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# STRUCTURE AND COSTRUCTURE FOR STRONGLY REGULAR RINGS

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## 0. Introduction and preliminaries

This paper was initiated by considering Michael Barr's conjecture that commutative regular rings form the equational completion of (i.e., are the result of applying structure semantics (see [5] and [6]) to) the category of products of fields and ring homomorphisms. This is shown to be true and also that strongly regular rings form the equational completion of products of skew fields. In the process we find a canonical way of representing a strongly regular ring with unit as an equalizer of maps between products of skew fields. A close look at this representation shows that strongly regular rings with unit are coalgebras under the category $\mathcal{K}$ of products of fields and "coordinated" unitary ring homomorphisms. The proof shows that the category of sheaves of skew fields over compact Hausdorff spaces is cotripleable under $\mathcal{K}$. Incidentally, $\mathcal{K}$ is tripleable over sets, see [5]. The Appelgate–Tierney iterated cotriple construction (see [1]) starting with the left adjoint of the forgetful functor from $\mathcal{K}$ to **Rings** arrives at strongly regular rings with unit after two stages.

The last two sections of this paper contain some somments about equational completions and describe the free regular ring generated by a given ring (in the commutative case).

### 0.1. Terminology and preliminary remarks

(1) In any ring $R$, we define $s$ to be the *semi-inverse* of $r$ if $r = r^2s$, $s = s^2r$ and $rs = sr$. It is easily shown (see [5]) that semi-inverses are unique and preserved by homomorphisms. We use $\bar{r}$ to denote the semi-inverse of $r$. Occassionally, for a longer expression such as $1 - x$ we use $(1 - x)^-$ to denote the semi-inverse.

(2) The word *ideal* means two-sided ideal. However, for strongly regular rings every left or right ideal is two-sided.

(3) The word *map* means "morphism in the appropriate category".

(4) In a topological space a subset is *clopen* if it is both closed and open. It is well known that a compact Hausdorff space is totally disconnected iff it has a base

of clopen sets. Also, the quotient of a compact Hausdorff space by the relation of "being in the same component" is Hausdorff and totally disconnected.

(5) A ring $R$ is *strongly regular* iff for every $a \in R$ there exists an $x$ with $a = a^2x$. This and other characterizations are given in [2]. Some known properties of strongly regular rings are listed below and for convenience a proof is sketched.

**0.2. Proposition.** *If R is a strongly regular ring, then:*

(1) *every nilpotent is* 0;

(2) $ab = 0$ *iff* $ba = 0$;

(3) *every idempotent is central*;

(4) *if* $a = a^2x$, *then* $ax = xa$ *is idempotent*;

(5) *every element has a semi-inverse*;

(6) *R is embeddable in a product of skew fields.*

**Proof.** (1) If $a = a^2x$, then $a = a^{n+1}x^n$ for all $n$.

(2) If $ab = 0$, then $(ba)^2 = 0$.

(3) Let $e^2 = e$. Then $(exe - ex)e = 0$, so $e(exe - ex) = 0$ or $exe = ex$. Similarly $exe = xe$.

(4) $a^2(x - xax) = 0$, so $(x - xax)a^2 = 0$, so $xa(a - xa^2) = 0$, so $(a - xa^2)xa = 0$ or $axa = xa^2$, so $(ax - xa)a = 0$ or $a(ax - xa) = 0$ or $a = axa = xa^2$. So $ax = xa^2x = xa$.

(5) Let $a = a^2x$. Then $x^2a$ is the semi-inverse of $a$.

(6) Given $a \in R$, find a maximal ideal not containing $a\bar{a}$. □

**0.3. Corollary.** *The strongly regular rings are algebraic over sets. The notions of prime ideal, primitive ideal, maximal ideal, and maximal left (or right) ideal coincide for such rings. Moreover, every left or right ideal is two-sided.*

## 1. Canonical representation with limit maps

**1.1. Notation and construction.** In what follows, $R$ denotes a strongly regular ring with unit. We let $\{M_x \mid x \in X\}$ be a one-to-one indexing of the maximal ideals of $R$. For each $r \in R$ we let

$$Z(r) = \{x \in X \mid r \in M_x\}, \qquad C(r) = X \smallsetminus Z(r).$$

(We think of $Z(r)$ and $C(r)$ as the "zero-set" and "cozero-set" of $r$.) The family $\{Z(r) \mid r \in R\}$ is a base for the closed sets of the spec topology on $X$. Since $C(r) = Z(1 - r\bar{r})$ and $Z(r) = C(1 - r\bar{r})$, we see that $\{Z(r)\}$ is a clopen base. Also, the spec topology is compact, Hausdorff and totally disconnected. [We write $X = \operatorname{spec} R$ even if $R$ is not commutative.]

For each $x \in X$, let $K_x = R/M_x$ and let $q_x : R \to K_x$ be the quotient map. We define $P(R)$ to be $\pi\{K_x \mid x \in X\}$. In the obvious way we identify $R$ as a subring of $P(R)$. That is, if $p_x : P(R) \to K_x$ is the projection map, then $p_x | R = q_x$.

Incidentally, $P$ can obviously be regarded as a functor on the category $\mathcal{R}$ of strongly regular rings with unit. To see this, let $t : R \to S$ be in $\mathcal{R}$, and let $\{M_x \mid x \in X\}$ be the maximal ideals of $R$ and let $\{N_y \mid y \in Y\}$ be those of $S$. For each $y \in Y$, note that $t^{-1}(N_y)$ is a prime, hence (by 0.3) maximal, ideal of $R$, so $t^{-1}(N_y) = M_x$ for a unique $x$. Let $t_y : R/M_x \to S/N_y$ be the induced map. Then $P(t) : P(R) \to P(S)$ is defined so that $p_y P(t) = t_y p_x$.

If $f \in P(R)$, then $Z(f) = \{x \mid p_x f = 0\}$. Clearly this extends the $Z(r)$ notation for $r \in R$. However, if $f \notin R$, then $Z(f)$ need not be open or closed. $C(f)$ is the complement of $Z(f)$.

We let $\beta X$ denote the set of ultrafilters on $X$. [Thus $\beta X$ can be regarded as the Stone–Čech compactification of $X$ when $X$ is given the discrete topology.] For each $U \in \beta X$ we define

$$M_U = \{f \in P(R) \mid Z(f) \in U\}, \qquad K_U = P(R)/M_U .$$

Let $q_U : P(R) \to K_U$ be the quotient map. Clearly we have

$$P^2(R) = \prod \{K_U \mid U \in \beta X\} .$$

We let $p_U : P^2(R) \to K_U$ be the projection.

We do *not* identify $P(R)$ as a subring of $P^2(R)$ but rather let $\eta : P(R) \to P^2(R)$ be the canonical injection (i.e., for which $p_U \eta = q_U$ for all $U \in \beta X$).

There is another "reasonable" map from $P(R)$ to $P^2(R)$. For each $U \in \beta X$, note that $M_U \cap R$ is a maximal ideal of $R$. It is readily seen that $M_U \cap R = M_x$, where $x = \lim U$ in the spec topology. By the first isomorphism theorem there exists a map $\lambda_U : K_x \to K_U$ for which $\lambda_U q_x = q_U | R$. We let $\lambda : P(R) \to P^2(R)$ be defined by the equations $p_U \lambda = \lambda_U p_x$ (where $x = \lim U$). The maps $\{\lambda_U\}$ are called the *limit maps* for $R$.

**1.2. Comment.** As shown in the next section, $P$ can be regarded as the left adjoint to an appropriate forgetful functor. Therefore $P$ can be thought of as part of a triple on the category of strongly regular rings with unit. In this context, $\eta$ and $\lambda$ would be denoted by $\eta_{PR}$ and $P(\eta_R)$, respectively.

**1.3. Theorem.** *$R$ is the* equalizer *of $\eta$ and $\lambda$. That is, for $f \in P(R)$ we have $f \in R$ iff $\eta(f) = \lambda(f)$ iff $q_U(f) = \lambda_U P_x(f)$ for all $U \in \beta X$ (in the last equation it is implicit that $x = \lim U$ in order for the composition $\lambda_U P_x$ to make sense).*

**Proof.** We define $f \in P(R)$ to be *continuous* if $q_U(f) = \lambda_U P_x(f)$ for all $U \in \beta X$ (it is understood that $x$ must be $\lim U$ in this kind of equation). Clearly every $r \in R$ is continuous and we must prove the converse, viz. that every continuous $f$ is in $R$. We proceed by a series of lemmas. (The term *continuous* can be given a topological context, see Remark 1.9.)

**1.4. Lemma.** *If $f$ is continuous, then $Z(f)$ is a closed subset of $X$.*

**Proof.** Choose $y$ in the closure of $Z(f)$. This means that there exists $U \in \beta X$ with $y = \lim U$ and $Z(f) \in U$. Since $Z(f) \in U$, we have $q_U(f) = 0$ or $\lambda_U P_y(f) = 0$ or that $P_y(f) = 0$ or that $y \in Z(f)$. (Note that $\lambda_U$ is one-to-one as it is a unitary ring homomorphism between skew fields.)

**1.5. Lemma.** *If $f$ is continuous, $Z(f - r)$ is clopen for all $r \in R$.*

**Proof.** Clearly the continuous elements of $P(R)$ form a strongly regular subring which includes $R$. By 1.4, each $Z(f - r)$ is closed. Also

$$X \setminus Z(f - r) = Z(1 - (f - r)(f - r)^-) .$$

**1.6. Lemma.** *For each $A \subseteq X$ define $\chi_A \in P(R)$ in the obvious way (that is, $p_x \chi_A = 1$ if $x \in A$ and 0 if $x \notin A$). Then $\chi_A \in R$ iff $A$ is clopen.*

**Proof.** Let $A$ be clopen. Since $A$ is open, it is a union of basic open sets of the form $Z(r)$ for $r \in R$. Since $A$ is compact, there exist $r_1, r_2, ..., r_n$ for which $A = \bigcup Z(r_i)$. Let $r = r_1 ... r_n$. Then $\chi_A = 1 - r\bar{r}$.

**1.7. Lemma.** *Let $f$ be continuous, and let $r, s \in R$ be given. Then there exists $t \in R$ such that*

$$Z(f - t) \supseteq Z(f - r) \cup Z(f - s) .$$

**Proof.** Let $A = Z(f - r) \setminus Z(f - s)$, and let $B = Z(f - s)$. Choose $t = \chi_A r + \chi_B s$.

**1.8. Corollary.** *Let $f$ be continuous, and let $r_1, ..., r_n \in R$ be given. Then there exists $t \in R$ such that $Z(f - t)$ contains $\bigcup Z(f - r_i)$.*

**Proof of 1.3.** Let $f$ be continuous. For all $x \in X$ there exists $r \in R$ with $x \in Z(f - r)$. By compactness there exist $r_1, ..., r_n \in R$ with $X = \bigcup Z(f - r_i)$. By 1.8 there exists $t \in R$ with $X = Z(f - t)$. This implies $f = t \in R$. □

**1.9. Remark.** The limit maps $\{\lambda_U\}$ can be used to define a topology on the disjoint union $\bigcup K_x$. For each $U \in \beta X$ we define a *U-section* to consist of a set $A \in U$ and $f \in \Pi \{K_x \mid x \in A\}$. For each $U$-section $f$ it makes sense to discuss $q_U(f) \in K_U$. Also the "range" of $U$-section can be regarded as a subset of $\bigcup K_x$.

Given $b \in K_x$ and $S \subseteq \bigcup K_x$, we say that $b$ is in the closure of $S$ if there exists $U \in \beta X$ with $x = \lim U$ and a $U$-section $f$ whose range is in $S$ and for which $q_U(f) = \lambda_u(b)$. Then the projection $\bigcup K_x \to X$ is the sheaf constructed in [4]. The continuous sections $X \to \bigcup K_x$ correspond to the continuous members of $P(R)$. We soon consider the related questions of given an "arbitrary" set of $\{\lambda_U\}$ when do they define a sheaf topology? First we establish some notation.

**1.10. Notation for ultrafilters**

For any set $S$ we let $\beta S$ be the set of ultrafilters on $S$. For each $A \subseteq S$ we let

$$I_A = \{U \in \beta S \mid A \in U\}.$$

The family $\{I_A \mid A \subseteq S\}$ is a clopen base for a compact Hausdorff topology on $\beta S$. (In fact this makes $\beta S$ the Stone–Čech compactification of $S$ with the discrete topology.) For each $s \in S$ we let $\langle s \rangle \in \beta S$ denote the "constant ultrafilter" consisting of all $A \subseteq S$ with $s \in A$. If $\Pi\{K_s \mid s \in S\}$ is a product of fields, then $\beta S$ canonically indexes the maximal ideals (as noted before). The spec topology on $\beta S$ trivially coincides with the topology generated by $\{I_A \mid A \subseteq S\}$. Note that $\beta^2 S$ is the set of ultrafilters on $\beta S$. For each $\Omega \in \beta^2 S$ we define $\Omega_0 \in \beta S$ by

$$\Omega_0 = \{A \subseteq S \mid I_A \in \Omega\}.$$

We note that $\Omega_0 = \lim \Omega$ in the $\{I_A\}$-topology on $\beta S$.

In the case of a compact Hausdorff space $X$, there is another map from $\beta^2 X$ to $\beta X$. First, for each family $\mathcal{A} \subseteq \beta X$ we define $\lim \mathcal{A}$ to be $\{\lim U \mid U \in \mathcal{A}\}$. Then if $\Omega \in \beta^2 X$, we define $\Omega_1 \in \beta X$ by

$$\Omega_1 = \{\lim \mathcal{A} \mid \mathcal{A} \in \Omega\}.$$

It can be shown that (in the topology on $X$) one always has $\lim \Omega_0 = \lim \Omega_1$.

**1.11. Problem.** When is a set of maps $\{\lambda_U\}$ the limit maps of a strongly regular ring? That is, let $X$ be a compact, Hausdorff, totally disconnected space. Let $T = \Pi\{K_x \mid x \in X\}$ be a product of a family of skew fields indexed by $X$. For each $U \in \beta X$ let $M_U$ be the corresponding maximal ideal; let $K_U = T/M_U$, and let $q_U : T \to K_U$ be the quotient map. We also adopt the other notation defined above (such as $p_x$ and $Z(f)$ for $f \in T$).

Whenever $\lim U = x$, suppose that a unitary ring homomorphism $\lambda_U : K_x \to K_U$ is given. When does there exist a strongly regular ring $R$ such that $T = P(R)$, spec $R = X$ and such that $\{\lambda_U\}$ are the limit maps of $R$ as in Theorem 1.3?

**Solution.** Note that $P(T) = \Pi\{K_U \mid U \in \beta X\}$. Define two maps $\lambda, \eta : T \to P(T)$ so that $p_U \eta = q_U$ for all $U$ and $p_U \lambda = \lambda_U p_x$ for all $U$. We define $R$ to be the equalizer of $\lambda$ and $\eta$. By 1.3 this is the only possible solution.

For each $\Omega \in \beta^2 X$ let $M_\Omega$ be the corresponding maximal ideal of $P(T)$, and let $q_\Omega : P(T) \to K_\Omega$ be the corresponding quotient map. Recall the above definitions of $\Omega_0$ and $\Omega_1$. Then it is readily shown that $\eta^{-1}(M_\Omega) = M_{\Omega_0}$, so $\eta$ induces a map $\eta_\Omega : K_{\Omega_0} \to K_\Omega$ for which $q_\Omega \eta = \eta_\Omega q_{\Omega_0}$. Similarly, $\lambda^{-1}(M_\Omega) = M_{\Omega_1}$, so $\lambda$ induces a map $\lambda_\Omega : K_{\Omega_1} \to K_\Omega$ for which $q_\Omega \lambda = \lambda_\Omega q_{\Omega_1}$. We can now state the required conditions on $\lambda$.

**1.12. Theorem.** *With the above notation,* $\{\lambda_U\}$ *is the set of limit maps of* $R$ *and* spec $R = X$ *and* $P(R) = T$ *in the obvious way iff the following* coherence conditions *are satisfied*:

(CC 1) *If $U = \langle x \rangle$, the constant ultrafilter, then $\lambda_U$ is the obvious isomorphism. That is, $\lambda_{\langle x \rangle} p_x = q_{\langle x \rangle}$.*

(CC 2) *For all $\Omega \in \beta^2 X$, we have*

$$\eta_\Omega \lambda_{\Omega_0} = \lambda_\Omega \lambda_{\Omega_1}.$$

**1.13. Comments.**

(a) These two conditions are suggestive of the rules of behavior for the limits of ultrafilters in a compact Hausdorff space. That is, the limit of a constant ultrafilter must be the obvious one and a "limit of limits" can be evaluated in two ways which must coincide.

(b) The maps $\{\lambda_U\}$ can be used to topologize $\mathbf{U}K_x$ as a space over $X$. The coherence conditions are what is needed to make this a sheaf of rings in the sense of [3].

(c) A concise way of stating (CC 1) and (CC 2) is to simply state that $\lambda$ is a *costructure map* for the cotriple associated with $P$. See Section 2 for details.

**Proof of 1.12.** Let $f \in T$. Then, by definition, $f \in R$ iff $\eta(f) = \lambda(f)$ iff $q_U(f) = \lambda_U p_x(f)$ for all $U \in \beta X$. We proceed by a series of lemmas.

**1.8. Lemma.** *For each $A \subseteq X$, let $\chi_A$ be defined in the obvious way (cf. 1.6). Then $\chi_A \in R$ iff $A$ is clopen.*

**Proof.** The kind of argument used in (1.6) works.

**1.15. Lemma.** *$R$ has $\{\lambda_U\}$ for the limit maps iff each $p_x$ maps $R$ onto $K_x$.*

**Proof.** Clearly it is necessary for each $p_x$ to map $R$ onto $K_x$. Conversely, assume that each $p_x$ is onto, and let $M$ be a maximal ideal of $R$. Define $\mathcal{F}$ as the set of all clopen $A \subseteq X$ for which $\chi_A \notin M$. Clearly $\mathcal{F}$ is closed under finite intersections, so by compactness there exists $x \in \bigcap \mathcal{F}$. As $X$ is totally disconnected, one can show that $x$ is unique. It is readily seen that $M$ is the kernel of $p_x \mid R$. Also the spec topology on $X$ coincides with the given topology since they have the same clopen sets (by 1.14 and 1.6). Since $p_x$ maps $R$ onto $K_x$, we see that $P(R) = T$, and since $\lambda_U(p_x \mid R) = (q_U \mid R)$ by choice of $R$ we see that $\{\lambda_U\}$ are the limit maps.

**1.16. Lemma.** (CC 1) *and* (CC 2) *are necessary.*

**Proof.** Since every element of $K_x$ would have to be the image under $p_x$ of an element of $R$ (if $R$ has the desired properties) one can verify (CC 1) and (CC 2) by diagram chasing.

**1.17. Lemma.** *Assume* (CC 1) *and* (CC 2). *Let $x \in X$ and $b \in K_x$ be given. Then there exists $f \in T$ with $p_x(f) = b$ and $q_U(f) = \lambda_U(b)$ for all $U$ with* $\lim U = x$.

**Proof.** Let

$$N_x = \{f \in T \mid Z(f) \text{ is a neighborhood of } x\}.$$

Let $T_x = T/N_x$, and let $s : T \to T_x$ be the quotient map. $T_x$ is a strongly regular ring to which we shall apply Theorem 1.3. We must find the maximal ideals and limit maps for $T_x$.

Clearly the maximal ideals of $T_x$ correspond to the maximal ideals of $T$ which contain $N_x$. But $N_x \subseteq M_U$ iff $\lim U = x$. Let

$$L_x = \{U \in \beta X \mid \lim U = x\}.$$

Then

$$P(T_x) = \Pi\{K_U \mid U \in L_x\}.$$

For each $U \in L_x$, let $\bar{q}_U : T_x \to K_U$ be the associated quotient map for which $\bar{q}_U s = q_U$. Let $\bar{p}_U : P(T_x) \to K_U$ be the projection map. Let $e : T_x \to P(T_x)$ be the natural embedding for which $\bar{p}_U e = \bar{q}_U$. Let $t : P(T) \to P(T_x)$ be determined by $\bar{p}_U t = p_U$ for all $U \in L_x$. It follows that $es = t\eta$.

The maximal ideals of $P(T_x)$ are indexed by $\beta L_x$ which can be easily identified with the set of $\Omega \in \beta^2 X$ for which $L_x \in \Omega$. For each such $\Omega$ there obviously exists $\bar{q}_\Omega : P(T_x) \to K_\Omega$ for which $\bar{q}_\Omega t = q_\Omega$. It is readily verified that $\{\eta_\Omega \mid L_x \in \Omega\}$ are the limit maps for $T_x$. Therefore by 1.3, in effect, we see that $e$ is an isomorphism between $T_x$ and the set of all $f \in P(T_x)$ for which $\bar{q}_\Omega(f) = \eta_\Omega \bar{p}_{\Omega_0}(f)$ for all $\Omega \in \beta^2 X$ with $L_x \in \Omega$.

Define $m : K_x \to P(T_x)$ by $\bar{p}_U m = \lambda_U$ for all $U \in L_x$. We claim that $m$ factors through $e$. It suffices to show that $\bar{q}_\Omega m = \eta_\Omega \bar{p}_{\Omega_0} m$ for all $\Omega$ with $L_x \in \Omega$. Since $p_x$ is onto, it suffices to show that $\bar{q}_\Omega m p_x = \eta_\Omega \bar{p}_{\Omega_0} m p_x$. But $m p_x = t\lambda$ (by compositions with $\bar{p}_u$) so

$$\bar{q}_\Omega m p_x = q_\Omega \lambda = \lambda_\Omega q_{\Omega_1}.$$

On the other hand,

$$\eta_\Omega \bar{p}_{\Omega_0} m p_x = \eta_\Omega \lambda_{\Omega_0} p_x = \lambda_\Omega \lambda_{\Omega_1} p_x$$

by using (CC 2). So it suffices to show that $\lambda_\Omega q_{\Omega_1} = \lambda_\Omega \lambda_{\Omega_1} p_x$. But $L_x \in \Omega$, so $\{x\} = \lim L_x \in \Omega_1$ which means that $\Omega_1 = \langle x \rangle$, therefore $\lambda_{\Omega_1} p_x = q_{\Omega_1}$ by (CC 1).

This verifies our claim, so there exists $n : K_x \to T_x$ for which $m = en$. Choose $f \in T$ for which $s(f) = n(b)$. Then, as is easily shown, $f$ is the desired element.

**Proof of 1.12.** Assume (CC 1) and (CC 2). Let $b \in K_x$ be given. By 1.15 we must find $h \in R$ with $p_x(h) = b$. Let $f \in T$ be as in 1.17, so that $p_x(f) = b$ and $q_U(f) = \lambda_U(b)$ whenever $x = \lim U$. Let

$$\mathcal{D} = \{U \in \beta X \mid q_U(f) \neq \lambda_U p_y(f), \text{ where } y = \lim U\}.$$

If $\mathcal{D}$ is empty, then $f \in R$, and we are finished. For each neighborhood $N$ of $x$ define

$$D_N = \{U \in \mathcal{D} \mid N \in U\}.$$

We claim that there exists $N$ for which $D_N$ is empty. Assume the contrary. Then since $D_N \cap D_L \supseteq D_{N \cap L}$, there exists $\Omega \in \beta^2 X$ such that $D_N \in \Omega$ for all neighborhoods $N$. Clearly

$$x = \lim \Omega_0 = \lim \Omega_1 .$$

Therefore by choice of $f$ we have

$$q_{\Omega_0}(f) = \lambda_{\Omega_0}(b), \qquad q_{\Omega_1}(f) = \lambda_{\Omega_1}(b).$$

It follows that $q_\Omega \eta(f) = q_\Omega \lambda(f)$ (using the choice of $f$ and (CC 2)). This means that $\eta(f) - \lambda(f)$ is in $M_\Omega$, so there exists $\mathcal{A} \in \Omega$ such that for all $U \in \mathcal{A}$ we have $p_U \eta(f) = p_U \lambda(f)$. In other words, $q_U(f) = \lambda_U p_y(f)$, where $y = \lim U$. This means that $\mathcal{A}$ is disjoint from $\mathcal{D}$ hence from each $D_N$, contradicting that $\mathcal{A} \in \Omega$ and $D_N \in \Omega$.

Using the verified claim, we can readily find a clopen neighborhood $N$ of $x$ such that whenever $N \in U$, then $q_U(f) = \lambda_U p_y(f)$ (for $y = \lim U$). Let $h = \chi_N f$. It is clear that $h \in R$ and $p_x(h) = b$. □

## 2. Coalgebras under products of skew fields

### 2.1. Notation and preliminary remarks

(1) A function $f$ from the product set $\Pi\{S_i \mid i \in I\}$ to the product set $\Pi\{T_j \mid j \in J\}$ is said to be *coordinated* iff for each $j \in J$ there exists $i = f^*(j) \in I$ and a map $f_j : S_i \to T_j$ such that $p_j f = f_j P_i$. The function $f^* : J \to I$ is an *index for* $f$ and is uniquely determined if the sets involved are fields and if $f$ is a unitary ring homomorphism. The maps $f_j$ are the *coordinates* of $f$.

(2) We let $\mathcal{K}$ be the category of products of skew fields and *coordinated* unitary ring homomorphisms. This category is tripleable over sets (see [5], and note that coordinated is equivalent to continuous in topology defined in [5]).

(3) Let $\mathcal{R}$ be the category of strongly regular rings with unit. There is an obvious forgetful functor $\mathcal{K} \to \mathcal{R}$ and it has left adjoint $P$, the functor described in the beginning of the previous section. We continue to use the notation $X$, $M_x$, $K_x$, $p_x$, $q_x$ developed in that section. The front adjunction $\eta_R : R \to P(R)$ was previously called the "obvious" embedding of $R$ into $P(R)$. Thus $\eta_R$ is defined by $p_x \eta_R = q_x$ for all $x \in X$. The maps called $\eta$ and $\lambda$ in Theorem 1.3 would be $\eta_{P(R)}$ and $P(\eta_R)$ respectively, in the present context.

(4) The above adjointness generates a cotriple $(P, \epsilon, \delta)$ on $\mathcal{K}$. If $K = \Pi\{K_x \mid x \in X\}$, then $P(K) = \Pi\{K_U \mid U \in \beta X\}$, and $\epsilon : P(K) \to K$ is defined so that $p_x \epsilon = p_{(x)}$. The map $\delta : P(K) \to P^2(K)$ is defined as $P(\eta_K)$. Notice that we are

gently abusing the language by letting "$P$" denote both a functor from $\mathcal{R}$ to $\mathcal{K}$ as well as the restriction to a functor from $\mathcal{K}$ to $\mathcal{K}$. For example $\eta_K : K \to P(K)$ is a unitary ring homomorphism but is not coordinated (unless $X$ is finite).

**2.1. Theorem.** *Let $K = \Pi\{K_x \mid x \in X\}$. Let $\lambda : K \to P(K)$ be a morphism of K. Then $\lambda$ is a* costructure *map for a coalgebra* with *respect to the cotriple $(P, \epsilon, \delta)$ iff the coordinates of $\lambda$ (that is, the induced maps $\lambda_U$) satisfy the coherence conditions of Theorem* 1.12. *In this case $\lambda$ defines a compact Hausdorff topology on X and a topology on $\bigcup K_x$ making $\bigcup K_x \to X$ a sheaf. Thus the category of sheaves of skew fields (i.e., the stalks are skew fields) over compact Hausdorff spaces is cotripleable under $\mathcal{K}$. The full subcategory of sheaves with totally disconnected base is coreflective and is equivalent to $\mathcal{R}$.*

[*Note*: We call a subcategory "coreflective" if its inclusion functor has a right adjoint. Other authors call such a category "reflective".]

**Proof.** That the property of $\lambda$ being a costructure map is equivalent to the coherence condition of 1.12 is a matter of straightforward verification.

If $\lambda$ is such a map, then the index for $\lambda$, namely $\lambda^* : \beta X \to X$ is easily seen to be a structure map for the triple $\beta$. (One easily verifies that $\lambda^*\nu^* = (\nu\lambda)^*$ and $[P(\lambda)]^* = \beta(\lambda^*)$.) This induces a compact Hausdorff topology on $X$. The "limit maps" $\{\lambda_U\}$ then define a sheaf topology because of the arguments in the proof of 1.12. For example Lemma 1.17 still holds and all but the last paragraph of the proof of 1.12 still applies.

If $X$ should be totally disconnected, then 1.12 says in effect that $X$ is the coalgebra arising from a strongly regular ring. If $X$ is not totally disconnected, it can be made totally disconnected by identifying components. If $A \subseteq X$ is a component, then the equalizer of $\lambda$ and $\eta$ "restricted" to $\Pi\{K_x \mid x \in A\}$ must be a skew field $J_A$. (For $J_A$ is certainly strongly regular and any non-trivial idempotent would give rise to a clopen subset of $A$.) It is a straightforward verification to show that $\Pi J_A$ can be given a compatible costructure map and coreflects $(K, \lambda)$ into the full subcategory of coalgebras with totally disconnected index sets. □

**2.3. Comments**

(1) The cotripleability of sheaves is suggestive of the cotripleability obtained by van Osdol [3]. However, there the base space $X$ was fixed. Here $X$ can vary among the compact Hausdorff spaces, but the stalks must all be skew fields. We should point out that the morphisms between two such sheaves must be the cohomomorphism from $p$ to $q$ turns out to be a pair $(f, h)$, where $f : Y \to X$ is continuous and where for each $y \in Y$ there exists $h_y$, a unitary ring homomorphism from the field $p^{-1}(f_y)$ to the field $q^{-1}(y)$, such that continuous sections over open sets are preserved. (That is, given $U$ open in $X$ and a continuous $s : U \to E$ with $ps = 1_U$, there exists $t : f^{-1}(U) \to F$ defined by $t(y) = h_y sf(y)$. It is required that $t$ be continuous.)

(2) If we apply the iterated cotriple machine of [1] to the functor $P : \mathcal{R} \to \mathcal{K}$, we arrive at sheaves of fields over compact Hausdorff spaces and then at the coreflective subcategory of sheaves with totally disconnected base, which is isomorphic to $\mathcal{R}$. Thus the machine terminates after two stages.

## 3. On equational completions

Theorem 1.3 easily enables us to conclude that the category of strongly regular rings with unit is the equational completion of the category of products of skew fields and *all* unitary ring homomorphisms. (Note that this is *not* the category $\mathcal{K}$, as maps are not required to be coordinated.) Clearly, commutative regular rings from the equational completion of the subcategory of products of fields.

It is also easily shown that the equational completion of the category of products of skew fields and ring homomorphisms is the category of strongly regular rings (not necessarily with unit). This is so, roughly because every such ring is a filtered union of strongly regular subrings with unit. The arguments are facilitated by using the propositions below. In this section we use the definitions from [5].

**3.1. Proposition.** *Let $T_0$ be a triple over $\mathcal{S}$ (= Sets), and let $\mathcal{M}$ be a full subcategory of $T_0$-algebras, closed under finite products. Let $I : \mathcal{M} \to \mathcal{S}$ be the underlying set functor, and let $T$ be the equational structure of $I$. Assume that every $T_0$-subobject of an object of $\mathcal{M}$ is also a $T$-subobject. Then $T_0$ is, in effect, a separating triple for $T$. That is, we can augment $\mathcal{M}$ by adjoining all $T_0$-subobjects and all $T_0$-homomorphisms without affecting the equational structure.*

(So using the augmented category $T_0$ would be a separating triple as defined in [5].)

**Proof.** The proof of [5, 1.3(a)] still enables us to deduce that $T_0(n)$ is dense in $T(n)$, so for Hausdorff algebras the $T$-homomorphisms are simply the continuous $T_0$-homomorphisms. For the augmented category, the topology is clearly discrete, so for these models the $T$-homomorphisms are just the $T_0$-homomorphisms. □

**3.2. Proposition.** *Let $\mathcal{M}$ be a full subcategory of $T_0$-algebras, closed under* all *products. Let $T$ be the equational structure of $\mathcal{M}$. If every $T_0$-subobject of an object of $\mathcal{M}$ is also a $T$-subobject, then $\mathcal{S}^T$ is the Birkhoff subcategory of $\mathcal{S}^{T_0}$ generated by $\mathcal{M}$.*

**Proof.** Since $\mathcal{M}$ is closed under the formation of all products, it can be seen that the limit topology on $T(n)$ is discrete, so all the $Q$-topologies are discrete. Since $T_0$ is in effect a separating triple, it follows that the $T$-homomorphisms coincide with the $T_0$-homomorphisms. The $T$-algebras are therefore a full subcategory of the $T_0$-algebras and closed under subobjects and quotients in view of [5, 1.3(c), 1.6]. □

**3.3. Examples**

(1) Let $\mathcal{M}$ be the category of products of skew fields and unitary ring homomorphisms. Let $T_0$ be the theory of strongly regular rings with unit. Then let $R$ be a $T_0$-subobject of a model $\Pi K_i$. Let $\theta : P(R) \to \Pi K_i$ be the obvious projection. Then $R$ is the image under $\theta$ of the equalizer of $\eta$ and $\lambda$ of 1.3. Applying 3.2 we see that $T_0$ is the equational structure of $\mathcal{M}$.

(2) Let $\mathcal{M}'$ have the same object as above and *all* ring homomorphisms for maps. Let $T_0'$ be the theory of strongly regular rings (not necessarily with unit). Let $T'$ be the equational structure of $\mathcal{M}$. Then $T'$ is a subtheory of $T_0$ (of the above example), so $T'$ is finitary. Also every $T_0'$-subobject is a filtered union of $T_0$-subobjects, so is a $T'$-subobject. Using 3.2 we can determine that $T_0' = T'$.

## 4. Free objects in the commutative case

Using [6, Proposition 5], we know that the forgetful functor from regular commutative rings to commutative rings has a left adjoint. Thus each commutative ring $R$ can be thought of as "freely generating" a regular commutative ring. In fact it suffices to adjoin to $R$ a semi-inverse for each element. First let us observe that it suffices to consider commutative rings without non-zero nilpotents:

**4.1. Lemma.** *A commutative ring $R$ can be embedded in a regular commutative ring iff $R$ has no non-zero nilpotents.*

**Proof.** If $R$ has no non-zero nilpotents, then $R$ can be embedded in a product of integral domains, hence in a product of (quotient) fields. □

**4.2. Proposition.** *Let $R$ be a commutative ring with unit and no non-zero nilpotents. Let $R^\#$ be the commutative ring obtained by adjoining a semi-inverse for each element of $R$. Then $R$ is a unitary subring of $R^\#$, and $R^\#$ is regular and is the regular commutative ring freely generated by $R$.*

**Proof.** Notice that $R^\#$ is generated by $R \cup \{\bar{r} \mid r \in R\}$ subject to the relations $r = r^2\bar{r}$, $\bar{r} = \bar{r}^2 r$, commutativity and that the existing operations on $R$ still hold. Using 4.1 it is readily shown that $R$ is a unitary subring of $R^\#$. A typical element $x \in R^\#$ can be written as $x = a_1\bar{b}_1 + \ldots + a_n\bar{b}_n$. We must show that $x$ has a semi-inverse. For each subset $S \subseteq \{1, 2, \ldots, n\}$ we let

$$B_S = \Pi\{b_i \mid i \in S\}.$$

Then $\bar{B}_S = \Pi\{\bar{b}_i \mid i \in S\}$. We define

$$E_S = B_S\bar{B}_S\,\Pi\{(1 - b_i\bar{b}_i \mid i \notin S\}.$$

Then $E_S$ is idempotent and $E_S E_T = 0$ for $S \neq T$. Also $\Sigma E_S = 1$ (which can be proved by induction on $n$). Finally we define

$$\Delta_S = \Sigma \{a_i B_{S \setminus \{i\}} \mid i \in S\}.$$

Notice that if $x$ is "evaluated" in a field and $S = \{i \mid b_i \neq 0\}$, then by adding fractions, $x = \Delta_S \bar{B}_S$. It can then be shown that $xE_S = \Delta_S \bar{B}_S E_S$. Let $y = \Sigma B_S \bar{\Delta}_S E_S$. Using the above rules one can readily show that $y = \bar{x}$. □

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