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A COUNTEREXAMPLE TO THE (UNSTABLE) GROMOV–LAWSON–ROSENBERG CONJECTURE

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Doing surgery on the 5-torus, we construct a five-dimensional closed spin-manifold M with $\pi_1(M) \cong \mathbb{Z}^4 \times \mathbb{Z}/3$, so that the index invariant in the KO-theory of the reduced C^*-algebra of $\pi_1(M)$ is zero. Then we use the theory of minimal surfaces of Schoen/Yau to show that this manifold cannot carry a metric of positive scalar curvature. The existence of such a metric is predicted by the (unstable) Gromov–Lawson–Rosenberg conjecture.

1. OBSTRUCTIONS TO POSITIVE SCALAR CURVATURE

We start with a discussion of the index obstruction for spin manifolds to admit a metric with scal >0, constructed by Lichnerowicz [7], Hitchin [4] and in the following refined version due to Rosenberg [10].

THEOREM 1.1. *Let M^m be a closed spin-manifold, $\pi := \pi_1(M)$. One can construct a homomorphism, called index, from the singular spin bordism $\Omega_*^{\mathrm{spin}}(B\pi)$ to the (real) KO-theory of the reduced real C^*-algebra of π:*

$$\mathrm{ind}: \Omega_*^{\mathrm{spin}}(B\pi) \to KO_*(C^*_{\mathrm{red}}\pi).$$

Let $u: M \to B\pi$ be the classifying map for the universal covering of M. If M admits a metric with positive scalar curvature, then

$$\mathrm{ind}([u: M \to B\pi]) = 0 \in KO_m(C^*_{\mathrm{red}}\pi).$$

Gromov and Lawson [3] and Rosenberg [9] conjectured that the vanishing of the index should also be sufficient for the existence of a metric with scal > 0 on M if $m \geqslant 5$. This was proven by Stefan Stolz [16] for $\pi = 1$, and subsequently by him and other authors also for a few other groups [9, 6, 1, 12].

In dimension $\geqslant 5$ there is only one known additional obstruction for positive scalar curvature metrics, the minimal surface method of Schoen and Yau, which we will recall now. (In dimension 4, the Seiberg–Witten theory yields additional obstructions.) The first theorem is the differential geometrical backbone for the application of minimal surfaces to the positive scalar curvature problem:

THEOREM 1.2. *Let (M^m, g) be a manifold with scal > 0, $\dim M = m \geqslant 3$. If V is a smooth $(m-1)$-dimensional submanifold of M with trivial normal bundle, and if V is a local minimum of the volume functional, then V admits a metric of positive scalar curvature, too.*

Proof. Schoen and Yau: [13, 5.1] for $m = 3$, [14, proof of Theorem 1] for $m > 3$. □

The next statement from geometric measure theory implies applicability of the previous theorem if $\dim(M) \leqslant 7$.

THEOREM 1.3. (Morgan [8, Ch. 8] and references therein, in particular Federer [2, 5.4.18]). *Suppose* M^m *is a smooth orientable closed manifold,* $\dim M = m \leqslant 7$. *Choose* $0 \neq x \in H_{m-1}(M, \mathbb{Z})$. *Then a smooth orientable closed* $(m-1)$*-dimensional submanifold* V *of* M *exists which represents* x *and which has minimal volume under all currents representing* x. *In particular,* V *is a local minimum of the volume functional with orientable* (*hence trivial*) *normal bundle.*

This implies the following statement about homology and cohomology which was observed by Stephan Stolz. Let X be any space.

Definition 1.4. For $m \geqslant 2$ we define

$$H_m^+(X, \mathbb{Z}) := \{f_*[M] \in H_m(X, \mathbb{Z}); f: M^m \to X \text{ and } M \text{ admits metric with scal } > 0\}.$$

COROLLARY 1.5. *Let* X *be any space,* $\alpha \in H^1(X, \mathbb{Z})$. *Cap-product with* α *induces a map*

$$\alpha\cap : H_m(X, \mathbb{Z}) \to H_{m-1}(X, \mathbb{Z}).$$

If $3 \leqslant m \leqslant 7$, *then* $\alpha\cap$ *maps* $H_m^+(X, \mathbb{Z})$ *to* $H_{m-1}^+(X, \mathbb{Z})$.

Proof. If $f: M^m \to X$ represents $x \in H_m^+(X, \mathbb{Z})$ and M admits a metric with positive scalar curvature, then by Theorems 1.2 and 1.3 the class $f^*\alpha\cap[M]$ is represented by $N^{m-1} \overset{j}{\hookrightarrow} M$ where N admits a metric with positive scalar curvature. In particular, $\alpha\cap x = f_*(f^*\alpha\cap[M])$ is represented by $f\circ j: N \to X$, i.e. $\alpha\cap x \in H_{m-1}^+(X, \mathbb{Z})$. □

2. COUNTEREXAMPLE TO THE GROMOV–LAWSON–ROSENBERG CONJECTURE

The aim of this paper is to present an example which shows that the conjecture is not true for arbitrary fundamental groups.

To produce the counterexample, we use the only other known obstruction for positive scalar curvature, namely the minimal surface method explained above.

The fundamental group will be $\pi := \mathbb{Z}^4 \times \mathbb{Z}/3$. We start with the computation of the KO-theory of $C^*_{\mathrm{red}}\pi$. Note that the reduced C^*-algebra of the product of two groups is the (minimal) tensor product of the individual C^*-algebras [15, pp. 14, 15]. By [15, pp. 14 and 1.5.4]

$$KO_n(C^*_{\mathrm{red}}(\mathbb{Z}^4 \times \mathbb{Z}/3)) \cong \bigoplus_{i=1}^{16} KO_{n-n_i}(C^*_{\mathrm{red}}(\mathbb{Z}/3)) \quad \text{for suitable } n_i \in \mathbb{N}.$$

For a finite group G, it is well known that $KO_*(C^*_{\mathrm{red}}(G))$ is a direct sum of copies of the KO-theories of $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{H}$. In particular, it is a direct sum of copies of $\mathbb{Z}$ and $\mathbb{Z}/2$. Therefore, the same is true for π:

PROPOSITION 2.1. $KO_*(C^*_{\mathrm{red}}\pi)$ *is a direct sum of copies of* $\mathbb{Z}$ *and* $\mathbb{Z}/2$. *In particular, its torsion is only* 2*-torsion.*

We will now construct a spin manifold M^5 with $\pi_1(M) = \pi$, so that the class $[u: M \to B\pi] \in \Omega_5^{\mathrm{spin}}$ is 3-torsion. Then, automatically

$$\mathrm{ind}(u: M \to B\pi) = 0 \in KO_5(C^*_{\mathrm{red}}\pi).$$

Example 2.2. Let $p: S^1 \to B\mathbb{Z}/3$ be a map so that $\pi_1(p)$ is surjective and equip S^1 with the spin structure induced from D^2. This is 3-torsion since $\tilde{\Omega}_1^{\mathrm{spin}}(B\mathbb{Z}/3) \cong H_1(B\mathbb{Z}/3, \mathbb{Z}) \cong \mathbb{Z}/3$ (use the Atiyah–Hirzebruch spectral sequence). Consider the singular manifold

$$f = \mathrm{id} \times p : S^1 \overbrace{\times \cdot \times}^{4} S^1 \times S^1 \to S^1 \overbrace{\times \cdot \times}^{4} S^1 \times B\mathbb{Z}/3 = B\pi$$

This is then 3-torsion in $\Omega_5^{\mathrm{spin}}(B\pi)$. Doing surgery we can construct a bordism $F: W \to B\pi$ in $\Omega_5^{\mathrm{spin}}(B\pi)$ from f to some $u: M \to B\pi$ where u is an isomorphism on π_1.

Now, M is a manifold with trivial index, and we have to show that it does not admit a metric with positive scalar curvature. Assume that the converse is true.

We study the homology and cohomology of π first. By the Künneth theorem

$$H_1(B\pi, \mathbb{Z}) = x_1\mathbb{Z} \oplus \cdots \oplus x_4\mathbb{Z} \otimes y\mathbb{Z}/3$$

$$H^1(B\pi, \mathbb{Z}) = a_1\mathbb{Z} \oplus \cdots \oplus a_4\mathbb{Z}$$

$$0 \neq w = x_1 \times \cdots \times x_4 \times y \in H_5(B\pi, \mathbb{Z})$$

$$0 \neq z = x_4 \times y = a_1 \cap (a_2 \cap (a_3 \cap w)) \in H_2(B\pi, \mathbb{Z}).$$

We use the map

$$B_* : \Omega_*^{\mathrm{spin}}(X) \to H_*(X, \mathbb{Z}) : [f: M \to X] \mapsto f_*[M]$$

which is an edge homomorphism in the Atiyah–Hirzebruch spectral sequence. Of course, $w = f_*([T^5]) = u_*[M]$ is the image of the considered singular manifold under this transformation.

If M would admit a metric with scal > 0, then

$$w \in H_5^+(B\pi).$$

Iterated application of Theorem 1.5 implies that

$$0 \neq z \in H_2^+(B\pi).$$

But there is only one two-dimensional oriented manifold with positive curvature, namely S^2. Since $\pi_2(B\pi) = 0$ any map $g: S^2 \to B\pi$ is null homotopic. In particular, $g_*[S^2] = 0 \in H_2(B\pi, \mathbb{Z})$, and therefore $H_2^+(B\pi, \mathbb{Z}) = 0$.

This is the desired contradiction and M does not admit a metric with positive scalar curvature.

Remark 2.3. There is also a twisted version of Rosenbergs obstruction [9, 11, 5, unpublished notes of Stolz] which can be applied if the universal covering of a closed manifold M is spin, even if M is not orientable. Our method can be developed to give examples where the twisted index of a non-orientable manifold is zero, but it does not admit a metric with scal >0.

The index map ind factorizes through topological periodic KO-homology of $B\pi$. Our method yields examples where even the image in this group is zero, although M has no metric with positive scalar curvature.

Acknowledgements—To work on the counterexample was inspired by talks of Stephan Stolz where he expressed his opinion that the original GLR-conjecture is false. The author wants to thank Stephan Stolz for useful and enlightening conversations on the subject. Stolz conjectures that a weaker form, the so-called stable GLR-conjecture, is true (cf. [17]) and shows [18] that this conjecture follows from the Baum–Connes conjecture. It is well known that our group π fulfills the Baum–Connes conjecture, although the proof for the real version, which is

required here, is not published anywhere (private communication of J. Rosenberg). Therefore, M is a counter-example to the unstable Gromov–Lawson–Rosenberg conjecture but if B is a Bott manifold and n is sufficiently large $M \times B^n$ admits a metric with scal >0.

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