}\) has a unique invariant measure \( \mu \) and \( C^\infty_c(\mathbb{R}^N) \) is a core for \((A_p, D_p)\) in \( L^p(\mu) \).

2. Auxiliary results

2.1. Second order differential operators in \( C_0(\mathbb{R}^N) \)

We consider second order elliptic partial differential operators

\[
Au(x) := \sum_{i,j=1}^N a_{ij}(x)D_{ij}u(x) + \sum_{i=1}^N b_i(x)D_iu(x) + c(x)u(x), \quad x \in \mathbb{R}^N, \quad (2.1)
\]

where, for all \( i, j = 1, \ldots, N \), \( a_{ij}, b_i, c \in C(\mathbb{R}^N) \), \( a_{ij} = a_{ji} \).

Set \( b(x) := (b_1(x), \ldots, b_N(x)) \) and \( a(x) := (a_{ij}(x))_{i,j=1,\ldots,N} \) for every \( x \in \mathbb{R}^N \). We also assume that \( c_0 := \sup_{\mathbb{R}^N} c(x) < \infty \) and that \( (a(x)\xi) \cdot \xi \geq \nu(x)|\xi|^2 \) for every \( x, \xi \in \mathbb{R}^N \), with \( \inf_K \nu > 0 \) for every compact subset \( K \subset \mathbb{R}^N \).

Along this section, we extend some results of [14] that were formulated for the case \( c(x) = 0 \) and we give suitable conditions on the coefficients of the differential operator \( A \) as in \((2.1)\) to ensure the space of test function \( C^\infty_c(\mathbb{R}^N) \) is a core for \((A, D_0(A))\) in \( C_0(\mathbb{R}^N) \), where \( D_0(A) \) is the Dirichlet domain in \( C_0(\mathbb{R}^N) \) defined by

\[
D_0(A) := \left\{ u \in \bigcap_{1 < p < \infty} W^{2,p}_{\text{loc}}(\mathbb{R}^N) \cap C_0(\mathbb{R}^N) \mid Au \in C_0(\mathbb{R}^N) \right\}.
\]

In order to do this, we first recall the following general result by Bony, Courrége and Priouret for reader’s convenience (see [5, Chapter 0, pp. 386–388]).

**Theorem 2.1.** Let \( A : D(A) \subseteq C_0(\mathbb{R}^N) \to C_0(\mathbb{R}^N) \) be a densely defined linear operator and \( \omega \in \mathbb{R} \). The following statements are equivalent:

(a) \((A, D(A))\) is closable and its closure generates a strongly continuous positive semigroup \((S(t))_{t \geq 0}\) on \( C_0(\mathbb{R}^N) \) satisfying \( \|S(t)\| \leq e^{\omega t} \) for every \( t \geq 0 \);

(b) (i) there exists \( \lambda > \omega \) such that \((\lambda - A)(D(A))\) is dense in \( C_0(\mathbb{R}^N) \),

(ii) if \( u \in D(A) \) and \( \sup_{\mathbb{R}^N} u(x) > 0 \), then for every \( x_0 \in \mathbb{R}^N \) such that \( u(x_0) = \sup_{\mathbb{R}^N} u(x) \) we have \( Au(x_0) \leq \omega u(x_0) \).
On the other hand, for second order elliptic partial differential operators as in (2.1) the following results hold. The proof of the first one is given by proceeding as in [14, Appendix] with some minor changes (see also [13, Theorem 3.1.10]).

Lemma 2.1. Let \( u \in \bigcap_{p>1} W^{2,p}_{\text{loc}}(\mathbb{R}^N) \) such that \( Au \in C(\mathbb{R}^N) \). If \( u \) has a relative positive maximum (respectively negative minimum) at the point \( x_0 \), then \( Au(x_0) \leq c_0 u(x_0) \) (respectively \( Au(x_0) \geq c_0 u(x_0) \)).

In particular, if \( \lambda > c_0 \) and \( Au \leq \lambda u \) (respectively \( Au \geq \lambda u \)), then \( u \) cannot have a strictly negative minimum (respectively strictly positive maximum).

With the proof of [14, Theorem 3.4], one also gets:

Theorem 2.2. For every \( f \in C_b(\mathbb{R}^N) \) and for every \( \lambda > c_0 \) there exists \( u \in \bigcap_{p>1} W^{2,p}_{\text{loc}}(\mathbb{R}^N) \cap C_b(\mathbb{R}^N) \) such that \( \lambda u - Au = f \) and \( \|u\|_{\infty} \leq \|f\|_{\infty} / (\lambda - c_0) \). Moreover, if \( f \geq 0 \) then \( u \geq 0 \).

We point out that, in general, \( u \) does not belong to \( C_0(\mathbb{R}^N) \) even if \( f \in C_0(\mathbb{R}^N) \). The next result gives us a sufficient condition in order that \( u \in C_0(\mathbb{R}^N) \), i.e., that \( \lambda - A \) is bijective from \( D_0(A) \) onto \( C_0(\mathbb{R}^N) \).

Theorem 2.3. Assume that there exists a positive function \( W \in C^2(\mathbb{R}^N \setminus B_\rho) \) for some \( \rho > 0 \), such that \( W(x) \to 0 \) as \( |x| \to +\infty \) and \( AW \leq \lambda W \) for some \( \lambda \geq c_0 \). Then \( (A,D_0(A)) \) generates a strongly continuous positive semigroup \( (S(t))_{t \geq 0} \) on \( C_0(\mathbb{R}^N) \) such that \( \|S(t)\| \leq e^{c_0 t} \) for every \( t \geq 0 \).

Proof. By proceeding as in the proof of [14, Theorem 3.17], we get that for every \( \lambda > c_0 \) and for every \( f \in C_c(\mathbb{R}^N) \) there exists \( u \in D_0(A) \) such that \( \lambda u - Au = f \). Hence \( (\lambda - A)(D_0(A)) \) is dense in \( C_0(\mathbb{R}^N) \). Moreover, analogously to [14, Lemma 3.1], we prove that \( (A,D_0(A)) \) is closed in \( C_0(\mathbb{R}^N) \) and, by Lemma 2.1, it satisfies condition (b)(ii) in Theorem 2.1. The assertion follows from Theorem 2.1. \( \square \)

In particular, under mild conditions on the coefficients of the differential operator \( A \) given in (2.1), the space of test functions \( C^\infty_c(\mathbb{R}^N) \) is a core for \( (A,D_0(A)) \) in \( C_0(\mathbb{R}^N) \). To prove this, we need the following lemma.

Lemma 2.2. Assume that, for all \( i, j = 1, \ldots, N \), \( a_{ij} \in C^1(\mathbb{R}^N) \) and \( b_i \) is locally Lipschitz continuous on \( \mathbb{R}^N \). If \( \mu \) is a locally finite regular Borel measure on \( \mathbb{R}^N \) such that, for some \( \lambda > 0 \) and for every \( \varphi \in C^\infty_c(\mathbb{R}^N) \),

\[
\int_{\mathbb{R}^N} (\lambda - A) \varphi \, d\mu = 0,
\]

then \( \mu = u \, dx \) with \( u \in W^{2,p}_{\text{loc}}(\mathbb{R}^N) \) for every \( p > N \). In particular, \( u \in C^1(\mathbb{R}^N) \).
Proof. By [2, Corollary 2.10], $\mu = u \, dx$ with $u \in W^{1,p}_{0}(\mathbb{R}^{N})$ for every $p > N$. Fix $\rho > 0$ and choose $\eta \in C_{c}^{\infty}(\mathbb{R}^{N})$ such that $\eta(x) = 1$ if $|x| \leq \rho$ and $\eta(x) = 0$ if $|x| \geq 2\rho$. Moreover choose $\gamma \in C_{c}^{\infty}(\mathbb{R}^{N})$ such that $\gamma(x) = 0$ if $|x| \leq 2\rho$ and $\gamma(x) = 1$ if $|x| \geq 3\rho$. Then $\gamma \eta = 0$ on $\mathbb{R}^{N}$.

Consider the elliptic differential operator $A_{1} := (1 - \gamma)A + \gamma \Delta$. It is easily proved that $A_{1}$ is locally uniformly elliptic, the coefficients of the diffusion part of $A_{1}$ are in $C_{b}^{1}(\mathbb{R}^{N})$, while the coefficients of the drift part are in $W^{1,\infty}(\mathbb{R}^{N})$. Set $v = u\eta \in W_{0}^{1,p}(\mathbb{R}^{N})$ and observe that, for every $\varphi \in C_{c}^{\infty}(\mathbb{R}^{N})$,
\[
v A_{1} \varphi = u\eta (1 - \gamma)A \varphi = u\eta A \varphi = u\left[ A(\eta \varphi) - \varphi (A\eta - c\eta) \right] - 2a(a\nabla \eta) \cdot \nabla \varphi
\]
\[
eq u\left[ A(\eta \varphi) - \varphi (A\eta - c\eta) \right] - 2(a\nabla \eta) \cdot \nabla (u\varphi) + 2\varphi (a\nabla \eta) \cdot \nabla u;
\]

hence,
\[
\int_{\mathbb{R}^{N}} v A_{1} \varphi \, dx = \int_{\mathbb{R}^{N}} \{ u\left[ A(\eta \varphi) - \varphi (A\eta - c\eta) \right] + 2u \varphi \text{div}(a\nabla \eta) - 2(u\varphi) \cdot \nabla u \varphi \} \, dx
\]
\[
= \int_{\mathbb{R}^{N}} \left[ \lambda u\eta - (A\eta - c\eta)u + 2u \text{div}(a\nabla \eta) - 2(u\varphi) \cdot \nabla u \varphi \right] \varphi \, dx.
\]

By the assumptions on the coefficients of $A$ and since $u \in W_{0}^{1,p}(\mathbb{R}^{N})$, the function $g := \lambda u\eta - (A\eta - c\eta)u + 2u \text{div}(a\nabla \eta) - 2(u\varphi) \cdot \nabla u \varphi \in L^{p}_{0}(\mathbb{R}^{N})$. Then, by [16, Lemma A2], $v \in W_{0}^{2,p}(\mathbb{R}^{N})$ for every $p > N$ and by the arbitrariness of $\rho$, we get the assertion. 

Theorem 2.4. Assume that, for all $i, j = 1, \ldots, N$, $a_{ij} \in C_{1}(\mathbb{R}^{N})$ and $b_{i}$ is locally Lipschitz continuous on $\mathbb{R}^{N}$. If there exists a strictly positive function $W \in C^{2}(\mathbb{R}^{N})$ such that $\lim_{|x| \rightarrow +\infty} W(x) = 0$, $AW \leq \lambda_{0}W$ for some $\lambda_{0} > c_{0}$ and $(a\nabla W) \cdot \nabla W \leq K W^{2}$ for some $K > 0$, then $C_{c}^{\infty}(\mathbb{R}^{N})$ is a core for $(A, D_{0}(A))$ in $C_{0}(\mathbb{R}^{N})$.

Proof. Observe that the closed operator $\lambda - A : D_{0}(A) \rightarrow C_{0}(\mathbb{R}^{N})$ is bijective for every $\lambda > c_{0} = \sup_{\mathbb{R}^{N}} c(x)$ by Theorem 2.3. Then it is enough to show that $(\lambda - A)(C_{c}^{\infty}(\mathbb{R}^{N}))$ is dense in $C_{0}(\mathbb{R}^{N})$ for some $\lambda > c_{0}$. Fix $\lambda > c_{0}$. Let $\mu \in (C_{0}(\mathbb{R}^{N}), \| \|_{\infty})'$ such that $(\lambda \varphi - A \varphi, \mu) = 0$ for every $\varphi \in C_{c}^{\infty}(\mathbb{R}^{N})$. By the Riesz Representation Theorem, $\mu$ is a finite regular Borel measure on $\mathbb{R}^{N}$. Hence, by Lemma 2.2, $\mu$ has density $u \in C_{1}(\mathbb{R}^{N})$ and since $\mu$ is finite, we get that $u \in L^{1}(\mathbb{R}^{N})$. We will prove that $u = 0$.

Let $\beta_{i} = b_{i} - \sum_{j=1}^{N} D_{j}a_{ij}$ for all $i = 1, \ldots, N$. By integrating by parts, we obtain that, for every $\varphi \in C_{c}^{\infty}(\mathbb{R}^{N})$,
\[
\lambda \int_{\mathbb{R}^{N}} u \varphi \, dx = \int_{\mathbb{R}^{N}} u A \varphi \, dx = \int_{\mathbb{R}^{N}} \left[ -\nabla \varphi \cdot (a\nabla u) + u \beta \cdot \nabla \varphi + cu \varphi \right] \, dx.
\]

By density, (2.2) holds also for every $\varphi \in C_{c}^{1}(\mathbb{R}^{N})$. 


Let $v \in C^1(\mathbb{R}^N)$, $\eta \in C_c^\infty(\mathbb{R}^N)$, with $\eta \geq 0$. By replacing $\phi$ with $\eta v$ in (2.2), we get that
\[
\lambda \int_{\mathbb{R}^N} \eta vu = \int_{\mathbb{R}^N} \left[ -v \nabla \eta (a \nabla u) - \eta \nabla v (a \nabla u) + uv \beta \cdot \nabla \eta + u \eta \beta \cdot \nabla v + cuv \right] dx.
\]
(2.3)

Let $F : \mathbb{R} \rightarrow [-1, 1]$ be an increasing $C^\infty$-function such that $F(s) = 0$ if $|s| \leq 1$, $F(s) = -1$ if $s \leq -2$ and $F(s) = 1$ if $s \geq 2$. For every $n \in \mathbb{N}$ and $x \in \mathbb{R}^N$, set $u_n(x) := F(nu(x))$. Then $u_n \in C^1(\mathbb{R}^N), |u_n| \leq 1$ and $u_n(x) \rightarrow \text{sign} u(x)$ as $n \rightarrow +\infty$ for every $x \in \mathbb{R}^N$. Moreover, $\nabla u_n(x) = nF'(nu(x))\nabla u(x)$ for every $x \in \mathbb{R}^N$ and $n \in \mathbb{N}$. By replacing $v$ with $u_n$ in (2.3) and by observing that $n\eta F'(nu(x))\nabla u(x) \cdot (a \nabla u(x)) \geq 0$ for all $x \in \mathbb{R}^N$, we obtain that, for each $n \in \mathbb{N}$,
\[
\lambda \int_{\mathbb{R}^N} \eta u_n dx = \int_{\mathbb{R}^N} \left[ -u_n \nabla \eta (a \nabla u) - n \eta F'(nu(x)) \nabla u \cdot (a \nabla u) \right] dx
\]
\[+ \int_{\mathbb{R}^N} \left[ u_n \beta \nabla \eta u + n \eta F'(nu(x)) \beta \cdot \nabla u + cu_n u \eta \right] dx \]
\[\leq \int_{\mathbb{R}^N} \left[ -\nabla \eta \cdot (a \nabla (u_n u)) + u \nabla \eta \cdot (a \nabla u_n) \right] dx
\]
\[+ \int_{\mathbb{R}^N} \left[ u_n u \beta \cdot \nabla \eta + n \eta F'(nu(x)) \beta \cdot \nabla u + cu_n u \eta \right] dx \]
\[\leq \int_{\mathbb{R}^N} \left[ u_n \nabla \eta u + n \eta F'(nu(x)) \beta \cdot \nabla u + cu_n u \eta \right] dx.
\]
Since $F'(y) \neq 0$ if and only if $n^{-1} \leq |y| \leq 2n^{-1}$, it holds that $|nF'(nu(x))| |u(x)| \leq 2$ for every $n \in \mathbb{N}$ and $x \in \mathbb{R}^N$. On the other hand, if $u(x) \neq 0$, then there exists $n_0 \in \mathbb{N}$ such that for every $n \geq n_0, |u(x)| > \frac{2}{n}$. Thus $nF'(nu(x)) = 0$ for all $n \geq n_0$ so that, for all $x \in \mathbb{R}^N, nF'(nu(x))u(x) \rightarrow 0$ as $n \rightarrow +\infty$. By passing to the limit and taking into account that the supports of all the involved functions are contained in the support of $\eta$, we get that
\[
\lambda \int_{\mathbb{R}^N} \eta |u| dx \leq \int_{\mathbb{R}^N} |u| \nabla (a \nabla \eta) dx + \int_{\mathbb{R}^N} \left[ \beta \nabla \eta |u| + c |u| \eta \right] dx = \int_{\mathbb{R}^N} |u| A \eta dx. \quad (2.4)
\]
Next, let $\psi : \mathbb{R} \rightarrow [0, 1]$ be a decreasing $C^\infty$-function such that $\psi(s) = 1$ if $s \leq 1$ and $\psi(s) = 0$ if $s \geq 2$ and define $\eta_n(x) := \psi(\frac{1}{nW(x)})$ for every $n \in \mathbb{N}$ and $x \in \mathbb{R}^N$. Then, for
every \( n \in \mathbb{N} \), there exists \( C_n > 0 \) such that \( W(x) \leq \frac{1}{2n} \) if \( |x| \geq C_n \). Hence \( \eta_n(x) = 0 \) if \( |x| \geq C_n \), i.e., \( \eta_n \in C_c^\infty(\mathbb{R}^N) \). Moreover, \( \eta_n \leq 1 \) and \( \lim_{n \to +\infty} \eta_n = 1 \) pointwise.

Standard calculations now give that, for every \( n \in \mathbb{N} \),

\[
A\eta_n = -\frac{AW - cW}{nW^2} \eta'\left(\frac{1}{nW}\right) + (a\nabla W) \times \nabla W \left(\eta'' \frac{1}{nW} \left(\frac{1}{nW}\right) + \frac{2}{nW^3} \eta' \left(\frac{1}{nW}\right)\right) + c\eta_n;
\]

thereby implying that \( \lim_{n \to +\infty} A\eta_n = c \) pointwise. Moreover, by observing that \( \eta' \leq 0 \) and that \( \eta' \left(\frac{1}{nW}\right) \) and \( \eta'' \left(\frac{1}{nW}\right) \) do not vanish only if \( 1 \leq \frac{1}{nW} \leq 2 \), we get

\[
A\eta_n \leq 2(\lambda - c_0) \|\eta'\|_\infty + 4K \left(\|\eta''\|_\infty + \|\eta'\|_\infty\right) + c_0,
\]

for every \( n \in \mathbb{N} \). Thus, by replacing \( \eta \) with \( \eta_n \) in (2.4) and by applying Fatou’s lemma (we recall that \( \lim_{n \to +\infty} \eta_n = 1 \) and \( \lim_{n \to +\infty} A\eta_n = c \) pointwise), we obtain that

\[
\lambda \int |u| \, dx \leq \limsup_{n \to +\infty} \int |A\eta_n| |u| \, dx \leq \int |c(x)| |u| \, dx \leq c_0 \int |u| \, dx.
\]

Since \( \lambda > c_0 \), we can conclude that \( u = 0 \). \( \square \)

2.2. Second order differential operators in the weighted space \( VC_0(\mathbb{R}^N) \)

We now assume that

\[
Au(x) = \sum_{i,j=1}^N a_{ij}(x) D_{ij} u(x) + \sum_{i=1}^N b_i(x) D_i u(x), \quad x \in \mathbb{R}^N, \tag{2.5}
\]

where for all \( i, j = 1, \ldots, N \), \( a_{ij}, \ b_i \in C_\text{loc}^\alpha(\mathbb{R}^N) \) for some \( 0 < \alpha < 1 \), \( a_{ij} = a_{ji} \), and \( (a(x)\xi) \cdot \xi \geq v(x) |\xi|^2 \) with \( \inf_K v > 0 \) for every compact subset \( K \subset \mathbb{R}^N \).

In this section we first give a result of generation of strongly continuous positive semigroups by the realization of the differential operator \( A \) given in (2.5) in the weighted space \( VC_0(\mathbb{R}^N) \) and then we prove that the space of test functions \( C_c^\infty(\mathbb{R}^N) \) is also a core for \( A \), endowed with a suitable domain, in \( VC_0(\mathbb{R}^N) \). The weighted function spaces \( VC_0(\mathbb{R}^N) \) are defined as it follows.

Let \( V \in C(\mathbb{R}^N) \) with \( \inf_{\mathbb{R}^N} V > 0 \) and \( V^{-1} \in C_0(\mathbb{R}^N) \). We define

\[
VC_0(\mathbb{R}^N) := \left\{ f \in C(\mathbb{R}^N) \mid \frac{f}{V} \in C_0(\mathbb{R}^N) \right\}
\]
and endow it with the weighted norm $\|f\|_V := \|f V^{-1}\|_\infty$. Then $VC_0(\mathbb{R}^N)$ is a Banach lattice and the following continuous inclusions hold:

$$C_0(\mathbb{R}^N) \hookrightarrow C_b(\mathbb{R}^N) \hookrightarrow VC_0(\mathbb{R}^N) \hookrightarrow (C(\mathbb{R}^N), \tau_c),$$

where $\tau_c$ is the topology of uniform convergence on compact subsets of $\mathbb{R}^N$. Moreover, it is easily seen that the space $C_c^\infty(\mathbb{R}^N)$ (and therefore also $C_0(\mathbb{R}^N)$ and $C_b(\mathbb{R}^N)$) is dense in $VC_0(\mathbb{R}^N)$.

**Remark 2.1.** Let $(f_n)_n$ be a $\|\cdot\|_\infty$-bounded sequence in $C_b(\mathbb{R}^N)$ and $f \in C_b(\mathbb{R}^N)$. Then $\tau_c - \lim_{n \to +\infty} f_n = f$ if and only if $\lim_{n \to +\infty} \|f_n - f\|_V = 0$. Indeed, since $(f_n)_n$ is $\|\cdot\|_\infty$-bounded,

$$\sup_n \|f_n - f\|_\infty \leq M/V.$$

Hence, for a fixed $\varepsilon > 0$, there exists a compact set $K \subset \mathbb{R}^N$ such that

$$\sup_n \sup_{x \in \mathbb{R}^N \setminus K} \frac{|f_n(x) - f(x)|}{V(x)} < \varepsilon.$$

The assertion follows by observing that $(f_n)_n$ converges to $f$ uniformly on $K$.

The converse is obvious because $\inf_{\mathbb{R}^N} V > 0$.

**Remark 2.2.** Let $A : D(A) \subset VC_0(\mathbb{R}^N) \to VC_0(\mathbb{R}^N)$ be a linear operator and define $A_0 : D(A_0) \subset C_0(\mathbb{R}^N) \to C_0(\mathbb{R}^N)$ by $A_0 u := \frac{1}{V} A(u V)$ for every $u \in D(A_0) = V^{-1} D(A)$. Then, by observing that $C_0(\mathbb{R}^N)$ and $VC_0(\mathbb{R}^N)$ are topologically isomorphic via the map

$$\Phi : C_0(\mathbb{R}^N) \to VC_0(\mathbb{R}^N), \quad f \mapsto f V,$$

it is clear that $(A, D(A))$ generates a strongly continuous positive semigroup $(T(t))_{t \geq 0}$ on $VC_0(\mathbb{R}^N)$ if and only if $(A_0, D(A_0))$ generates a strongly continuous positive semigroup $(S(t))_{t \geq 0}$ on $C_0(\mathbb{R}^N)$. In this case, for every $f \in VC_0(\mathbb{R}^N)$, $T(t) f = S(t)(f V)$.

Let us consider differential operators $A$ as in (2.5) and define

$$D_V(A) := \left\{ u \in \bigcap_{1 < p < +\infty} W^{2,p}_{loc}(\mathbb{R}^N) \cap VC_0(\mathbb{R}^N) \mid Au \in VC_0(\mathbb{R}^N) \right\},$$

where $V \in C(\mathbb{R}^N)$ with $\inf_{\mathbb{R}^N} V > 0$ and $V^{-1} \in C_0(\mathbb{R}^N)$. Then the following result holds.

**Proposition 2.1.** Let $V \in C^2(\mathbb{R}^N)$ be a strictly positive function such that $\lim_{|x| \to +\infty} V(x) = +\infty$ and $AV \leq \lambda V$ for some $\lambda > 0$. Then the differential operator $(A, D_V(A))$ generates a strongly continuous positive semigroup $(T_V(t))_{t \geq 0}$ on $VC_0(\mathbb{R}^N)$ such that
∥TV(t)∥VC0(RN) ≤ eλt and TV(t)f = T(t)f for every t ≥ 0 and for every f ∈ Cb(RN), where (T(t))t⩾0 is the semigroup of positive contractions on Cb(RN) generated by (A, Dmax(A)).

Proof. By [14, Theorem 4.4] and [15, Lemma 3.9], (A, Dmax(A)) generates a semigroup (T(t))t⩾0 of positive contractions on Cb(RN) that can be represented in the form

\[ T(t)f(x) = \int_{\mathbb{R}^N} p(t, x, y)f(y)dy \]

for all t ≥ 0, f ∈ Cb(RN) and x ∈ RN, where p is a positive kernel that satisfies, for every t ≥ 0 and x ∈ RN,

\[ \int_{\mathbb{R}^N} p(t, x, y)V(y)dy \leq e^{\lambda t}V(x). \]

It follows that, for every f ∈ Cb(RN) and x ∈ RN,

\[ |T(t)f(x)| \leq e^{\lambda t}∥f∥_V V(x), \]

that is ∥T(t)f∥V ≤ eλt∥f∥V. Since Cb(RN) is dense in VC0(RN), each operator T(t) can be extended to an operator TV(t) on the whole space VC0(RN) for which ∥TV(t)∥VC0(RN) ≤ eλt. The semigroup property clearly extends to (TV(t))t⩾0 by density.

For every f ∈ Cb(RN), (T(t)f) is ∥·∥∞-bounded and limt→0+ T(t)f = f uniformly on all compact sets of RN, thereby implying that limt→0+ ∥T(t)f − f∥V = 0. Since supT∈[0,T]∥T(t)∥VC0(RN) < +∞ for every T > 0, by density we can again conclude that the semigroup (TV(t))t⩾0 is strongly continuous on VC0(RN).

Let (AV, D(AV)) be the generator of (TV(t))t⩾0 in VC0(RN). By the same previous considerations, we obtain that

\[ \lim_{t \to 0^+} \left\| \frac{T(t)u - u}{t} - Au \right\|_V = 0 \quad \text{for every } u \in D_{\text{max}}(A). \]

This implies that D_{\text{max}}(A) ⊆ D(AV) and AVu = Au for every u ∈ D_{\text{max}}(A). Since for every f ∈ Cb(RN) there exists a ∥·∥∞-bounded sequence (fn)n ∈ D_{\text{max}}(A) such that τc - limn→+∞ fn = f, D_{\text{max}}(A) is clearly ∥·∥V-dense in Cb(RN) and then in VC0(RN). Moreover, it is invariant under the semigroup (T(t))t⩾0, thereby implying that it is a core for (AV, D(AV)).

We now consider the differential operator A0 on C0(RN) defined by A0u := \frac{1}{V}A(Vu) for every u ∈ VD(A) = D0(A0). Then

\[ A0u = \sum_{i,j} a_{ij}D_{ij}u + \frac{1}{V} \sum_{i=1}^N \left( \sum_{j=1}^N 2a_{ij}D_j V + Vb_i \right) D_i u + \frac{AV}{V} u \quad (2.6) \]
and $A_0(\frac{1}{V}) = 0$. By Theorem 2.3 it generates a strongly continuous positive semigroup on $C_0(\mathbb{R}^N)$. From Remark 2.2 it follows that $(A, D_V(A))$ generates a strongly continuous positive semigroup on $VC_0(\mathbb{R}^N)$ and it coincides with $A_V$ on $D_{\text{max}}(A)$.

Let $f \in D(A_V)$. Then there exists a sequence $(f_n)_n$ in $D_{\text{max}}(A)$ such that $f_n \to f$ and $A f_n = A_V f_n \to A_V f$ in $VC_0(\mathbb{R}^N)$. Since $(A, D_V(A))$ is closed, we get that $f \in D_V(A)$ and $A f = A_V f$. Therefore $D(A_V) \subseteq D_V(A)$. Then, since for some $\mu > 0 (\mu - A, D(A_V))$ and $(\mu - A, D(A_V))$ are invertible, we get that $(A, D_V(A)) = (A_V, D(A_V))$. □

We are now able to state and prove the following result.

**Theorem 2.5.** Assume that, for all $i, j = 1, \ldots, N$, $a_{ij} \in C^1(\mathbb{R}^N)$ and $b_i$ is locally Lipschitz continuous on $\mathbb{R}^N$ and there exists a strictly positive function $V \in C^2(\mathbb{R}^N)$ such that $\lim_{|x| \to +\infty} V(x) = +\infty$, $AV \leq \lambda V$ for some $\lambda > 0$ and $(a \nabla V) \cdot \nabla V \leq KV^2$ for some $K > 0$.

Then $C^\infty_c(\mathbb{R}^N)$ is a core for $(A, D_V(A))$.

**Proof.** Observe that all conditions in Theorem 2.4 are satisfied by $A_0 = \frac{1}{V} A(V \cdot)$ (see (2.6)), with in particular $W = \frac{1}{V}$. Then $C^\infty_c(\mathbb{R}^N)$ is a core for $(A_0, D_0(A_0))$ in $C_0(\mathbb{R}^N)$, thereby implying that $VC^\infty_c(\mathbb{R}^N)$ is a core for $(A, D_V(A))$. Finally, a classical argument of regularization by convolutions yields the thesis. □

**Remark 2.3.** Recall that a subspace $D$ of the domain of a linear operator $A : D(A) \subseteq C_b(\mathbb{R}^N) \to C_b(\mathbb{R}^N)$ is said to be a bicore for $(A, D(A))$ if for all $u \in D(A)$ there exists a sequence $(u_n)_n \subseteq D$ such that $(u_n)_n$ and $(A u_n)_n$ are $\| \cdot \|_\infty$-bounded, and $u_n \stackrel{\text{loc}}{\to} u$, $A u_n \stackrel{\text{loc}}{\to} Au$. It is worth noting that, if we consider the operator $A u = u'' - x^3 u'$ on $\mathbb{R}$, then $C^\infty_c(\mathbb{R})$ is not a bicore for $(A, D_{\text{max}}(A))$ (see [1, Theorem 2.1]), but it is a core for $(A, D_V(A))$ where $V(x) = 1 + x^2$.

3. Proof of Theorem 1.1

Let $1 \leq q \leq p$. By the remarks before Theorem 1.1, if $f \in D(A_q)$, then there is a sequence $(f_n)_n \subseteq D(A_q)$ such that

$$\| f_n - f \|_{L^q(\mu)} \to 0 \quad \text{and} \quad \| A f_n - A_q f \|_{L^q(\mu)} \to 0 \quad \text{as} \quad n \to \infty. \quad (3.1)$$

On the other hand, $VC_0(\mathbb{R}^N) \hookrightarrow L^q(\mu)$ continuously as $V \in L^p(\mu) \subseteq L^q(\mu)$ and $C^\infty_c(\mathbb{R}^N)$ is a core for $(A, D_V(A))$ in $C V_0(\mathbb{R}^N)$ by Theorem 2.5. Since $D_{\text{max}}(A) \subseteq D_V(A)$ (see Proposition 2.1), for each $n \in \mathbb{N}$ there is $g_n \in C^\infty_c(\mathbb{R}^N)$ such that $\| f_n - g_n \|_V \leq n^{-1}$ and $\| A f_n - A g_n \|_V \leq n^{-1}$. On the other hand, for each $n \in \mathbb{N}$, it holds

$$\| g_n - f \|_{L^q(\mu)} \leq \| f_n - f \|_{L^q(\mu)} + \| f_n - g_n \|_{L^q(\mu)}$$

$$\leq \| f_n - f \|_{L^q(\mu)} + \| V \|_{L^q(\mu)} \| f_n - g_n \|_V$$

and
\[ \|A g_n - A q f\|_{L^q(\mu)} \leq \|A f_n - A q f\|_{L^q(\mu)} + \|A f_n - A g_n\|_{L^q(\mu)} \]
\[ \leq \|A f_n - A f\|_{L^q(\mu)} + \|V\|_{L^q(\mu)} \|A f_n - A g_n\| V , \]
thus \( \|g_n - f\|_{L^q(\mu)} \to 0 \) and \( \|A g_n - A q f\|_{L^q(\mu)} \to 0 \) as \( n \to \infty \). 

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**References**