JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS 38, 174–186 (1972)

Generalized Invariant Variational Problems

JOHN DAVID LOGAN*

Research Institute, University of Dayton, Dayton, Ohio 45409

Submitted by Richard Bellman

Received November 12, 1970

1. INTRODUCTION

In the early part of this century, Emmy Noether initiated the study of so-called invariant variational problems. These problems deal with the invariance properties of functionals of the form

$$W[y(x)] = \int_{x_0}^{x_1} L[x, y(x), y'(x)]\, dx$$

under certain continuous groups of transformations of the variables x and y. The results of Emmy Noether were published in 1918 in the form of two theorems [1]. The first of these theorems states that if W is invariant under a ρ-parameter infinitesimal group of transformations, then ρ linearly independent combinations of the variational derivatives are divergences. The second theorem requires a different group, the so-called infinite continuous group. It states that if W is invariant under an infinitesimal group of transformations depending linearly upon τ arbitrary functions and their derivatives up to some order σ, then there exists τ identities between the variational derivatives and their derivatives up to order σ.

Physically, these theorems have far-reaching consequences. For example, if L is the Lagrangian of some physical system, then the invariance of W under a group of the type in the first theorem results in conservation laws for the system. E. Bessel–Hagen [2] in 1921 applied the first theorem in this way to obtain conservation laws for the N-body problem and for electrodynamics by using the ten-parameter Galilean group and the fifteen-parameter conformal group, respectively. In 1924, D. Hilbert [3] applied the second theorem to obtain equations in electrodynamics and general relativity by requiring that the action integral be invariant with respect to arbitrary transformations of the spatial variables. Since then, a rather large number

* Present address: Department of Mathematics, University of Arizona, Tucson, Arizona 85721.

of papers have appeared relating Noether's theorems to particular physical theories.

Only recently, however, has an effort been made toward generalizations and extensions of these theorems. In 1967, A. Trautman [4] presented a derivation of the first Noether theorem in the language of modern differential geometry (vector bundles, jets, etc.) and, in much the same manner, J. Komorowski [5, 6] in 1969 gave a modern version of both theorems. H. Rund [7], D. G. B. Edelen [8], and V. R. Tihomirov [9] have attempted to enlarge the range of application of the theorems by considering functionals defined on geometric objects. In still another direction, N. Kuharcuk [10] in 1967, gave a generalization of the first Noether theorem by investigating the invariance properties of a functional defined on functions which take their values in a Banach space.

In this paper, the Noether theorems are extended to include constraining relations on the variations. These relations take the form of linear differential operators and Fredholm-type integral operators acting on a set of fundamental variations. Operators of these types readily admit the definition of adjoint operators, and this fact becomes essential in performing calculations analogous to the classical integration by parts used in obtaining usual variational formulae, e.g., the Euler–Lagrange Equations. In addition, these constraint conditions on the variations afford the investigation of variational problems in which all quantities which occur in the Lagrangian are allowed to vary independently. Physically, this means, for example, that the variation of the velocity need not be the derivative of the variation of position, as is classically assumed. Also, these constraints on the variations lead to a new variational principle from which generalized equations of motion and conservation laws can be obtained.

Geometrically, the setting for the investigations in this paper is a differentiable manifold. Although the results are of a local character, the setting of a manifold points out certain geometric features that are not clearly distinguished in Euclidean space (e.g., the tangent bundle), and this provides some insight into the nature of invariant variational problems. The summation convention is used throughout, often without explicitly mentioning it.

2. The Fundamental Variational Formula

In this section, we derive an expression representing the variation or differential of the Lagrangian. Such an expression will be fundamental to the formulation of variational and invariance principles. The expression given here will be a generalization of the classical expression in that the variations of the arguments of the Lagrangian will be constrained to satisfy certain characteristic relations which permit the derivation of generalized Euler–

Lagrange equations. No derivatives are initially assumed in the Lagrangian; they may be introduced through the constraints or after the basic formula for the differential has been obtained.

Let M be an m-dimensional differentiable manifold of class C^∞ and denote by T_*M the tangent bundle of M. We assume that there is given a C^∞ function

$$L : M \to R^1$$

defined on the manifold, and we let

$$dL : T_*M \to R^1$$

be the differential of L. On the tangent space $T_{u_0}M$ at $u_0 \in M$, the mapping $dL : T_{u_0}M \to R^1$ is a linear, real-valued function; if U_0 is a vector in $T_{u_0}M$, then we shall write

$$dL(u_0\,,\, U_0) \in R^1.$$

In terms of a local coordinate system $(u^1, u^2,..., u^m)$ at u_0, the mapping dL is given by

$$dL(u_0\,,\, U_0) = \frac{\partial L}{\partial u^i}(u_0)\, U_0{}^i, \tag{2.1}$$

where the summation convention is assumed, and where $U_0{}^1,..., U_0{}^m$ are the components of U_0 in the coordinate system $(u^1,..., u^m)$. Inherent in (2.1) is the transformation condition on the $U_0{}^i$; namely, if

$$u^i \to \bar{u}^i(u), \qquad i = 1, 2,..., m$$

represents a change to a new coordinate system $(\bar{u}^1,..., \bar{u}^m)$, then the new components of U_0 are given by

$$\bar{U}_0{}^h = \frac{\partial \bar{u}^h}{\partial u^i}\, U_0{}^i, \qquad h = 1, 2,..., m.$$

For an arbitrary point u in the local coordinate neighborhood at u_0, we write (2.1) symbolically as

$$dL(u, U) = \frac{\partial L}{\partial u}(u) \cdot U, \tag{2.2}$$

where $U \in T_uM$.

Now let $v : [s_0\,,\, s_1] \to M$ be a C^∞ mapping of the real interval $[s_0\,,\, s_1]$ into the manifold (more precisely, into the local coordinate neighborhood at u_0) given explicitly by

$$u = v(s), \qquad s \in [s_0\,,\, s_1].$$

In addition, let $V : [s_0\,,\, s_1] \to T_{v(s)}M$ be a C^∞ mapping given by

$$U = V(s), \qquad s \in [s_0\,,\, s_1],$$

where $U \in T_{v(s)}M$. Conceptually, $v(s)$ represents a curve in the manifold and $V(s)$ represents a vector field along the curve. We will call $V(s)$ the *variation* of the curve $v(s)$.

We now define the differential of L along the curve $v(s)$ in the direction $V(s)$ by the mapping

$$dL \circ (v \times V) : [s_0, s_1] \to R^1$$

where

$$v \times V : [s_0, s_1] \to M \times T_{v(s)}M$$

is defined by $(v \times V)(s) = (v(s), V(s))$. According to (2.2), this differential is given by

$$dL(v(s), V(s)) = \frac{\partial L}{\partial u}(v(s))\, V(s). \tag{2.3}$$

Denoting $\partial L/\partial u\,(v(s))$ by $\partial L/\partial v$, equation (2.3) becomes

$$dL(v(s), V(s)) = \frac{\partial L}{\partial v} \cdot V(s). \tag{2.4}$$

This differential is interpreted as the variation of the Lagrangian L at $v(s)$ in the direction $V(s)$. Since we eventually wish to consider Lagrangians L which depend upon n functions and their derivatives up to the N-th order, we will now index $v(s)$ and $V(s)$ as follows:

$$v(s) = (v_k{}^J(s)) \qquad \text{and} \qquad V(s) = (V_k{}^J(s)),$$

where $k = 1,..., n$; $J = 0, 1,..., N$ and $m = n(N + 1)$. Therefore, equation (2.4) becomes

$$dL(v(s), V(s)) = \frac{\partial L}{\partial v_k{}^J} V_k{}^J(s) \tag{2.5}$$

with summation over J and k being understood.

Before proceeding with the derivation of an expression for $dL(v(s), V(s))$, we make some remarks regarding the nature of the Lagrangian L and manifold M. Classically, the Lagrangian is usually taken to depend upon functions and their derivatives up to some given order. However, there are some problems in precisely defining this dependence [11, 12]. A. Trautman [4] and J. Komorowski [5, 6] define the Lagrangian using the concept of a jet bundle, but this degree of sophistication can be avoided by noticing that derivatives in the Lagrangian play no role in the derivation of variational formulae; indeed, the constraining relations on the variations assume this role. Consequently, in the present analysis we postpone the introduction of derivatives into the Lagrangian and define the Lagrangian on a manifold large enough in dimension to accommodate derivatives after the calculations are performed. We shall offer further comment on this point in Section 4.

The fundamental step in obtaining a final form for $dL(v(s), V(s))$ is the integration by parts which isolates the variational derivatives. Classically, this parts integration depends explicitly on the fact that the local variation of a derivative is the derivative of the local variation, i.e., the variation and derivative commute. We now generalize this notion by assuming the variations $V_k{}^J(s)$ of $v_k{}^J(s)$ can be written as differential operators acting on a set of functions

$$\eta_k : [s_0, s_1] \to R^1, \qquad k = 1, 2, ..., n.$$

We call the η_k the *fundamental variations*. More precisely, if $A_\Gamma{}^J : [s_0, s_1] \to R^1$, $J = 0, 1, ..., N$, $\Gamma = 1, 2, ..., \Omega$ are real-valued functions, then we assume

$$V_k{}^J(s) = A_\Gamma{}^J(s)\, D^\Gamma \eta_k(s), \tag{2.6}$$

where $D^\Gamma = d^\Gamma/ds^\Gamma$. Symbolically, we write (2.6) as

$$V(s) = \mathcal{O}\!\ell\,\eta(s).$$

Now, substituting (2.6) into (2.5), we have

$$dL(v(s), \mathcal{O}\!\ell\,\eta(s)) = \frac{\partial L}{\partial v_k{}^J} A_\Gamma{}^J(s)\, D^\Gamma \eta_k(s). \tag{2.7}$$

Integrating by parts, we obtain

$$\begin{aligned} dL(v(s), \mathcal{O}\!\ell\,\eta(s)) &= \mathcal{D}^\Gamma \left[\frac{\partial L}{\partial v_k{}^J} A_\Gamma{}^J\right] \eta_k(s) \\ &\quad + D \sum_{\Lambda=1}^{\Omega} \sum_{\Delta=0}^{\Lambda-1} \eta_k^{\Lambda-\Delta-1}(s)\, \mathcal{D}^\Delta \left[\frac{\partial L}{\partial v_k{}^J} A_\Lambda{}^J\right], \end{aligned} \tag{2.8}$$

where $\mathcal{D}^\Gamma = (-1)^\Gamma D^\Gamma$, and where summation over J, k, Γ are understood. If we denote the boundary term by

$$B\left(\frac{\partial L}{\partial v}, \mathcal{O}\!\ell, \eta, D^\eta, ..., D^{\Omega-1}\eta\right) \equiv \sum_{\Lambda=1}^{\Omega} \sum_{\Delta=0}^{\Lambda-1} \eta_k^{\Lambda-\Delta-1}(s)\, \mathcal{D}^\Delta \left[\frac{\partial L}{\partial v_k{}^J} A_\Lambda{}^J\right], \tag{2.9}$$

and if we denote the generalized variational derivatives by

$$\frac{\delta L}{\delta v_k}(\mathcal{O}\!\ell) \equiv \mathcal{D}^\Gamma \left[\frac{\partial L}{\partial v_k{}^J} A_\Gamma{}^J\right], \tag{2.10}$$

then (2.8) can be written as

$$dL(v(s), \mathcal{A}\eta(s)) = \frac{\delta L}{\delta v_k}(\mathcal{A})\,\eta_k(s) + DB\left(\frac{\partial L}{\partial v}, \mathcal{A}, \eta, D\eta, \ldots, D^{\Omega-1}\eta\right). \tag{2.11}$$

This is the fundamental formula for the variation of L at $v(s)$ in the direction $V(s)$. If desired, derivatives can be introduced by taking

$$v_k{}^J(s) = x_k^{(J)}(s),$$

where (J) denotes the J-th derivative, and where $x_1, \ldots, x_n$ are functions on $[s_0, s_1]$. In this case, we write the Lagrangian as

$$L(x_1(s), \ldots, x_n(s); x_1^{(1)}(s), \ldots, x_n^{(1)}(s); \ldots; x_1^{(N)}(s), \ldots, x_n^{(N)}(s)).$$

Equation (2.11) reduces to the classical case, where the variation of a derivative is the derivative of the variation, if

$$(A_k{}^J(s)) = (\delta_k{}^J), \qquad \Omega = N,$$

that is, when $(A_k{}^J(s))$ is taken to be the square identity matrix.

3. A Generalized Variational Principle

In this section, we calculate the variation or differential of the action functional while at the same time showing that a Fredholm-type integral operator with continuous kernel can be added to the constraining relations (2.6). Let $L : M \to R^1$ and $v : [s_0, s_1] \to M$ be as in Section 2, and define the functional

$$W[v(s)] = \int_{s_0}^{s_1} (L \circ v)(s)\, ds. \tag{3.1}$$

We define the variation or differential of W at $v(s)$ in the direction $V(s)$ by

$$dW[v(s), V(s)] = \int_{s_0}^{s_1} dL(v(s), V(s))\, ds.$$

Using (2.5) this becomes

$$dW[v(s), V(s)] = \int_{s_0}^{s_1} \frac{\partial L}{\partial v_k{}^J} V_k{}^J\, ds. \tag{3.2}$$

Now, if we assume that the conditions on the variations take the form

$$V_k{}^J(s) = A_\Gamma{}^J(s)\, D^\Gamma \eta_k(s) + \int_{s_0}^{s_1} K^J(s, t)\, \eta_k(t)\, dt, \tag{3.3}$$

or symbolically the form

$$V(s) = \mathcal{O}\!\ell\eta(s) + \mathcal{K}\eta(s),$$

then (3.2) becomes

$$dW[v(s), \mathcal{O}\!\ell\eta + \mathcal{K}\eta]$$
$$= \int_{s_0}^{s_1} \frac{\partial L}{\partial v_k{}^J}(s)\, A_\Gamma{}^J(s)\, D^\Gamma \eta_k(s)\, ds + \int_{s_0}^{s_1} \frac{\partial L}{\partial v_k{}^J}(s) \int_{s_0}^{s_1} K^J(s, t)\, \eta_k(t)\, dt\, ds. \tag{3.4}$$

The second integral may be rewritten so that (3.4) becomes

$$dW[v(s), \mathcal{O}\!\ell\eta + \mathcal{K}\eta]$$
$$= \int_{s_0}^{s_1} \frac{\partial L}{\partial v_k{}^J}(s)\, A_\Gamma{}^J(s)\, D^\Gamma \eta_k(s)\, ds + \int_{s_0}^{s_1} \eta_k(s) \int_{s_0}^{s_1} K^J(t, s) \frac{\partial L}{\partial v_k{}^J}(t)\, dt\, ds.$$

Therefore using (2.8), (2.9) and collecting terms,

$$dW[v(s), \mathcal{O}\!\ell\eta + \mathcal{K}\eta]$$
$$= \int_{s_0}^{s_1} \left\{ \mathcal{D}^\Gamma \left[\frac{\partial L}{\partial v_k{}^J} A_\Gamma{}^J(s) \right] + \int_{s_0}^{s_1} K^J(t, s) \frac{\partial L}{\partial v_k{}^J}(t)\, dt \right\} \eta_k(s)\, ds$$
$$+ \int_{s_0}^{s_1} DB \left(\frac{\partial L}{\partial v}, \mathcal{O}\!\ell, \eta, D\eta, \ldots, D^{\Omega-1}\eta \right) ds. \tag{3.5}$$

If we denote the generalized variational derivative by

$$\frac{\delta L}{\delta v_k}(\mathcal{O}\!\ell, \mathcal{K}) \equiv \mathcal{D}^\Gamma \left[\frac{\partial L}{\partial v_k{}^J} A_\Gamma{}^J(s) \right] + \int_{s_0}^{s_1} K^J(t, s) \frac{\partial L}{\partial v_k{}^J}(t)\, dt, \tag{3.6}$$

then we can state a new variational principle in the following form:

THEOREM 1. *If* $dW[v(s), \mathcal{O}\!\ell\eta + \mathcal{K}\eta] = 0$ *for all* $\eta_k(s)$, $k = 1, 2, \ldots, n$, *which are of class* $C^{\Omega-1}$ *on* $[s_0, s_1]$, *then*

$$\frac{\delta L}{\delta v_k}(\mathcal{O}\!\ell, \mathcal{K}) = 0, \qquad k = 1, 2, \ldots, n. \tag{3.7}$$

The proof of this theorem follows immediately from the Fundamental Lemma of the Calculus of Variations and the fact that B is linear in η_k and its derivatives up to order $\Omega - 1$.

Equations (3.7) represent generalized equations of motion. The classical Euler–Lagrange equations, which are a special case of (3.7), follow from (3.7) by assuming

$$v_k^{\ J}(s) = x_k^{(J)}(s), \qquad (A_\Gamma^{\ J}(s)) = (\delta_\Gamma^{\ J}), \qquad N = \Omega,$$

and

$$K^J(s, t) = 0, \qquad J = 0, 1, \ldots, N.$$

Also, we note that the presence of the Fredholm operator does not affect the boundary terms; it enters only into the generalized variational derivatives. Therefore, the presence of such operators, as we shall see, cannot affect conservation laws that are obtained from invariance assumptions.

All of the calculations given in this paper hold true for the higher dimensional case, i.e., the case where the Lagrangian depends upon n functions defined on R^ν and their partial derivatives up to some order N. The details are given in [12].

4. Invariance with Respect to a Finite Group

We now apply the results of the previous section in order to obtain relations among the variational derivatives and certain divergences. Let $H_{\mu k} : M \to R^1$, $\mu = 1, \ldots, \rho$; $k = 1, \ldots, n$ be functions defined on the manifold and let ω^1, $\omega^2, \ldots, \omega^\rho$ be independent (essential) parameters. Further, suppose that the fundamental variations $\eta_k(s)$ are given by

$$\eta_k(s) = \omega^\mu(H_{\mu k} \circ v)\,(s), \tag{4.1}$$

i.e., they are determined by a finite continuous group. Now we give precise meaning to the statement that the functional W defined by (3.1) is invariant with respect to the variations given by (4.1).

Definition 1. W is divergence invariant with respect to

$$\eta_k(s) = \omega^\mu(H_{\mu k} \circ v)\,(s)$$

if there exist functions $G_\mu : M \to R^1$, $\mu = 1, 2, \ldots, \rho$ such that

$$dW[v(s), \mathcal{O}\omega^\mu(H_{\mu k} \circ v) + \mathcal{K}\omega^\mu(H_{\mu k} \circ v)] = \int_{s_0}^{s_1} \omega^\mu \frac{d}{ds}(G_\mu \circ v)\, ds \tag{4.2}$$

for all s_0 and s_1. If $G_\mu = 0$ for all μ, then we say that W is absolutely invariant.

A generalized form of the first Noether theorem can now be stated and proved. Briefly, it gives identities satisfied by the variational derivatives under hypothesis that the functional W is invariant according to Definition 1. Of particular importance for the determination of conservation laws is the fact that these identities reduce to divergences under the assumption that the generalized equations of motion (3.7) hold true. First, let us denote

$$\mathscr{B}\langle H_\mu; G_\mu\rangle \equiv (G_\mu \circ v)(s) - B\left(\frac{\partial L}{\partial v}, \mathcal{O}\!\!\!l, H_\mu \circ v, D(H_\mu \circ v), \ldots, D^{\Omega-1}(H_\mu \circ v)\right) \tag{4.3}$$

where B is given by (2.9).

THEOREM 2. *If W is divergence invariant with respect to the variations given by* (4.1), *then*

$$\frac{\delta L}{\delta v_k}(\mathcal{O}\!\!\!l, \mathscr{K})(H_{\mu k} \circ v)(s) = D\mathscr{B}\langle H_\mu; G_\mu\rangle, \qquad \mu = 1, \ldots, \rho. \tag{4.4}$$

Proof. By hypothesis with (4.2) and (3.5), it follows that

$$\omega^\mu \int_{s_0}^{s_1} \frac{\delta L}{\delta v_k}(\mathcal{O}\!\!\!l, \mathscr{K})(H_{\mu k} \circ v)(s)\, ds = \omega^\mu \int_{s_0}^{s_1} \frac{d}{ds}\mathscr{B}\langle H_\mu; G_\mu\rangle\, ds.$$

Since the parameters $\omega^1, \ldots, \omega^\rho$ are independent, and since the integration holds for all s_0, s_1, relations (4.4) follow.

COROLLARY. *If W is divergence invariant with respect to the variations given by* (4.1), *and if $\delta L/\delta v_k(\mathcal{O}\!\!\!l, \mathscr{K}) = 0$, $k = 1, 2, \ldots, n$, then*

$$\mathscr{B}\langle H_\mu; G_\mu\rangle = \text{constant}, \qquad \mu = 1, \ldots, \rho. \tag{4.5}$$

Equations (4.5) *will be interpreted as conservation laws since they represent first integrals of the generalized equations of motion.*

At this point we comment again on the occurrence of derivatives in variational and invariance problems in the calculus of variations. The present analysis shows that derivatives can enter variational problems in two independent ways, through the Lagrangian and through the constraint relations; if, in addition, invariance problems are considered, then derivatives can enter via two more independent routes—through the infinitesimal transformations and through the divergence invariance assumption. This is accomplished by taking

$$v_k{}^J(s) = x_k^{(J)}(s),$$

where $J = 0, 1,..., N$; $k = 1, 2,..., n$, and where $x_1(s),..., x_n(s)$ are functions defined on $[s_0, s_1]$. Herein lies the advantage of defining the Lagrangian on a manifold large enough to accommodate derivatives, but initially assuming no derivatives; one can observe precisely the role and occurrence of derivatives.

5. Invariance with Respect to an Infinite Group

We now assume a different type of invariance, namely one in which the fundamental variations $\eta_k(s)$ depend upon arbitrary functions and their derivatives up to some given order. In particular, we assume that

$$\eta_k(s) = \mathscr{H}_{\sigma k}[p^\sigma(s)], \qquad \sigma = 1, 2,..., \tau, \tag{5.1}$$

where $p^1(s),..., p^\tau(s)$ are arbitrary, independent, real-valued functions defined on $[s_0, s_1]$, and where the $\mathscr{H}_{\sigma k}$ are linear differential operators. For conciseness, we denote (5.1) symbolically by $\eta(s) = \mathscr{H}(p)$.

Definition 2. The functional W defined by (3.1) is invariant with respect to the variations (5.1) if

$$dW[v(s), \mathfrak{A}\mathscr{H}(p) + \mathscr{K}\mathscr{H}(p)] = 0. \tag{5.2}$$

If $\tilde{\mathscr{H}}_{\sigma k}$ denotes the adjoint of the operator $\mathscr{H}_{\sigma k}$, then we have the following generalization of the second Noether theorem.

Theorem 3. *If W is invariant with respect to the variations given by* (5.1), *then*

$$\tilde{\mathscr{H}}_{\sigma k}\left[\frac{\delta L}{\delta v_k}(\mathfrak{A}, \mathscr{K})\right] = 0, \qquad \sigma = 1,..., \tau. \tag{5.3}$$

Proof. By (5.2) and (3.5),

$$\int_{s_0}^{s_1} \frac{\delta L}{\delta v_k}(\mathfrak{A}, \mathscr{K})\, \mathscr{H}_{\sigma k}[p^\sigma(s)]\, ds$$
$$+ \int_{s_0}^{s_1} \frac{d}{ds} B\left(\frac{\partial L}{\partial v}, \mathfrak{A}, \mathscr{H}(p), D\mathscr{H}(p),..., D^{\Omega-1}\mathscr{H}(p)\right) ds = 0.$$

Since this equation holds for arbitrary $p^\sigma(s)$, in particular it holds for $p^\sigma(s)$ which vanish at s_0 and s_1 along with its derivatives up to order $\Omega - 1$. Also, since the operators $\mathscr{H}_{\sigma k}$ are linear, and since it follows from (2.9) that B is linear in $\mathscr{H}(p), D\mathscr{H}(p),..., D^{\Omega-1}\mathscr{H}(p)$, we have the second integral above vanishing. Therefore,

$$\int_{s_0}^{s_1} \frac{\delta L}{\delta v_k}(\mathcal{O}\!\!l, \mathscr{K})\, \mathscr{H}_{\sigma k}[p^\sigma]\, ds = 0.$$

Rewriting this equation in terms of the adjoint operators,

$$\int_{s_0}^{s_1} \tilde{\mathscr{H}}_{\sigma k}\left[\frac{\delta L}{\delta v_k}(\mathcal{O}\!\!l, \mathscr{K})\right] \cdot p^\sigma(s)\, ds = 0.$$

Again, by the independence of the $p^\sigma(s)$, and by the Fundamental Lemma of the Calculus of Variations, the relations (5.3) follow.

6. Invariance of Mixed Type

In this section we shall indicate how variations of the types (4.1) and (5.1) can be considered simultaneously in invariant variational problems in order to obtain identities satisfied by the generalized variational derivatives. Classically, this would mean that the group of transformations under which the action is invariant depends upon ρ parameters and upon τ arbitrary functions and their derivatives up to some order.

Definition 3. The functional W is divergence invariant with respect to the variations defined by (4.1) and (5.1) if there exist functions $G_\mu : M \to R^1$ such that

$$\begin{aligned} &dW[v(s), \mathcal{O}\!\!l(\omega^\mu(H_\mu \circ v) + \mathscr{H}(p)) + \mathscr{K}(\omega^\mu(H_\mu \circ v) + \mathscr{H}(p))] \\ &\quad = \int_{s_0}^{s_1} \omega^\mu \frac{d}{ds}(G_\mu \circ v)(s)\, ds \end{aligned} \tag{6.1}$$

for all s_0, s_1. Writing (6.1) out using (3.5), we obtain

$$\begin{aligned} &\int_{s_0}^{s_1} \left\{\frac{\delta L}{\delta v_k}(\mathcal{O}\!\!l, \mathscr{K})\right\} [\omega^\mu(H_{\mu k} \circ v) + \mathscr{H}_{\sigma k}[p^\sigma]]\, ds \\ &\quad + \int_{s_0}^{s_1} \frac{d}{ds} B\left(\frac{\partial L}{\partial v}, \mathcal{O}\!\!l, \omega^\mu(H_\mu \circ v) + \mathscr{H}(p),..., D^{\Omega-1}(\omega^\mu(H_\mu \circ v) + \mathscr{H}(p)\right) ds \\ &\quad = \omega^\mu \int_{s_0}^{s_1} \frac{d}{ds}(G_\mu \circ v)\, ds. \end{aligned}$$

Again, using the linearity of B and the arbitrariness of the $p^\sigma(s)$, we obtain

$$\int_{s_0}^{s_1} \frac{\delta L}{\delta v_k}(\mathcal{A}, \mathcal{K})\, \omega^\mu(H_{\mu k} \circ v)(s)\, ds + \int_{s_0}^{s_1} \frac{\delta L}{\delta v_k}(\mathcal{A}, \mathcal{K})\, \mathcal{H}_{\sigma k}[p^\sigma(s)]\, ds$$

$$= \omega^\mu \int_{s_0}^{s_1} \frac{d}{ds} \mathcal{B}\langle H_\mu; G_\mu\rangle\, ds$$

Upon writing the second integral above in terms of the adjoint operators $\tilde{\mathcal{H}}_{\sigma k}$ and upon appealing to the independence of the parameters $\omega^1, \ldots, \omega^\rho$ and the functions $p^1(s), \ldots, p^\tau(s)$, we conclude that

$$\int_{s_0}^{s_1} \frac{\delta L}{\delta v_k}(\mathcal{A}, \mathcal{K})\,(H_{\mu k} \circ v)(s)\, ds = \int_{s_0}^{s_1} \frac{d}{ds} \mathcal{B}\langle H_\mu; G_\mu\rangle\, ds, \qquad \mu = 1, \ldots, \rho,$$

$$\int_{s_0}^{s_1} \tilde{\mathcal{H}}_{\sigma k}\left[\frac{\delta L}{\delta v_k}(\mathcal{A}, \mathcal{K})\right] p^\sigma(s)\, ds = 0, \qquad \sigma = 1, \ldots, \tau.$$

Therefore, using the Fundamental Lemma of the Calculus of Variations and the arbitrariness of the region of integration, we have the following theorem.

THEOREM 4. *If W is divergence invariant with respect to the variations given by* (4.1) *and* (5.1), *then the following $\tau + \rho$ relations among the generalized variational derivatives hold true:*

$$\frac{\delta L}{\delta v_k}(\mathcal{A}, \mathcal{K})\,(H_{\mu k} \circ v) = D\mathcal{B}\langle H_\mu; G_\mu\rangle, \qquad \mu = 1, \ldots, \rho,$$

$$\tilde{\mathcal{H}}_{\sigma k}\left[\frac{\delta L}{\delta v_k}(\mathcal{A}, \mathcal{K})\right] = 0, \qquad \sigma = 1, \ldots, \tau.$$

7. AN EXAMPLE

The general concepts presented above are not without example. In fact, the generalized variational principle given in Section 3 was motivated by a specific variational principle in hydromechanics given by S. Drobot and A. Rybarski [13] in 1959. They considered an action integral

$$W = \int L(x, p(x))\, dx,$$

where the Lagrangian did not depend upon derivatives, but derivatives were introduced via the so-called hydromechanical variations

$$\delta p^\alpha = \frac{\partial}{\partial x^\beta}(p^\beta(x)\, \delta x^\alpha - p^\alpha(x)\, \delta x^\beta),$$

where $x = (x^1,..., x^4)$, $p(x) = (p^1(x),..., p^4(x))$, and α, $\beta = 1, 2, 3, 4$. Such considerations led to the discovery of which mathematical facts cause the hydromechanical equations of motion to follow from a variational principle independently of the form of the Lagrangian, and which particular facts lead to the conservation laws.

The general results given in this paper characterize the conditions on the variations under which equations of motion can be obtained. The form of the Lagrangian or its initial dependence upon derivatives does not play an essential role; however, a significant role is played by relations on the variations. Further examples from electrodynamics are given in [12].

REFERENCES

1. E. NOETHER, Invariante Variationsprobleme, *Nachr. Akad. Wiss. Gottingen, Math.-Phys. Kl. II* **1918** (1918), 235–257.
2. E. BESSEL-HAGEN, Über die Erhaltungssätze der Electrodynamik, *Math. Ann.* **84** (1921), 258–276.
3. D. HILBERT, Gundlagen der Physik, *Math. Ann.* **92** (1924), 258–289.
4. A. TRAUTMAN, Noether's equations and conservation laws, *Comm. Math. Phys.* **6** (1967), 248–261.
5. J. KOMOROWSKI, A modern version of the E. Noether's theorems in the calculus of variations, I, *Studia Math.* **29** (1968), 261–273.
6. J. KOMOROWSKI, A modern version of the E. Noether's theorems in the calculus of variations, II, *Studia Math.* **32** (1969), 181–190.
7. H. RUND, Invariant theory of variational problems for geometric objects, *Tensor* **18** (1967), 240–257.
8. D. G. B. EDELEN, "Nonlocal Variations and Local Invariance of Fields," American Elsevier, New York, 1969.
9. V. R. TIHOMIROV, A generalization of the Noether theorem for tensor fields of arbitrary rank, given in a Riemannian space (Russian), *Karbardino-Balkarsk Gos. Univ. Ucen, Zap.* **24** (1965), 271–274.
10. N. M. KUHARČUK, Certain questions of the calculus of variations for functionals on vector functions with values in a Banach space (Russian), *Ukrain. Mat. Z.* **19** (1967), No. 1, 95–98.
11. S. MACLANE, Hamiltonian mechanics and geometry, *Amer. Math. Monthly* **77**, No. 6 (1970), 570–586.
12. J. D. LOGAN, "Noether's Theorems and the Calculus of Variations," Ph.D. Dissertation, The Ohio State University, September 1970.
13. S. DROBOT AND A. RYBARSKI, A variational principle in hydromechanics, *Arch. Rational Mech. Anal.* **2** (1958–59), 393–410.