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LANGUAGES FOR MONOIDAL CATEGORIES

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Proofs of propositions about ordinary categories, e.g. the Yoneda Lemma, may often be reinterpreted to yield proofs of the equivalent propositions about enriched categories, without recourse to diagram-chasing. Elements of the homset are replaced by variables of the hom-type. The language may also be used to lift definitions to the enriched context in a natural way, e.g., natural transformation, strong monic.

1. Introduction

When working with a symmetric monoidal category the common practice is to explore problems and formulate hypotheses without reference to the associativity, unit and symmetry morphisms, as if one is working in a strict monoidal category. The coherence theorem of MacLane ([28], see also Kelly [14]) provides reassurance that hypotheses so formulated hold in the general situation. But now arises the problem: the coherence theorem gives no hint as to how to *prove* theorems. Proofs which are entirely elementary in **Sets**, or at least familiar, become irritating exercises in the technique of diagram-chasing (e.g. [7,9,27]) or equivalently, computations employing strings of horizontal and vertical composites (e.g. [6,13,32]). The difficulty of constructing these proofs is perhaps the greatest barrier to the continued development of enriched category theory.

The solution is a language for monoidal categories in which proofs can be constructed just as in **Sets**, using variables.

Consider the following example, due to R. Wood. Let $(\mathscr{V}, \otimes, I, a, l, r, c)$ be a symmetric monoidal category and (X, m) a semigroup in $\mathscr{V}$, i.e.

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$$\begin{tikzcd}
(X\otimes X)\otimes X \arrow[r, "m\otimes X"] \arrow[dd, "a"'] & X\otimes X \arrow[dr, "m"] & \\
& & X \\
X\otimes(X\otimes X) \arrow[r, "X\otimes m"'] & X\otimes X \arrow[ur, "m"'] &
\end{tikzcd}$$

commutes, with a right unit $e:I\rightarrow X$ such that

$$\begin{tikzcd}
X \arrow[r, "r^{-1}"] \arrow[d, "1"'] & X\otimes I \arrow[d, "X\otimes e"] \\
X & X\otimes X \arrow[l, "m"]
\end{tikzcd}$$

commutes. If $\mathscr{V}=$ **Sets**, then define $f:X\rightarrow X$ by

$$x \mapsto e\cdot x \tag{1.1}$$

where $e:1\rightarrow X$ is identified with its image and $x\cdot y=m(x,y)$. Now f is idempotent since

$$\begin{aligned} e\cdot(e\cdot x) &= (e\cdot e)\cdot x \quad \text{(associativity)} \\ &= e\cdot x. \qquad\quad \text{(right unit)} \end{aligned} \tag{1.2}$$

The comparable result for an arbitrary $\mathscr{V}$ is the idempotence of f, now defined by

$$X \xrightarrow{l} I\otimes X \xrightarrow{e\otimes X} X\otimes X \xrightarrow{m} X.$$

The proof, however, becomes far more complicated.

$$\begin{tikzcd}
X \arrow[rrr, "f"'] \arrow[d, "l^{-1}"'] \arrow[drrr, phantom, "\text{(I)}"] & & & X \arrow[d, "l^{-1}"] \\
I\otimes X \arrow[rrr, "I\otimes f"] \arrow[d, "e\otimes X"'] \arrow[drrr, phantom, "\text{(II)}"] & & & I\otimes X \arrow[d, "e\otimes X"] \\
X\otimes X \arrow[r, "X\otimes l^{-1}"] \arrow[dr, "r^{-1}\otimes X"'] \arrow[ddrr, "X\otimes X"', rounded corners, to path={|- (\tikztotarget)}] \arrow[dr, phantom, bend left, "\text{(III)}"] \arrow[ddrr, phantom, "\text{(VI)}"] & X\otimes(I\otimes X) \arrow[r, "X\otimes(e\otimes X)"] \arrow[dr, phantom, "\text{(IV)}"] & X\otimes(X\otimes X) \arrow[r, "X\otimes m"] \arrow[ddr, phantom, "\text{(V)}"] & X\otimes X \arrow[dd, "m"] \\
& (X\otimes I)\otimes X \arrow[u, "a"] \arrow[r, "(X\otimes e)\otimes X"] & (X\otimes X)\otimes X \arrow[u, "a"'] \arrow[d, "m\otimes X"] & \\
& & X\otimes X \arrow[r, "m"] & X
\end{tikzcd} \tag{1.3}$$

Cells (I)–(IV) in Diagram (1.3) commute because of the monoidal structure of $\mathscr{V}$: (I) and (IV) by the naturality of l and a; (II) because tensor is a functor; and (III) by the triangle law for coherence. The semigroup axioms only appear in cells (V) and (VI), which correspond to the two lines of (1.2). Theorem 2.1 below, which is a consequence of coherence, allows the commutativity of (1.3) to be deduced from (1.2).

Simpler proofs are not the only benefit: morphisms in $\mathscr{V}$ may be determined by their action on variables as in (1.1), and structures specified by axioms. For example, $\mathscr{V}$-categories can be defined in this way and much of $\mathscr{V}$-category theory, including the theory of triples, the construction of functor categories and the Yoneda lemma can be developed just as in **Sets**.

The use of typed languages to describe the internal logic of a category is well known in the study of toposes and Cartesian closed categories, e.g. [3,5,12,22].

The power of this language is that it manipulates, not only the formal monoidal structure, which the coherence theorem also does, but the data specific to $\mathscr{V}$, such as the semigroup X above. This is accomplished by creating, in Section 3, the *theory* of a monoidal category (more precisely, the theory of a pseudo-monoid) whose *models* in **Cat** are the monoidal categories. The language depends on both the theory and the model.

The theories are symmetric monoidal 2-categories, a generalisation of props [29], called 2-props. The 2-cells of the theory of commutative pseudo-monoids are generated by the associativity $a : \otimes(\otimes,1) \Rightarrow \otimes(1,\otimes) : 3 \to 1$, the left and right unit, and the symmetry. In the 2-prop whose models in **Cat** yield the data for a monoidal functor $(\phi, \tilde{\phi}, \phi^\circ)$ the 2-cells include $\tilde{\phi}$ and ϕ°. A model of a 2-prop in **Cat** is a strong, symmetric, monoidal 2-functor which sends the 2-cells to *canonical* natural transformations.

The languages introduced in Section 4 allow suppression of canonical transformations. This may, in general, result in loss of information, though in practice, coherence theorems provide conditions (preferably none) which prevent this.

Section 5 shows how the language for the monoidal structure of a monoidal closed category is sufficiently rich to manipulate the closed structure. In Section 6 is shown how to manipulate limits in a monoidal category. Section 7 extends the possibilities of languages using components which allow arbitrary terms s of type $X \otimes Y$ to be written as $s_1 \otimes s_2$. Finally, Section 8 provides a language, and the accompanying coherence theorem, for a pair of monoidal functors and a monoidal natural transformation between them.

Pseudo-monoids appear in other 2-categories. In [11] it is shown that triples, bicategories and braided monoidal categories arise in this way and appropriate languages can be constructed. Certainly, there are many other possible uses for such languages.

2. The language for a monoidal category

Let $(\mathcal{V}, \otimes, I, a, l, r, c)$ be a given symmetric monoidal category (see e.g. [17,30]). All monoidal categories, functors and transformations are assumed symmetric unless otherwise stated. Results for non-symmetric $\mathcal{V}$ follow by deleting all references below to symmetry. Construct a typed language for $\mathcal{V}$ as follows.

The *types* are the objects of $\mathcal{V}$. A term s comes equipped with an associated type X, written $s \in X$. To each type X is associated a countable set of *variables* $x_n \in X$ $(x_0 = x)$. Also, there is a constant $* \in I$. A *basic term* is a bracketed, formal tensor of variables and $*$ in which no variable appears twice, though $*$ may appear often and different variables of the same type may occur. For example, $x \otimes (y \otimes z)$ and $* \otimes ((x_1 \otimes *) \otimes x_2)$ are terms while $x_1 \otimes x_1$ is not. The *function symbols* are the morphisms of $\mathcal{V}$. Finally, a *term* s consists of a basic term $x \in X$ and a function symbol $f : X \to Y$, written $s = f(x) \in Y$. If no variables occur in x, then s is a *constant*.

That $x \otimes x$ is not a term reflects the fact that the tensor product, unlike its Cartesian cousin, does not have diagonals, $X \to X \otimes X$. Note that a term $s \in X$ may be thought of as a 'polynomial' $s : I \to X$ in the sense of Lambek and Scott [22, II.3.5].

Let $s = f(x)$ and $t = g(y)$ be terms and h be a function symbol. Define operations on terms by

$$s \otimes t = f \otimes g(x \otimes y), \qquad h(s) = (hf)(x)$$

whenever $x \otimes y$ is well defined as a basic term and hf is a morphism of $\mathcal{V}$.

Now define an equivalence relation on terms which exploits the monoidal structure of $\mathcal{V}$. For $s \in X$, $t \in Y$ and $u \in Z$ it is generated by

$$a_{X,Y,Z}((s \otimes t) \otimes u) \equiv s \otimes (t \otimes u) \tag{2.1}$$

$$l_X(* \otimes s) \equiv s, \tag{2.2}$$

$$r_X(s \otimes *) \equiv s, \tag{2.3}$$

$$c_{X,Y}(s \otimes t) \equiv t \otimes s. \tag{2.4}$$

Note that, since $\otimes$ is not, in general, Cartesian, $s \otimes t = u \otimes v$ does not imply that $s = u$ or $t = v$.

The power of the language lies in this: the equivalence relation can be used to simplify calculations in a particular monoidal category by suppressing the components of the associativity and unit isomorphisms. The resulting expressions are easier to manipulate, while the coherence theorem guarantees (as will be shown below) that no information is lost in the process. In this way coherence is put to effective use.

Theorem 2.1. *Let $f, g : X \to Y$ be morphisms of $\mathcal{V}$ and let $x \in X$ be a basic term, e.g., a variable. Then $f(x) \equiv g(x)$ iff $f = g$.* □

The proof is a simple induction, but it requires closer analysis of the properties of canonical transformations (Section 3) and the structure of basic terms (Section 4). First consider some applications. Often equivalence of terms will be replaced by equality to heighten the analogy with classical results.

Example 2.2. Let $\mathscr{V}=R$-**mod** for R a commutative ring, with the usual tensor and R as unit. Then, if X is an R-module, variables $x\in X$ behave like elements of X and $*$ like $1\in R$. Theorem 2.1 shows that if linear maps f, $g:X\otimes Y\to Z$ agree on elements of the form $x\otimes y$, then they are equal.

Example 2.3. Using the language it is possible to describe objects with algebraic structure in $\mathscr{V}$ e.g. monoids, semi-rings, semi-modules etc. by means of axioms. Consider, for example, the semigroup (X, m, e) of Section 1. That there is a morphism f satisfying (1.1) is obvious, while its uniqueness follows from Theorem 1.2. Similarly, (1.2) shows that $f^2=f$.

There are limitations on the kinds of axioms, $s\equiv t$ which can be imposed, though: since there are no projections, s and t must employ the same variables; since there are no diagonals each variable occurs exactly once in each term. Thus, $x\cdot y=x$ is unacceptable as an axiom (unless y is a constant), as is $x\cdot x=x$. In particular, the axiom for an inverse, $x\cdot x^{-1}=e$ fails on both counts.

Example 2.4. Let R be a monoid in $\mathscr{V}$ and let left and right actions of R on an object M be written $r\otimes m\mapsto r\cdot m$ and $m\otimes r\mapsto m\cdot r$. If R is commutative, then a left R-action induces a right R-action such that $r\cdot(m\cdot s)=(r\cdot m)\cdot s$. The diagrammatic proof is given by

$$\begin{tikzcd}
 & & M\otimes R \arrow[drr, "c"] \arrow[d, phantom, "\text{(IV)}"] & & & \\
(R\otimes M)\otimes R \arrow[urr, "\bullet\otimes 1"] \arrow[rr, "c"'] \arrow[dddd, "a"'] & & R\otimes(R\otimes M) \arrow[rr, "1\otimes\bullet"'] \arrow[d, "a^{-1}"'] & & R\otimes M \arrow[ddr, "\bullet"] & \\
 & & (R\otimes R)\otimes M \arrow[drr, "\bullet\otimes 1"] & & \text{(I)} & \\
 & \text{(V)} & & \text{(II)} & R\otimes M \arrow[r, "\bullet"] & M \\
 & & (R\otimes R)\otimes M \arrow[uu, "c\otimes 1"] \arrow[urr, "\bullet\otimes 1"'] & & \text{(III)} & \\
R\otimes(M\otimes R) \arrow[rr, "1\otimes c"] & & R\otimes(R\otimes M) \arrow[rr, "1\otimes\bullet"] \arrow[u, "a^{-1}"] & & R\otimes M \arrow[uur, "\bullet"'] &
\end{tikzcd}$$

Cells (I)–(III) are axioms, (IV) commutes by the naturality of c, and (V) commutes since

$$[(c\otimes 1)a^{-1}(1\otimes c)]a = (a^{-1}ca^{-1})a = a^{-1}c.$$

One may assert that (V) commutes by coherence, provided it is checked that the two natural transformations have the same domain and codomain.

The variable proof is syntactically the same as that for R-modules, where R is a commutative ring.

$$\begin{aligned}(r\cdot m)\cdot s &= s\cdot(r\cdot m)\\ &= (s\cdot r)\cdot m \quad \text{(I)}\\ &= (r\cdot s)\cdot m \quad \text{(II)}\\ &= r\cdot(s\cdot m) \quad \text{(III)}\\ &= r\cdot(m\cdot s).\end{aligned}$$

Example 2.5. The fundamentals of category theory may be developed in $\mathscr{V}$ exactly as in **Sets**: a $\mathscr{V}$-category $\mathscr{A}$ has composition and units

$$M_{A,B,C}:\mathscr{A}(B,C)\otimes\mathscr{A}(A,B)\to\mathscr{A}(A,C), \qquad J_A:I\to\mathscr{A}(A,A).$$

Let $f\in\mathscr{A}(A,B)$, $g\in\mathscr{A}(B,C)$ and $h\in\mathscr{A}(C,D)$ be variables, and write $g\circ f$ for $M(g\otimes f)$ and 1_A for $J_A(*)$. Then the category axioms are

$$h\circ(g\circ f)=(h\circ g)\circ f, \qquad f\circ 1_A=f=1_B\circ f.$$

Similarly, if $T:\mathscr{A}\to\mathscr{B}$ is a $\mathscr{V}$-functor, then $T_{A,B}(f)=Tf\in\mathscr{B}(TA,TB)$ and we have

$$Tg\circ Tf=T(g\circ f), \qquad T(1_A)=1_{TA}.$$

$\mathscr{V}$-natural transformations $\alpha:S\Rightarrow T$ are given by families of morphisms $\alpha_A:I\to\mathscr{B}(SA,TA)$. Write α_A for $\alpha_A(*)$. Then the axiom is

$$Tf\circ\alpha_A=\alpha_B\circ Sf. \tag{2.5}$$

Seen from this viewpoint, the fundamentals of category theory, including the Yoneda Lemma (see Lemma 6.3 below) may be lifted from ordinary category theory to the enriched context merely by re-interpreting the ordinary proofs.

Example 2.6. Many authors (e.g. [6,13,19,27]) have studied triples and cotriples in $\mathscr{V}$-**Cat**. However, the axioms for a triple make sense in $\mathscr{V}$ and the functorial semantics lifts directly if $\mathscr{V}$ has equalizers. If $T=(T,\eta,\mu)$ is a triple on $\mathscr{A}$, then a T-algebra is a pair (A,a) where A is an object of $\mathscr{A}$ and $a\in\mathscr{A}(TA,A)$ is a 'constant' (i.e. $a:I\to A(TA,A)$) which satisfies $a\circ\eta=1$ and $a\circ Ta=a\circ\mu$. The algebras form a $\mathscr{V}$-category $\mathscr{A}^T$ with homs given by equalizers

$$\begin{tikzcd}\mathscr{A}^T((A,a),(B,b)) \arrow[r, "i"] & \mathscr{A}(A,B) \arrow[r, shift left, "\alpha"] \arrow[r, shift right, "\beta"'] & \mathscr{A}(TA,B)\end{tikzcd}$$

where $\alpha(f)=f\circ a$ and $\beta(f)=b\circ Tf$. The composition law is given by $i(g\circ f)=(ig)\circ(if)$. More precisely, given $f:(A,a)\to(B,b)$ and $g:(B,b)\to(C,c)$

$$\begin{aligned}\alpha((ig)\circ(if)) &= ig\circ if\circ a=(ig)\circ b\circ T(if)=c\circ T(ig)\circ T(if)\\ &= c\circ T((ig)\circ(if))=\beta((ig)\circ(if)).\end{aligned}$$

Hence, $\alpha M(i,i)=\beta M(i,i)$ (M is the composition and so $M(i,i)$ factorises as iM^T, making M^T the composition for $\mathscr{A}^T$. The identities are defined similarly and the axioms easily verified. This method is simpler than that of, say, [27] and has the advantage of being familiar. The proof would be shorter if terms of $\mathscr{A}^T((A,a),(B,b))$ could be identified with their image under i. This is done in Section 6.

When $\mathscr{V}$ is a strict monoidal category, the language may still clarify an argument, even though the canonical transformations play no role. For example, in [2] it is stated that an action of one triple on another $\sigma:TT'\Rightarrow T'$ induces a morphism of triples $\alpha:T\Rightarrow T'$. Formally, we have

Proposition 2.7. *Let (T,η,μ) and (T',η',μ') be triples on $\mathscr{A}$ and $\sigma:TT'\Rightarrow T'$ be a natural transformation such that the diagrams*

$$\begin{tikzcd}
T' \arrow[r, "\eta T'"] \arrow[dr, "T'"'] & TT' \arrow[d, "\sigma"] & T^2T' \arrow[r, "\mu T'"] \arrow[d, "T\sigma"'] & TT' \arrow[d, "\sigma"] & TT'^2 \arrow[r, "T\mu'"] \arrow[d, "\sigma T'"'] & TT' \arrow[d, "\sigma"] \\
 & T' & TT' \arrow[r, "\sigma"'] & T' & T'^2 \arrow[r, "\mu'"'] & T'
\end{tikzcd}$$

commute. Then $\alpha=\sigma\cdot T\eta':T\Rightarrow T'$ is a morphism of triples. □

Observe now that $\mathscr{V}=\mathrm{End}(\mathscr{A})$ is a strict (non-symmetric) monoidal category and a triple is exactly a monoid object in $\mathscr{V}$. Thus Proposition 2.7 is a special case of

Proposition 2.8. *Let $(R,\cdot,e)$ and $(S,\cdot,e)$ be two monoids in a monoidal category $\mathscr{V}$, and let $T:R\otimes S\to S$ be an action i.e. (writing $r\cdot s$ for $\sigma(r\otimes s)$) we have*

$$e\cdot s=s,\qquad (r\cdot r')\cdot s=r\cdot(r'\cdot s),\qquad r\cdot(s\cdot s')=(r\cdot s)\cdot s'.$$

Then there is a monoid morphism $\alpha:R\to S$ given by $\alpha(x)=x\cdot e$.

Proof.

(i) $\alpha(e)=e\cdot e'=e'$,

(ii) $$\begin{aligned}\alpha(x)\cdot\alpha(y) &= (x\cdot e')\cdot(y\cdot e')=x\cdot(e'\cdot(y\cdot e'))=x\cdot(y\cdot e')\\ &= (x\cdot y)\cdot e'=\alpha(x\cdot y).\end{aligned}$$ □

Although this proof contains as many steps as the shortest diagrammatic one [11], it is more straightforward, first because of the ease of the language, and second because the proof is standard in the case $\mathscr{V}=\mathbf{Ab}$, for then R is a ring, S is an R-algebra via σ and α the resulting ring homomorphism. This approach may simplify some calculations in homology (see, e.g. [1]).

3. 2-Props

As has been often observed, the coherence theorem for monoidal categories is a statement about natural transformations, rather than their components, since components may form diagrams which do not commute. Here, the abstract data and its coherence will be embodied in the *theory* $\mathscr{T}_0$ of monoidal categories. Calculations, however, require one to come to grips with the idiosyncracies of particular monoidal categories, so that languages are built for a *model* $\mathscr{V}$ of the theory; its types and function symbols are the objects and the morphisms of $\mathscr{V}$. It is the controlled interplay of the theory and its model that allows the full power of coherence to be employed.

To see what kind of structure $\mathscr{T}_0$ has, consider the monoidal data. Representing $\mathscr{V}^n$ by n, we have, for example, the associativity isomorphism

$$\begin{tikzcd} 3 \arrow[r, "{(1,\otimes)}"] \arrow[d, "{(\otimes,1)}"'] \arrow[dr, phantom, "\overset{a}{\Rightarrow}"] & 2 \arrow[d, "\otimes"] \\ 2 \arrow[r, "\otimes"'] & 1 \end{tikzcd} \tag{3.1}$$

Clearly, $\mathscr{T}_0$ should be a 2-category. Equally clearly, $\mathscr{T}_0$ has a tensor product, namely addition. It is possible to impose a Cartesian product on the theory, since the tensor on **Cat** is cartesian, but there are several reasons for choosing instead a tensor product. Chief among them is that the language for a Cartesian theory would be stronger than that constructed in Section 2, since no use was made there of projections etc. Also, there are pseudo-monoids of interest, e.g. bicategories, in 2-categories which are not Cartesian [11].

In order to place these theories in context, consider the one-dimensional tensor product theories, i.e., MacLane's props [29] (see also the operads of May [31]). A *prop* is a strict (symmetric) monoidal category $\mathscr{T}$, together with a strict monoidal functor which is bijective on objects

$$\theta : P \to \mathscr{T}$$

$P=\Sigma P_n$ is the permutation category (P_n is the group of permutations on n elements, regarded as a category whose only object is n) with tensor given by addition of objects, and the obvious action on morphisms. The objects of $\mathscr{T}$ may be identified, via θ, with the natural numbers. The morphisms of $\mathscr{T}$ are called *operations*. A *model* of a prop is a strong monoidal functor, just as a model of a finite product theory is a product-preserving functor. For example, there is a prop $\mathscr{S}$ whose models are (equivalent to) commutative monoids. $\mathscr{S}$ is equivalent to $\mathbf{Sets}_{\mathrm{f}}$, the category of finite sets (every function is a permutation followed by an order-preserving map i.e. a monoid operation).

Operations in the image of P are called *permutations*. An *expansion* of an operation $F: p \to q$ is one of the form $(m,F,n): m+p+n \to m+q+n$. An *iterate* F of a set $\{F_i\}$ of operations is a composite of expansions of F_i's. Say F is *left-biased* if the value of m in the expansions, in their order of application, is increasing, e.g. $F(F,1)$ is a left-biased iterate of F while $F(1,F)$ is not. Where bias is not important (m,F,n) may be written as $(1,F,1)$ etc. If $(1,G,1)(1,F,1)=(1,F,1)(1,G,1)$, then F and G are said to *commute*.

Let $\mathscr{T}$ be a 2-category. It is a *strict*, *symmetric monoidal* 2-*category* if there is a 2-functor $\oplus : \mathscr{T}^2 \to \mathscr{T}$ called *tensor*, together with a *unit object* 0 satisfying the monoid axioms, and a *symmetry* which is a 2-natural transformation $C: \oplus \to \oplus S$, (where $S: \mathscr{T}^2 \to \mathscr{T}^2$ is the switch functor) which is a natural involution and satisfies the hexagonal condition. Explicitly, for objects X, Y and Z and 1-cells F, G, H and K, and 2-cells α, β, γ and δ (with horizontal composition denoted by $*$), the axioms are

$$(H\oplus K)(F\oplus G) = (HF\oplus KG), \tag{3.2}$$

$$(F\oplus G)\oplus H = F\oplus(G\oplus H) \quad \text{and} \quad \mathrm{id}_0\oplus F = F = F\oplus \mathrm{id}_0, \tag{3.3}$$

$$C^2 = 1, \tag{3.4}$$

$$(1\oplus C_{X,Z})(C_{X,Y}\oplus 1) = C_{X,(Y,Z)}, \tag{3.5}$$

$$C(F\oplus G) = (G\oplus F)C, \tag{3.6}$$

$$(\alpha\oplus\beta)\oplus\gamma = \alpha\oplus(\beta\oplus\gamma) \quad \text{and} \quad 1_{\mathrm{id}_0}\oplus\alpha = \alpha = \alpha\oplus 1_{\mathrm{id}_0}, \tag{3.7}$$

$$(\gamma\oplus\delta)(\alpha\oplus\beta) = (\gamma\alpha\oplus\delta\beta), \tag{3.8}$$

$$(\gamma\oplus\delta)*(\alpha\oplus\beta) = (\gamma*\alpha\oplus\delta*\beta), \tag{3.9}$$

whenever both sides are defined. Monoidal 2-functors and 2-natural transformations are defined similarly.

Make P a monoidal 2-category by adding in identity 2-cells. By abuse of notation, the same names will be used for categories and their corresponding 2-categories. Define a 2-*prop* to be a strict (symmetric) monoidal 2-category $\mathscr{T}$ together with a strict monoidal 2-functor $\theta: P \to \mathscr{T}$. Call its 1-cells *operations* and its 2-cells *canonical transformations*. The definitions of permutations, and of expansions, iterates, left-bias and commutativity of operations are as before, and extended to canonical transformations relative to vertical composition.

A *model* of $\mathscr{T}$ in a 2-category $\mathscr{A}$ is a strong monoidal 2-functor $M=(M,\tilde{M}, M^\circ): \mathscr{T} \to \mathscr{A}$. Let $\mathscr{A}$ be **Cat** with chosen products and $M(1)=\mathscr{V}$. Define $\mathscr{V}^n$ by (i) $\mathscr{V}^0=1$, (ii) $\mathscr{V}^1=\mathscr{V}$ and, (iii) $\mathscr{V}^{n+1}=\mathscr{V}^n\times\mathscr{V}$ for $n\geq 1$. Define $\tilde{M}_n: \mathscr{V}^n \to M(n)$ by

(i) $\tilde{M}_0 = M^\circ : 1 \to M(0)$,

(ii) $\tilde{M}_1 = 1 : \mathscr{V} \to M(1)$,

(iii) $\tilde{M}_2 = \tilde{M} : \mathscr{V}^2 \to M(2)$,

(iv) $\tilde{M}_{n+1} = \tilde{M}(\tilde{M}_n \times 1) : \mathscr{V}^n \times \mathscr{V} \to M(n+1)$.

A model $M' : \mathscr{T} \to$ **Cat** is *standard* if $M'(n) = \mathscr{V}^n$. Given a model M, construct a standard model as follows. Let $\alpha : F \Rightarrow G : m \to n$. Then $M'(\alpha)$ is

$$\begin{tikzcd}
\mathscr{V}^m \arrow[r, "\tilde{M}_m"] & M(m) \arrow[r, bend left, "M(F)"] \arrow[r, bend right, "M(G)"'] \arrow[r, phantom, "M(\alpha)\Downarrow"] & M(n) \arrow[r, "\tilde{M}_n^{-1}"] & \mathscr{V}^n
\end{tikzcd}$$

with $(M')^\circ = 1$ and $\tilde{M}'$ is

$$\begin{tikzcd}
\mathscr{V}^m \times \mathscr{V}^n \arrow[r, "\tilde{M}_m \times \tilde{M}_n"] & M(m) \times M(n) \arrow[r, "\tilde{M}"] & M(m+n) \arrow[r, "\tilde{M}_{m+n}^{-1}"] & \mathscr{V}^{m+n}.
\end{tikzcd}$$

Clearly $\tilde{M} : M' \to M$ is a monoidal 2-natural isomorphism, and hence an isomorphism of models (however this is defined). Thus,

Theorem 3.1. *In* **Cat** *every model is isomorphic to a standard model.* ☐

The 2-prop $\mathscr{T}_0$ for monoidal categories is constructed from a 2-dimensional sketch (see, for example [2]). $\mathscr{T}_0$ has models in other 2-categories so a more apt name for it is the theory of *commutative pseudo-monoids*, since a model satisfies the monoid axioms up to isomorphism. For 1-cells F and G denote $F \oplus G$ by (F, G). The data for $\mathscr{T}_0$ are given by 3.1 and

$$\begin{tikzcd}
1 \arrow[d, "{(I,1)}"'] \arrow[dr, "1"] \arrow[dr, phantom, shift right=3ex, "\overset{l}{\Rightarrow}"] & \\
2 \arrow[r, "\otimes"'] & 1
\end{tikzcd} \tag{3.10}$$

$$\begin{tikzcd}
1 \arrow[d, "{(1,I)}"'] \arrow[dr, "1"] \arrow[dr, phantom, shift right=3ex, "\overset{r}{\Rightarrow}"] & \\
2 \arrow[r, "\otimes"'] & 1
\end{tikzcd} \tag{3.11}$$

$$\begin{tikzcd}
2 \arrow[d, "C"'] \arrow[dr, "\otimes"] \arrow[dr, phantom, shift right=3ex, "\overset{c}{\Leftarrow}"] & \\
2 \arrow[r, "\otimes"'] & 1
\end{tikzcd} \tag{3.12}$$

The *shapes* [18] form the smallest set E of 1-cells containing $\otimes$, I and C and closed under iteration. The set of operations is the quotient of E by the equivalence relation generated by the set of axioms of a strict monoidal 2-category, i.e., the axioms of a 2-category and (3.2)–(3.6), closed under iteration. Expressions for canonical transformations are defined similarly: they form the smallest set of 2-cells containing a, l, r and c, which is closed under vertical and horizontal composition, and tensoring. Equality of expressions for transformations is generated by (3.7)–(3.9) and the axioms of the theory, namely

$$\begin{tikzcd}
& & \otimes(\otimes,\otimes) \arrow[drr, "a(1,\otimes)"] & & \\
\otimes(\otimes,1)(\otimes,1) \arrow[urr, "a(\otimes,1)"] \arrow[dr, "\otimes a"'] & & & & \otimes(1,\otimes)(1,\otimes) \\
& \otimes(1,\otimes)(1,I,1) \arrow[rr, "a(1,\otimes,1)"'] & & \otimes(1,\otimes)(1,\otimes,1) \arrow[ur, "\otimes a"'] &
\end{tikzcd} \tag{3.13}$$

$$\begin{tikzcd}
& \otimes(\otimes,1)(1,I,1) \arrow[dl, "a(1,I,1)"'] \arrow[dr, "\otimes(r,1)"] & \\
\otimes(1,\otimes)(1,I,1) \arrow[rr, "\otimes(1,l)"'] & & \otimes
\end{tikzcd} \tag{3.14}$$

$$\begin{tikzcd}
& \otimes \arrow[dl, "c"'] \arrow[dr, "1"] & \\
\otimes C \arrow[rr, "c_C"'] & & \otimes
\end{tikzcd} \tag{3.15}$$

$$\begin{tikzcd}
\otimes(\otimes,1) \arrow[r, "a"] \arrow[d, "\otimes(c,1)"'] & \otimes(1,\otimes) \arrow[r, "c(1,\otimes)"] & \otimes C(1,\otimes)=\otimes(1,\otimes)C_{1,2} \arrow[d, "aC_{1,2}"] \\
\otimes(\otimes C,1) \arrow[r, "a(C,1)"'] & \otimes(1,\otimes)(C,1) \arrow[r, "\otimes(1,c)(C,1)"'] & \otimes(1,\otimes)(1,C)(C,1)
\end{tikzcd} \tag{3.16}$$

(where $C_{1,2}=a^{-1}(1,c)a(c,1)a^{-1}$) and closed under vertical and horizontal composition, and tensoring. The 2-categorical and monoidal structures on $\mathscr{T}_0$ are defined in the obvious way. Notice that by forcing a, l, r and c to be identities, one obtains a strict monoidal 2-functor (a morphism of 2-props), which is surjective, from $\mathscr{T}_0$ to $\mathscr{S}$ i.e. from the theory of pseudo-monoids to that of monoids.

An object A of a category $\mathscr{A}$ is *sub-terminal* if, for each object X of $\mathscr{A}$, there is at most one morphism from X to A. If every object of $\mathscr{A}$ is sub-terminal, then it follows that $\mathscr{A}$ is a (pre-)order. Say a 1-cell f of a bicategory $\mathscr{B}$ is sub-terminal if it is such as an object of its hom-category. To be *sub-initial* is defined dually. A

2-prop is *coherent* if every operation is sub-terminal, i.e. there is at most one canonical transformation from an operation to any other.

MacLane's coherence theorem for monoidal categories, as a statement about $\mathcal{T}_0$, asserts that every operation with codomain 1 is sub-terminal. This is sufficient for a proof of Theorem 2.1, since the basic terms there only employ these operations. More, however, is true.

Theorem 3.2. *$\mathcal{T}_0$ is coherent.*

Proof. Let $F: m \to n$ be an operation. By induction there is a unique permutation P on m and a unique family (F_i) of operations, for $i = 1, \dots, n$, such that $F = (F_i)P$. Similarly, if there is a canonical transformation $\alpha : F \Rightarrow G$, then $G = (G_i)P$ and $\alpha = (\alpha_i)P$. Now MacLane's original result shows that each α_i is unique. Hence, so is α. □

The data for a monoidal category has been described by a club [16], which employs a calculus of substitution. The replacement of substitution by products and composition made here, exactly mimics that made by Lawvere for 1-dimensional theories.

Without pausing to develop the theory of 2-categorical structure-semantics, let us return to the consideration of languages.

4. Languages for 2-props

Let $\mathcal{T}$ be any 2-prop and $M : \mathcal{T} \to \mathbf{Cat}$ be a standard model with $M(1) = \mathcal{V}$. Whenever possible, explicit reference to M will be avoided. Construct a language $\mathcal{L}(M) = \mathcal{L}$ whose *types* are objects of $\mathcal{V}^n$ for all n. Note that the language constructed in Section 2 only had objects of $\mathcal{V}$ as types, and so will be a fragment of the language constructed here. If V is an object of $\mathcal{V}$ $(V \in \mathcal{V})$, then there are countably many variables v^k of type V $(k \in N)$, written $v^k \in V$. Given variables $v_i \in V_i$ $(1 \leq i \leq n)$, there is a *sequence of variables* $v = (v_1, v_2, \dots, v_n)$ of type $V = (V_1, V_2, \dots, V_n)$. The empty sequence is a constant, denoted $! \in 1$, where 1 is the unique object of the terminal category **1**. The *function symbols* are the morphisms of $\mathcal{V}^n$. A *term* s consists of a sequence of variables $v \in V$ as above, an operation $F: n \to p$ in $\mathcal{T}$, and a function symbol $f: FV \to V'$ in $\mathcal{V}^p$. Denote s by $f(Fv) \in V'$. s is a *basic term* if $f = 1$ or a *constant* if $v = !$. Let $t = g(Gw)$ be another term, H an operation and h a function symbol. Constructions on terms are given by

$$(s, t) = (f, g)(F, G)(v, w), \tag{4.1}$$

$$Hs = Hf((HF)v), \tag{4.2}$$

$$h(s) = (hf)(Fv), \tag{4.3}$$

whenever the right-hand side is defined.

Define an order $\geq$ on terms: say s *reduces* to t if $s \geq t$ (say $s \equiv t$ iff $s \geq t$ and $t \geq s$). It is generated by:

$$\text{If } \alpha : F \Rightarrow G \text{ in } \mathscr{T} \text{ and } s \in X, \text{ then } \alpha_X(Fs) \geq Gs, \tag{4.4}$$

$$C(s,t) \equiv (t,s), \tag{4.5}$$

$$\text{If } s_i \geq t_i \text{ for } i = 1,2, \text{ then } (s_1,s_2) \geq (t_1,t_2), \tag{4.6}$$

$$\text{If } s \geq t, \text{ then } Fs \geq Ft \text{ for } F \text{ an operation}, \tag{4.7}$$

$$\text{If } s \geq t, \text{ then } f(s) \geq f(t) \text{ for } f \text{ a function symbol.} \tag{4.8}$$

Reduction removes canonical transformations from terms. If α is an isomorphism in $\mathscr{T}$, then (4.4) is an equivalence, e.g. (2.1)–(2.4). However, there are cases where canonical transformations are not isos, e.g., the natural transformations in the structure of a monoidal functor cf. Section 8.

Theorem 4.1. *Let $s = f(Fv)$ and $t = g(Gw)$. If $s \geq t$ then there is a permutation P (necessarily unique) such that $Pv \equiv w$ and a canonical transformation $\alpha : F \Rightarrow GP$ such that $f = g \circ \alpha_v$ (where $v \in V$).*

Proof. Since $\mathscr{T}$ is closed under vertical composition, and permutations compose, it is sufficient to check the conclusion for (4.4)–(4.8). For (4.4), (4.5) and (4.8) this is trivial, while the other two steps follow since canonical transformations are closed under horizontal composition and products. □

Corollary 4.2. *Let f and g be function symbols and let x be a basic term. Further, let $u = h(Hw)$ be a term such that H is sub-terminal in $\mathscr{T}$. If $f(x) \geq u$ and $g(x) \geq u$, then $f = g$.*

Proof. Let $x = Fv$ where $v \in V$. Then $f = h \circ \alpha_V$ and $g = h \circ \beta_V$ for some $\alpha, \beta : F \Rightarrow HP$ in $\mathscr{T}$. Since P is an isomorphism, HP is sub-terminal. Hence $\alpha = \beta$ and $f = g$. □

Of course there is a dual theorem about sub-initial operations.

Corollary 4.3. *Let $\mathscr{T}$ be coherent. Then $f(x) \equiv g(x)$ iff $f = g$.*

Proof. Every operation is sub-terminal. □

Theorem 2.1 is a consequence of taking $\mathscr{T}$ to be $\mathscr{T}_0$.

5. Closed categories

Let $\mathscr{V}$ be a monoidal category. If $\mathscr{V}$ is closed, then the internal hom is determined by the monoidal structure. Consequently, the language of $\mathscr{V}$ as a monoidal category is sufficiently complex to allow easy manipulation of the closed structure. More precisely, *no* new canonical transformations are introduced, such as the evaluation $\varepsilon_{X,Y} : [X,Y] \otimes X \to Y$ since their appearance as function symbols in $\mathscr{L}$ is sufficient. The opposite approach is taken in previous works (e.g. [18,20,21,34]) which results in qualified coherence theorems. As full coherence for monoidal categories is available here, no extra restrictions on the use of variables are necessary.

Some new notation is introduced to $\mathscr{L}$ in the closed case. A morphism $f : X \otimes Y \to Z$ corresponds to $\check{f} : X \to [Y,Z]$. $(\hat{\ })$ is inverse to $(\check{\ })$. Also $g : X \to Y$ corresponds to $(gl)\check{\ } : I \to [X,Y]$ which, when applied to $*$, yields $\ulcorner g \urcorner$, the *name* of g. Finally, given terms $\phi \in [X,Y]$ and $s \in X$, let $\phi(s) = \varepsilon_{X,Y}(\phi \otimes s)$. Directly from these definitions we have

Lemma 5.1. *Let f and g be as above with $s, t \in X$ and $\phi, \psi \in [X,Y]$. Then*

(i) $\check{f}(s)(t) = f(s \otimes t)$,

(ii) $\ulcorner f \urcorner(s) = f(s)$,

(iii) *if* $\phi \equiv \psi$ *and* $s \equiv t$, *then* $\phi(s) \equiv \psi(t)$. □

Of more interest is

Theorem 5.2. *If $\phi, \psi \in [X,Y]$ and $\phi(x) \equiv \psi(x)$ for a basic term $x \in X$,* then $\phi \equiv \psi$.

Proof. Let $\phi = f(Fv)$ and $\psi = g(Gv)$. Then

$$\phi(x) = f(Fv)(x) = \varepsilon(f \otimes 1)(F \otimes 1)(v \otimes x) = \hat{f}(\otimes(F,1)(v,x)).$$

Hence $\hat{f} = \hat{g} \circ \alpha_{(V,X)}$ where $\alpha : \otimes(F,1) \Rightarrow \otimes(G,1)$ is canonical. Since F is canonically isomorphic to $\otimes(F,1)(1,I)$, it follows that $\alpha(1,I)$ yields a canonical $\beta : F \Rightarrow G$. Now, α and $\otimes(\beta,1)$ are a parallel pair of 2-cells in $\mathscr{T}_0$ and so equal by coherence. Hence $f = g \cdot \beta_V$ and $\phi = \psi$. □

Example 5.3 (Kelly, MacLane). Let $\mathscr{V}$ be a monoidal closed category with $d : X \to [Y, X \otimes Y]$ being the unit for the closed structure. Write A^* for $[A,I]$ and define $k : A \to A^{**}$ by $k(a)(f) = f(a)$.

In [18] it is shown that

$$A^* \xrightarrow{k} A^{***} \xrightarrow{[k,1]} A^*$$

is the identity while

$$A^{***} \xrightarrow{[k,1]} A^* \xrightarrow{k} A^{***}$$

need not be. These facts are reflected in the following calculations:

$$([k,1]k(f))(a) = k(f)(k(a)) = k(a)(f) = f(a).$$

Thus, $[k,1]k(f) = f$ and so $[k,1]k = 1$. However, for $g \in A^{***}$ and $f \in A^{**}$,

$$(k[k,1](g))(f) = f([k,1]g)$$

which does not reduce to $g(f)$. Consequently, there is no conflict between the existence of non-commuting diagrams of natural transformations and Theorem 2.1.

6. Limits in $\mathscr{V}$

Let $\mathscr{V}$ be a monoidal category and $D:\mathscr{I}\to\mathscr{V}$ a small diagram. In order to show that $\{f_i:X\to D_i\}$ forms a cone over D, one must establish the commutativity of certain diagrams, for which the language may be useful. Calculations with cocones may be dealt with similarly. For example, to define a morphism $X+Y\to Z$ simply specify its effect on variables of type X and type Y.

Example 6.1. Let $\mathscr{V}$ be a monoidal category having finite sums with tensors distributing over them. Then a unit may be freely adjoined to a semigroup X in $\mathscr{V}$ by defining a monoid structure on $X+I$.

There is, however, a more satisfying approach to limits, since terms $s\in X$ behave like morphisms $s:I\to X$. Let $D:\mathscr{I}\to\mathscr{V}$ be a diagram. A family of *compatible terms* for D consists of a sequence of variables υ and terms $s_i\in D_i$ whose variables are those of υ such that $Du(s_i)\equiv s_j$ for all $u:i\to j$ in $\mathscr{I}$.

Theorem 6.2. *Let (L,λ_i) be the limit of $D:\mathscr{I}\to\mathscr{V}$. Then there is a bijection between terms $s\in L$ and families of compatible terms $\{s_i\in D_i\}$, given by $s\mapsto\{\lambda_i(s)\}$.*

Proof. Consider such a family $\{s_i\}$ and choose a basic term x which is a tensor of their variables. Then there is a family of morphisms $\{f_i\}$ such that $s_i\equiv f_i(x)$. Thus, by Theorem 2.1, we have $Du\circ f_i=f_j$ i.e. $\{f_i\}$ forms a cone over D. Hence there is a unique f such that $\lambda_i\circ f=f_i$. Let $s=f(x)$. $\square$

Thus, in Example 2.6, the terms of $\mathscr{A}^T((A,a),(B,b))$ may be identified with those $s\in\mathscr{A}(A,B)$ such that $\alpha(s)\equiv\beta(s)$, i.e.

$$\mathscr{A}^T((A,a),(B,b))=\{f\in\mathscr{A}(A,B)\mid f\cdot a=b\cdot Tf\}$$

describes the equalizer and results in simpler arguments.

Aside from the computational benefits of this result (e.g. [4]) there are conceptual simplifications, too. In [8] strong monics are defined using a pullback. On applying Theorem 6.2 the usual diagonal fill-in condition appears. Ends may be constructed in terms of limits. Hence the end-based definition of the centre of an enriched category [25] reduces to the usual one, on application of Theorem 6.2. More fundamentally, the enriched hom $[A,B](S,T)$ of $\mathscr{V}$-functors S and T is given by the end $\int_A B(SA,TA)$ [30]. Hence a term $s\in[A,B](S,T)$ (i.e. a generalised natural transformation $\alpha:S\Rightarrow T$) is a family $\alpha_A\in B(SA,TA)$ satisfying (2.5). Note that if α is a constant, then it is an ordinary natural transformation. As an example, consider the enriched Yoneda Lemma [17, §2.4].

Lemma 6.3. (Yoneda). *Given a monoidal closed category $\mathscr{V}$ and a $\mathscr{V}$-functor $F:\mathscr{A}\to\mathscr{V}$ then, for each object K of $\mathscr{A}$, the cone (or wedge* [30]*)*

$$\lambda_A : FK \to [\mathscr{A}(K,A), FA]$$

given by $\lambda_A(x)(f)=Ff(x)$ is an end, i.e. $FK=[\mathscr{A},\mathscr{V}](\mathscr{A}(K,-),F)$.

Proof. The argument is the standard one, repeated here in its new context. Let $x\in FK$ and $g\in\mathscr{A}(A,B)$ be variables. Then

$$\begin{aligned}(Fg\circ\lambda_A(x))(f) &= Fg(Ff(x)) = F(g\circ f)(x) = \lambda_B(x)(g\circ f)\\ &= \lambda_B(x)\mathscr{A}(K,g)(f).\end{aligned}$$

Hence, $Fg\circ\lambda_A(x)=\lambda_B(x)\circ\mathscr{A}(K,g)$ by Theorem 5.2 and so $\{\lambda_A(x)\}$ is a compatible family. Let $\lambda(x)\in[\mathscr{A},\mathscr{V}](\mathscr{A}(K,-),F)$ be the corresponding natural transformation where λ is a morphism into the end. Conversely, let α be a term of this end-type (a natural transformation). Define $\varrho(\alpha)=\alpha_K(1_K)$. Then $\varrho(\lambda(x))=\lambda_K(x)(1_K)=F1(x)=x$ and so $\varrho\circ\lambda=1$. Also,

$$\lambda_A(\varrho(\alpha))(f) = Ff(\alpha_K(1_K)) = \alpha_A(f).$$

Hence $\lambda_A(\varrho(\alpha))=\alpha_A$ and so $\lambda\circ\varrho=1$. □

7. Components

Sometimes a term $s\in X\otimes Y$ appears in calculations which would be much easier to manipulate if it were a tensor of terms of types X and Y. Consider, for example, an adjunction in a monoidal category $\mathscr{V}$. Recall X is *left adjoint* to Y in $\mathscr{V}$ if there are $\eta: I\to Y\otimes X$ and $\varepsilon: X\otimes Y\to I$ such that

$$X \xrightarrow{r^{-1}} X\otimes I \xrightarrow{X\otimes\eta} X\otimes(Y\otimes X) \xrightarrow{a^{-1}} (X\otimes Y)\otimes X \xrightarrow{\varepsilon\otimes X} I\otimes X \xrightarrow{l} X \tag{7.1}$$

$$Y \xrightarrow{l^{-1}} I\otimes Y \xrightarrow{\eta\otimes Y} (Y\otimes X)\otimes Y \xrightarrow{a} Y\otimes(X\otimes Y) \xrightarrow{Y\otimes\varepsilon} Y\otimes I \xrightarrow{r} Y \tag{7.2}$$

are both identities. The following abbreviations will be employed:

(i) $\varepsilon(s\otimes t)=s(t)$,
(ii) $l(s\otimes t)=s\cdot t$,
(iii) $r(t\otimes s)=t\cdot s$.

Assume now that $\eta(*)$ may be written as $\eta_1\otimes\eta_2$. Then (7.1) and (7.2) become, for $x\in X$ and $y\in Y$,

$$x(\eta_1)\cdot\eta_2 = x, \tag{7.3}$$

$$\eta_1\cdot\eta_2(y) = y, \tag{7.4}$$

which are much more concise. A particular case of this approach is Sweedler's

Σ-notation for a coalgebra (C, Δ, ε) [33] which, after removing the Σ's, says $\Delta(c) = c_1 \otimes c_2$. Of course, η_1 and η_2 do not exist in $\mathscr{L}$ in general, so a new language is required.

Extend $\mathscr{L}$ to a language $\mathscr{L}'$ as follows: for each term $s \in X_1 \otimes X_2$ introduce new terms $s_1 \in X_1$ and $s_2 \in X_2$ called *components* of s. Terms still consist of a function symbol, an operation and a sequence, but now the distinct items in the sequence may be components, as well as variables. The *atomic* terms are those of $\mathscr{L}$. It may happen that two distinct components in the sequence use the same variable in their construction. In this sense components may be thought of as variables, though they are not basic terms, which are tensors of variables and $*$'s, as before. $\geq$ is extended to a new order $\gg$ by introducing the generating reductions:

$$s \gg s_1 \otimes s_2, \tag{7.5}$$

$$s_1 \otimes s_2 \gg s, \tag{7.6}$$

and closing under (4.1)-(4.3) provided both sides are well defined (watch for duplication of variables!). (7.5) is an *s-expansion*, (7.6) an *s-contraction*. Together they are called *s-bonding*. Write $s \succeq t$ if there is a proof of $s \gg t$ in which there is no bonding. Clearly Theorem 4.1 fails for (7.5) and (7.6). Equally clearly, it applies to $\succeq$ in $\mathscr{L}'$. Theorem 7.2 will show how to eliminate bonding from a proof using properties of $\otimes$ in $\mathscr{T}_0$.

A *bi-pushout* [32] of a span (F, G) in $\mathscr{T}_0$ is determined by a 2-cell isomorphism $\alpha : HF \Rightarrow KG$ such that given an iso $\beta : H'F \Rightarrow K'G$ there is an operation L (unique up to isomorphism) and isomorphisms $\gamma : H' \Rightarrow LH$ and $\delta : LK \Rightarrow K'$ such that γ, α and δ patch to give β.

$$\begin{tikzcd}
{} \arrow[r, "F"] \arrow[d, "G"'] & {} \arrow[d, "H"] \arrow[ddr, "H'"] \arrow[dl, phantom, "\alpha\Swarrow"] & \\
{} \arrow[r, "K"'] \arrow[drr, "K'"'] & {} \arrow[dr, "L"] \arrow[r, phantom, "\gamma\Swarrow"] \arrow[d, phantom, "\delta\Swarrow"] & {} \\
& {} & {}
\end{tikzcd}$$

(The 2-dimensional aspect of the universal property and the uniqueness of γ and δ are guaranteed by the coherence of $\mathscr{T}_0$.)

Lemma 7.1. *$\mathscr{T}_0$ has all bi-pushouts, they are preserved by tensoring, and include*

$$\begin{tikzcd}
2 \arrow[r, "\otimes"] \arrow[d, "\otimes"'] & 1 \arrow[d, "1"] \\
1 \arrow[r, "1"'] & 1
\end{tikzcd}$$

Proof. Recall (Section 3) that there is a surjective, strict monoidal functor $\mathscr{T}_0 \to \mathscr{S}$, the prop for commutative monoids. Write $[F]$ for the isomorphism class of F in $\mathscr{S}$. Now $\mathscr{S}$ is equivalent to $\mathbf{Sets}_{\mathrm{f}}$ which has all pushouts, including

$$\begin{tikzcd}
2 \arrow[r] \arrow[d] & 1 \arrow[d] \\
1 \arrow[r] & 1
\end{tikzcd}$$

and they are preserved by tensor $(+)$. Thus $\mathscr{S}$ has these properties, which are lifted to $\mathscr{T}_0$ as follows. Consider a span (F, G) in $\mathscr{T}_0$ and choose F' and G' so that

$$\begin{tikzcd}
{} \arrow[r, "{[F]}"] \arrow[d, "{[G]}"'] & {} \arrow[d, "{[G']}"] \\
{} \arrow[r, "{[F']}"'] & {}
\end{tikzcd}$$

is a pushout in $\mathscr{S}$. Now operations of $\mathscr{T}_0$ have equal images in $\mathscr{S}$ only if they are isomorphic. Hence there is an isomorphism $\alpha : G'F \Rightarrow F'G$ in $\mathscr{T}_0$ with $[\alpha] = 1$ which is the bi-pushout. $\square$

Theorem 7.2. *Let* $s \gg t$ *in* $\mathscr{L}'$. *Then there is a proof of it in which all contractions occur before any expansions. Hence, if* s *and* t *are atomic, then* $s \geq t$ *in* $\mathscr{L}$.

Proof. It is sufficient to show the result for a proof beginning with its sole expansion of, say, v, and ending with its sole contraction of, say, u. If $u \neq v$, then without loss of generality the proof is of the form

$$\begin{aligned}
f(F(u_1, u_2, v, x)) &\gg f(F(u_1, u_2, v_1 \otimes v_2, x)) \\
&\equiv g(G(u_1 \otimes u_2, v_1, v_2, x)) \\
&\gg g(G(u, v_1, v_2, x)).
\end{aligned}$$

Thus $f = g \circ \alpha$ for some $\alpha : F(2, \otimes, n) \to G(\otimes, n+2)$ where the type of x lies in $\mathscr{V}^n$. Now

$$\begin{tikzcd}
{} \arrow[r, "{(2,\otimes,n)}"] \arrow[d, "{(\otimes,2,n)}"'] & {} \arrow[d, "{(\otimes,1,n)}"] \\
{} \arrow[r, "{(1,\otimes,n)}"'] & {}
\end{tikzcd}$$

is a bi-pushout and so there is an operation H of $\mathscr{T}$ and canonical transformations $\beta : F \Rightarrow H(\otimes, 1, n)$ and $\gamma : H(1, \otimes, n) \Rightarrow G$ with $\alpha = \gamma\beta$. Here, then is a proof in which

v-expansion follows u-contraction:

$$\begin{aligned} f(F(u_1,u_2,v,x)) = g\gamma\beta(F(u_1,u_2,v,x)) &\equiv g\gamma(H(u_1\otimes u_2,v,x)) \\ &\geqslant g\gamma(H(u,v_1,v_2,x)) \geqslant g(G(u,v_1,v_2,x)). \end{aligned}$$

Alternatively, let $u=v$. Then the proof has the form

$$f(F(u,x)) \geqslant f(F(u_1\otimes u_2,x)) \equiv g(G(u_1\otimes u_2,x)) \geqslant g(G(u,x)).$$

Hence $f=g\circ\alpha$ where $\alpha : F(\otimes, n) \Rightarrow G(\otimes, n)$. The lemma shows that $\alpha=\beta(\otimes,n)$ for some $\beta : F \Rightarrow G$. Thus $f=g\circ(\beta,n)$ and so there is a proof without contraction. □

Clearly, the techniques used here can be applied more generally. The chief requirement for expanding $s\in Fv$ as Fs' is that the bi-pushout of F against itself be the identity, i.e. F is bi-epi.

In [24] is the following proposition, generalizing the result of R. Paré for ordinary adjunctions, which is used to illustrate the techniques:

Theorem 7.3 (Lindner). *Let* $\eta : I \to Y\otimes X$ *and* $\varepsilon : X\otimes Y \to I$ *satisfy* (7.4). *Define* $i : X \to X$ *by* $i(x) = x(\eta_1)\cdot\eta_2$. *Then i is idempotent. It splits iff Y has a left adjoint.*

Proof. Here it is shown that $i^2=i$. The rest is left to the reader.

$$\begin{aligned} i^2(x) &= i(x(\eta_1)\cdot\eta_2) = [x(\eta_1)\cdot\eta_2](\eta_1)\cdot\eta_2 \\ &= [x(\eta_1)\cdot\eta_2(\eta_1)]\cdot\eta_2 \quad (x(\eta_1)\in I) \\ &= x(\eta_1\cdot\eta_2(\eta_1))\cdot\eta_2 \quad (\eta_2(\eta_1)\in I) \\ &= x(\eta_1)\cdot\eta_2 = i(x). \end{aligned}$$

□

8. Monoidal functors and transformations

Recall that a monoidal functor $(\phi,\tilde{\phi},\phi^\circ) : \mathcal{V} \to \mathcal{W}$ consists of a functor $\phi : \mathcal{V} \to \mathcal{W}$, a natural transformation with components $\tilde{\phi}_{X,Y} : \phi X\otimes\phi Y \to \phi(X\otimes Y)$ and a morphism $\phi^\circ : I \to \phi I$ of $\mathcal{W}$. More precisely, $\tilde{\phi} : \phi\otimes\phi \Rightarrow \phi\otimes$ and $\phi^\circ : I \to \phi I$ are natural transformations such that

$$\phi a\cdot\tilde{\phi}\cdot(\tilde{\phi}\otimes 1) = \tilde{\phi}\cdot(1\otimes\tilde{\phi})\cdot a, \tag{8.1}$$

$$\phi l\cdot\tilde{\phi}\cdot(\phi^\circ\otimes 1) = l, \tag{8.2}$$

$$\phi r\cdot\tilde{\phi}\cdot(1\otimes\phi^\circ) = r, \tag{8.3}$$

$$\phi c\cdot\tilde{\phi} = \tilde{\phi}\cdot c. \tag{8.4}$$

If there is a theory in which ϕ is an operation and $\tilde{\phi}$ and ϕ° are canonical transformations, then we have the reductions

$$\tilde{\phi}(\phi s\otimes\phi t) \geq \phi(s\otimes t), \qquad \phi^\circ(*) \geq \phi *.$$

Here $\geq$ is not an equivalence relation unless $\tilde{\phi}$ and ϕ° are isomorphisms, i.e. ϕ is a strong functor. Further, if $\mathscr{V}$ and $\mathscr{W}$ are closed, then so is ϕ. Let $s \in [X, Y]$ and $t \in X$ with $e : [X, Y] \otimes X \to Y$ the evaluation. Then $\check{\phi} : \phi[X, Y] \to [\phi X, \phi Y]$ is determined by $\check{\phi}(\phi s)(\phi t) = (\phi \varepsilon)\tilde{\phi}(\phi s \otimes \phi t)$. It follows that

$$\check{\phi}(\phi s)(\phi t) \geq \phi(s(t)).$$

Recall that a natural transformation $\eta : \phi \Rightarrow \psi$ is monoidal if it satisfies the equations

$$\tilde{\psi}(\eta \otimes \eta) = \eta\tilde{\phi}, \tag{8.5}$$

$$\eta_I \phi^\circ = \psi^\circ. \tag{8.6}$$

It may be desirable to make η canonical, too. Then we have

$$\eta(\phi s) \geq \psi s.$$

Since this structure is two-sorted ($\mathscr{V}$ and $\mathscr{W}$) we must generalise the concept of theory from a 2-prop (which is one-sorted) to a monoidal 2-category. Models will remain strong monoidal functors.

Define $\mathscr{T}_1$ to be the strict monoidal 2-category generated by two copies of $\mathscr{T}_0$ (with inclusions i and i'), operations ϕ, $\psi : 1 \to 1'$ and canonical transformations $\tilde{\phi}$, ϕ°, $\tilde{\psi}$, ψ° and η with domains and codomains as above, and satisfying (8.1)–(8.6). The objects and cells in the image of i' are distinguished, when necessary, by primes, e.g. $\otimes' : 2' \to 1'$. The symmetry for $\mathscr{T}_1$ is generated by C, C' and $\mathrm{id} : 1 + 1' \to 1' + 1$, i.e. $C_{m+n', p+q'} = (C_{m,p}, C'_{n,q})$. Then a standard model M of $\mathscr{T}_1$ in **Cat** (i.e. $M(m, p) = \mathscr{V}^m \times \mathscr{W}^p$) is just an example of a monoidal natural transformation.

A language for a standard model of $\mathscr{T}_1$ can be constructed just as before, so that the reductions above appear in the order on the terms. Theorem 4.1 and Theorem 7.2 carry over into this context, too. However, not every operation of $\mathscr{T}_1$ is locally sub-terminal. For example, there are two distinct canonical transformations $\phi I \to \phi I \otimes \phi I$; when $\phi : \mathbf{Ab} \to \mathbf{Sets}$ is the forgetful functor, they take $n \in \mathbb{Z}$ to $(n, 1)$ and $(1, n)$ in $\mathbb{Z}^2$. The theory could be easily extended to cover a language in which vertical composition of monoidal transformations occurs, but horizontal composition seems to be more difficult. Accordingly, the general situation is left until another occasion. For a club approach to coherence for a monoidal functor see [23].

An operation F of $\mathscr{T}_1$ has a *normal form* if it can be expressed as

$$m + n' \xrightarrow{(F_1, 1)} p + r + n' \xrightarrow{(1, \Phi, 1)} p + r' + n' \xrightarrow{(1, F_2)} p + q'$$

(also denoted (F_2, Φ, F_1)) where F_1 and F_2 are in the images of $\mathscr{T}_0$, and $\Phi = (\phi_i)$ where ϕ_i is ϕ or ψ. Let (G_2, Ψ, G_1) be another normal form for F. If domain$(\Phi) = r$ and there is a permutation $P: r \to r$ such that

$$\begin{tikzcd}
 & p+r+n' \arrow[r, "{(1,\Phi,1)}"] \arrow[dd, "{(1,P,1)}"] & p+r'+n' \arrow[dr, "{(1,F_2)}"] \arrow[dd, "{(1,P,1)}"'] & \\
m+n' \arrow[ur, "{(F_1,1)}"] \arrow[dr, "{(G_1,1)}"'] & & & p+q' \\
 & p+r+n' \arrow[r, "{(1,\Psi,1)}"'] & p+r'+n' \arrow[ur, "{(1,G_2)}"'] &
\end{tikzcd}$$

commutes, then say that the two forms are *congruent*. A canonical tranmsformation $\omega: F \Rightarrow G$ has a *normal form* if its domain and codomain have normal forms and it can be expressed as

$$\begin{tikzcd}
 & p+r+n' \arrow[r, "{(1,\Phi,1)}"] \arrow[dd, "{(1,H,1)}"] \arrow[ddr, phantom, "{(1,\beta,1)\Downarrow}"] & p+r'+n' \arrow[dr, "{(1,F_2)}"] \arrow[dd, "{(1,H,1)}"'] & \\
m+n' \arrow[ur, "{(F_1,1)}"] \arrow[dr, "{(G_1,1)}"'] \arrow[r, phantom, "{(\alpha,1)\Downarrow}"] & {} & {} \arrow[r, phantom, "{\Downarrow(1,\gamma)}"] & p+q' \\
 & p+s+n' \arrow[r, "{(1,\Psi,1)}"'] & p+s'+n' \arrow[ur, "{(1,G_2)}"'] &
\end{tikzcd} \tag{8.7}$$

with α and γ in the images of $\mathcal{T}_0$ and β given by

$$\begin{tikzcd}
r \arrow[r, shift left=2, "{(\phi_i)}"] \arrow[r, shift right=2, "{(\psi_i)}"'] \arrow[r, phantom, "{(\eta_i)\Downarrow}"] \arrow[d, "\tilde{H}"'] \arrow[dr, phantom, "\tilde{\beta}\Downarrow"] & r' \arrow[d, "\tilde{H}"] \\
t \arrow[r, "{(\psi_k)}"'] \arrow[d, "H^\circ"'] \arrow[dr, phantom, "\beta^\circ\Downarrow"] & t' \arrow[d, "H^{\circ\prime}"] \\
s \arrow[r, "{(\psi_j)}"'] & s'
\end{tikzcd}$$

where each η_i is either η or an identity, and $\tilde{\beta}$, β°, $\tilde{H}$ and H° are left-biased iterates of $\tilde{\phi}$, ϕ°, $\otimes$ and I respectively. Note that the choice of the operations F_1, F_2, H etc. in the normal form determines α, β and γ.

The reader may suspect that the use of diagrams in these definitions and the calculations below is contrary to the spirit of this work. However, variables are not always superior: their virtue lies in shortening arguments (when many cells of a diagram commute for tiresome reasons) or making them more familiar. Neither criterion is satisfied here.

Theorem 8.1. *Every operation and canonical transformation has a normal form: any two such for an operation F are congruent.*

Proof. The operations of $\mathcal{T}_1$ are obtained by first constructing all possible expressions for operations from the generating operations, under composition and tensoring, and then imposing the axioms for a strict, symmetric, monoidal 2-category closed under composition and tensoring. Assign to each expression F another, called $N(F)$, as follows:

(a) If F is a generating operation, then $N(F)$ is 'F with identities' e.g. $N(\phi)=(1,\phi,1)$.

Let $N(F)=(F_2,\Phi,F_1)$ and $N(G)=(G_2,\Psi,G_1)$. Then

(b) $N(F,G)=((F_2,G_2),(\Phi,\Psi),(F_1,G_1))$.

(c) $N(GF)$ is

$$m+n' \xrightarrow{((G_1,1)F_1,1)} s+u+p+n' \xrightarrow{(1,\Psi,\Phi,1)} s+u'+p'+n' \xrightarrow{(1,G_2(1,F_2))} s+t'.$$

If $F=G$ as operations, then induction on the length of the proof establishes $N(F)=N(G)$ as expressions.

The existence of a normal form for ω is established in the same way. Given normal forms for the domain and codomain and H which can be used in a normal form for ω (not any forms will do!), coherence in $\mathcal{T}_0$ ensures the uniquess of α and γ, while H° and $\tilde{H}$ determine β and (η_i). □

Corollary 8.2. *Operations of the form* $G=G_2'\psi G_1$ *or* $G=G_2'\phi G_1$ *are sub-terminal in* $\mathcal{T}_1$.

Proof. It is sufficient to work with $G_2'\psi G_1$. The theorem shows that this is the unique normal form equal to G since the only permutation $P:1\to 1$ is the identity. Also, given a canonical transformation $\omega:F\Rightarrow G$ with a normal form as in the theorem, then $H^\circ=\mathrm{id}$ and $\tilde{H}=\otimes(\otimes,1)(\otimes,1)\dots(\otimes,1)$ is the unique left-biased iterate of $\otimes$ from p to 1. $\tilde{\beta}=\tilde{\psi}(\tilde{\psi},1)\dots(\tilde{\psi},1)$ and each η_i is the unique 2-cell with codomain ψ. Now, however, every normal form for F determines a canonical transformation: they must be shown to be equal. Without loss of generality, replace Φ by Ψ, whose components are all ψ. Express ω as in (8.7) and let $P:r\to r$ be a permutation. Then, ω is

$$\begin{tikzcd}
 & r \arrow[r, "\Psi"] \arrow[d, "P"] & r' \arrow[d, "P"'] \arrow[dr, "F_2P"] & \\
m \arrow[ur, "P^{-1}F_1"] \arrow[r, "F_1"] \arrow[dr, "G_1"'] \arrow[dr, phantom, "\alpha\Downarrow"] & r \arrow[r, "\Psi"] \arrow[d, "H"] \arrow[dr, phantom, "\beta\Downarrow"] & r' \arrow[r, "F_2"] \arrow[d, "H"'] \arrow[dr, phantom, "\gamma\Downarrow"] & q' \\
 & 1 \arrow[r, "\psi"] & 1' \arrow[ur, "G_2"'] &
\end{tikzcd}$$

We must prove that it has a normal form whose domain is $(P^{-1}F_1,\Psi,F_2P)$. It is sufficient to show that the canonical transformation given by the middle pair of cells has a normal form whose domain and codomain are $(1,\Psi,J_2)$ and $(K_2,\psi,1)$, respectively, and this for $P=(1,C,1)$. Now P interacts with only one of the cells of

$\tilde{\beta}$, and so the problem reduces to two cases. If P is $(C,1)$ then, by (8.4),

$$\begin{tikzcd}
2 \arrow[r, "{(\psi,\psi)}"] \arrow[d, "C"'] & 2' \arrow[d, "C"] \\
2 \arrow[r, "{(\psi,\psi)}"] \arrow[d, "\otimes"'] \arrow[dr, phantom, "\tilde{\psi}\Downarrow"] & 2' \arrow[d, "\otimes"] \\
1 \arrow[r, "\psi"] & 1'
\end{tikzcd}$$

can be replaced by

$$\begin{tikzcd}
 & 2 \arrow[rr, "{(\psi,\psi)}"] \arrow[dd, "\otimes"] & & 2' \arrow[dd, "\otimes"'] \arrow[dr, "\otimes C"] & \\
2 \arrow[ur, "1"] \arrow[dr, "\otimes C"'] \arrow[r, phantom, "c\Downarrow"] & & \tilde{\phi}\Downarrow & & 1' \arrow[l, phantom, "\Downarrow c^{-1}"] \\
 & 1 \arrow[rr, "\psi"'] & & 1' \arrow[ur, "1"'] &
\end{tikzcd}$$

Otherwise, using (8.1) (twice) and (8.4), replace

$$\begin{tikzcd}
3 \arrow[r, "{(\psi,\psi,\psi)}"] \arrow[d, "{(1,C)}"'] & 3' \arrow[d, "{(1,C)}"] \\
3 \arrow[r, "{(\psi,\psi,\psi)}"] \arrow[d, "{\otimes(\otimes,1)}"'] \arrow[dr, phantom, "{\tilde{\psi}(\tilde{\psi},1)\Downarrow}"] & 3' \arrow[d, "{\otimes(\otimes,1)}"] \\
1 \arrow[r, "\psi"] & 1'
\end{tikzcd}$$

by

$$\begin{tikzcd}
 & 3 \arrow[rr, "{(\psi,\psi,\psi)}"] \arrow[dd, "{\otimes(\otimes,1)}"] & & 3' \arrow[dd, "{\otimes(\otimes,1)}"'] \arrow[dr, "{\otimes(\otimes,1)(1,C)}"] & \\
3 \arrow[ur, "1"] \arrow[dr, "{\otimes(\otimes,1)(1,C)}"'] \arrow[r, phantom, "\kappa\Downarrow"] & & {\phi(\phi,1)\Downarrow} & & 1' \arrow[l, phantom, "\Downarrow\kappa^{-1}"] \\
 & 1 \arrow[rr, "\psi"'] & & 1' \arrow[ur, "1"'] &
\end{tikzcd}$$

where $\kappa = a_{(1,C)}^{-1} \cdot (1,c) \cdot a$. $\square$

It is easy now to establish the effect of monoidal functors on structures in $\mathscr{V}$ (e.g. [10,15]) in particular on categories and triples. For example, consider

Theorem 8.3 (Eilenberg-Kelly [9, III.3]). *Let* $\eta:\phi\Rightarrow\psi:\mathscr{V}\to\mathscr{W}$ *be a monoidal transformation. Then* $\phi,\psi:\mathscr{V}\text{-}\mathbf{Cat}\to\mathscr{W}\text{-}\mathbf{Cat}$ *are* 2*-functors and* $\eta:\phi\Rightarrow\psi$ *is a* 2*-natural transformation.*

Proof. First, show that ϕ maps $\mathscr{V}$-categories to $\mathscr{W}$-categories. Let $\mathscr{A}$ be a $\mathscr{V}$-category with variables $f\in\mathscr{A}(A,B)$ and $g\in\mathscr{A}(B,C)$. Then $\phi\mathscr{A}$ has the same objects as $\mathscr{A}$ with $(\phi\mathscr{A})(A,B)=\phi\mathscr{A}(A,B)$. Composition is determined by $\phi g\circ\phi f\geq\phi(g\circ f)$ since ϕf and ϕg are basic. The identities are determined by $1_A\geq\phi(1_A)$. Now

$$\phi f\circ 1_A\geq\phi f\circ\phi(1_A)\geq\phi(f\circ 1_A)=\phi f.$$

Since ϕ is sub-terminal, the morphism whose action on ϕf yields $\phi f\circ 1_A$ is the identity, i.e. $\phi f\circ 1_A=\phi f$. The other axioms for $\phi\mathscr{A}$ and the rest of the proof are handled similarly. □

Extend $\mathscr{L}$ to a language $\mathscr{L}'$ in which bonding occurs as in (7.5) and (7.6) for both $\otimes$ and $\otimes'$.

Theorem 8.4. *Let* $s\gg t$ *in* $\mathscr{L}'$. *Then there is a proof of it in which all contractions occur before any expansions. Hence, if* s *and* t *are atomic, then* $s\geq t$ *in* $\mathscr{L}$.

Proof. The proof follows that of Theorem 7.2. It is sufficient to note that the inclusions i and i' of $\mathscr{T}_0$ in $\mathscr{T}_1$ preserve cocomma squares. For i' the result is trivial. For i the result follows from Theorem 8.1. □

Thus, for example, the effect of monoidal functors on adjunctions [24] is easy to calculate.

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