Averaging Methods for Differential Equations with Retarded Arguments and a Small Parameter*

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1. Introduction and Notation

In the last few years, there has been an increasing interest in the theory of differential equations with retarded arguments or systems in which the rate of change of a system may depend upon its past history. This is partially due to the fact that such equations arise in a natural manner in certain types of control problems. Much of the recent literature has been devoted to the extension of known results for ordinary differential equations to differential equations with retarded arguments. The present paper is another step in this direction.

More specifically, we shall indicate in what manner a particular form of the method of averaging of Krylov-Bogoliubov-Mitropolski-Diliberto can be extended to differential equations with hereditary dependence.

For ordinary differential equations, this method is well understood by most people who are concerned with either the computational aspects or the qualitative theory of nonlinear oscillations. In the development of this method for retarded systems, the basic difficulty lies in the fact that motions defined by the solutions of the equations cannot be described adequately in a finite dimensional space. The proper setting seems to be in an infinite dimensional space and, in the particular formulation given below, in a Banach space. To the author's knowledge, Krasovskii [10] was the first to exploit such equations in this setting in the extension of Lyapunov's second method.

The extension of the method of averaging to differential equations with retardation relies heavily upon the theory of linear equations with constant coefficients as developed by Shimanov [13, 14] and the author [6]. We will not give the details of this theory of linear systems, but merely apply the

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results to our problem. The results are first stated in general form and then specific examples are discussed in detail to illustrate the application of the results.

The following notation will be used. $R^n$ is the linear space of complex $n$-vectors and for $x$ in $R^n$, $|x|$ is any vector norm. For any given numbers $\alpha$, $\beta$, $\alpha \leq \beta$, $C([\alpha, \beta], R^n)$ will denote the space of continuous functions mapping the interval $\alpha, \beta$ into $R^n$ and for any $\varphi$ in $C([\alpha, \beta], R^n)$,

$$||\varphi|| = \sup_{\alpha \leq \theta \leq \beta} |\varphi(\theta)|.$$  

For any $r \geq 0$, any continuous function $x(u)$ defined on $\sigma - r \leq t \leq \sigma + A$, $A > 0$, $\sigma$ given, we shall let the symbol $x_t$, $t \geq \sigma$, denote the function $x_t(\theta) = x(t + \theta)$, $-r \leq \theta \leq 0$; that is, $x_t$ is in $C([-r, 0], R^n)$ and corresponds to that “segment” of the function $x(u)$ defined by letting $u$ range in the interval $t - r \leq u \leq t$.

Let $F(t, \varphi)$ be a function defined for all $\varphi$ in $C([-r, 0], R^n)$, $||\varphi|| \leq H$, $H > 0$, $t$ in $[0, \infty)$. Let $x(t)$ denote the right hand derivative of a function $x(u)$ at $t$ and consider the functional-differential equation

$$x(t) = F(t, x_t). \tag{1.1}$$

It should be clear that equations of the form (1.1) include the usual differential-difference equations

$$x(t) = f(t, x(t), x(t - r))$$

or equations with any finite number of lags as well as ordinary differential equations ($r = 0$) and even more general equations of the form

$$x(t) = g(t, \int_{-r}^{0} [d\eta(\theta)] x(t + \theta))$$

where $\eta$ is a matrix whose elements are functions of bounded variation.

Let $\sigma \geq 0$ be a given real number, let $\varphi$ be a given continuous function defined on $[\sigma - r, \sigma]$, and let $x_{\sigma}$ designate the function in $C([-r, 0], R^n)$ given by $x_{\sigma}(\theta) = x(\sigma + \theta)$, $-r \leq \theta \leq 0$, $||x_{\sigma}|| \leq H$. A function $x_{\sigma}(\varphi)$ is said to be a solution of (1.1) with initial function $\varphi$ at $\sigma$ if there is an $A > 0$ such that $x_{\sigma}(\varphi)$ is in $C([-r, 0], R^n)$, $||x_{\sigma}(t_0, \varphi)|| \leq H$, for $\sigma \leq t < \sigma + A$, $x_{\sigma}(\sigma, \varphi) = \varphi_{\sigma}$ and $x(\sigma, \varphi)$ satisfies (1.1) for $\sigma \leq t < \sigma + A$.

It is not difficult to prove that $F(t, \varphi)$ continuous in $\varphi$ implies the existence of a solution of (1.1) with initial function $\varphi$ at $\sigma$. Furthermore, if $F(t, \varphi)$ is locally Lipschitzian in $\varphi$, then for any given $\sigma$, $\varphi$, the solution is unique. In the following, we shall always suppose that $F(t, \varphi)$ is continuous in $t$, $\varphi$ and locally Lipschitzian in $\varphi$ in $||\varphi|| \leq H$, $t \geq 0$. 
In any differential system, it seems reasonable to suppose that the state of a system at any given time $t$ should be that part of the system which uniquely defines the behavior of the system for all future time $\tau \geq t$. Consequently, we define our state space for (1.1) to be the space $C([-r, 0], \mathbb{R}^n)$, which in the following we will designate by $C$ except when confusion may arise.

An integral curve or trajectory of system (1.1) which passes through $(\sigma, \varphi_0)$ will be that collection of points in $\mathbb{R} \times C$ given by $\{(t, x_t(\sigma, \varphi)), \ t \geq \sigma\}$, where $x(\sigma, \varphi)$ is the solution of (1.1) with initial function $\varphi$ at $\sigma$. System (1.1) is said to be autonomous if $F(t, \varphi)$ is independent of $t$. In such a case, it is sufficient to always choose $\sigma = 0$ in the definition of a solution since our system (1.1) is invariant under translations in $t$. Also, one can define orbits or paths of autonomous systems (1.1) as the collection of points in $C$ defined by $\{x(\sigma, \varphi), \ t \geq 0\}$ where $x(0, \sigma)$ is the solution of the autonomous system (1.1) with initial value $\varphi$ at 0. It is easy to see that periodic solutions correspond to closed curves in $C$.

The above definition of trajectories and orbits of (1.1) yields a situation which is analogous to ordinary differential equations. However, the reader should realize that the situation here is more complicated. First of all, trajectories in general are only defined to the right of $\sigma$ and the mapping $x_t(\sigma, \varphi)$ taking $C$ into $C$ for each fixed $t \geq 0$, is a smoothing operator if $r > 0$. In fact, for any $\sigma$, the mapping $x_t(\sigma, \varphi), \ t \geq \sigma + r$, takes closed bounded subsets of $C$ into compact subsets of $C$. This shows that $x_t(\sigma, \varphi)$ cannot be a homeomorphism for $r > 0$ even if it is one-to-one. Secondly, the mapping $x_t(\sigma, \varphi)$ need not be one-to-one even when the uniqueness property holds. In fact, for the scalar equation

$$x(t) = x(t - r)(1 - x(t)),$$

the solution $x(0, \varphi)$ corresponding to an initial function $\varphi$ with $\varphi(0) = 1$ is such that $x_t(0, \varphi) = 1$ for $t \geq r$. Therefore, a subset of $C([-r, 0], \mathbb{R})$ which is the translate of a subspace of codimension 1 is such that the corresponding trajectories all coincide after $r$ units of time.

2. A CONVENIENT COORDINATE SYSTEM

In this paper, we are interested in the oscillatory properties of perturbations of linear autonomous functional-differential equations. In ordinary differential equations experience has shown an understanding of oscillations in perturbations of linear equations with constant coefficients is most easily accomplished by the introduction of a coordinate system which exhibits in an explicit manner the behavior of the unperturbed equation on the subspaces which correspond respectively to the eigenvalues with positive real parts,
zero real parts and negative real parts. In this section, we indicate how this same end can be accomplished for hereditary functional-differential equations.

In particular, we consider the general linear autonomous system

\[ \dot{u}(t) = \int_{-r}^{0} [d\eta(\theta)] u(t + \theta) \]  

(2.1)

where \( \eta \) is an \( n \times n \) matrix whose elements are functions of bounded variation on the interval \([-r, 0]\), and the perturbed linear system

\[ \dot{x}(t) = \int_{-r}^{0} [d\eta(\theta)] x(t + \theta) + ef(t, x_t), \]  

(2.2)

where \( f(t, x_t) \) is a continuous function of \( t, x_t \) for \( t \in (-\infty, \infty) \) and \( x_t \) in \( C \), \( \| x_t \| \leq H \).

Our coordinate system for (2.1) and (2.2) is obtained by an application of the general theory of (2.1) in [6, 13, 14]. The characteristic values of (2.1) are the roots of the characteristic equation

\[ \det [\lambda I - \int_{-r}^{0} e^{\lambda \theta} d\eta(\theta)] = 0 \]  

(2.3)

and, to any characteristic value \( \lambda \), there is a solution of (2.1) of the form \( e^{\lambda t} b \) for some constant vector \( b \) and all \( t \) in \((-\infty, \infty)\). To any characteristic value \( \lambda \) of (2.1) of multiplicity \( m(\lambda) \), there are exactly \( m(\lambda) \) linearly independent solutions of (2.1) of the form \( p(t) e^{\lambda t} \) where \( p(t) \) is a polynomial in \( t \). If \( p_1(\lambda, t) e^{\lambda t}, \ldots, p_{m(\lambda)}(\lambda, t) e^{\lambda t}, -\infty < t < \infty \) is a basis for the solutions of this form, then we define functions \( \varphi_j(\lambda) \) in \( C \) by the relations

\[ \varphi_j(\lambda)(\theta) = p_j(\lambda, \theta) e^{\lambda \theta}, -r \leq \theta \leq 0. \]  

If \( \Phi(\lambda) = (\varphi_1(\lambda), \ldots, \varphi_{m(\lambda)}(\lambda)) \), then it follows directly from the differential equation (2.1) that there is a matrix \( B_\lambda \) of order \( m(\lambda) \) whose only characteristic value is \( \lambda \) such that

\[ \Phi(\lambda)(\theta) = \Phi(\lambda)(0) e^{B_\lambda \theta}, \quad -r \leq \theta \leq 0. \]  

(2.4)

Furthermore, if \( u(\varphi), \varphi = \Phi(\lambda) a \) for some vector \( a \), is the solution of (2.1) with initial function \( \varphi \) at \( 0 \), then \( u(\varphi) \) is defined on \((-\infty, \infty)\) and \( u(\varphi) = \Phi e^{B_\lambda t} a, -\infty < t < \infty \); that is, the equation (2.1) on the generalized eigenspace of \( \lambda \) behaves essentially as an ordinary differential equation.

Along with system (2.1), we consider the equation\(^1\)

\[ \psi(s) = -\int_{-r}^{0} [d\eta^T(\theta)] \psi(s - \theta) \]  

(2.5)

\[^1\text{If } A \text{ is a matrix } A^T \text{ denotes the transpose of } A.\]
"adjoint" to (2.1) with respect to the bilinear form

\[(\psi, \varphi) = \psi^T(0) \varphi(0) - \int_{-r}^{0} \int_{-r}^{0} \psi^T(\xi - \theta) [d\eta(\theta)] \varphi(\xi) \, d\xi, \quad (2.6)\]

defined for all \(\psi\) in \(C^* = C([-r, r], R^n), \) \(\varphi\) in \(C = C([-r, 0], R^n).\) The characteristic values of the adjoint equation are the roots of the equation

\[\det [\lambda I - \int_{-r}^{0} e^{\theta \eta} d\eta(\theta)] = 0, \quad (2.7)\]

and to each such root \(\lambda\), there is a solution of (2.5) of the form \(e^{-\lambda t} b\) for some constant vector \(b\) and all \(s\) in \((-\infty, \infty).\) Notice that the solutions of (2.3) and (2.7) are the same. If \(\lambda\) is a characteristic value of (2.5) of multiplicity \(m(\lambda),\) let \(q_j(\lambda, s) e^{-\lambda s}, j = 1, 2, \ldots, m(\lambda),\) be a basis for the solutions of (2.5) of the form \(q(s) e^{-\lambda s}, q(s)\) a polynomial, and define functions \(\psi_j(\lambda)\) in \(C^*\) by the relation \(\psi_j(\lambda)(\theta) = q_j(\lambda, \theta) e^{-\lambda \theta}, 0 \leq \theta \leq r.\) If \(\Psi(\lambda) = (\psi_1, \ldots, \psi_{m(\lambda)})\) and \(\Phi(\lambda)\) is defined as in the previous paragraph, then it is shown in \([6]\) that the matrix \((\Psi(\lambda), \Phi(\lambda)) = (\psi_j(\lambda), \varphi_k(\lambda)), j, k = 1, 2, \ldots, m(\lambda)\) is nonsingular and, therefore, without any loss in generality, can be taken to be the identity.

Suppose now that \(\lambda_1, \ldots, \lambda_k\) are the characteristic values of (2.1) and (2.5) with real parts \(\geq 0.\) Let \(\Phi(\lambda_1), \ldots, \Phi(\lambda_k)\) be the corresponding sets of functions in \(C\) defined above and let \(\Psi(\lambda_1), \ldots, \Psi(\lambda_k)\) be those in \(C^*.\) If we let \(\Phi = (\Phi(\lambda_1), \ldots, \Phi(\lambda_k)), \) \(\Psi = (\Psi(\lambda_1), \ldots, \Psi(\lambda_k)),\) then the matrix \((\Psi, \Phi)\) is nonsingular and may be chosen as the identity matrix. Furthermore, the matrix \(B = \text{diag}(B_{\lambda_1}, \ldots, B_{\lambda_k}),\) where the \(B_{\lambda_j}\) are defined by (2.4) is such that \(\Phi(\theta) = \Phi(0) e^{\theta A},\) \(u_j(t) = Pe^{at}, \) \(-\infty < t < \infty,\) where \(\varphi = \Phi \alpha\) and \(u(\varphi)\) is the solution of (2.1) with initial value \(\varphi\) at \(0.\)

We are now in a position to introduce a coordinate system in \(C.\) In fact, for any \(\varphi\) in \(C,\) we let

\[\varphi = \Phi c + \tilde{\varphi}, \quad c = (\Psi, \varphi), \quad (2.8)\]

where \(\Psi, \Phi\) are defined as above. For any element \(\varphi\) in \(C\) this decomposition is unique.

Now let \(x\) be a solution of (2.2) with initial value \(\varphi\) at \(\sigma\) and let \(\tilde{x}_t, y(t)\) be defined by

\[x_t = \Phi y(t) + \tilde{x}_t, \quad y(t) = (\Psi, x_t), \quad \varphi = \Phi a + \tilde{\varphi}, \quad a = (\Psi, \varphi). \quad (2.9)\]

Is it possible to find differential equations for \(y(t), \tilde{x}_t\) given the differential equation (2.2) for \(x\)? From the definition of \(y(t)\) it is easy to see that

\[\dot{y}(t) = By(t) + e\Psi^T(0) f(t, \Phi y(t) + \tilde{x}_t), \quad y(\sigma) = a,\]
which is an equation not involving any delay terms in $y$. In general, the function $x_t$ satisfies a differential equation in a Banach space (see [14]) but not a functional differential equation of the form (2.2). The basic reason for this is that the function $x_t$ does not satisfy the property that $x_t(\theta) = x(t + \theta)$ for $\theta$ in $[-r, 0]$. To avoid discussing differential equations in a Banach space, we make use of the equivalent integral equation for the solutions of (2.2). If $X_0$ is a matrix function on $[-r, 0]$ defined by $X_0(\theta) = 0$ for $-r \leq \theta < 0$, $X_0(0) = I$, the identity matrix of order $n$, and $u(X_0)$ is the solution of (2.1) with initial value $X_0$ at 0, then it is known (see [1] or [5]) that the solution $x$ of (2.2) with initial value $\varphi$ at $\sigma$ is given by

$$x_t(\theta) = u_{t-\theta}(\varphi)(0) + \epsilon \int_{\sigma}^{t+\theta} u_{t-\theta-r}(X_0)(0)f(\tau, x_\tau)d\tau, \quad \text{for} \quad t + \theta \leq \sigma$$

or

$$x_t(\theta) = \varphi(t + \theta), \quad \sigma - r \leq t + \theta \leq \sigma.$$  

A few simple observations then show that these equations can be written as

$$x_t(\theta) = u_{t-\theta}(\varphi)(\theta) + \epsilon \int_{\sigma}^{t} u_{t-\theta-\tau}(X_0)(\theta)f(\tau, x_\tau)d\tau, \quad -r \leq \theta \leq 0,$$

or

$$x_t = u_{t-\theta}(\varphi) + \epsilon \int_{\sigma}^{t} U_{t-\tau}(X_0)f(\tau, x_\tau)d\tau. \quad (2.10)$$

Since each side of this relation belongs to $C$, we can decompose the equation into two parts according to the relation (2.8).

If we let $x_t$, $\varphi$ be decomposed as in (2.9) and let $X_0 = \Phi Y_0 + X_0$, $Y_0 = (\Psi, X_0) = \Phi^{\Psi T}(0)$, then Eq. (2.10) can be written as the two equations

$$\Phi y(t) = u_{t-\theta}(\Phi a) + \epsilon \int_{\sigma}^{t} u_{t-\theta}(\Phi^{\Psi T}(0))f(\tau, \Phi y(\tau) + \tilde{x}_\tau)d\tau$$

$$\tilde{x}_t = u_{t-\theta}(\tilde{\varphi}) + \epsilon \int_{\sigma}^{t} u_{t-\theta}(X_0)f(\tau, \Phi y(\tau) + \tilde{x}_\tau)d\tau.$$

Using the fact that $u_{t-\theta}(\Phi) = \Phi e^{\Omega(t-\theta)}$ and $\Phi b = 0$ implies $b = 0$, we obtain the following system of equations

$$y(t) = By(t) + \epsilon Y^{T}(0)f(t, \Phi y(t) + \tilde{x}_t), \quad y(\sigma) = a,$$

$$\tilde{x}_t = u_{t-\theta}(\tilde{\varphi}) + \epsilon \int_{\sigma}^{t} u_{t-\theta}(X_0)f(\tau, \Phi y(\tau) + \tilde{x}_\tau)d\tau, \quad (2.11)$$

which are equivalent to (2.10) and, therefore, are equivalent to (2.2). For the particular way in which $\Phi$ was constructed (namely, as a basis for all the
generalized eigenspaces of the characteristic roots of (2.1) with real part \( \geq 0 \), it follows that there are positive constants \( K, \alpha \) such that

\[
\| u_t(\bar{\phi}) \| \leq Ke^{-\alpha t}, \quad t \geq 0,
\]

(2.12)

for all \( \bar{\phi} \) in \( C \) with \((\Psi, \bar{\phi}) = 0\). The matrix \( \bar{X}_0 \) does not belong to \( C \) but \( u_t(\bar{X}_0) \) does for \( t \geq \tau \) and \( u_t(\bar{X}_0) \) is bounded for \( 0 \leq t \leq \tau \). Therefore, the constants \( K, \alpha \) can be assumed to be such that (2.12) is also satisfied by \( \bar{X}_0 \).

Before passing to perturbation theory, let us do the above decomposition for a specific example to show that it is not impossible. Consider the simple special case of Eq. (2.1),

\[
\dot{u}(t) = -\alpha u(t - \tau), \quad \alpha r = \frac{\pi}{2}, \quad r > 0,
\]

(2.13)

and the perturbed equation

\[
\dot{x}(t) = -\alpha x(t - \tau) + \epsilon f(t, x_t).
\]

(2.14)

The characteristic equation of (2.13) is

\[
\lambda = -\alpha e^{-\alpha \tau}.
\]

(2.15)

Since \( \alpha r = \pi/2 \), it is not difficult to show that the roots of (2.15) all have negative real parts except for two which are \( \pm i\alpha \) and both \( \pm i\alpha \) are simple roots.

If we let \( \lambda_1 = i\alpha, \lambda_2 = -i\alpha \), then the functions \( \Phi(\lambda_1), \Phi(\lambda_2) \) defined in the beginning of this section are \( e^{i\alpha \theta}, e^{-i\alpha \theta}, -r \leq \theta \leq 0 \), respectively and \( \Phi = (\Phi(\lambda_1), \Phi(\lambda_2)) \). Since we are only interested in real functions and in order to avoid discussing complex solutions of (2.13), we define

\[
\Phi = (\varphi_1, \varphi_2), \quad \varphi_1(\theta) = \sin \alpha \theta, \quad \varphi_2 = \cos \alpha \theta, \quad -r \leq \theta \leq 0.
\]

(2.16)

A simple computation shows that

\[
\Phi(\theta) = \Phi(0) e^{B \theta}, \quad -r \leq \theta \leq 0,
\]

(2.17)

where

\[
B = \begin{pmatrix} 0 & -\alpha \\ \alpha & 0 \end{pmatrix}.
\]

The adjoint system is

\[
v(s) = \alpha v(s + r)
\]

(2.18)

and the bilinear for \((\psi, \varphi)\) for this particular case is

\[
(\psi, \varphi) = \psi(0) \varphi(0) - \alpha \int_{-r}^{0} \psi(\xi + r) \varphi(\xi) d\xi.
\]

(2.19)
If we let \( \Psi = (\psi_1, \psi_2) \), \( \psi_1 = \sin \alpha \theta, \psi_2 = \cos \alpha \theta, \) \( 0 \leq \theta \leq r \), then

\[
(\Psi, \Phi) = \frac{1}{2} \begin{bmatrix} 1 & -\pi/2 \\ \pi/2 & 1 \end{bmatrix}
\]

which is nonsingular. We now change the basis \( \Psi \) in order to make this expression the identity. If

\[
\Psi^* = \Psi(\Psi, \Phi)^{-1} = \Psi \frac{2}{1 + (\pi^2/4)} \begin{bmatrix} 1 & -\pi/2 \\ \pi/2 & 1 \end{bmatrix}
\]

then \( (\Psi^*, \Phi) = I \), the identity and we are in a position to apply the transformation (2.9) to (2.14). The corresponding equivalent system is then (2.11) with \( B \) defined in (2.17) and \( \Psi \) replaced by \( \Psi^* \) defined in (2.20).

3. Perturbation Theory

It was indicated in the previous section that the transformation (2.9) takes the general system (2.2) into an equivalent system of the form

\[
\dot{y}(t) = B y(t) + \Psi^T(0) f(t, \Phi y(t) + \hat{x}_i), \quad y(0) = b
\]

\[
\hat{x}_i = u_{i-s}(\tilde{\varphi}) + \epsilon \int_0^t u_{i-r}(\tilde{X}_0) f(\tau, \Phi y(\tau) + \hat{x}_s) d\tau
\]

(3.1)

where \( \varphi = \Phi b + \tilde{\varphi}, b = (\Psi, \varphi) \), the eigenvalues of \( B \) have nonnegative real parts, \( u(\varphi) \) is the solution of (2.1) with initial value \( \varphi \) at 0 and there are positive constants \( K, \alpha \) such that

\[
\| u_{i}(\tilde{\varphi}) \| \leq Ke^{-\alpha t} \| \tilde{\varphi} \|, \quad t \geq 0,
\]

for any \( \tilde{\varphi} \) in \( C \) such that \( (\Psi, \tilde{\varphi}) = 0 \) and also the same relation holds with \( \tilde{\varphi} \) replaced by \( \tilde{X}_0 \).

Equations (3.1) are now in a form which is very similar to that which is encountered in the theory of oscillations in ordinary differential equations. One can show that any solution of (3.1) which is bounded on \( (-\infty, \infty) \) must be of such a nature that \( \hat{x}_i = O(\epsilon) \) as \( \epsilon \to 0 \). Consequently, if our analysis is based upon an approximation procedure which can be justified to be correct by investigating only the terms of order \( \epsilon \), then the basic problem lies in the investigation of the ordinary differential equation

\[
\dot{y}(t) = B y(t) + \epsilon \Psi^T(0) f(t, \Phi y(t)).
\]

(3.2)

The analysis of (3.2) is well understood and usually proceeds by the introduction of convenient combinations of polar coordinates and rectangular coordi-
nates and the application of averaging procedures and successive approximations.

For simplicity, let us make the assumption that all the characteristic values of (2.1) have nonpositive real parts. Then $B$ in (3.1) has all eigenvalues purely imaginary and a combination of polar and rectangular changes of coordinates in the components of $y$ (see the examples in Section 4 for the types of coordinates involved) leads to a set of equations of the form

$$
\dot{\zeta} = \bar{d} + \varepsilon \Theta(t, \zeta, \rho, \bar{x}_t),
\dot{\rho} = \varepsilon \bar{R}(t, \zeta, \rho, \bar{x}_t)
$$

where $\zeta$ is a $p$-dimension vector, $\rho$ is a $q$-dimension vector, $\bar{x}_t$ is an element of the Banach space $C$, the vector $\bar{d}$ is a constant vector with positive components and the functions $\Theta, \bar{R}, F$ are multiply periodic in the vector $\zeta$.

Assume that the functions $\Theta, \bar{R}, F$ with arguments $t, \zeta, \rho, \bar{\phi}$ have continuous second derivatives with respect to $\zeta, \rho, \bar{\phi}$ in some set. Let

$$\zeta + \tau = (\zeta_1 + \tau, \ldots, \zeta_p + \tau)$$

and assume that

$$
\lim_{t \to \infty} \frac{1}{T} \int_0^T \bar{R}(t + \tau, \zeta + \tau, \rho, 0) \, d\tau \equiv R_0(\rho)
$$

is independent of $t, \zeta$. We define the averaged equations associated with (3.3) to be the equations

$$\dot{\rho} = \varepsilon R_0(\rho). \quad (3.5)$$

Notice that the averaged equations (3.5) are obtained from $R(t, \zeta, \rho, 0)$ and, therefore in a specific problem, they arise from an investigation of the ordinary differential equation (3.2).

**Theorem 3.1.** If system (3.3) satisfies the conditions enumerated above and if there exists a vector $\rho_* \in \mathbb{R}^q$ such that $R_0(\rho_*) = 0$ and the eigenvalues of the matrix $\partial R_0(\rho_*)/\partial \rho$ have nonzero real parts, then there exists an $\varepsilon > 0$ and functions $g(t, \zeta, \epsilon), h(t, \zeta, \epsilon), 0 \leq \epsilon \leq \epsilon_0, g \in L^\infty, h \in C, g(t, \zeta, 0) = \rho_0, h(t, \zeta, 0) = 0$, multiply periodic in $\zeta$ and almost periodic in $t$ such that the set $S_{\varepsilon}, 0 \leq \epsilon \leq \epsilon_0$, defined by

$$S_{\varepsilon} = \{(t, \zeta, \rho, \bar{\phi}) : \rho = g(t, \zeta, \epsilon), \bar{\phi} = h(t, \zeta, \epsilon), -\infty < t < \infty, -\infty < \zeta_j < \infty, j = 1, 2, \ldots, p\}$$

is an integral manifold of system (3.3). If the functions $\Theta, \bar{R}, F$ are independent of $t$ (or periodic in $t$ of period $\omega$), then the functions $g, h$ are independent of $t$ or periodic in $t$ of period $\omega$. Furthermore, if all eigenvalues of $\partial R_0(\rho_*)/\partial \rho$ have

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negative real parts, $S_\epsilon$ is asymptotically stable for $0 < \epsilon \leq \epsilon_0$ and if one eigenvalue has a positive real part then $S_\epsilon$ is unstable for $0 < \epsilon \leq \epsilon_0$.

We merely give an indication of the proof of this theorem since it is so analogous to the proof for the case of ordinary differential equations given in Bogoluibov and Mitropolski [2] and Hale [7].

In [7, Chap. 12], it is shown that there are functions $u(t, \zeta, \rho, \epsilon), \omega(t, \zeta, \rho, \epsilon)$, multiply periodic in $\zeta$ and almost periodic in $t$ such that the transformation

$$
\zeta \to \zeta + \epsilon u(t, \zeta, \rho, \epsilon), \quad \rho \to \rho + \epsilon \omega(t, \zeta, \rho, \epsilon)
$$

applied to $\dot{\zeta} = d + \epsilon \Theta(t, \zeta, \rho, 0), \dot{\rho} = \epsilon R(t, \zeta, \rho, 0)$ yields new equations of the form

$$
\dot{\zeta} = d(t, \zeta, \rho, \epsilon) + \epsilon \Theta_1(t, \zeta, \rho, \epsilon), \quad \dot{\rho} = \epsilon R_0(\rho) + \epsilon R_1(t, \zeta, \rho, \epsilon)
$$

where $\Theta_1, R_1$ are zero for $\epsilon = 0$. Consequently, if this transformation is applied to (3.3), we obtain a system of the form

$$
\dot{\zeta} = d + \epsilon \Theta_1(t, \zeta, \rho, \epsilon), \quad \dot{\rho} = \epsilon R_0(\rho) + \epsilon R_1(t, \zeta, \rho, \epsilon)
$$

where $F_1$ is the same type of function $F$ and $\Theta_1, R_1, R_2$ satisfy the following conditions:

$$
R_1(t, \zeta, \rho, 0) = 0, \quad R_2(t, \zeta, \rho, 0, \epsilon) = 0,
$$

$$
| R_2(t, \zeta, \rho, \bar{\psi}, \epsilon) - R_2(t, \zeta, \rho, \tilde{\psi}, \epsilon) | \leq K \| \bar{\psi} - \tilde{\psi} \|
$$

for some constant $K$ and $\rho, \bar{\psi}, \tilde{\psi}$ in a bounded set.

One now proceeds in a manner completely analogous to that given in [2], [7] to show that the functions $g, h$ mentioned in the theorem are the fixed points of an integral operator. The stability of the integral manifold must be investigated separately and is supplied using the ideas developed in [8] in connection with a saddle point for functional-differential equations.

As in [2], [7], one can also prove

**Theorem 3.2.** Suppose the averaged equations (3.5) have a nonconstant periodic solution $\rho = \rho^0(t)$ of period $T$ such that $g - 1$ of the characteristic exponents of the associated linear variational equations have nonzero real parts. Then there exists an $\epsilon_0 > 0$ and functions $g(t, \zeta, \psi, \epsilon), h(t, \zeta, \psi, \epsilon), 0 \leq \epsilon \leq \epsilon_0$, $g$ in $R^g$, $h$ in $C, g(t, \zeta, \psi, 0) = \rho^0(\psi), 0 \leq \psi \leq T, h(t, \zeta, \psi, 0) = 0$, multiply periodic in $\zeta$, periodic in $\psi$ of period $T$ and almost periodic in $t$ such that the set $S_\epsilon, 0 \leq \epsilon \leq \epsilon_0$, defined by

$$
S_\epsilon = \{ (t, \zeta, \rho, \bar{\psi}) : \rho = g(t, \zeta, \psi, \epsilon), \bar{\psi} = h(t, \zeta, \psi, \epsilon), \epsilon_0 \}
$$

$$
-\infty < t < \infty; -\infty < \zeta, \epsilon \leq \infty, j = 1, 2, \ldots, \rho; 0 \leq \psi \leq T
$$
is an integral manifold of system (3.3.) If $\Theta$, $R$, $F$ are independent of $t$ (or periodic in $t$ of period $\omega$), then the functions $g$, $h$ are independent of $t$ (or periodic in $t$ of period $\omega$). Furthermore, the stability properties of $S$ are the same as those of the periodic solution $\rho^p(t)$ of (3.5).

We now state some important corollaries of these theorems before turning to specific examples. Consider the equation

$$\dot{x}(t) = ef(t, x_t)$$

(3.6)

where $\epsilon > 0$ is a parameter, $f(t, \varphi)$ is almost periodic in $t$ uniformly with respect to $\varphi$ in some subset of $C([-\pi, \pi], \mathbb{R}^n)$, and has a continuous second Frechét derivative with respect to $\varphi$. Let

$$f_0(\varphi) = \lim_{T \to \infty} \frac{1}{T} \int_0^T f(\tau, \varphi) \, d\tau.$$  

(3.7)

Halanay [4] has discussed for small $\epsilon$, some of the relationships between the solutions [on the interval $(0, \infty)$], of $\dot{x}(t) = ef_0(x_t)$ and the solutions of (3.6) for the case in which the retardation interval is of order $\epsilon$. We now show that Theorems 3.1 and 3.2 imply that his results and even more are valid without any restriction on the retardation interval. In fact, we can prove the following two theorems. In the statement of these theorems, $y$ sometimes denotes a vector in $n$-dimensional Euclidean space and sometimes a vector of constant functions in $C([-\pi, \pi], \mathbb{R}^n)$, but it is clear from the context which meaning is implied. The averaged equations of (3.6) are then defined to be the ordinary differential equations

$$\dot{y} = ef_0(y).$$

(3.8)

**Theorem 3.3.** If the averaged equations (3.8) have an equilibrium point $y_0$ such that the matrix of coefficients of the linear variational equations has no eigenvalues on the imaginary axis, then, for $\epsilon$ sufficiently small, (3.6) has a unique almost periodic solution $x = g(t, \epsilon)$ in a neighborhood of $x = y_0$, $g(t, 0) = y_0$, and the stability properties of $g$ are the same as the stability properties of $y_0$.

**Theorem 3.4.** If (3.8) has a nonconstant periodic solution $y = y^{(0)}(t)$ of period $T$, such that the linear variation equation has $n - 1$ of its characteristic exponents not on the imaginary axis, then, for $\epsilon$ sufficiently small, there exists a function $g(t, \zeta, \epsilon)$ in $C$, almost periodic in $t$ uniformly with respect to $\zeta$, periodic in $\zeta$ of period $T$, $g(t, \zeta, 0) = y^{(0)}_\zeta$, $y^{(0)}_\zeta(\theta) = y^{(0)}(\zeta + \theta)$, $0 \leq \theta \leq T$, such that the surface $S_\epsilon$ in $C \times (-\infty, \infty)$ defined by

$$S_\epsilon = \{(\varphi, t) : \varphi = g(t, \zeta, \epsilon), 0 \leq \zeta \leq T, -\infty < t < \infty\}$$
is an integral manifold of (3.6). Furthermore, $S_\xi$ is unique in a neighborhood of $S_0 = \{(\varphi, t) : \varphi = y^{(0)}_\zeta, 0 \leq \zeta \leq T, -\infty < t < \infty\}$ and has the same stability properties as $S_0$. If $f$ in (3.6) is independent of $t$, then $g$ is independent of $t$, and if $f$ is periodic in $t$, then $g$ is periodic in $t$ with the same period.

To show that these results are consequences of Theorems 3.1 and 3.2, we proceed as follows. For any $\varphi$ in $C([r - r, \infty], \mathbb{R}^n)$, the decomposition

$$\varphi = b + \tilde{\varphi},$$

is unique if $b$ is the constant function whose value is $\varphi(0)$. If, in (3.6),

$$x_t = y(t) + \tilde{x}_t$$

then

$$\dot{y}(t) = ef(t, y(t) + \tilde{x}_t)$$

$$\tilde{x}_t = u_{t-s}(\tilde{\varphi}) - \int_0^t u_{t-s}(\tilde{X}_0) f(r, y(r) + \tilde{x}_r) \, dr$$

(3.9)

and

$$\|u_t(\tilde{\varphi})\| \leq Ke^{-\alpha t}\|\tilde{\varphi}\|, \quad t \geq 0$$

for some positive $K, \alpha$. This last relation is obviously true in this case since $u_t(\tilde{\varphi}) = 0$ for all $\tilde{\varphi}$ and $t \geq r$. System (3.9) is a special case of (3.3) and one obtains Theorems 3.3 and 3.4 from Theorems 3.1 and 3.2.

4. SOME SPECIFIC EXAMPLES

Let us first discuss the oscillatory properties of Eq. (2.13); namely, the equation

$$\dot{x}(t) = -\alpha x(t - r) + ef(t, x_t)$$

$$\alpha r = \pi/2, \quad \epsilon > 0.$$  (4.1)

We have seen in Section 2 that this equation is equivalent to the system

$$\dot{y}_1 = -\alpha y_2 + \pi \epsilon \mu^2 f(t, \Phi y + \tilde{x}_t)$$

$$\dot{y}_2 = \alpha y_1 + 2\pi \mu \varepsilon f(t, \Phi y + \tilde{x}_t)$$

$$\tilde{x}_t = u_{t-r}(\tilde{\varphi}) + \int_0^t u_{t-s}(\tilde{X}_0) f(r, \Phi y(r) + \tilde{x}_r) \, dr$$

$$\Phi(\theta) = (\varphi_1(\theta), \varphi_2(\theta)) \overset{\text{def}}{=} (\sin \alpha \theta, \cos \alpha \theta), \quad -r \leq \theta \leq 0$$

$$y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \quad \mu^2 = \frac{1}{1 + (\pi^2/4)}.$$  (4.2)

If we let

$$y_1 = \rho \sin \alpha \zeta$$

$$y_2 = -\rho \cos \alpha \zeta$$  (4.3)
then system (4.2) becomes
\[
\begin{align*}
\dot{\zeta} &= 1 + \frac{\varepsilon \mu^2}{\rho} (\pi \mu \cos \alpha \zeta + 2 \mu \sin \alpha \zeta) = 1 + \varepsilon \Theta(t, \zeta, \rho, \xi_i) \\
\dot{\rho} &= \varepsilon \mu^2 (\pi \mu \sin \alpha \zeta - 2 \mu \cos \alpha \zeta) = \varepsilon \mathcal{R}(t, \zeta, \rho, \xi_i) \\
\ddot{x}_i &= u_{t-o}(\bar{\rho}) + \varepsilon \int_0^t u_{t-\tau}(\bar{X}_0) f(\tau, \Phi y(\tau) + \ddot{x}_i) \, d\tau
\end{align*}
\]  
where \( f = f(t, \Phi y + \ddot{x}_i) \) and \( y \) is given in (4.3).

If the function \( f \) is almost periodic in \( t \) uniformly with respect to the other arguments, then the averaged equations for this particular case are
\[
\begin{align*}
\dot{\rho} &= \varepsilon R_o(\rho) \\
R_o(\rho) &= \lim_{T \to \infty} \frac{1}{T} \int_0^T \mu^2 [\pi \mu \sin \alpha (\zeta + \tau) - 2 \mu \cos \alpha (\zeta + \tau)] \, d\tau \\
f &= f(t + \tau, \rho [\varphi_1 \sin \alpha (\zeta + \tau) - \varphi_2 \cos \alpha (\zeta + \tau)]) \\
\varphi_1(\theta) &= \sin \alpha \theta, \quad \varphi_2(\theta) = \cos \alpha \theta, \quad -\pi \leq \theta \leq 0,
\end{align*}
\]  
where we always assume that this limit is independent of \( t, \zeta \).

Notice that Eqs. (4.5) are the same equations that are obtained by introducing the polar coordinate transformation (4.3) to the ordinary differential equation
\[
\begin{align*}
\dot{y}_1 &= -\alpha y_2 + \pi \mu^2 f(t, \Phi y) \\
\dot{y}_2 &= \alpha y_1 + 2 \pi \mu^2 f(t, \Phi y)
\end{align*}
\]  
and then taking the average.

Let us now take some specific functions \( f \) in (4.1) to show that important information is obtained by this method.

**Example 4.1** (Pinney [12]). \( f = -\gamma x(t - 1) + \beta x^3(t - 1), \alpha = \pi/2, \gamma = 1 \). Since
\[
x(t - 1) = \Phi(-1) y(t) = -y_1(t) = -\rho \sin \frac{\pi}{2} \zeta,
\]
it is easy to check that the average \( R_o(\rho) \) in (4.5) is
\[
R_o(\rho) = K_{\gamma \rho} \left( 1 - \frac{3 \beta \rho^2}{4 \gamma} \right),
\]
where \( K \) is a positive constant, and the averaged equations (4.5) are
\[
\dot{\rho} = \varepsilon K_{\gamma \rho} \left( 1 - \frac{3 \beta \rho^2}{4 \gamma} \right).
\]
If $\gamma \beta > 0$ the averaged equation has an equilibrium point $\rho_0 = \sqrt{4\gamma/3\beta}$ which is asymptotically stable if $\gamma > 0$ and unstable if $\gamma < 0$. Consequently, for $\epsilon > 0$ and sufficiently small, Theorem 3.1 asserts for $\gamma \beta > 0$ the existence of functions $g(\xi, \epsilon), h(\xi, \epsilon)$ (these functions are independent of $t$ since $f$ is), $g(\xi, 0) = \sqrt{4\gamma/3\beta}, h(\xi, 0) = 0$, periodic in $\xi$ of period 4 such that

$$S_\epsilon = \{(t, \xi, \rho, \varphi) : \rho = g(\xi, \epsilon), \varphi = h(\xi, \epsilon), -\infty < t < \infty, 0 \leq \xi \leq 4\}$$

is an integral manifold of (4.4) which is stable for $\gamma < 0$. From (2.9) and (4.3), this implies that

$$T_\epsilon = \{(t, \varphi) : \varphi = g(\frac{\pi \xi}{2} - \varphi_2 \cos \frac{\pi \xi}{2} + h(\xi, \epsilon), 0 \leq \xi \leq 4\}$$

is an integral manifold of our original system. Such a cylinder $T_\epsilon$ in $R \times C$ obviously corresponds to a periodic solution of our system which is stable if $\gamma > 0$ and unstable if $\gamma < 0$ and has an amplitude approximately equal to $\sqrt{4\gamma/3\beta}$. The approximate period $\omega$ is obtained by solving the equation

$$\xi = 1 + \epsilon \mu a \left(\gamma \rho_0 \sin \frac{\pi \xi}{2} - \beta \rho_0 - \beta \rho_0 \sin \frac{\pi \xi}{2}\right), \quad \rho_0 = \sqrt{4\gamma/3\beta},$$

and determining $\omega$ so that $\xi(t + \omega) = \xi(t) + 4$.

It is interesting to note that the second order system (4.6) for this example is actually equivalent to a second order scalar differential equation. The method of averaging should then allow one to obtain an "equivalent" linear second order equation in the sense of Krylov-Bogoliubov. This should in turn lead to methods which will yield important information about equations with retardation when $\epsilon$ is not small—describing functions, etc. So far, this has not been exploited.

**Example 4.2.** Consider the equation

$$\dot{x}(t) = -\left[-\frac{\pi}{2} + \epsilon \eta(t)\right] x(t - 1) (1 - x^2(t))$$  \hspace{1cm} (4.7)

where $\eta$ is almost periodic in $t$.

This is a special case of (4.1) with $\alpha = \pi/2, r = 1$, and

$$f = \frac{\pi}{2} x^2(t) x(t - 1) - \eta(t) x(t - 1) x(t - 1) + \epsilon \eta(t) x^2(t) x(t - 1).$$

Using the fact that

$$x(t) = \Phi(0) y(t) = y_2(t) = -\rho \cos \left(\frac{\pi \xi}{2}\right)$$

$$x(t - 1) = \Phi(-1) y(t) = -y_1(t) = -\rho \sin \left(\frac{\pi \xi}{2}\right)$$
the averaged equation (4.5) becomes
\[ \rho = \epsilon \mu^2 \rho \left( \eta_0 - \frac{\pi}{8} \rho^2 \right), \quad \eta_0 = \lim_{T \to \infty} \frac{1}{T} \int_0^T \eta(t) \, dt \]
provided
\[ \lim_{T \to \infty} \frac{1}{T} \int_0^T \eta(t) \left( \frac{\cos \pi (\zeta + t)}{\cos \pi (\zeta + t)} \right) \, dt = 0. \]
Consequently, if \( \eta_0 > 0 \) and \( \epsilon > 0 \) is sufficiently small there exists a stable integral manifold of solutions of (4.7) whose parametric representation in \( C \) is almost periodic in \( t \) and periodic in \( \zeta \) and for \( \epsilon = 0 \) is given by
\[ \Phi \left( \frac{\rho_0 \sin \pi \zeta/2}{- \rho_0 \cos \pi \zeta/2} \right), \quad \rho_0 = \sqrt{8 \eta_0 / \pi}. \]
If \( \eta(t) \) is independent of \( t \), then the parametric representation of the integral manifold is independent of \( t \) and one obtains a nonconstant periodic solution of (4.7) with amplitude approximately \( \sqrt{8 \eta / \pi} \). Jones [9] has discussed periodic solutions of (4.5) with \( \eta \) independent of \( t \) without assuming \( \epsilon \) is small. He has also discussed more general equations.

**Example 4.3.** Consider the system
\[
\begin{align*}
\dot{x}_1(t) &= x_2(t) \\
\dot{x}_2(t) &= -\alpha^2 x_1(t) + \epsilon [1 - x_1^2(t - r)] x_2(t)
\end{align*}
\]
where \( \alpha > 0, r \geq 0, \epsilon \geq 0 \) are parameters. For \( r = 0 \), this is van der Pol's equation. By using the preceding theory, we will investigate the existence and stability of limit cycles of (4.8) for \( \epsilon \) small.

If \( \varphi_1, \varphi_2 \) are the vectors in \( C([-r, 0], R^2) \) defined by
\[ \varphi_1(\theta) = \left( \begin{array}{c} \cos \alpha \theta \\ - \alpha \sin \alpha \theta \end{array} \right), \quad \varphi_2(\theta) = \left( \begin{array}{c} \alpha^{-1} \sin \alpha \theta \\ \cos \alpha \theta \end{array} \right), \quad -r \leq \theta \leq 0, \]
and \( \Phi = (\varphi_1, \varphi_2) \), then the transformation
\[ x_t = \Phi y(t) + \bar{x}_t, \quad y(t) = x(t), \quad y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \quad x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \]
applied to (4.8) yields the equivalent system
\[
\begin{align*}
\dot{y}_1 &= y_2 \\
\dot{y}_2 &= -\alpha^2 y_1 + \epsilon \left[ 1 - \left( y_1 \cos \alpha r - \frac{1}{\alpha} y_2 \sin \alpha r + \bar{x}_t(-r) \right)^2 \right] y(t) \\
\bar{x}_t &= u_{t-\varphi} + \epsilon \int_0^t u_{t-\tau}(\bar{X}_0) F(y(\tau), \bar{x}_\tau) \, d\tau
\end{align*}
\]
where \( u_i(\phi) \) has the same meaning as in the previous sections and \( F \) is a two-vector whose specific form is of no particular interest here.

If we introduce the polar coordinates

\[
\begin{align*}
y_1 &= \rho \sin \alpha \zeta \\
y_2 &= \rho \alpha \cos \alpha \zeta
\end{align*}
\]

into (4.9) and set \( \dot{x}_i = 0 \), we obtain

\[
\begin{align*}
\dot{\zeta} &= 1 - \frac{\varepsilon}{2\alpha} \sin 2\alpha \zeta [1 - \rho^2 (\sin \alpha (\zeta - r))^2] \\
\dot{\rho} &= -\varepsilon \rho \cos^2 \alpha \zeta [1 - \rho^2 (\sin \alpha (\zeta - r))^2].
\end{align*}
\]

The averaged equation is

\[
\dot{\rho} = \frac{\varepsilon \rho}{2} [1 - \frac{1}{2} (1 - \frac{1}{2} \cos 2\alpha \phi) \rho^2].
\]

This equation has an equilibrium point

\[
\rho_0 - \sqrt{\frac{2}{\gamma}}, \gamma = 1 - \left( \frac{\cos 2\alpha \phi}{2} \right) > 0
\]

for every value of \( r \) and the linear variational equation relative to \( \rho_0 \) is \( \dot{\rho} = -2\varepsilon \rho \). Consequently, Theorem 3.1 implies as in Example 4.1 the existence of a stable periodic solution of (4.8) with amplitude approximately \( \sqrt{2/\gamma} \). The existence of a periodic solution for this equation was discussed in the thesis of W. Fuller [3].

For physical examples of retarded equations and the importance of oscillatory phenomenon, see Chapter 21, Minorsky [11].

References


