JOURNAL OF ALGEBRA, 3, 187-205 (1966)

Locally Finite Groups Embeddable in Stability Groups

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Communicated by I. N. Herstein

Received January 5, 1965

1. INTRODUCTION

This paper is essentially a continuation of [*4*], in which various forms of the question, "What groups can be embedded in the stability group of a suitable series (or normal system) of some group?" were considered. Here we shall determine those locally finite groups which can be embedded in the stability group of some (not necessarily well-ordered) invariant series.

For the most part we shall continue to use the notation of [*4*], some of which we now briefly recall. If Ω is a totally ordered set, then by a *series of type* Ω of a group G we shall mean a set

$$(\Lambda_\sigma, V_\sigma; \sigma \in \Omega) \tag{1}$$

of pairs of subgroups Λ_σ, V_σ of G, such that

(i) $V_\sigma \lhd \Lambda_\sigma$ for all $\sigma \in \Omega$

(ii) $\Lambda_\sigma \leqslant V_\tau$ if $\sigma < \tau$

(iii) $G - 1 = \bigcup_{\sigma \in \Omega} (\Lambda_\sigma - V_\sigma)$,

where $G - 1$ denotes the set of elements $\neq 1$ of G and $\Lambda_\sigma - V_\sigma$ the set of elements of Λ_σ which do not lie in V_σ. (1) is an *ascending* series if Ω is well-ordered, and a *descending* series if Ω is inversely well-ordered. The series is said to be *invariant* if each Λ_σ and V_σ is normal in G, and *central* if the commutator subgroup $[\Lambda_\sigma, G]$ is contained in V_σ for all $\sigma \in \Omega$. We refer to [*4*], section 1.2, for a fuller discussion of these terms.

As in [*4*], we define the stability group of (1) to be

$$\Gamma = \bigcap_{\sigma \in \Omega} C_A(\Lambda_\sigma / V_\sigma), \tag{2}$$

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where $A = \operatorname{Aut} G$ and $C_A(\Lambda_\sigma/V_\sigma)$ consists of all $\alpha \in A$ such that $[x, \alpha^{\pm 1}] \in V_\sigma$ whenever $x \in \Lambda_\sigma$. If we are given a homomorphism φ of a group B into Γ, then we shall say that B ***stabilizes*** (1) with respect to φ; usually it will be clear which homomorphism is meant and φ will not be explicitly mentioned. If φ is (1,1), then we shall say that B faithfully stabilizes (1). The statement that B ***can faithfully stabilize*** a series of a certain kind thus means that there is an isomorphism of B into the stability group of a suitable series of this kind.

As usual, when we speak of a *class* of groups $\mathfrak{C}$, we shall understand that $\mathfrak{C}$ is closed under isomorphisms and contains all the groups of order 1. In this paper we shall principally be concerned with the four classes of groups

$$\mathfrak{Y},\ \mathfrak{Y}^D,\ \mathfrak{T},\ \mathfrak{T}^D. \tag{3}$$

$\mathfrak{Y}$ and $\mathfrak{Y}^D$ are defined by: $A \in \mathfrak{Y}$ (respectively $\mathfrak{Y}^D$) iff A can faithfully stabilize some invariant series (respectively invariant descending series). of some group. $\mathfrak{T}$ and $\mathfrak{T}^D$ are correspondingly defined with "group" replaced by "abelian group", the word "invariant" becoming redundant of course. We note the relations

$$\mathfrak{T}^D \leqslant \mathfrak{T} \cap \mathfrak{Y}^D \leqslant \mathfrak{Y}. \tag{4}$$

We shall use throughout the notation for standard group classes and closure operations as set out in [*4*], section 1.3, (cf. also [*3*], p. 533). Briefly, the group classes we shall use are: $\mathfrak{F}$ = finite, $\mathfrak{F}_\varpi$ = finite ϖ-groups (where ϖ is always a set of primes), $\mathfrak{A}$ = abelian, $\mathfrak{G}$ = finitely-generated, $\mathfrak{N}$ = nilpotent, $\mathfrak{N}_k$ = nilpotent groups of class $\leqslant k$ (where k is an integer $\geqslant 0$). As for closure operations, if $\mathfrak{C}$ is any class of groups, then $\mathrm{S}\mathfrak{C}$ consists of all groups which can be embedded in some $\mathfrak{C}$-group, $\mathrm{Q}\mathfrak{C}$ consists of all homomorphic images of $\mathfrak{C}$-groups, $\mathrm{L}\mathfrak{C}$ is the class of locally-$\mathfrak{C}$ groups, and $\mathrm{R}\mathfrak{C}$ is the class of residually-$\mathfrak{C}$ groups. Finally, if $\mathfrak{C}$ and $\mathfrak{D}$ are two group classes, then $\mathfrak{C}\mathfrak{D}$ is defined to be the class of all extensions of a $\mathfrak{C}$-group by a $\mathfrak{D}$-group.

(Note that our use of the symbol $\mathfrak{T}$ differs from its use in [4], however.) With this notation, we have

$$\mathfrak{C} = \mathrm{RS}\mathfrak{C} \tag{5}$$

where $\mathfrak{C}$ is any of the classes (3) (cf. [*4*], Lemma 1).

For the statement of our main results we require to define another group class. For any periodic group G and integer $k > 0$, let $\pi_k(G) = \cap\, G_{\varpi'}$, where the intersection is taken over all sets ϖ consisting of k distinct primes. ϖ', as usual, denotes the set of primes not lying in ϖ, and $G_{\varpi'}$ is the subgroup generated by the ϖ'-elements of G. Now if L is a normal subgroup of the periodic group G, then G/L is a ϖ-group iff $G_{\varpi'} \leqslant L$; the class of periodic groups G satisfying $\pi_k(G) = 1$ may therefore be naturally described as the class of periodic residually k-prime groups. A finite residually one-prime group is clearly nilpotent, and this shows that the class of periodic residually one-prime groups is just $\mathrm{L}(\mathfrak{F} \cap \mathfrak{N})$. We shall need to show that, like this class, *the class of periodic residually two-prime groups is* L*-closed* (Lemma 5). Our main results are stated in terms of the class $\mathfrak{N}^{(2)}$ defined by: $G \in \mathfrak{N}^{(2)}$ iff $G \in \mathfrak{F} \cap \mathfrak{A}\mathfrak{N}$ and $\pi_2(G) = 1$. In Lemma 2, we shall deduce some interesting alternative characterizations of the class $\mathfrak{N}^{(2)}$ from a recent paper of B. Huppert [5].

Our main result is

THEOREM 1. $\mathrm{L}\mathfrak{F} \cap \mathfrak{Y} = \mathrm{L}\mathfrak{F} \cap \mathfrak{T} = \mathrm{L}\mathfrak{N}^{(2)}$.

Thus a locally finite group can faithfully stabilize an invariant series if and only if it can faithfully stabilize a series of some abelian group, and all such groups lie in a certain class of abelian-by-locally nilpotent groups. It does not seem to be known whether the first part of this assertion continues to hold without the restriction of local finiteness. But for finite groups we have the following even stronger result.

THEOREM 2. *Every $\mathfrak{N}^{(2)}$-group can faithfully stabilize a descending series of type $(\omega + n)^*$ of some finitely-generated abelian group.*

Here ω is the first infinite ordinal, n an integer $\geqslant 0$, and $(\omega + n)^*$ denotes the inverse ordered set of the ordered set $\omega + n$.

Theorems 1 and 2 together give the following result.

COROLLARY. *The intersection with $\mathfrak{F}$ of any of the classes* (3) *is just the class $\mathfrak{N}^{(2)}$.*

We should perhaps point out here that $\mathfrak{N}^{(2)}$ is not the whole of the class $\mathfrak{F} \cap \mathfrak{A}\mathfrak{N}$. To see this, consider the wreath product $G = H \wr K$, where H and K are cyclic groups of orders p and qr respectively, p, q and r being distinct primes. This is a metabelian group. If $H = \{x\}$ and α and β are elements of K of orders q and r respectively, then $[x, \alpha, \beta]$ is a nontrivial element of $\pi_2(G)$. Thus $G \notin \mathfrak{N}^{(2)}$.

The intersections with $\mathrm{L}\mathfrak{F}$ of the classes (3) do not all coincide. In fact

$$\mathrm{L}\mathfrak{F} \cap \mathfrak{Y}^D < \mathrm{L}\mathfrak{F} \cap \mathfrak{T} \tag{6}$$

To see this, it is only necessary to consider any perfect $\mathrm{L}\mathfrak{F}_p$-group $A \neq 1$, for example, the one constructed by D.H. MacLain [7]. By Theorem 1, $A \in \mathfrak{T}$. But A cannot lie in $\mathfrak{Y}^D$, since all groups of this class have a descending series with abelian factors. (Cf. [4], section 2.6.) The precise relationship between the classes $\mathrm{L}\mathfrak{F} \cap \mathfrak{T}^D$ and $\mathrm{L}\mathfrak{F} \cap \mathfrak{Y}^D$ is not known, so far as we are aware.

The following special case of Theorem 1 is perhaps of interest in its own right. It will follow from our proof of Theorem 1.

THEOREM 3. *Suppose that the group G has an invariant series faithfully stabilized by the locally finite group A. If G is either torsion-free or locally finite, then A is locally nilpotent.*

In Section 2 we shall give some alternative characterizations of the class $\mathfrak{N}^{(2)}$ and obtain some properties of this class which we shall need. In Section 3 we prove Theorem 2 and show that every $\mathrm{L}\mathfrak{N}^{(2)}$-group lies in $\mathfrak{T}$. The remainder of the paper is devoted to the other half of Theorem 1, which consists in showing that $\mathfrak{F} \cap \mathfrak{Y} \leqslant \mathfrak{N}^{(2)}$.

2. FINITE TWO-PRIME GROUPS

LEMMA 1. *Let A be a finite group. For each prime p, let B_p be the group generated by the p-elements of $A_{p'}$. Then $A \in \mathfrak{N}^{(2)}$ iff, for each prime p, $B_p \in \mathfrak{A}$ and for each $1 \neq x \in B_p$, $\exists\, K \lhd A_{p'}$ such that $x \notin K$ and $A_{p'}/K$ is a two-prime group.*

Proof. The condition is clearly necessary, since the class $\mathfrak{N}^{(2)}$ is S-closed and so will contain each $A_{p'}$ if it contains A. Thus B_p, which lies in the last term of the lower central series of $A_{p'}$, will be abelian.

Now suppose the last condition to be satisfied. Then, in the first place, $A \in \mathfrak{A}\mathfrak{N}$. This was effectively proved in [4], Lemma 16, but we have thought it desirable to give the proof here. Let $B = \cap A_{p'}$ over all primes p. Then $B \lhd A$. Now every p-element of $A_{p'}$ lies in $A_{q'}$ for every $q \neq p$ and so lies in B; consequently $B_p \leqslant B$. Furthermore, every p-element of B lies in $A_{p'}$ and so in B_p. Thus, since B_p is by assumption abelian and so a p-group, B_p is precisely the set of p-elements of B. It follows that B is the direct product of the subgroup B_p and so is abelian. Now every Sylow p-subgroup of A/B is of the form PB/B for a suitable Sylow p-subgroup P of A. But if Q is a Sylow q-subgroup of A, where $q \neq p$, then $P \leqslant A_{r'}$ for every $r \neq p$, and $Q \leqslant A_{p'}$, whence $[P, Q] \leqslant \cap_r A_{r'} = B$. Thus, in A/B any two Sylow subgroups of coprime orders commute elementwise, and so $A/B \in \mathfrak{N}$.

To show that $A \in \mathfrak{N}^{(2)}$, it is clearly sufficient to show that, if $1 \neq x \in B$, then $\exists\, L \lhd A$ such that $x \notin L$ and A/L is a two-prime group. Since x has a nontrivial power lying in some subgroup B_p, we may suppose at the outset that $x \in B_p$. However, by hypothesis, $\exists\, L \lhd A_{p'}$ such that $x \notin L$ and $A_{p'}/L$ is a two-prime group. Now since xL is a nontrivial p-element of $A_{p'}/L$, it follows that $| A_{p'}/L |$ is divisible only by p and possibly one other prime q. By replacing L by the subgroup of $A_{p'}$ generated by the $(p, q)'$-elements of $A_{p'}$, we may suppose that L is characteristic in $A_{p'}$ and so normal in A. But $A/A_{p'}$ is a p-group and so A/L is a (p, q)-group. This concludes the proof of Lemma 1.

The following properties of the class $\mathfrak{N}^{(2)}$ may perhaps be regarded as generalizations of similar properties of the class of finite nilpotent groups. I am indebted to Professor B. Huppert for a helpful discussion of these results.

LEMMA 2. *Let $G \in \mathfrak{F} \cap \mathfrak{A}\mathfrak{N}$. Then the following conditions are equivalent.*

(i) $\pi_2(G) = 1$, *that is*, $G \in \mathfrak{N}^{(2)}$.

(ii) *In G, any two Sylow subgroups of coprime orders permute.*

(iii) *For each prime p, each chief p-factor of G is centralized by some $G_{\varpi'}$, where ϖ consists of p and one other prime.*

(iv) *Each chief factor H/K of G is avoided by some $G_{\varpi'}$ ($|\varpi| = 2$), in the sense that $H \cap G_{\varpi'} \leqslant K$.*

Proof. By Satz 1 of [5], (ii) and (iii) are equivalent and are implied by (i). We shall first show that (ii) implies (i).

Suppose the contrary, and let G be an abelian-by-nilpotent group of minimal order satisfying (ii) but not (i). Let B be a minimal normal subgroup of G, and $1 \neq C \lhd G$. Now by [5], Hilfsatz 1, the property (ii) passes to quotient groups, hence both G/B and G/C lie in $\mathfrak{N}^{(2)}$. Therefore

$$\pi_2(G) \leqslant B \cap C,$$

and so $B \cap C \neq 1$. Hence $C \geqslant B$, and B is the unique minimal normal subgroup of G.

Let K be the last term of the lower central series of G. By assumption, $K \in \mathfrak{A}$, and since A has only one minimal normal subgroup we must in fact have $K \in \mathfrak{A}_p$ for some prime p (where $\mathfrak{A}_p$ denotes the class of abelian p-groups), and $B \leqslant K$.

Now if $q \neq p$, then $G_{q'}$ is evidently a q'-group. If $G_{p'} < G$, then $G_{p'} \in \mathfrak{N}^{(2)}$, and G would then satisfy the hypotheses of Lemma 1 and so would itself lie in $\mathfrak{N}^{(2)}$. Hence $G_{p'} = G$. G/K, being nilpotent and generated by p'-elements, therefore lies in $\mathfrak{F}_{p'}$. By a well known theorem of Schur (cf. [6], p. 201), G splits over K, that is, $G = KL$, $K \cap L = 1$, for some subgroup L of G. $L \cong G/K$ is a nilpotent p'-group and so is the direct product of its Sylow subgroups, say $L = Dr_{q \neq p} L_q$.

Let P be the subgroup of elements of order p in K. Since G acts on P as a p'-group, it follows that P is completely reducible under the action of G, that is, P is the direct product of minimal normal subgroups of G. Hence $P = B$. Hence by condition (iii) there is a prime q such that $[P, L_{q'}] = 1$.

It follows that $[K, L_{q'}] = 1$. For if not, let y be an element of K of minimal order p^n not centralized by $L_{q'}$. Let $\alpha \in L_{q'}$. Now y^p is of order p^{n-1}, hence $1 = [y^p, \alpha] = [y, \alpha]^p$, since $K \in \mathfrak{A}$. Therefore $[y, \alpha]$ is centralized by α. But $\alpha^m = 1$ for some p'-number m and so $1 = [y, \alpha^m] = [y, \alpha]^m$. It follows that $[y, \alpha] = 1$, and $L_{q'}$ centralizes y, contrary to hypothesis .Hence $[K, L_{q'}] = 1$. Consequently $L_{q'} \lhd G$, and so $L_{q'} = 1$. Therefore G is a (p, q)-group and so lies in $\mathfrak{N}^{(2)}$. This contradiction concludes the proof that (ii) implies (i).

To see that (iv) implies (iii), let H/K be a chief p-factor of G and suppose $G_{\varpi'}$ avoids it, where $|\varpi| = 2$. Now $KG_{\varpi'}/K$ is the subgroup generated by the ϖ'-elements of G/K, and since $KG_{\varpi'} \cap H \leqslant K$, there are nontrivial p-elements of G/K not lying in $KG_{\varpi'}/K$. Hence $p \in \varpi$. But

$$[H, G_{\varpi'}] \leqslant H \cap G_{\varpi'} \leqslant K.$$

Hence $G_{\varpi'}$ centralizes H/K.

Furthermore, (iv) is implied by the other properties. For as already pointed out, these properties all pass to quotient groups, hence it is sufficient to show that every minimal normal subgroup M of G is avoided by some $G_{\varpi'}$ with $|\varpi| = 2$. But by property (i), if $1 \neq Z \in M$, then z lies outside some such $G_{\varpi'}$, and so, as $G_{\varpi'} \lhd G$, we must have $G_{\varpi'} \cap M = 1$. This concludes the proof of Lemma 2.

We define the closure operation R_0 to be the "finite version" of R—a class $\mathfrak{C}$ is R_0-closed iff, whenever a group G has normal subgroups K and L with $K \cap L = 1$ and both G/K and G/L lie in $\mathfrak{C}$, then $G \in \mathfrak{C}$. Then

$$\mathfrak{N}^{(2)} = \{\mathrm{R}_0, \mathrm{Q}, \mathrm{S}\}\mathfrak{N}^{(2)} \tag{7}$$

where $\{\mathrm{X}, \mathrm{Y}, ...\}$ denotes the *closure join* of the closure operations $\mathrm{X}, \mathrm{Y}, ...$. (cf. [3], p. 533). The Q-closure follows from Lemma 2 and Hilfsatz 1 of [5], and the others are immediate.

We shall require the following fact.

LEMMA 3. *Let $G \in \mathfrak{F} \cap \mathfrak{A}\mathfrak{N}$ and suppose L is a normal subgroup of G. Suppose further that $1 = L_0 \leqslant L_1 \leqslant \cdots \leqslant L_k = L$ is a series of normal subgroups of G such that, for some prime q, $G_{q'}$ centralizes every L_{i+1}/L_i. If $G/L \in \mathfrak{N}^{(2)}$, then $G \in \mathfrak{N}^{(2)}$.*

Proof. Let $1 = G_0 \leqslant G_1 \leqslant \cdots \leqslant G_n = G$ be a chief series of G containing all the L_i. It is enough to show that, if G_{i+1}/G_i is an r-factor, then it is centralized by some $G_{\varpi'}$, where ϖ is a set of two primes containing r. For by the Jordan-Hölder theorem, every chief factor of G is G-isomorphic to some G_{i+1}/G_i (where G is regarded as a group of operators for itself by conjugation), and Lemma 2 would then give the result.

However, if $G_i \geqslant L$ this is true since G/L, which lies in $\mathfrak{N}^{(2)}$ by assumption, satisfies (iii) of Lemma 2. But if $G_i < L$, then G_{i+1}/G_i is a factor of some L_{j-1}/L_j and so is even centralized by $G_{q'}$.

We should perhaps point out that Lemmas 2 and 3 depend rather strongly on the fact that we are working with abelian-by-nilpotent groups, as does the rather surprising fact that, for abelian-by-nilpotent finite groups, the property $\pi_2(G) = 1$ is preserved under homomorphisms. This will be clear from the following example.

LEMMA 4. *There is a finite group $K \in \mathfrak{N}_2\mathfrak{A}$ which satisfies* (ii) *and* (iii) *of Lemma* 2, *but for which $\pi_2(K) \neq 1$. $|K|$ is divisible by only three primes, the center Z of K is of order* 2, *and $K/Z \in \mathfrak{N}^{(2)}$. Further, K is a homomorphic image of a $\mathfrak{N}_2\mathfrak{A}$-group N with $\pi_2(N) = 1$.*

Proof. Let q be a prime $\neq 2$. Let G be the group of class 2 and exponent 4 generated by the elements $x_1, x_2, \ldots, x_{q-1}$ subject to the realtions

$$x_i^2 = [x_j, x_k, x_l] = 1$$

for i, j, k, l running from 1 to $q - 1$. Then G' is of exponent 2 and has a basis consisting of the $\frac{1}{2}q(q-1)$ commutators $[x_j, x_k]$ with $j < k$. We identify these to obtain a group H_q of class 2 and exponent 4 generated by the elements $x_1, x_2, \ldots, x_{q-1}$ with $[x_i, x_j] = z$ if $i \neq j$, where $z^2 = 1$ and z lies in the center of H_q. Let $Z = \{z\}$. Then Z is the unique minimal normal subgroup of H_q. Also, H_q has an automorphism α of order q which maps $x_i \to x_{i+1}$ for $1 \leqslant i < q - 1$, and $x_{q-1} \to x_1 x_2 \ldots x_{q-1}$.

Let K_q be the natural split extension of H_q by $\{\alpha\}$. Then Z is the center of K_q, and also its unique minimal normal subgroup. Choose an odd prime $r \neq q$ and let K_r be constructed in the same way as K_q. Let K be the central product (direct product with identified centres) of K_q and K_r. Then $K = K_q K_r$, $[K_q, K_r] = 1$, $K_q \cap K_r = Z$, and it is easy to verify that Z is the unique minimal normal subgroup of K. But $|K| = 2^{q+r-1}qr$ is divisible by three distinct primes. Hence $K_{\varpi'} \geqslant Z$ whenever $|\varpi| = 2$, and so $\pi_2(K) \geqslant Z$. In particular, $\pi_2(K) \neq 1$.

Now K is a homomorphic image of $N = K_q \times K_r$ and $\pi_2(N) = 1$. By Satz 1 of [5], it follows that both N and K have properties (ii) and (iii) of Lemma 1. Also, both N and K lie in $\mathfrak{N}_2\mathfrak{A}$. Further, Z is a central subgroup of order 2 of K, and $K/Z \cong K_q/Z \times K_r/Z \in \mathfrak{N}^{(2)}$. K thus has all the properties which we required of it.

3. EMBEDDING L$\mathfrak{N}^{(2)}$-GROUPS IN STABILITY GROUPS

Before showing that every L$\mathfrak{N}^{(2)}$-group lies $\mathfrak{T}$, we need the following lemma about the structure of L$\mathfrak{N}^{(2)}$-groups.

Let $\mathfrak{P}$ be the class of all groups of the form $B \,\bar{\wr}\, \Gamma$, where $B \in \mathfrak{A}_p$ and Γ is a locally nilpotent (p, q)-group for suitable primes p and q. Here $\bar{\wr}$ denotes the complete wreath product.

LEMMA 5.

(i) $\mathrm{L}(\mathfrak{A}\mathfrak{N}) = \mathfrak{A}(\mathrm{L}\mathfrak{N})$.

(ii) *If G is periodic and $\pi_2(H) = 1$ for every $\mathfrak{G}$-subgroup H of G, then* $\pi_2(G) = 1$.

(iii) $\mathrm{L}\mathfrak{N}^{(2)} \leqslant \mathrm{RS}\mathfrak{P}$.

(iv) $\mathfrak{N}^{(2)} \leqslant \mathrm{R}_0\mathrm{S}(\mathfrak{F} \cap \mathfrak{P})$.

Proof. (i) Clearly $\mathfrak{A}(\mathrm{L}\mathfrak{N}) \leqslant \mathrm{L}(\mathfrak{A}\mathfrak{N})$. Suppose $G \in \mathrm{L}(\mathfrak{A}\mathfrak{N})$. Let $L = \{\bar{\gamma}(H)\}$, where H runs over the $\mathfrak{G}$-subgroups of G and $\bar{\gamma}(H)$ denotes the last term of the lower central series of H; thus $\bar{\gamma}(H) \in \mathfrak{A}$. If H and K are two $\mathfrak{G}$-subgroups of G and $J = \{H, K\}$, then $\{\bar{\gamma}(H), \bar{\gamma}(K)\} \leqslant \bar{\gamma}(J) \in \mathfrak{A}$. This makes it clear that L is an abelian subgroup of G. Also every conjugate of $\bar{\gamma}(H)$ under K lies in $\bar{\gamma}(J) \leqslant L$, and so $L \lhd G$. Clearly $G/L \in \mathrm{L}\mathfrak{N}$, whence $G \in \mathfrak{A}(\mathrm{L}\mathfrak{N})$.

(ii) Let $P = \pi_2(G) = \cap\, G_{\varpi'}$ over all sets ϖ of two primes, and suppose if possible that $P \neq 1$. Since G is periodic, it follows that P contains an element u of prime order p, say. Now u certainly lies in $G_{p'}$, and so u can be expressed as a product of a finite number of elements whose orders are all prime to p. Thus $u \in N_{\varpi_1}$, where N is a $\mathfrak{G}$-subgroup of G and ϖ_1 a finite subset of p'. However, for each $q \in \varpi_1$, $\exists$ $\mathfrak{G}$-subgroup $N^{(q)}$ of G such that u lies in the subgroup generated by the $(p, q)'$-elements of $N^{(q)}$. Let $Q = \{N, N^{(q)}; q \in \varpi_1\}$, and let ϖ be any set of two primes. If $p \notin \varpi$, then u is a ϖ'-element and so lies in $Q_{\varpi'}$. If $\varpi = (p, r)$ and $r \notin \varpi_1$, then $u \in N_{\varpi_1} \leqslant Q_{\varpi'}$. If $r \in \varpi_1$, then $u \in N^{(r)}_{\varpi'} \leqslant Q_{\varpi'}$. Consequently $u \in \pi_2(Q)$. But Q is finitely-generated and so $\pi_2(Q) = 1$ by assumption. This contradiction shows that $P = 1$, as required.

(iii) Let $G \in \mathrm{L}\mathfrak{N}^{(2)}$. Then $G \in \mathfrak{A}(\mathrm{L}\mathfrak{N})$ and $\pi_2(G) = 1$, by parts (i) and (ii). Let $W = G/G_{\varpi'}$, where $\varpi = (p, q)$. Then $W \in \mathfrak{A}(\mathrm{L}\mathfrak{N})$, since this class is Q-closed, and W is a (p, q)-group. Thus W has an abelian normal subgroup V such that $W/V = \Gamma \in \mathrm{L}\mathfrak{N}$. Now $V = V_p \times V_q$ is the direct product of its Sylow subgroups V_p and V_q, and W/V_p is an extension of V_q by Γ. It is well known that every extension of a group A by a group B can be embedded in $A \,\bar{\wr}\, B$. Consequently $W/V_p \in \mathrm{S}\mathfrak{P}$. Since $V_p \cap V_q = 1$, we find that $W \in \mathrm{RS}\mathfrak{P}$, and so, since $\pi_2(G) = 1$, this class also contains G.

If G is finite, then clearly all the relevant members of $\mathfrak{P}$ may be supposed finite, whence (iv) also follows.

LEMMA 6.

(i) $\mathrm{L}\mathfrak{N}^{(2)} \leqslant \mathfrak{T}$.

(ii) *Every $\mathfrak{N}^{(2)}$-group can faithfully stabilize a descending series of type $(\omega + n)^*$ of some finitely-generated abelian group.*

Part (ii), of course, is precisely Theorem 2.

Proof. By Lemma 5 and the relations (5) of Section 1, it is enough to show that $\mathfrak{P} \leqslant \mathfrak{T}$ and that every finite $\mathfrak{P}$-group has the property required in (ii).

Accordingly, let $W = B \,\bar{\wr}\, \Gamma$ be any $\mathfrak{P}$-group. Thus $B \in \mathfrak{A}_q$ for some prime q, and Γ is a locally nilpotent (p, q)-group, so that $\Gamma = \Gamma_p \times \Gamma_q$ is the direct product of its Sylow p- and q-subgroups Γ_p and Γ_q. Let $b \to b^*$ be an isomorphism of B onto a group B^*, let $\{x\}$ be an infinite cyclic group, and let X and Y be the base groups of $\{x\} \wr \Gamma_p$ and the complete wreath product $B^* \,\bar{\wr}\, \Gamma_q$ respectively. We shall first embed W in Aut G, where G is the abelian group

$$G = X \times Y. \tag{8}$$

In the first place, Γ_p is to transform X as in $\{x\} \wr \Gamma_p$, and Γ_q is to transform Y as in $B^* \,\bar{\wr}\, \Gamma_q$. If we impose the further conditions

$$[X, \Gamma_q] = [Y, \Gamma_p] = 1, \tag{9}$$

we shall clearly obtain an embedding of Γ in Aut G. Now any element of Γ has a unique representation in the form $\xi\eta(\xi \in \Gamma_p, \eta \in \Gamma_q)$, consequently the base group $\bar{B}$ of $B \,\bar{\wr}\, \Gamma$ may be written as the cartesian product

$$Cr(B_{\xi\eta}\,;\, \xi \in \Gamma_p, \eta \in \Gamma_q).$$

To represent $\bar{B}$ by automorphisms of G we require that, if $\xi \in \Gamma_p$ and $\bar{b} \in \bar{B}$, then

$$[x^{\xi}, \bar{b}] = y, \tag{10}$$

where $y \in Y$ and if, for $\eta \in \Gamma_q$, y_η denotes the ηth coordinate of y, then

$$y_\eta = (\bar{b}_{\xi\eta})^*. \tag{11}$$

If we also require

$$[\bar{B}, Y] = 1, \tag{12}$$

then we may clearly regard $\bar{B}$ as a subgroup of Aut G. Transforming (10) with an element $\xi'\eta'$ of Γ, we obtain

$$[x^{\xi\xi'}, \bar{b}^{\xi'\eta'}] = y^{\eta'} \tag{13}$$

and the $\eta\eta'$th coordinate of $y^{\eta'}$ is $y_\eta = (\bar{b}_{\xi\eta})^*$. Thus $\bar{b}^{\xi'\eta'}$ is that element of $\bar{B}$ whose $\xi\xi'\eta\eta'$-coordinate is $\bar{b}_{\xi\eta}$. This makes it clear that the subgroup $\{\bar{B}, \Gamma\}$ of Aut G is isomorphic with $B \wr \Gamma$ and so may be identified with W.

It remains to show that there is a series of G whose stability group contains W. We shall first consider the action of Γ_q on Y. Now since Y is the cartesian product of a number of abelian q-groups, every periodic element of Y is a q-element. Let Δ be any finitely-generated subgroup of Γ_q and let U be any finitely-generated subgroup of Y. Since $\Gamma_q \in \mathrm{L}\mathfrak{F}_q$, Δ is a finite q-group, and so U lies in a finitely-generated Δ-invariant subgroup V of Y. Since V has no elements of order prime to q, we have $\bigcap_{n=0}^{\infty} V^{q^n} = 1$, where V^{q^n} denotes the subgroup consisting of the q^nth powers of the elements of V. Consequently the natural semidirect product $V\Delta$ is a $\mathrm{R}\mathfrak{F}_q$-group, and hence is a Z-group, that is, it has a central series. Let T be the natural semidirect product $Y\Gamma_q$; $Y \triangleleft T$, $Y \cap \Gamma_q = 1$. Now every finitely-generated subgroup of T lies in one of the form $V\Delta$, and so is a Z-group. Since the class of Z-groups is L-closed, by a well known theorem of Mal'cev ([*6*], p. 218), T itself is a Z-group. The intersections with Y of any central series of T then furnish a series of Y stabilized by Γ_q (cf. also Plotkin [*8*], p. 1389). Let $(\Lambda_\sigma, V_\sigma; \sigma \in \Omega)$ be such a series. We note that if Γ_q and B are finite, then $T \in \mathfrak{F}_q$, and so this series will be finite.

By a similar argument we obtain a series $(\sum_\alpha, \Phi_\alpha; \alpha \in \Psi)$ of X stabilized by Γ_p; this will be of type ω^* if $\Gamma_p \in \mathfrak{F}$. Now write $\sum_\alpha^* = \sum_\alpha \times Y$; $\Phi_\alpha^* = \Phi_\alpha \times Y$ if $\alpha \in \Psi$. The subgroups Λ_σ and V_σ $(\sigma \in \Omega)$ form a series of Y which is extended to a series of G by the addition of the subgroups $\sum_\alpha^*$ and Φ_α^*. It is clear that $\bar{B}$ lies in the stability group of this series, and therefore so does W. Further, if W is finite, then the series will be of type $(\omega + n)^*$ where n is an integer ≥ 0. This concludes the proof of Lemma 6.

4. Some Useful Lemmas

To conclude the proof of Theorem 1, we have now to prove

Theorem 1*. *Suppose A is a finite subgroup of the stability group of some invariant series of the group G. Then $A \in \mathfrak{N}^{(2)}$.*

This will occupy the remainder of the paper. We shall first prove some miscellaneous lemmas.

We recall that a group G is said to be *finitely-presented* if it has a finite number of generators in terms of which it can be defined by a finite number of relations. Let us define the closure operator E by saying that a class $\mathfrak{C}$ is E-closed whenever every extension of a $\mathfrak{C}$-group by a $\mathfrak{C}$-group again lies in $\mathfrak{C}$. It is well known that *the class $\mathfrak{R}$ of finitely-presented groups is E-closed.* In fact, suppose G has a normal subgroup K such that K and $G/K = \Gamma$ both lie in $\mathfrak{R}$. Then K is generated by a finite number of elements $a_1, a_2, ..., a_k$ with relations $f_1(a) = \cdots = f_l(a) = 1$, and Γ is generated by $\beta_1, ..., \beta_m$ with relations $g_1(\beta) = \cdots = g_n(\beta) = 1$. Choose $b_i \in \beta_i$ for $1 \leqslant i \leqslant m$. Then we have $g_i(b) = h_i(a)$ for certain words $h_1, ..., h_n$ in the a's. Also $a_i^{b_j} \in K$ and so $a_i^{b_j} = u_{ij}(a)$ $(1 \leqslant i \leqslant k, 1 \leqslant j \leqslant m)$. The group G is then generated by $a_1, ..., a_k, b_1, ..., b_m$ with the defining relations $f_i(a) = 1$, $g_s(b) = h_s(a)$, $a_p^{b_j} = u_{pj}(a)$ for $1 \leqslant i \leqslant l$, $1 \leqslant s \leqslant n$, $1 \leqslant j \leqslant m$, $1 \leqslant p \leqslant k$, and thus lies in $\mathfrak{R}$.

For any group G, let $f(G)$ consist of those normal subgroups of G which are generated by a finite number of classes of conjugate elements in G.

LEMMA 7. *Suppose $G \in \mathfrak{G}$ and that K and L are normal subgroups of G with $K \leqslant L$. If $L \in f(G)$ and $L/K \in \mathfrak{R}$, then $K \in f(G)$.*

Proof. Suppose $G = \{x_1, ..., x_n\}$, where n is finite, and let G^* be freely generated by elements $x_1^*, ..., x_n^*$. Let θ be the homomorphism of G^* onto G which maps $x_i^* \rightarrow x_i$ for $1 \leqslant i \leqslant n$, and let M^* be its kernel. Write $K^* = \theta^{-1}(K)$, $L^* = \theta^{-1}(L)$. Now since $L \in f(G)$, we have $L = \{u_1^G, ..., u_k^G\}$ for some finite k. Choose elements $u_1^*, ..., u_k^* \in G^*$ such that $u_i^{*\theta} = u_i$ $(1 \leqslant i \leqslant k)$ and let $U^* = \{u_1^{*G^*}, ..., u_k^{*G^*}\}$. Now $L^* \lhd G^*$ and $U^*M^* \leqslant L^*$. Further, if $x^* \in L^*$, then $x^{*\theta} \in L$. But $U^{*\theta} = L$, and so $x^{*\theta} = u^{*\theta}$ for some $u^* \in U^*$. Consequently $x^* \equiv u^* \bmod M^*$ and so we have $L^* = U^*M^*$.

Now $M^* \leqslant K^* \leqslant U^*M^*$ and so

$$K^* = (K^* \cap U^*)M^* \tag{14}$$

But $U^*/K^* \cap U^* \simeq U^*K^*/K^* = L^*/K^* \simeq L/K \in \mathfrak{R}$. Also $G^*/U^* \in \mathfrak{R}$, clearly, since $G^* \in \mathfrak{G}$ and $U^* \in f(G^*)$. Hence, since $\mathfrak{R} = \text{E}\mathfrak{R}$, we have $G^*/K^* \cap U^* \in \mathfrak{R}$. But if we take any finite set of generators of an $\mathfrak{R}$-group W, then W can be defined by a finite number of relations between those generators ([6], p. 74). Hence $K^* \cap U^* \in f(G^*)$, and so

$$K^* \cap U^* = \{k_1^{*G^*}, ..., k_s^{*G^*}\}$$

for some finite s. It is now clear from (14) that $K = \{(k_1^{*\theta})^G, ..., (k_s^{*\theta})^G\}$, as required.

LEMMA 8. *Suppose $(\Lambda_\sigma, V_\sigma; \sigma \in \Omega)$ is an invariant series of G with stability group Γ, and let K be a finite normal subgroup of G. Then*

$$(\Lambda_\sigma K/K, V_\sigma K/K; \sigma \in \Omega)$$

is an invariant series of G/K. If A is a subgroup of Γ under which K is invariant, then A stabilizes this series of G/K.

This situation contrasts with the fact that nonascending series are not usually preserved by homomorphisms.

Proof. It is only necessary to verify that $G - K = \bigcup_{\sigma \in \Omega} (A_\sigma K - V_\sigma K)$, since the other conditions are clearly satisfied. Accordingly, let $x \in G - K$. Now the coset xK is a finite set of elements each of which lies in just one layer $A_\sigma - V_\sigma$ of the given series. Therefore only a finite number of layers meet this coset, and so there is a smallest σ such that $(A_\sigma - V_\sigma) \cap xK \neq \phi$. Then clearly $x \in A_\sigma K$. However $V_\sigma \cap xK = \phi$, and so $x \notin V_\sigma K$.

Case (i) of the following lemma establishes one half of Theorem 3.

LEMMA 9. *Suppose that the group G has a series whose stability group contains the finite group A. Then*

(i) *If $G \in \mathrm{L}\mathfrak{F}$, then $A \in \mathfrak{N}$.*

(ii) *If $G \in \mathrm{L}\mathfrak{F}_p$, then $A \in \mathfrak{F}_p$.*

(iii) *If G is abelian of finite exponent, then G has a finite series stabilized by A.*

Proof. (i) Let $x \in G$. Then x lies in a finitely-generated A-invariant subgroup H of G. Since $G \in \mathrm{L}\mathfrak{F}$, $H \in \mathfrak{F}$. The intersections with H of the given series of G constitute a finite series of H stabilized by A. Consequently by a well known result of P. Hall ([2], p. 787), A acts on H as a nilpotent group, that is, $\bar{\gamma}(A)$ centralizes H. Since x was an arbitrary element of G, it follows that $\bar{\gamma}(A) = 1$.

(ii) In this case, if $x \in G$, then x lies in an A-invariant $\mathfrak{F}_p$-subgroup H of G which has a finite series stabilized by A. It follows that A acts on H as a finite p-group. For if α is a p'-element of A, then the semidirect product $H\{\alpha\}$ is nilpotent and so H commutes with α. Consequently $A \in \mathfrak{F}_p$.

(iii) Here G is an abelian group of finite exponent and so we may write $G = \mathrm{Dr}_p\, G_p$ as the direct product of a finite number of Sylow p-subgroups G_p. Each G_p is characteristic in G and so invariant under A; consequently G_p has a series stabilized by A. By (ii), A acts on G_p as a finite p-group. It is clearly enough to show that G_p has a finite series stabilized by A, and so we may suppose that G is an abelian p-group of finite exponent p^n and $A \in \mathfrak{F}_p$. Suppose $|A| = p^m$. Now if $x \in G$, then x has at most p^m images under A, and each of these is of order at most p^n, so that x lies in an A-invariant subgroup G_x of order at most p^{m+n}. Now G_x and A are both finite p-groups, and so G_x has a finite series stabilized by A; this can be of length at most $m + n$. Let it be

$$1 = G_{x,0} \leqslant G_{x,1} \leqslant \cdots \leqslant G_{x,m+n} = G_x$$

where the factors need not all be nontrivial, of course. Put $G_i = \{G_{x,i}\ ;\ x \in G\}$ for $0 \leqslant i \leqslant m + n$.

Then

$$1 = G_0 \leqslant G_1 \leqslant \cdots \leqslant G_{m+n} = G.$$

is a finite series of G stabilized by A.

We shall also need the following well known fact (cf [1], Theorem 10.3.5).

LEMMA 10. *Let H, K, L be subgroups of a group G. Then any normal subgroup of G which contains two of $[H, K, L]$, $[K, L, H]$, $[L, H, K]$ also contains the third.*

We use the convention that repeated commutators are to be bracketed from the left; thus $[H, K, L] = [[H, K], L]$.

5. Proof of Main Theorem

We now investigate more closely the situation of Theorem 1*. Let G be a group having an invariant series $(\Lambda_\sigma, V_\sigma; \sigma \in \Omega)$ the stability group of which contains the group A, which we do not yet assume finite. We write W for the natural semidirect product GA; $G \lhd W$, $G \cap A = 1$. Also let $U = [G, A]$ and $A^* = \{A^W\} = \{A^G\} = [G, A]A$, so that both U and A^* are normal in W. We shall use this notation throughout the rest of the paper. We require some results about the structure of W.

LEMMA 11.

(i) *U has a series with factors central in A^*.*

(ii) *If $K \neq 1$ and L are subgroups of W with $L \leqslant A^*$, $K \leqslant U \cap L$, and $K \in f(L)$, then $[K, L] < K$.*

Proof. (i) If $\sigma \in \Omega$, the subgroups $\Lambda_\sigma \cap U$ and $V_\sigma \cap U$ are each normal in W and hence so is the centralizer in W of the factor $\Lambda_\sigma \cap U/V_\sigma \cap U$. Since this centralizer contains A, it must therefore contain A^*. Thus $(\Lambda_\sigma \cap U, V_\sigma \cap U; \sigma \in \Omega)$ is the required series of U.

(ii) Write $\Gamma_\sigma = \Lambda_\sigma \cap U$, $\Delta_\sigma = V_\sigma \cap U$ and let $K = \{x_1^L, ..., x_n^L\}$ where $n \neq 0$ is finite and no x_i is 1. Since $K \leqslant U$, $\exists\, \sigma_i$ such that $x_i \in \Gamma_{\sigma_i} - \Delta_{\sigma_i}$ $(i = 1, 2, ..., n)$. Let $\sigma = \max \sigma_i$. Then $x_i \in \Gamma_\sigma$ for $i = 1, 2, ..., n$ and so, since L normalizes Γ_σ, we have $K \leqslant \Gamma_\sigma$. However $\exists\, j$ such that $x_j \notin \Delta_\sigma$ and so $K \nleqslant \Delta_\sigma$. Since L centralizes $\Gamma_\sigma/\Delta_\sigma$ we obtain

$$[K, L] \leqslant [\Gamma_\sigma, L] \cap K \leqslant \Delta_\sigma \cap K < K,$$

as required.

LEMMA 12. *Suppose $A = \mathrm{Dr}_p A_p$ is the direct product of its Sylow p-subgroups A_p for various primes p. Then $[G, A_p, A_q] = 1$ if $p \neq q$.*

Proof. Write $A_p^* = \{A_p^G\} = [G, A_p]A_p$. Now $[G, A_p] \leqslant A_p^*$, which is generated by p-elements, the conjugates of elements of A_p under G. It will thus be enough to take a finite set $x_1, \ldots, x_r$ of p-elements of A_p^* and an element β of A_q and show that $K = [X, B] = 1$, where $X = \{x_1, \ldots, x_r\}$ and $B = \{\beta\}$. Let $L = \{X, B\}$. Then $K = \{[x_i, \beta]^L;\ 1 \leqslant i \leqslant r\}$, and so $K \in f(L)$. Also A_p^* and A_q commute elementwise mod U, so that $K \leqslant U$, and $L \leqslant A^*$. Suppose now that $K \neq 1$. Then by Lemma 11, $[K, L] = K_1 < K$. However K/K_1 is generated by the cosets of the elements $[x_i, \beta]$. The subgroup of L/K_1 generated by x_iK_1 and βK_1 is clearly nilpotent of class $\leqslant 2$, since $[x_i, \beta] \in K$ and K/K_1 is centralized by L. Since x_iK_1 and βK_1 are elements of coprime order of this subgroup they therefore commute, that is, $[x_i, \beta] \in K_1$. Thus $K = K_1$, which is a contradiction. Hence $K = 1$ and the lemma is established.

We now suppose that A is finite.

LEMMA 13. *For each prime p, let $A_{p'}$ denote the group generated by the p'-elements of A and let B_p denote the group generated by the p-elements of $A_{p'}$. Then*

(i) $[G, B_p, A_{p'}] = 1$,

(ii) *$[G, B_p]$ is an abelian p-group of finite exponent lying in the center of $[G, A_{p'}]$,*

(iii) *$B_p \in \mathfrak{A}$ for each prime p.*

Proof. (i) We prove this result by induction on $|A|$. Let p be any prime. If $A_{p'} < A$ the result is true by induction, and so we need only consider the case $A = A_{p'}$.

We shall first show that $[G, A, B_p, A] = 1$. Now any element of $[G, A, B_p]$ lies in a subgroup of the form $[H, B_p]$ where H is a finitely-generated subgroup of $[G, A]$. But if T is the subgroup generated by all elements $[x, \alpha]$, where x runs through G and α through the p'-elements of A, then $T = [G, A]$. For let $[x, \alpha]$ be such an element, and let $y \in G$ and $\beta \in A$. Then $[x, \alpha]^y = [xy, \alpha][y, \alpha]^{-1}$ and $[x, \alpha]^\beta = [x^\beta, \alpha^\beta]$. Since α^β is a p'-element of A, it follows that $T \lhd W = GA$. But, modulo T, G commutes with a set of generators of A. Hence $[G, A] \leqslant T$ and we clearly have equality. Therefore we may suppose

$$H = \{[x_i, \alpha_i];\ i = 1, 2, \ldots, n\} \tag{15}$$

where the x_i belong to G, the α_i are p'-elements of A, and, since A is finite, we may suppose that H is A-invariant. Let $K = [H, B_p]$. Since B_p is characteristic in A and H is normalized by A, we have $K \lhd HA = L$. In fact, if

$y_i = [x_i, \alpha_i]$ $(1 \leqslant i \leqslant n)$, then K is generated by the conjugates in L of the elements $[y_i, \beta]$ $(\beta \in B_p)$, and so $K \in f(L)$. Define $K_1 = [K, L]$, $K_2 = [K_1, L]$. Then $K_1 \lhd L$, $K_2 \lhd L$, and $K \geqslant K_1 \geqslant K_2$. We shall show that $K_1 = 1$.

Suppose if possible that $K_1 \neq 1$. Then certainly $K \neq 1$. Since $K \leqslant U$ and $L \leqslant A^*$, Lemma 11 gives $K_1 < K$. Since $K \in f(L)$ and L centralizes K/K_1, it follows that $K/K_1 \in \mathfrak{G}$. Also $K \leqslant L$ and so $K' \leqslant K_1$. Hence $K/K_1 \in \mathfrak{G} \cap \mathfrak{A}$. But the class $\mathfrak{R}$ of finitely-presented groups, which is E-closed and contains every cyclic group, therefore contains $\mathfrak{G} \cap \mathfrak{A}$. Consequently $K/K_1 \in \mathfrak{R}$. Since $K \in f(L)$ and $L \in \mathfrak{G}$, Lemma 7 gives $K_1 \in f(L)$. Since $K_1 \neq 1$ by hypothesis, Lemma 11 now shows that

$$K_2 < K_1 . \tag{16}$$

Now identity $[x, \alpha\beta] = [x, \beta][x, \alpha]^\beta$ shows that, for fixed $x \in H$, the map $x \to [x, \alpha]K_1$ maps B_p homomorphically into the abelian group K/K_1. Since the kernel of this homomorphism contains B_p', the image is an abelian p-group. But K/K_1 is generated by all such images as x runs through H, and so is an abelian p-group.

Also, for fixed $u \in L$, the map $y \to [y, u]K_2$ maps K homomorphically into the abelian group K_1/K_2, with kernel containing K_1. The image is therefore an abelian p-group, and since K_1/K_2 is generated by such images, $K_1/K_2 \in \mathfrak{A}_p$. K/K_2 is therefore a p-group of class at most two. Now A stabilizes the series $K/K_2 > K_1/K_2 > K_2/K_2$ of K/K_2 and therefore, by Lemma 9, acts on K/K_2 as a p-group. Since A is generated by p'-elements, it must therefore act trivially on K/K_2, that is,

$$[K, A] \leqslant K_2 . \tag{17}$$

But K and K_2 are each normal in L. Therefore K/K_2, being centralized by A, is also centralized by $[H, A]$, that is,

$$[K, [H, A]] \leqslant K_2 . \tag{18}$$

Let $y_i = [x_i, \alpha_i]$ be one of the generators of H, and let $z \in K$. Now $\{zK_2, y_iK_2\}$ is a nilpotent subgroup of H/K_2 of class at most two. But $\exists\, p'$-number m such that $\alpha_i^m = 1$. Consequently, $1 = [x_i, \alpha_i^m] \equiv [x_i, \alpha_i]^m$ mod$[H, A]$. Thus $y_i^m \in [H, A]$, and (18) shows that z commutes with y_i^m mod K_2. Since zK_2 is a p-element and m is a p'-number, it follows that zK_2 and y_iK_2 commute, that is, $[z, y_i] \in K_2$. Therefore $[K, H] \leqslant K_2$. In conjunction with (17), this shows that $[K, L] \leqslant K_2$, that is, $K_1 \leqslant K_2$. This contradiction to (16) shows that K_1 must in fact have been 1.

It follows that $[H, B_p]$ is centralized by A. Since every element of $[G, A, B_p]$ lies in a subgroup of the form $[H, B_p]$, we now have

$$[G, A, B_p, A] = 1. \tag{19}$$

Let $M = [G, A, B_p]$. Then M is certainly centralized by B_p and so, being normalized by $[G, A]$, M is centralized by $[G, A, B_p]$. In other words, $M \in \mathfrak{A}$. Furthermore, for fixed $x \in [G, A]$, the map $\alpha \to [x, \alpha]$ maps B_p homomorphically into the abelian group M; the image therefore lies in $\mathfrak{A}_p$, and so does M.

Now let $N = [G, B_p]$ and let $M^* = \{M^x; x \in G\} = \{M^G\}$. Now $M \lhd [G, A]$ and so $M \lhd N \lhd G$. Hence $M^x \lhd G$ for each $x \in G$. M^* is thus generated by normal $\mathfrak{A}_p$-subgroups of N, and so lies in $\mathrm{L}\mathfrak{F}_p$. Now N/M^* is centralized by B_p and so, since N and M^* are each normal in G, it is centralized by $[G, B_p] = N$. That is, $N/M^* \in \mathfrak{A}$. Also, if $x \in G$, then the map $\alpha \to [x, \alpha]M^*$ maps B_p homomorphically into N/M^*, and the argument which we have used several times already shows that $N/M^* \in \mathfrak{A}_p \leqslant \mathrm{L}\mathfrak{F}_p$. But the class $\mathrm{L}\mathfrak{F}_p$ is E-closed, and so $N \in \mathrm{L}\mathfrak{F}_p$. However N is A-invariant and so the intersections with it of the given series of G form a series of N stabilized by A. Therefore by Lemma 9, A acts on N as a p-group. Since A is generated by p'-elements, we therefore have $[N, A] = 1$. This concludes the proof of (i).

(ii) In the first place, $[G, B_p]$ is a normal subgroup of G centralized by $A_{p'}$ and so by $[G, A_{p'}]$. In particular, $[G, B_p]$ is abelian. As usual, for $x \in G$, the map $\alpha \to [x, \alpha]$ maps B_p homomorphically into $[G, B_p]$. The image has exponent at most $| B_p |$, hence so does $[G, B_p]$ itself.

(iii) By (i), $[G, B_p, B_p] = 1$ for each prime p. By Lemma 10, $[G, B'_p] = 1$ and so $B_{p'} = 1$, that is, $B_p \in \mathfrak{A}$.

We can now conclude the proof of Theorem 3. For if G is torsion-free, and even if G has no nontrivial periodic abelian normal subgroups, then (ii) of Lemma 13 shows that $[G, B_p] = 1$ and so $B_p = 1$ for each prime p. The argument of Lemma 1 now makes it clear that $A \in \mathfrak{N}$, as required.

Proof of Theorem 1*. We shall suppose the theorem false, and take A to be a counterexample of minimal order.

1. We first show, with the notation of Lemma 13, that $A/B_q \in \mathfrak{N}^{(2)}$ if $B_q \neq 1$. Let H be any A-invariant $\mathfrak{G}$-subgroup of $[G, A_{q'}]$ and let $J = [H, B_q]$. By Lemma 13 (ii), J is an abelian q-group of finite exponent lying in the center of H. Now J is generated by the conjugates in H of the set of elements $[x_i, \alpha]$, where x_i runs through a finite set of generators of H and α through B_q. Since J is central in H, it follows that $J \in \mathfrak{G}$, and so $J \in \mathfrak{F} \cap \mathfrak{A}_q$. Now H is A-invariant and so has an invariant series stabilized by A. Since J is a finite normal subgroup of H, Lemma 8 shows that H/J also has such a series. However $1 \neq B_q \leqslant C_A(H/J)$ and so by induction, $A/C_A(H/J) \in \mathfrak{N}^{(2)}$.

Let $C = \cap\, C_A(H/J)$, where H runs over the finitely-generated A-invariant subgroups of $[G, A_{q'}]$. Now by Lemma 13 (iii), $B_r \in \mathfrak{A}$ for each prime r, and the argument of Lemma 1 shows that the subgroup B generated by the B_r

is their direct product, and $A/B \in \mathfrak{N}$. Since $\mathfrak{N} \leqslant \mathfrak{N}^{(2)} = \mathrm{R}_0\mathfrak{N}^{(2)}$, we therefore have

$$A/C \cap B \in \mathfrak{N}^{(2)}. \tag{20}$$

Now $B_q \leqslant C$ and so $C \cap B = B_q \times R$, say, where $R \lhd A$ is an abelian q'-group. Let $\alpha \in R$ and $x \in [G, A_{q'}]$. Let H be an A-invariant $\mathfrak{G}$-subgroup of $[G, A_{q'}]$ containing x. Then $[x, \alpha] \in [H, B_q]$. By Lemma 13 (i), $[x, \alpha]$ is centralized by the q'-element α. Now $\alpha^s = 1$ for some q'-number s, and so $1 = [x, \alpha^s] = [x, \alpha]^s$. But by Lemma 13 (ii), $[H, B_q]$ is a q-group, and so $[x, \alpha] = 1$. Since x and α were arbitrary elements of $[G, A_{q'}]$ and R respectively, it follows that R centralizes $[G, A_{q'}]$.

Now $V = [G, B]$ is generated by the subgroups $[G, B_p]$, only a finite number of which are nontrivial. By Lemma 13 (ii), these are of finite exponent and commute elementwise in pairs. Hence V is abelian of finite exponent. But V is A-invariant and so has a series stabilized by A. Therefore by Lemma 9, V has a finite series

$$1 = V_0 \leqslant V_1 \leqslant \cdots \leqslant V_n = V$$

stabilized by A.

For $i = 0, 1, \ldots, n$, let R_i be the set of elements $\alpha \in R$ such that $[x, \alpha] \in V_i$ for all $x \in G$. The equations $[x, \alpha\beta] = [x, \beta][x, \alpha]^\beta$ and $[x, \alpha^{-1}] = [x, \alpha]^{-\alpha^{-1}}$ show, since V_i is A-invariant, that R_i is a subgroup of A. Also, if $\alpha \in R_i$ and $\beta \in A$ then $[x, \alpha^\beta] = [x^{\beta^{-1}}, \alpha]^\beta \in V_i$ and so $R_i \lhd A$ for $i = 0, 1, \ldots, n$. Now let $i > 0$, $x \in G$, $\alpha \in A_{q'}$, $\beta \in R_i$. Then

$$[\alpha, x^{-1}, \beta]^x[x, \beta^{-1}, \alpha]^\beta[\beta, \alpha^{-1}, x]^\alpha = 1$$

by a well known identity. The first factor is 1 since R centralizes $[G, A_{q'}]$. The second lies in $[G, R_i, A]^\beta \leqslant [V_i, A]^\beta \leqslant V_{i-1}$. Hence

$$[\beta, \alpha^{-1}, x] \in V_{i-1}^{\alpha^{-1}} = V_{i-1}.$$

Since this is true for any $x \in G$, and since $R \lhd A$, it follows that $[\beta, \alpha^{-1}] \in R_{i-1}$. R_i/R_{i-1} is thus centralized by all the q'-elements of A. Since $R_0 = 1$, Lemma 3 and (20) now show that $A/B_q \in \mathfrak{N}^{(2)}$.

2. Consequently at most one of the subgroups B_q can be nontrivial. For if $B_p \neq 1$ and $B_q \neq 1$ $(q \neq p)$, then A/B_p and A/B_q both lie in $\mathfrak{N}^{(2)}$, and $B_p \cap B_q = 1$. Therefore $A \in \mathrm{R}_0\mathfrak{N}^{(2)} = \mathfrak{N}^{(2)}$, which was supposed not to be the case.

3. Since not all the B_q can be trivial, it follows that, for some prime p, A has a normal $\mathfrak{A}_p$-subgroup B such that $A/B \in \mathfrak{N}$. This implies that $A = A_{p'}$. For if not, then $A_{p'} \in \mathfrak{N}^{(2)}$. But if $q \neq p$, then $A_{q'}$ is a q'-group, and therefore A satisfies the hypotheses of Lemma 1, and so lies in $\mathfrak{N}^{(2)}$.

Lemma 13 (ii) now shows that $[G, B]$ is an abelian p-group of finite exponent lying in the center of $[G, A]$. Let S be a finitely-generated A-invariant subgroup of $[G, A]$, and let $T = [S, B]$. Then T is central in S, and is generated by a finite number of classes of conjugate elements in S; consequently T is finitely-generated and so finite. But S is A-invariant and so has an invariant series stabilized by A. Since $T \in \mathfrak{F}$, Lemma 8 shows that S/T also has such a series.

However, B centralizes S/T, and so A acts on S/T as a finite nilpotent group. Let $D = C_A(S/T)$, and, for each prime q, let A_q be the group generated by the q-elements of A. Then A/D is the direct product of its Sylow subgroups A_qD/D. By Lemma 12,

$$[S, A_q, A_r] \leqslant T \qquad \text{if} \qquad q \neq r. \tag{21}$$

For each prime q, let $W_q = [S, A_q]T$. Then W_q is A-invariant, and by (21), $A/C_A(W_q/T)$ is a q-group. But $T \in \mathfrak{F}_p$ and is centralized by A. If $x \in W_q$ and $\varphi \in C_A(W_q/T)$, then $x^{\varphi} = xu$ for some $u \in T$, and for any integer n, $x^{\varphi^n} = xu^n$. Consequently, if $p^t = | T |$, then φ^{p^t} centralizes W_q. This makes it clear that $A/C_A(W_q)$ is a (p, q)-group and so lies in $\mathfrak{N}^{(2)}$. Now $[S, A] = \Pi_q [S, A_q]$ and so $\bigcap_q C_A(W_q) = C_A([S, A]) = E$, say, and $A/E \in \mathfrak{N}^{(2)}$. Since $A/B \in \mathfrak{N}$, we even have $A/E_0 \in \mathfrak{N}^{(2)}$, where $E_0 = E \cap B$. But $1 = [S, A, E_0] = [S, E_0, A]$ and so Lemma 10 gives $[S, [E_0, A]] = 1$, that is, $[E_0, A] \leqslant C_A(S)$. Lemma 3 now shows that $A/C_A(S) \in \mathfrak{N}^{(2)}$. Since this is true for every finitely-generated A-invariant subgroup S of $[G, A]$, we therefore have $A/F \in \mathfrak{N}^{(2)}$, where $F = B \cap C_A([G, A])$. But $1 = [G, A, F] = [G, F, A]$ and so another application of Lemma 10 gives $[G, [F, A]] = 1$. Therefore F lies in the center of A, and Lemma 3 now shows that $A \in \mathfrak{N}^{(2)}$. This contradiction to the assumption that A was a minimal counterexample to Theorem 1* shows that no such counterexamples exist. Therefore the proof of Theorem 1* is complete, and with it Theorem 1.

Acknowledgments

This work was begun during a Research Fellowship at Clare College, Cambridge, and completed with the assistance of the National Science Foundation. I am deeply grateful to these bodies for their support.

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