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Completeness is determined by any non-algebraic trajectory

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Abstract

It is proved that any polynomial vector field in two complex variables which is complete on a non-algebraic trajectory is complete.

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1. Introduction and statement of results

Let X be a holomorphic vector field on $\mathbb{C}^2$. For any $z \in \mathbb{C}^2$, the local solution $\varphi_z(T)$ of the associated ordinary differential equation $dz/dT = X(z(T))$ with the initial condition $z(0) = z \in \mathbb{C}^2$ can be extended by analytic continuation along paths in $\mathbb{C}$, to a maximal domain Ω_z, which may not be an open set of $\mathbb{C}$, but rather a Riemann domain over $\mathbb{C}$. The map

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$\varphi_z : \Omega_z \to \mathbb{C}^2$ is the *solution* of X through z, and its image $\varphi_z(\Omega_z)$, that will be denoted by C_z (or L_z, R_z, S_z), is the *trajectory* of X through z.

The vector field X is complete on C_z if $\Omega_z = \mathbb{C}$, and X is *complete* if it is complete on C_z, for every $z \in \mathbb{C}^2$. Each trajectory C_z on which X is complete (complete trajectory) is defined by an abstract Riemann surface uniformized by $\mathbb{C}$, and by the maximum principle, analytically isomorphic to $\mathbb{C}$ or $\mathbb{C}^*$.

Extrinsically, the topology of a trajectory can be very complicated. The simplest trajectories from this point of view are the analytic ones. One says that the trajectory C_z is *analytic* if it is contained in an analytic curve in $\mathbb{C}^2$ (but not necessarily equal to it, due to the possible presence of singularities). Otherwise C_z is a *non-analytic* trajectory.

An interesting remark (due to R. Moussu) is that two vector fields with a common non-analytic trajectory have to be collinear in any point. In this sense a non-analytic trajectory determines the vector field up to multiplication by a non-vanishing holomorphic function.

In this work we will consider *polynomial vector fields with at most isolated zeros*. The above remark for two polynomial vector fields can be restated. For a trajectory, it is enough not to be contained in an algebraic curve (that is, to be a *non-algebraic* trajectory) to determine the vector field up to multiplication by a nonzero constant.

In [1], Brunella studied foliations in $\mathbb{C}^2$ given by polynomial vector fields with a trajectory containing a planar isolated end (proper Riemann sub-surface isomorphic to $\{z : r < \|z\| \leq 1\}$, where $r \in [0, 1)$), properly embedded in $\mathbb{C}^2$ and whose closure in $\mathbb{CP}^2$ contains the line at infinity. He proved that these foliations can be determined in terms of a polynomial whose generic fiber is of type $\mathbb{C}$ or $\mathbb{C}^*$ and transversal to the foliation. As a remarkable corollary, he obtained that if the trajectory is a non-algebraic analytic plane, the foliation is given by the constant vector field after an analytic automorphism. Therefore, the trajectory in this case is determining the completeness of the vector field up to multiplication by a nowhere vanishing function.

Then if one attends to the completeness of a non-algebraic trajectory (not necessarily analytic) the following natural question arises [6, Question 5.1]:

Question 1. *If X is a polynomial vector field in $\mathbb{C}^2$ with the property of being complete on a single non-algebraic trajectory, is it complete?*

The main result of this work says that Question 1 has an affirmative answer:

Theorem 1. *Let us consider a polynomial vector field X on $\mathbb{C}^2$ which is complete on a non-algebraic trajectory. Then X is complete.*

Note that our theorem implies that any entire solution of a polynomial vector field can be determined up to an algebraic automorphism of $\mathbb{C}^2$. As the vector field is complete, the solution must correspond to one of the vector fields of Brunella's classification in [4] after a polynomial automorphism.

It could be very interesting to study if a non-analytic trajectory of a (non-polynomial) holomorphic vector field determines the completeness of the vector field.

1.1. About the proof of Theorem 1

For the sake of completeness, throughout the paper we include some definitions and results taken from Brunella's papers [2–4]. Let us begin by recalling some definitions. Consider the foliation $\mathcal{F}$ generated by X on $\mathbb{C}^2$ extended to $\mathbb{CP}^2$. According to Seidenberg's theorem, the minimal resolution of $\mathcal{F}$ is a new foliation $\tilde{\mathcal{F}}$ defined on a rational surface M after pulling back $\mathcal{F}$ by a birational morphism $\pi : M \to \mathbb{CP}^2$, that is a finite composition of blowing ups. Along with this resolution one has: (1) the Zariski open set $U = \pi^{-1}(\mathbb{C}^2)$ of M, over which X can be lifted to a holomorphic vector field $\tilde{X}$, (2) *the exceptional divisor* E of U, and (3) the *divisor at infinity*

$$D = M \setminus U = \pi^{-1}(\mathbb{CP}^2 \setminus \mathbb{C}^2) = \pi^{-1}(L_\infty),$$

that is a tree of smooth rational curves. The vector field $\tilde{X}$ can be extended to M, although it may have poles along one or more components of D. Let us still denote this extension by $\tilde{X}$. As only singularities of the foliation in $\mathbb{C}^2$ are blown up, and they are in the zero set of X, the vector field $\tilde{X}$ is holomorphic on the full U and it has the complete trajectory $\tilde{C}_z$ defined by $\pi^{-1}(C_z)$.

Therefore the reduced foliation $\tilde{\mathcal{F}}$ has at least one tangent entire curve: the one defined by $\tilde{C}_z$, which is Zariski dense in M. It implies that *the Kodaira dimension* $\mathrm{kod}(\tilde{\mathcal{F}})$ of $\tilde{\mathcal{F}}$ is 1 or 0 [13, Section IV] (*see* also [2, p. 131]).

In the case $\mathrm{kod}(\tilde{\mathcal{F}}) = 1$, [4] allows one to conclude that $\tilde{\mathcal{F}}$ is a Riccati foliation adapted to a fibration $g : M \to \mathbb{P}^1$, whose projection to $\mathbb{C}^2$ by π defines a rational function R of type $\mathbb{C}$ or $\mathbb{C}^*$. We can apply the study of [6] although R is not a polynomial (*see* also [8]) and deduce the completeness of X. We will analyze this case in Section 2.

In the case $\mathrm{kod}(\tilde{\mathcal{F}}) = 0$, we know that $\tilde{\mathcal{F}}$ is generated by a vector field on a smooth compact projective surface S, up to contractions of $\tilde{\mathcal{F}}$-invariant curves and covering maps [4]. However, we need to go a bit further to know if these models restrict to our open U a complete vector field. This is accomplished via the description of the irreducible components of $D \cup E$ that are not $\tilde{\mathcal{F}}$-invariant. When S is rational, we show that in fact $D \cup E$ must be invariant if $\tilde{\mathcal{F}}$ is not Riccati with respect to a fibration $g : M \to \mathbb{P}^1$ that is projected to $\mathbb{C}^2$ by π as a rational function R of type $\mathbb{C}$ or $\mathbb{C}^*$. For the remaining cases, i.e. when S is a $\mathbb{CP}^1$-bundle over an elliptic curve or a complex 2-torus, we prove that $D \cup E$ is always invariant by $\tilde{\mathcal{F}}$. For the proof of this last fact we will consider S as a differential manifold with a certain Riemannian metric. It will enable us to compute the distance from the complete trajectory to a compact set containing the components of $D \cup E$ that are not $\tilde{\mathcal{F}}$-invariant. As a consequence of the discussion above one obtains that the lifting of $\tilde{X}$ by a certain covering map can be decomposed in the product of a complete rational vector field by a second integral of it. It allows us to conclude that the projection $\pi_* \tilde{X}$ restricted to U i.e. X must be complete. We will analyze this case in Section 3.

Finally, we point out that [6,7] imply that Question 1 has an affirmative answer for a non-algebraic analytic trajectory. In those works, Brunella's results [1] are used as the main tool. The proof of our theorem is mainly based on Brunella's approach to the classification complete polynomial vector fields in the plane [4], since they can be applied to the foliation $\mathcal{F}$ although X could be in principle not complete. Theorem 1 is not only the generalization of the previous results mentioned above ([6,7]), but its proof also implies them.

2. kod($\mathcal{F}$) = 1

According to [13, Section IV] the absence of a first integral implies that $\tilde{\mathcal{F}}$ is *a Ricatti or a Turbulent foliation*, that is to say, the existence of a fibration

$$g : M \to B$$

whose generic fibre is a rational curve or an elliptic curve transverse to $\tilde{\mathcal{F}}$, respectively. *Remark that* B *is* $\mathbb{CP}^1$ *since* M *is a rational surface.*

2.1. Nef models and canonical models [13, Section III], [3, Section 4], [4, Section 3]

Existence of a nef model. As $\tilde{\mathcal{F}}$ is not a rational fibration it has a model $\hat{\mathcal{F}}$ *which is* reduced and nef. More concretely, after a contraction $s : M \to \hat{M}$ of the $\tilde{\mathcal{F}}$-invariant rational curves on M over which the canonical bundle $K_{\tilde{\mathcal{F}}}$ has a negative degree one obtains (*see* [3, Section 4], [4, Section 3]):

(1) a new surface $\hat{M}$, maybe with *cyclic quotient singularities;* and
(2) a reduced foliation $\hat{\mathcal{F}} = s_* \tilde{\mathcal{F}}$ on $\hat{M}$ such that its canonical $\mathbb{Q}$-bundle $K_{\hat{\mathcal{F}}}$ is *nef* (i.e. $K_{\hat{\mathcal{F}}} \cdot C \geq 0$ for any curve $C \subset \hat{M}$).

Recall that a cyclic quotient singularity p of $\hat{M}$ is locally defined by $\mathbb{B}^2/\Gamma_{k,h}$ where $\mathbb{B}^2 \subset \mathbb{C}^2$ is the unit ball and $\Gamma_{k,h}$ is the cyclic group generated by a map of the form $(z, w) \to (e^{\frac{2\pi i}{k}} z, e^{\frac{2\pi i}{k} h} w)$ with k, h positive coprime integers such that $0 < h < k$. *These singularities of $\hat{M}$ are not singularities of $\hat{\mathcal{F}}$.* That is, the foliation can be lifted locally to $\mathbb{B}^2 \setminus \{(0, 0)\}$ and extended to a foliation on $\mathbb{B}^2$ with a non-vanishing associated vector field.

Remark 1. The possible cyclic singularities of $\hat{M}$ are in the image of the exceptional divisor of s. Any rational curve C_0 of that divisor is $\tilde{\mathcal{F}}$-invariant, it has a unique singularity p of the foliation of type $d(x^n y^m)$ with $n, m \in \mathbb{N}^+$, where $C_0 = \{y = 0\}$, and it may also contain one cyclic quotient singularity q of order m (regular point if $m = 1$). After contracting C_0 by s (since $C_0^2 = -n/m$) we obtain a new quotient singularity of order n (regular if $n = 1$) [4, pp. 443–444].

Existence of a minimal model. After possibly additional contractions on $\hat{M}$ of rational curves, $q : \hat{M} \to N$, one obtains a reduced foliation $\mathcal{H} = q_* \hat{\mathcal{F}}$ (birational to $\tilde{\mathcal{F}}$) on a surface N regular on the (cyclic quotient) singularities of N whose canonical bundle $K_{\mathcal{H}}$ is nef and such that it verifies this property: if $K_{\mathcal{H}} \cdot C = 0 \Rightarrow C^2 \geq 0$ for any curve $C \subset N$. It is important to note that we can assume that q is given by contractions of *curves which are invariant by the foliation:* if C is not $\hat{\mathcal{F}}$-invariant it follows from the formula $(K_{\hat{\mathcal{F}}} + C) \cdot C \geq 0$ [3, Section 2] that $K_{\hat{\mathcal{F}}} \cdot C = 0 \Rightarrow C^2 \geq 0$. This model is the *minimal model* of $\tilde{\mathcal{F}}$.

$$\begin{array}{ccc} M & \xrightarrow{\ \pi\ } & \mathbb{CP}^2 \\ \Big\downarrow{\scriptstyle s} & & \\ \hat{M} & \xrightarrow[\ q\ ]{} & N \end{array}$$

Remark 2. In general the minimal model of $\tilde{\mathcal{F}}$ *is not unique*. However if we have another minimal model $\mathcal{H}'$ of $\tilde{\mathcal{F}}$ defined on N' and $p : N \to N'$ is an algebraic map defined everywhere with $p_* \mathcal{H} = \mathcal{H}'$ then p is an isomorphism [13, Lemma III.3.1].

Remark 3. As s and q are given by contractions of rational curves which are invariant by the foliation neither $\tilde{C}_z$ meets the exceptional divisor of s nor $s(\tilde{C}_z)$ meets the exceptional divisor of q. It implies that there must be a parabolic leaf of $\mathcal{H}$: the leaf that contains the Riemann surface $q(s(\tilde{C}_z))$ that supports the complete vector field $q_*(s_*(\tilde{X}_{|\tilde{C}_z}))$.

2.2. Turbulent case

When X is complete the case of a Turbulent $\tilde{\mathcal{F}}$ can be excluded as it is proved in [4, Lemma1]. We now prove that it continues being valid in a more general situation.

Lemma 1. *$\tilde{\mathcal{F}}$ is not a Turbulent foliation.*

Proof. Suppose that $\tilde{\mathcal{F}}$ is Turbulent. The description of models around each fibre of g after a birational morphism $\alpha : M \to M^*$ is known [3, Section 7]. The resulting foliation $\mathcal{G} = \alpha_* \tilde{\mathcal{F}}$ on M^* is regular on the (cyclic quotient) singularities of M^*, it is Turbulent with respect to $\bar{g} = g \circ \alpha^{-1}$, and each fiber of $\bar{g}$ is of one of the following classes:

(a) (resp. (d)): the fibre is smooth elliptic, transversal (resp. tangent) to $\mathcal{G}$ and may be multiple.
(b) (resp. (e)): the fibre is rational with three quotient singularities of orders k_1, k_2 and k_3 satisfying $\frac{1}{k_1} + \frac{1}{k_2} + \frac{1}{k_3} = 1$, transversal (resp. tangent) to $\mathcal{G}$ and of multiplicity 3, 4 or 6.
(c) (resp. (f)): the fibre is rational with four quotient singularities of order 2; transversal (resp. tangent) to $\mathcal{G}$ and of multiplicity 2.

We will call classes (a), (b) and (c) (resp. (d), (e) and (f)) *transversal fibres (resp. tangent fibres) of $\bar{g}$.*

For any leaf L of $\mathcal{G}$ outside tangent fibres of $\bar{g}$, $\bar{g}_{|L} : L \to B_0$, with B_0 defined as B minus the points over tangent fibres of $\bar{g}$, is a regular covering (in orbifold's sense). The orbifold structure in B_0 is the natural structure inherited from the orbifold structure on B induced by (the local models of) $\bar{g}$ [3, Section 7].

Claim 1. *There must be at least one tangent fibre G_0 of $\bar{g}$.*

We suppose that all the fibres are transversal and obtain a contradiction. Since $B_0 = B = \mathbb{CP}^1$, the orbifold universal covering of any leaf L, $\tilde{L}$, is equal to the one of B, $\tilde{B}$.

Let us suppose that $\tilde{B}$ is $\mathbb{C}$ or $\mathbb{CP}^1$. By pulling back sections of K_B under $\bar{g}$ we obtain sections of $K_{\mathcal{G}}$. We can in this way compute $K_{\mathcal{G}}$ and obtain that $deg(\bar{g}_* K_{\mathcal{G}}) = -\chi_{orb}(B)$ (see, [3, Section 7]). On the other hand since $\mathrm{kod}(\tilde{\mathcal{F}}) = \mathrm{kod}(\mathcal{G}) = 1$ then $deg(\bar{g}_* K_{\mathcal{G}}) > 0$. It follows that $\chi_{orb}(B) < 0$, which is impossible if B is parabolic (see Appendix E, Lemma E.4, in [14]). Thus $\tilde{B}$ is a disk.

As all the leaves of $\mathcal{G}$ are hyperbolic and the singularities are isolated (in fact $\mathcal{G}$ is regular), $K_{\mathcal{G}}$ is nef [5, Remark 8.8]. Moreover, it is clear that $K_{\mathcal{G}} \cdot C = 0 \Rightarrow C^2 \geq 0$: if C is not $\mathcal{G}$-invariant it follows from the formula $(K_{\mathcal{G}} + C) \cdot C \geq 0$ [3, Section 2]. If C is $\mathcal{G}$-invariant the Camacho–Sad formula [3, Section 2] implies that $C^2 = 0$ because $\mathcal{G}$ is regular on C. Therefore $\mathcal{G}$ is a minimal model of $\tilde{\mathcal{F}}$. But then it has necessarily a parabolic leaf (Remarks 2 and 3), which is a contradiction.

Claim 2. *If there is an irreducible component D_1 of $D \cup E$ that is not $\tilde{\mathcal{F}}$-invariant and which is not contained in any fiber of g, then $D_1 \subset \{\tilde{X} = 0\}$.*

It is important to note that the strict transform of D_1 by α, that we also denote by D_1, is a rational curve. Otherwise it is a point with infinitely many punctured disks invariant by $\mathcal{G}$ through it and then a singularity of $\mathcal{G}$, which is not possible. Hence $D_1 \cap G_0 \neq \emptyset$. Let us denote by J the leaf of $\mathcal{G}$ that defines the non algebraic component of $\alpha(\tilde{C}_z)$. There is at least one accumulation point of J on G_0 because $\bar{g}(J) = g(\tilde{C}_z)$ is $\mathbb{C}$ or $\mathbb{C}^*$. It must be a regular point of the foliation by the absence of singularities of the foliation on tangent fibers. Thus J must accumulate on G_0. It implies that $\tilde{C}_z \cap D_1 \neq \emptyset$ and then $D_1 \subset \{\tilde{X} = 0\}$ by the completeness of $\tilde{X}_{|\tilde{C}_z}$.

Let us take the generic fiber G of g, which is transverse to $\tilde{\mathcal{F}}$. Obviously, $D \cap G \neq \emptyset$. In the contrary case we have an elliptic curve contained in $\mathbb{C}^2$, which is impossible (maximum principle). Among the irreducible components of D cutting G at least one, say D_2, is $\tilde{\mathcal{F}}$-invariant. Otherwise $\tilde{X}$ would be holomorphic in a neighborhood of G and it vanishes on at least one component of D transversal to G, which implies that $\tilde{X}$ is identically zero by Claim 2. The existence of D_2 is enough to construct a rational integral for a Turbulent $\tilde{\mathcal{F}}$ as can be seen in [4, p. 438]. □

2.3. Ricatti case

Lemma 2. *$g_{|U}$ is projected by π as a rational function R of type $\mathbb{C}$ or $\mathbb{C}^*$. Moreover, $\tilde{\mathcal{F}}$ is R-complete.*

Proof. Up to contraction of rational curves inside fibers of g, which can produce cyclic quotient singularities of the surface but on which the foliation is always regular, one has that there are five possible models for the fibers of g [3, Section 7], [4, p. 439]. Let L_0 be the leaf of the foliation defined by $\tilde{C}_z$. One can conclude that the orbifold universal covering $\tilde{L}_0$ of L_0 is equal to the one of B_0, $\tilde{B}_0$, where B_0 is defined as $\mathbb{CP}^1$ minus the points over tangent fibres of g with the natural orbifold structure inherited from the orbifold structure on $\mathbb{CP}^1$ induced by (the local models of) g. Since X is complete on C_z, $\tilde{L}_0$ is biholomorphic to $\mathbb{C}$ and then L_0 is parabolic. This fact along with $\mathrm{kod}(\tilde{\mathcal{F}}) = 1$ implies by Brunella [4, Lemma 2] that there must be at least one fibre G_0 tangent to the foliation of class:

(d): the fibre is rational with two saddle-nodes of the same multiplicity m, with strong separatrices inside the fibre, or of class

(e): the fibre is rational with two quotient singularities of order 2, and a saddle-node of multiplicity l, with a strong separatrix inside the fibre.

Firstly one observes that there are irreducible components of $D \cup E$ that are not contained in any fiber of g. Let us take the generic fiber G of g, which is transverse to $\tilde{\mathcal{F}}$. Obviously, $D \cap G \neq \emptyset$. In the contrary case we have a rational curve contained in $\mathbb{C}^2$, which is impossible (maximum principle).

Let D_1 be one of these components. Then $D_1 \cap G_0 \neq \emptyset$ and there is at least one accumulation point of $\tilde{C}_z$ on G_0, say p, because $g(\tilde{C}_z)$ is $\mathbb{C}$ or $\mathbb{C}^*$. If p is a regular point of the foliation, $\tilde{C}_z$ must accumulate on G_0. If p is singular, it is a saddle-node with a strong separatrix defined by G_0, and therefore $\tilde{C}_z$ must also accumulate on all G_0 [12], in particular in the other saddle-node if it exists. There are two possibilities:

(i) *If D_1 is $\tilde{\mathcal{F}}$-invariant, D_1 is not in the divisor of poles of $\tilde{X}$.* Otherwise, $D_1 \cap G_0 \neq \emptyset$ is a saddle-node q. Let us take the rational section ω of $K_{\tilde{\mathcal{F}}}$ dual to $\tilde{X}$ that restricts to $\tilde{C}_z$ as the *differential of times* given by the flow of $\tilde{X}$ on $\tilde{C}_z$. One can construct a path $\gamma : (0, \epsilon] \to \tilde{C}_z$, with $\epsilon \in \mathbb{R}^+$ and $\gamma(t) \to q$ as $t \to 0$, such that $\int_\gamma \omega$ is finite (*see* [4, proof of Lemma 3]), which contradicts the completeness of $\tilde{X}$ on $\tilde{C}_z$.

(ii) *If D_1* is not *$\tilde{\mathcal{F}}$-invariant, necessarily $\tilde{C}_z \cap D_1 \neq \emptyset$ and $D_1 \subset \{\tilde{X} = 0\}$.* Otherwise, as $\tilde{C}_z \cap D_1 = \emptyset$ one has that $D_1 \cap G_0 \neq \emptyset$ is a saddle-node with D_1 defining its weak separatrix, which is $\tilde{\mathcal{F}}$-invariant [1, Lemma 11].

It follows from (i) and (ii) that $\tilde{X}$ is holomorphic in a neighborhood of G, which implies as in the above lemma that (ii) does not really occur. Thus D_1 is $\tilde{\mathcal{F}}$-invariant.

Therefore D must cut G at one or two points, and the projection R of $g_{|U}$ via π is of type $\mathbb{C}$ or $\mathbb{C}^*$. Moreover, the invariancy of the components of $D \cup E$ which are not contained in fibers of g implies that generically R is a fibration trivialized by the leaves of $\tilde{\mathcal{F}}$, and then $\tilde{\mathcal{F}}$ is R-complete. □

We will study the two possibilities after the previous lemma.

2.4. *R of type* $\mathbb{C}$

By Suzuki (*see* [17]) we may assume that $R = x$, up to a polynomial automorphism. Hence $\mathcal{F}$ is a Riccati foliation adapted to x and X is a complete vector field of the form

$$Cx^N \frac{\partial}{\partial x} + [A(x)y + B(x)]\frac{\partial}{\partial y}, \tag{1}$$

with $C \in \mathbb{C}$, $N = 0, 1$ and $A, B \in \mathbb{C}[x]$ (*see* [6, Proposition 4.2]).

2.5. *R of type* $\mathbb{C}^*$

By Suzuki (*see* [18]) we may assume that

$$R = x^m(x^\ell y + p(x))^n,$$

where $m \in \mathbb{N}^*$, $n \in \mathbb{Z}^*$, with $(m, n) = 1$, $\ell \in \mathbb{N}$, $p \in \mathbb{C}[x]$ of degree $< \ell$ with $p(0) \neq 0$ if $\ell > 0$ or $p(x) \equiv 0$ if $\ell = 0$, up to a polynomial automorphism.

New coordinates. According to relations $x = u^n$ and $x^\ell y + p(x) = v\,u^{-m}$ it is enough to take the rational map H from $u \neq 0$ to $x \neq 0$ defined by

$$(u, v) \mapsto (x, y) = (u^n, u^{-(m+n\ell)}[v - u^m p(u^n)]) \tag{2}$$

in order to get $R \circ H(u, v) = v^n$.

Although R is not necessarily a polynomial ($n \in \mathbb{Z}$), it is a consequence of the proof of [6, Proposition 3.2] that $H^*\mathcal{F}$ is a Riccati foliation adapted to v^n having $u = 0$ as an invariant line. Thus

$$\begin{aligned} H^*X &= u^k \cdot Z \\ &= u^k \cdot \left\{ a(v)u\frac{\partial}{\partial u} + c(v)\frac{\partial}{\partial v} \right\}, \end{aligned} \tag{3}$$

where $k \in \mathbb{Z}$, and $a, c \in \mathbb{C}[v]$.

At this point one could apply the techniques of [7] to analyze the possible global 1-forms of times associated to X in order to prove the existence of an invariant line. However, applying directly the local models of [4], it follows from [8, Lemma 2] that at least one of the irreducible components of R over 0 must be a $\mathcal{F}$-invariant line. Hence the polynomial $c(v)$ of (3) is in fact a monomial, and thus of the form cv^N with $c \in \mathbb{C}$ and $N \in \mathbb{N}$.

Finally, according to [6, pp. 661–662] we know that $X_{|C_z}$ complete implies $k = 0$ and $N = 0, 1$. Hence X is complete.

3. kod($\mathcal{F}$) = 0

According to [13, Section IV] we can contract $\tilde{\mathcal{F}}$-invariant rational curves on M via a contraction s to obtain a new surface $\hat{M}$ (maybe singular with cyclic quotient singularities), a reduced foliation $\hat{\mathcal{F}}$ on this surface, and a finite covering map r from a smooth compact projective surface S to $\hat{M}$ such that: (1) r ramifies only over cyclic (quotient) singularities of $\hat{M}$ and (2) the foliation $r^*(\hat{\mathcal{F}})$ is generated by a complete holomorphic vector field Z_0 on S with isolated zeros [4, p. 443].

$$\begin{array}{ccc} \mathbb{CP}^2 & \xleftarrow{\ \pi\ } & M \\ & & \Big\downarrow{\scriptstyle s} \\ & & \hat{M} & \xleftarrow{\ r\ } & S \end{array}$$

Remark 4. Note that $\tilde{C}_z$ does not meet the exceptional divisor of the contraction s. Let us set $\hat{C}_z$ as $s(\tilde{C}_z)$. Since $\hat{C}_z$ does not contain singularities of $\hat{M}$ then $\hat{C}_z$ is a Riemann surface, $s_{|\tilde{C}_z} : \tilde{C}_z \to \hat{C}_z$ is a biholomorphism and $r_{|r^{-1}(\hat{C}_z)} : r^{-1}(\hat{C}_z) \to \hat{C}_z$ is a non-ramified finite covering map. Thus $s_*(\tilde{X}_{|\tilde{C}_z})$ is complete on $\hat{C}_z$ and $r^*(s_*(\tilde{X}_{|\tilde{C}_z}))$ is complete on the connected components $\mathscr{M}_i$ of $r^{-1}(\hat{C}_z) = \cup_{i=0}^{l} \mathscr{M}_i$. Hence each $\mathscr{M}_i$ is a Riemann surface contained in a complete trajectory T_z of Z_0 that supports the complete vector field $r^*(s_*(\tilde{X}_{|\tilde{C}_z}))_{|\mathscr{M}_i}$, which does not necessarily coincide with $Z_{0|\mathscr{M}_i}$. It is convenient to observe that if T_z is isomorphic to $\mathbb{C}^*$ then, necessarily $\mathscr{M}_i = T_z$, and the vector field $r^*(s_*(\tilde{X}_{|\tilde{C}_z}))_{|\mathscr{M}_i}$ coincides with Z_0 on T_z, up to a multiplicative constant. The discrepancy between the two complete vector fields can occur only if T_z is isomorphic to $\mathbb{C}$, in which case $r^*(s_*(\tilde{X}_{|\tilde{C}_z}))_{|\mathscr{M}_i}$ could have one (and only one) zero at some point $p = T_z \setminus \mathscr{M}_i$.

It follows from [4, p. 443] that the covering r can be lifted to M via a birational morphism $g : W \to S$ and a ramified covering $h : W \to M$ such that $s \circ h = r \circ g$.

$$\begin{array}{ccc} M & \xleftarrow{\ h\ } & W \\ {\scriptstyle s}\Big\downarrow & {\scriptstyle s\circ h}\ \swarrow\ {\scriptstyle r\circ g} & \Big\downarrow{\scriptstyle g} \\ \hat{M} & \xleftarrow[\ r\ ]{} & S \end{array}$$

Let Y be the lift of Z_0 on W via g. Then Y must be a *rational vector field* on W generating the foliation $\bar{\mathcal{F}}$ given by $g^*(r^*(\hat{\mathcal{F}})) = h^*\tilde{\mathcal{F}}$. On the other hand, $\bar{\mathcal{F}}$ is also generated by the rational vector field $\bar{X}$ on W given by $h^*\tilde{X}$. Hence there is a rational function F on W such that

$$\bar{X} = F \cdot Y. \tag{4}$$

Remark 5. We remark from the above construction:

(1) The map g is a composition of blowing-ups over a finite set $\Theta = \{\theta_i\}_{i=1}^{s} \subset S$ of regular points of Z_0. The poles of Y are in $g^{-1}(\Theta)$ and they define a divisor $P \subset W$ invariant by $\bar{\mathcal{F}}$. Hence Y is holomorphic on $W \setminus P$. Note that in $W \setminus P$, Y has only isolated zeros.

(2) Since P is the exceptional divisor of g, $h(P)$ is the exceptional divisor of s and is $\tilde{\mathcal{F}}$-invariant. Then

$$h_{|W\setminus P} : W \setminus P \to M \setminus h(P)$$

is a regular covering map.

(3) Let C_{θ_i} be the trajectory of Z_0 through θ_i. Y is a complete holomorphic vector field on $W \setminus \{g^{-1}(C_{\theta_i})\}_{i=1}^{s}$. Each $g^{-1}(C_{\theta_i}) \setminus P$ is contained in a trajectory R_{z_i} of Y. Let us fix one of them, say C_{θ_j}. Let us set $\Theta \cap C_{\theta_j} = \{\theta_{j_l}\}_{l=0}^{h}$ taking $j_0 = j$. Note that $R_{z_{j_l}} = R_{z_j}$ for

any l. For every θ_{j_l} there is a point $\bar{\theta}_{j_l} \in P$ such that $R_{z_j} \cup \{\bar{\theta}_{j_l}\}$ defines a separatrix of $\bar{\mathcal{F}}$ through $\bar{\theta}_{j_l}$. Note that $\bar{\theta}_{j_l}$ is the unique singular point of $\bar{\mathcal{F}}$ in P such that $g(\bar{\theta}_{j_l}) = \theta_{j_l}$. We can take around $\bar{\theta}_{j_l}$ a neighbourhood U and coordinates (z, w) such that $\bar{\mathcal{F}}$ is generated by $z\partial/\partial z - w\partial/\partial w$ where $(R_{z_j} \cup \{\bar{\theta}_{j_l}\}) \cap U = \{w = 0\}$ and $g^{-1}(\theta_{j_l}) \cap U = \{z = 0\}$. As Y has a pole of order one along $\{z = 0\}$, it follows that

$$Y_{|R_{z_j}} = \frac{\partial}{\partial z} - \frac{w}{z}\frac{\partial}{\partial w}$$

is not complete. However, it extends on $R_{z_j} \cup \{\bar{\theta}_{j_l}\}_{l=0}^{h}$ as a complete vector field because g restricted to R_{z_j} extends to $R_{z_j} \cup \{\bar{\theta}_{j_l}\}_{l=0}^{h}$ as a biholomorphism onto C_{θ_j} and

$$\left(g_{|R_{z_j} \cup \{\bar{\theta}_{j_l}\}_{l=0}^{h}}\right)^{*} Z_{0|C_{\theta_j}} = Y_{|R_{z_j} \cup \{\bar{\theta}_{j_l}\}_{l=0}^{h}}. \tag{5}$$

Global holomorphic vector fields [2]. The list of holomorphic vector fields with isolated singularities on compact complex surfaces is well known. In [2, Chapter 6] we can find the details when the surface is projective. In particular, for Z_0 on S we have one of the following possibilities:

(I) S has an elliptic fibration $f : S \to B$, and Z_0 is a nontrivial holomorphic vector field on S tangent to the fibres of f. Each fibre of f is a smooth elliptic curve which can be multiple, and outside multiple fibers f is a locally trivial fibration. Moreover Z_0 has an empty zero set.

(II) $S = \mathbb{C}^2/\Lambda$ is a 2-torus and Z_0 is a linear vector field on it, that is, the quotient of a constant vector field on $\mathbb{C}^2$.

(III) S is a $\mathbb{CP}^1$-bundle over an elliptic curve $\mathcal{E}$, and Z_0 is transverse to the fibers and projects on $\mathcal{E}$ to a constant vector field. In this case Z_0 is the suspension of $\mathcal{E}$ via the representation $\rho : \pi_1(\mathcal{E}) \to Aut(\mathbb{CP}^1)$ associated to the bundle structure, and it generates a Riccati foliation without invariant fibres and whose monodromy map is ρ.

(IV) S is a rational surface, and up to a birational map we have $Y = \mathbb{CP}^1 \times \mathbb{CP}^1$ and $Z_0 = v_1 \oplus v_2$, where v_1 and v_2 are holomorphic vector fields on $\mathbb{CP}^1$.

In the course of the proof we will consider S in some cases as a differentiable manifold with a given Riemannian metric g. If (N, g) is a Riemannian manifold, we denote by d the distance given by the metric, and by $B_r^d(p)$ the open ball centered at p. For the basic notions of Riemannian geometry that will be used in the rest of the paper, see [16].

We will analyze the possible cases for Z_0 and S. First note that Case (I) does not really occur since $\mathcal{F}$ does not have a rational first integral.

3.1. *Cases* (II) *and* (III)

Proposition 1. *If Z_0 and S are as in* (II) *or* (III) *any irreducible component of $D \cup E$ is invariant by $\tilde{\mathcal{F}}$.*

Proof. Let D_0 be an irreducible component of $D \cup E$ that is not invariant by $\tilde{\mathcal{F}}$. There is a compact curve Q_0 (possibly singular) in S generically transversal to Z_0. It is enough to define Q_0 as one of the connected components of $r^{-1}(s(D_0))$. Note that $s(D_0)$ is not a point.

Case (II). Let us take S as the quotient manifold $\mathbb{C}^2/\Lambda$. We identify $\mathbb{C}^2$ with $\mathbb{R}^4$, and Λ is an integral lattice of rank four.

Remark 6. Let $\mu : \mathbb{C}^2 \to \mathbb{C}^2/\Lambda$ denote the canonical submersion map. If we consider $\mathbb{R}^4$ with the usual euclidean g, taking $g' = \mu_* g$ as the metric on $\mathbb{C}^2/\Lambda$, the map μ becomes a Riemannian covering map. We will denote by d and d' the distances in $(\mathbb{R}^4, g)$ and $(\mathbb{R}^4/\Lambda, g')$, respectively.

The vector field Z_0 is the projection by μ of a constant vector field on $\mathbb{C}^2$, and thus its trajectories must be of the form $\mu(L_t)$ where $\{L_t\}_{t\in\mathbb{C}}$ is the family of lines parallel to a given direction. Note that Z_0 *does not have singularities*.

Lemma 3. *There is a compact $K \subsetneq S$ such that $Q_0 \subset \mathring{K}$.*

Proof. Since Z_0 is complete and without singularities we can define

$$K = \{\varphi(T, z) \mid |T| \leq 1,\ z \in Q_0\} \tag{6}$$

where $\varphi : \mathbb{C} \times S \to S$ is the complex flow of Z_0. If we apply the Flow Box Theorem to the points of Q_0 we easily deduce that $Q_0 \subset \mathring{K}$. □

We define the following function

$$\begin{aligned} &\alpha : \mathbb{C} \to [0, +\infty) \\ &t \mapsto d(L_t, \mu^{-1}(Q_0)). \end{aligned} \tag{7}$$

Remark 7. *α is continuous*. For any sequence $\{t_n\}_{n\in\mathbb{N}} \subset \mathbb{C}$ converging to t_* as $n \to \infty$, one sees that $\alpha(t_n) \leq d(L_{t_n}, L_{t_*}) + \alpha(t_*)$ and $\alpha(t_*) \leq d(L_{t_*}, L_{t_n}) + \alpha(t_n)$. Then, $\lim_{n\to\infty} \alpha(t_n) = \alpha(t_*)$.

We will use that α has the following property with respect to K.

Lemma 4. *$\alpha(t) \neq 0$ if and only if $\mu(L_t) \cap \mathring{K} = \emptyset$.*

Proof. If $\alpha(t) \neq 0$, it is clear from (6) that $\mu(L_t) \cap \mathring{K} = \emptyset$. Otherwise $\mu(L_t) \cap Q_0 \neq \emptyset$, which is not possible with our assumptions.

If $\alpha(t) = 0$, we suppose $\mu(L_t) \cap \mathring{K} = \emptyset$ and obtain a contradiction.

Fact 1. *There is $\delta \in \mathbb{R}^+$ such that $d'(\mu(L_t), Q_0) \geq \delta$.*

Otherwise we can determine a sequence $\{x_n\}_{n\in\mathbb{N}} \subset Q_0$ converging to $x_* \in Q_0$ and such that $d'(\mu(L_t), x_n) < 1/n$ because Q_0 is compact and $(\mathbb{R}^4/\Lambda, g')$ is complete. But it implies that for any ball $\mathbb{B}_r^{d'}(x_*)$ there exists $n(r) \in \mathbb{N}^+$ such that $\mathbb{B}_{1/n(r)}^{d'}(x_{n(r)}) \subset \mathbb{B}_r^{d'}(x_*)$, and hence $\mathbb{B}_r^{d'}(x_*) \cap \mu(L_t) \neq \emptyset$, which contradicts our assumption $\mu(L_t) \cap Q_0 = \emptyset$.

Fact 2. $d(L_t, \mu^{-1}(Q_0)) \geq \delta$.

By contradiction, suppose that $d(L_t, \mu^{-1}(Q_0)) < \delta$. Then there are $z \in L_t$ and $\bar{z} \in \mu^{-1}(Q_0)$ with $d(z, \bar{z}) < \delta$. Note that $\mu^{-1}(Q_0)$ is an analytic variety (non necessarily compact) of $\mathbb{C}^2$ and that $\mu(z) \neq \mu(\bar{z})$ by Fact 1. Let c be a segment from z to $\bar{z}$. As μ defines a local isometry from $(\mathbb{R}^4, g)$ to $(\mathbb{R}^4/\Lambda, g')$, we can take $\mathbb{B}_{r_i}^d(z_i) \subset \mathbb{R}^4$, $0 \leq i \leq s$, centered at $z_i \in c$, where $z_0 = z$ and $z_s = \bar{z}$, and in such a way that μ restricted to each $\mathbb{B}_{r_i}^d(z_i)$ defines an isometry over its image. Moreover, we can assume that $\mathbb{B}_{r_i}^d(z_i) \cap \mathbb{B}_{r_j}^d(z_j) \neq \emptyset$ if and only if $j = i + 1$, and thus fix $s - 1$ points $z_{i,i+1}$ in these intersections. As the isometries preserve intrinsic distance, Fact 1 and the triangle inequality imply the following contradiction

$$\delta > d(z, \bar{z}) = \sum_{i=1}^{s-1} d(z_i, z_{i,i+1}) + d(z_{i,i+1}, z_{i+1})$$

$$= \sum_{i=1}^{s-1} d'(\mu(z_i), \mu(z_{i,i+1})) + d'(\mu(z_{i,i+1}), \mu(z_{i+1}))$$
$$\geq d'(\mu(z), \mu(\bar{z})) \geq \delta. \quad \square$$

Lemma 5. *$\mu(L_t) \cap Q_0 \neq \emptyset$ for any $t \in \mathbb{C}$, and then $\alpha \equiv 0$.*

Proof. Suppose, by contradiction, that $\mu(L_t) \cap Q_0 = \emptyset$. It implies that $\mu(L_t) \cap \mathring{K} = \emptyset$. On the other hand, by Lemma 4, $\alpha(t) \neq 0$. Then $\alpha^{-1}(0)$ is a closed set strictly contained in $\mathbb{C}$, and if we take $\tilde{t}$ on its boundary we can fix a sequence $\{t_n\}_{n\in\mathbb{N}}$ with $\alpha(t_n) \neq 0$ converging to $\tilde{t} \in \mathbb{C}$ with $\alpha(\tilde{t}) = 0$. Note that $\mu(L_{\tilde{t}}) \cap Q_0 \neq \emptyset$ due to $\mu(L_{\tilde{t}}) \cap \mathring{K} \neq \emptyset$ since $\alpha(\tilde{t}) = 0$ (Lemma 4).

Let us take $\tilde{x} \in \mu(L_{\tilde{t}}) \cap Q_0$ with $\mu(\tilde{z}) = \tilde{x}$, and set $\{z_n\}_{n\in\mathbb{N}}$ converging to $\tilde{z}$ with $z_n \in L_{t_n}$. By continuity, $\{\mu(z_n)\}_{n\in\mathbb{N}}$ must converge to $\tilde{x}$. However, as $z_n \in L_{t_n}$ for any n, it holds $\mu(z_n) \notin \mu(L_{t_n}) \cap \mathring{K}$ since $\alpha(t_n) \neq 0$ (Lemma 4), what is a contradiction. Then $\mu(L_t) \cap Q_0 \neq \emptyset$. $\square$

It follows from Remark 4 that $\mathscr{M}_i$ is contained in a trajectory of Z_0. Hence there is L_{s_i} such that $\mu(L_{s_i}) \supset \mathscr{M}_i$.

Lemma 6. *$\mu(L_{s_i}) \cap Q_0 = \{p_i\}$, where p_i is the unique point in $\mu(L_{s_i}) \setminus \mathscr{M}_i$. In particular, $\mu(L_{s_i})$ and $\mathscr{M}_i$ are respectively biholomorphic to $\mathbb{C}$ and $\mathbb{C}^*$.*

Proof. Lemma 5 implies that $\mu(L_{s_i}) \cap Q_0 \neq \emptyset$. Moreover it is clear that $\mu(L_{s_i}) \cap Q_0 \subset \mu(L_{s_i}) \setminus \mathscr{M}_i$. It follows from Remark 4 that $r^*(s_*(\tilde{X}_{|\tilde{C}_z}))_{|\mathscr{M}_i}$ is complete and then it extends holomorphically by zeros on $\mu(L_{s_i}) \setminus \mathscr{M}_i$. Since C_z is not algebraic, $\mathscr{M}_i$ is biholomorphic to $\mathbb{C}^*$ and $\mu(L_{s_i}) \cap Q_0 = \mu(L_{s_i}) \setminus \mathscr{M}_i$ is a unique point p_i. Thus $\mu(L_{s_i})$ must be biholomorphic to $\mathbb{C}$. $\square$

Since the foliation defined by Z_0 on S has codimension 1 and it does not have singularities, the closure of $\mu(L_{s_i})$ in the open set $U' \subset S$ of non-compact leaves, that we will denote by L', is a subvariety of real codimension 0, 1 or 2 [9, Théorème 1.4]. *It holds $U' = S$ and then L' is the closure of $\mu(L_{s_i})$ in S*. If there were one compact leaf J, [9, Théorème 1.4] also assures that any non-compact leaf must accumulate J. In particular $\mu(L_{s_i})$ accumulates J. On the other side, as Q_0 cuts any leaf (Lemma 5), it must cut J, and $\mu(L_{s_i})$ accumulates the points of $J \cap Q_0$, which is not possible since p_i is the unique point in $\mu(L_{s_i}) \setminus \mathscr{M}_i$ (Lemma 6). Note that *L' is S or a real compact subvariety of dimension three*. If L' had real codimension 2, it would define a real compact subvariety of dimension two of S (L' is closed in S) containing $\mu(L_{s_i})$, which is a non-algebraic leaf. One concludes that $\mu(L_{s_i})$ must intersect infinitely many times Q_0, and then one obtains again a contradiction with Lemma 6.

Case (III). Let us consider S as a $\mathbb{CP}^1$-bundle over an elliptic curve $\mathcal{E}$ with bundle projection $p : S \to \mathcal{E}$. The structure of S can be lifted as $\mathbb{CP}^1$-bundle $\tilde{S}$ over $\mathbb{C}$ via the universal covering map $\varGamma : \mathbb{C} \to \mathcal{E}$: we can determine a complex surface $\tilde{S}$, a holomorphic covering $F : \tilde{S} \to S$ and a bundle projection $\tilde{p} : \tilde{S} \to \mathbb{C}$ such that $p \circ F = \varGamma \circ \tilde{p}$.

$$\begin{array}{ccc} \tilde{S} & \xrightarrow{F} & S \\ \downarrow{\scriptstyle \tilde{p}} & & \downarrow{\scriptstyle p} \\ \mathbb{C} & \xrightarrow[\varGamma]{} & \mathcal{E} \end{array}$$

Moreover as $\mathbb{C}$ is contractible this $\mathbb{CP}^1$-bundle is trivial. Thus $\tilde{S} = \mathbb{C} \times \mathbb{CP}^1$ and $\tilde{p}(x, y) = x$ is the projection over the first factor.

Lemma 7. *There is a holomorphic automorphism σ of $\mathbb{C} \times \mathbb{CP}^1$ such that $\sigma^*(F^*Z_0)$ is the horizontal vector field.*

Proof. It is clear that F^*Z_0 generates a Riccati foliation adapted to $\tilde{p}$ and without invariant fibres. If $\sigma(t, y) = \tilde{\varphi}(t, 0, y)$, with $\tilde{\varphi}$ the complex flow of F^*Z_0, σ is bijective, since each trajectory of F^*Z_0 intersects each fibre of p in only one point, and $\sigma(\mathbb{C} \times \{y\})$ are the trajectories of F^*Z_0. □

After Lemma 7, the trajectories of Z_0 are of the form $(F \circ \sigma)(L_t)$ where $\{L_t\}_{t \in \mathbb{CP}^1}$ is now the family of lines $L_t = \mathbb{C} \times \{t\}$.

Remark 8. As S is compact, S (as real manifold) admits a Riemannian metric g'. Let us set $\bar{\mu} = F \circ \sigma$. The map $\bar{\mu}$ from $(\mathbb{R}^2 \times \mathbb{S}^2, \bar{\mu}^* g')$ to (S, g') is a local Riemannian isometry. But still more, F is a covering map and σ is a biholomorphism; hence $F \circ \sigma$ is also a covering map and $\bar{\mu}$ is a Riemannian covering map. As (S, g') is compact, it is complete, and $(\mathbb{R}^2 \times \mathbb{S}^2, \bar{\mu}^* g')$ is complete. We will denote by d and d' the distances in $(\mathbb{R}^2 \times \mathbb{S}^2, \bar{\mu}^* g')$ and (S, g'), respectively.

The vector field Z_0 is complete and without zeros. We will consider as in case (II) an irreducible component D_0 of $D \cup E$ that is not invariant by $\tilde{\mathcal{F}}$, and the compact curve Q_0 (possibly singular) in S generically transversal to Z_0, defined by one of the connected components of $r^{-1}(s(D_0))$. As in Lemma 3 we can determine a compact set $K \subsetneq S$ such that $Q_0 \subset \mathring{K}$.

We will consider the continuous map (it follows as in Remark 7)

$$\begin{aligned} &\bar{\alpha} : \mathbb{CP}^1 \to [0, +\infty) \\ &t \mapsto d(L_t, \bar{\mu}^{-1}(Q_0)). \end{aligned} \tag{8}$$

Once we have fixed $\bar{\mu} = F \circ \sigma$, the complete metrics in Remark 8, the compact set K and the map $\bar{\alpha}$ as (8), we can prove similar Lemmas to Lemmas 3–6 of Case (II), where μ and α must be substituted by $\bar{\mu}$ and $\bar{\alpha}$ in their statements.

Let L_{s_i} be such that $\bar{\mu}(L_{s_i}) \supset \mathcal{M}_i$. Since $p_{|\bar{\mu}(L_{s_i})} : \bar{\mu}(L_{s_i}) \to \mathcal{E}$ is a covering map, and $\bar{\mu}(L_{s_i})$ is biholomorphic to $\mathbb{C}$ (Lemma 6), $\bar{\mu}(L_{s_i})$ *must cut almost all the fibres of p infinitely many times*. Let $\kappa \in \mathcal{E}$ such that $p^{-1}(\kappa)$ contains an infinite sequence of different points in $p^{-1}(\kappa) \cap \bar{\mu}(L_{s_i})$. By compactness of $p^{-1}(\kappa)$, the above sequence converges to $q_1 \in p^{-1}(\kappa)$. Note that q_1 is a regular point of Z_0. If $\bar{\mu}(L_{\tilde{s}})$ is the trajectory of Z_0 through q_1, $\bar{\mu}(L_{s_i})$ must accumulate $\bar{\mu}(L_{\tilde{s}})$ (flow-box theorem). On the other hand, $\bar{\mu}(L_{\tilde{s}}) \cap Q_0 \neq \emptyset$ (Lemma 5) implies a contradiction with the fact that p_i is the unique point in $\bar{\mu}(L_{s_i}) \setminus \mathcal{M}_i$ (Lemma 6).

Remark 9. One can also obtain a contradiction by distinguishing several cases, according to the (abelian) monodromy $\Gamma \subset Aut(\mathbb{CP}^1)$. If Γ has rank 1, then the non-algebraic leaves of Z_0 are isomorphic to $\mathbb{C}^*$, and one gets a contradiction by using Remark 5, and the fact that the intersection with algebraic curves is nonempty. If Γ has rank 2 then $\Gamma = \langle f, g \rangle$ with $f(z) = \alpha z$, $g(z) = \beta z$ or $f(z) = z + 1$, $g(z) = z + w$. In the first case (where, moreover, $\alpha^n \beta^m \neq 1$ for every $(m, n) \neq (0, 0)$) the non-algebraic leaves are sufficiently dense to apply the same argument as in case (II). In the second case one can prove that every algebraic curve $C \subset S$ different from the elliptic curve $E = \{z = \infty\}$ must intersect E, and from this fact it follows again that every non-algebraic trajectory of Z_0 intersects C infinitely many times. □

3.2. *Case* (IV)

There is a birational transformation $G : S \to \mathbb{CP}^1 \times \mathbb{CP}^1$ sending Z_0 to $G_*Z_0 = v_1 \oplus v_2$, where v_1 and v_2 are holomorphic vector fields on $\mathbb{CP}^1$. The description of G can be found in [2, p. 87]. In particular, G is a finite sequence of birational transformations which are contractions of curves invariant by Z_0 or blowing-ups at zeros of Z_0. Hence the exceptional divisor of G does not meet $\mathscr{M}_i$, and as a consequence $G(\mathscr{M}_i)$ is biholomorphic to $\mathscr{M}_i$. But still more, as $\mathscr{M}_i$ supports a complete vector field according to Remark 4, we can define an entire curve $f : \mathbb{C} \to G(\mathscr{M}_i)$. In the absence of rational first integrals, we may assume that G_*Z_0 is not constant. *Note that $G(\mathscr{M}_i)$ is contained in a trajectory L_z of G_*Z_0 and that $L_z \setminus G(\mathscr{M}_i)$ is empty or one point.* There are two cases for G_*Z_0:

(a) v_1 *and* v_2 *with zeros of order one at* 0 $(\lambda z\partial/\partial z + \mu w\partial/\partial w)$.

(b) v_1 *with a zero of order one at* 0 *and* v_2 *constant* $(\lambda z\partial/\partial z + \mu\partial/\partial w)$.

Proposition 2. *There exists at most an irreducible component D_0 of $D \cup E$ that is not invariant by $\tilde{\mathcal{F}}$. If D_0 exists:*

(1) *G_*Z_0 is as in* (**b**)*;*
(2) *$r^{-1}(s(D_0)) = Q_0$;*
(3) *The strict transform A_0 of Q_0 by G is $\overline{\{w = 0\}}$; and*
(4) *$\tilde{\mathcal{F}}$ is a Riccati foliation adapted to a rational map that projects by π as a rational function R of type $\mathbb{C}$ or $\mathbb{C}^*$.*

Proof. Let D_0 be a component of $D \cup E$ not invariant by $\tilde{\mathcal{F}}$, and Q_0 be the curve in S defined as in Proposition 1. If G_*Z_0 is as **(b)** let us suppose that either there is one component D_j of $D \cup E$ not invariant by $\tilde{\mathcal{F}}$ and different from D_0 or there is another component of $r^{-1}(s(D_0))$ different from Q_0.

Lemma 8. *There exist an open set $B \subset \mathbb{CP}^1 \times \mathbb{CP}^1$ biholomorphic to $\mathbb{C}^2$ and an entire curve $\bar{f} : \mathbb{C} \to G(\mathscr{M}_i) \cap B$ tangent to $G_*Z_{0|B}$ whose image avoids at least three algebraic curves contained in B.*

Proof. We analyze the two cases:

G_*Z_0 as in (**a**): Let B be $\mathbb{CP}^1 \times \mathbb{CP}^1$ minus $\{z = \infty\} \cup \{w = \infty\}$. As $\{z = \infty\}$ and $\{w = \infty\}$ are invariant by G_*Z_0, $G(\mathscr{M}_i) \subset B$. Note that G_*Z_0 on B is complete. If $\bar{f} = f$, $\bar{f} : \mathbb{C} \to G(\mathscr{M}_i) \subset B$ is an entire map whose image avoids at least $\{z = 0\}$, $\{w = 0\}$ and $A_0 \cap B$.

G_*Z_0 is as in (**b**): Let B be $\mathbb{CP}^1 \times \mathbb{CP}^1$ minus $\{z = 0\} \cup \{w = 0\}$. In this case $\{z = 0\}$ is invariant by G_*Z_0 but $\{w = 0\}$ is not. Remark that any non-algebraic trajectory of G_*Z_0 is of type $\mathbb{C}$ and intersects $\{w = c\}$, with $c \neq \infty$, in a unique point. More still, *one can suppose that* $A_0 \neq \overline{\{w = 0\}}$. Otherwise one defines Q_0 as any other component of $r^{-1}(s(D_0))$ or $r^{-1}(s(D_j))$, where D_j is a component of $D \cup E$ not invariant by $\tilde{\mathcal{F}}$ and $D_j \neq D_0$.

(b.1) If $L_z \setminus G(\mathscr{M}_i) = \emptyset$, we take $G(\mathscr{M}_i) \cap \{w = 0\} = p$ and the trajectory $G(\mathscr{M}_i) \cap B = L_z \setminus \{p\} \simeq \mathbb{C}^*$ of G_*Z_0 on B. As the universal covering of $G(\mathscr{M}_i) \cap B$ is $\mathbb{C}$, there exists $\bar{f} : \mathbb{C} \to G(\mathscr{M}_i) \cap B$ whose image avoids at least the algebraic curves: $\{z = \infty\}$, $\{w = \infty\}$ and $A_0 \cap B$.

(b.2) If $L_z \setminus G(\mathscr{M}_i) = q \in \{w = 0\}$, the argumentation is similar to (b.1) since $G(\mathscr{M}_i) \cap B = G(\mathscr{M}_i) = L_z \setminus \{q\} \simeq \mathbb{C}^*$ is a trajectory of G_*Z_0 on B.

(b.3) If $L_z \setminus G(\mathscr{M}_i) = q \notin \{w = 0\}$, we take the automorphism of $\mathbb{CP}^1 \times \mathbb{CP}^1$, $(z, w) \mapsto \delta(z, w) = (z, w - q_2)$, where $q = (q_1, q_2)$. As δ leaves invariant $G_* Z_0$ since $\delta_* G_* Z_0 = G_* Z_0$ and $\delta(q) \in \{w = 0\}$, it is enough to apply (b.2) to $L_z \setminus \delta(G(\mathscr{M}_i)) = \{\delta(q)\}$. □

Let $\mathbb{CP}^2$ be the compactification of B. The image of $\bar{f} : \mathbb{C} \to G(\mathscr{M}_i) \cap B$ is contained in $\mathbb{CP}^2$ minus at least four hypersurface sections, that is, three sections defined by the algebraic curves of Lemma 8 along with the line at infinity $\mathbb{CP}^2 \setminus B$. According to Green's Theorem [11, p. 199], $\bar{f}(\mathbb{C})$ must be contained in some algebraic curve, which contradicts our assumptions. Hence (1), (2) and (3) of the statement of Proposition follows.

Note that C_z, $\tilde{C}_z$, $\hat{C}_z$, $G(\mathscr{M}_i)$ and $\mathscr{M}_i$ are biholomorphic to $\mathbb{C}^*$, and that $L_z \simeq \mathbb{C}$ and $G(\mathscr{M}_i) = L_z \setminus \{q\} \simeq \mathbb{C}^*$ with $q \in \{w = 0\} \setminus \{(0, 0), (\infty, 0)\}$. $G(\mathscr{M}_i)$ has two parabolic ends, which are properly embedded in the complementary set of $\{z = \infty\} \cup \{w = \infty\}$ in $\mathbb{CP}^1 \times \mathbb{CP}^1$: one Σ_1 defined by a punctured disk centered at q, that is algebraic; and the other Σ_2 defined by $G(\mathscr{M}_i) \setminus \Sigma_1$, that is transcendental and accumulates $\{w = \infty\}$. Note that $G_* Z_0$ has two saddle-nodes as singularities: one at $(0, \infty)$, with a strong separatrix inside $\{w = \infty\}$ and a weak separatrix inside $\{z = 0\}$; and the other one at (∞, ∞), with a strong separatrix inside $\{w = \infty\}$ and a weak separatrix inside $\{z = \infty\}$. On the other hand $G_* Z_0$ defines a Riccati foliation adapted to $\beta(z, w) = w$. One may assume (maybe after blowing-up reduced singularities) that $\bar{\mathcal{F}}$ is Riccati with respect to $\beta_W = \beta \circ G \circ g$ and that $G \circ g$ is the contraction of curves inside fibers of β_W that produces the local models of [4, p. 439]. In this case all the fibers $\{w = c\}$, with $c \neq \infty$, are transversal minus one that is tangent, $\{w = \infty\}$, and of class (d).

Since h is an algebraic covering map from W to M, the proper mapping theorem allows one to define the trace of β_W as a rational function β_M on M [10]. Moreover, one can assume that β_M is a fibration after eliminating its base points. Recall that the property of being reduced is stable by blowing ups. Moreover, the possible dicritical components of the resolution of the pencil given by β_M must be transversal to the corresponding foliation.

By construction, the generic fiber F of β_M is a curve transverse to $\tilde{\mathcal{F}}$. Note that D_0 must be contained in a fiber F_0 of β_M as a consequence of (3) in the statement of this Proposition. Let us consider the following cases according to the genus of F.

- If F is of genus ≥ 2, it follows from [15, Theorem III.6.1] that $\tilde{\mathcal{F}}$ has a rational first integral, which is not possible.
- If F is of genus 1, $\tilde{\mathcal{F}}$ is a Turbulent foliation. Let us see that this case neither occurs because it would imply the existence of a rational first integral as before. Indeed, note that F does not cut F_0 since β_M is a fibration. On the other hand, $D \cap F \neq \emptyset$ by the maximum principle. As D_0 is the unique irreducible component of $D \cup E$ that is not invariant by $\tilde{\mathcal{F}}$, there must be one $\tilde{\mathcal{F}}$-invariant component D_2 of D such that $D_2 \cap F \neq \emptyset$. The existence of D_2 implies that $\tilde{\mathcal{F}}$ has a first integral (Lemma 1).
- If F is of genus 0, $\tilde{\mathcal{F}}$ is a Riccati foliation. Let us see that $\tilde{\mathcal{F}}$ satisfies (4) of the statement of Proposition 2. After contraction of rational curves each fiber of β_M admits one of the five possible models in [3, Section 7], [4, p. 439]. If there is one fiber F_1 tangent to $\tilde{\mathcal{F}}$, as D_0 is the unique irreducible component of $D \cup E$ which is not invariant by $\tilde{\mathcal{F}}$ and it is contained in F_0, then F must cut D in only one or two points near F_0. Then we can conclude as in Lemma 2 that β_M projects by π as a rational function R of type $\mathbb{C}$ or $\mathbb{C}^*$. Finally, one shows that all the fibers of β_M are not transverse to $\tilde{\mathcal{F}}$. In the contrary case, if L_0 is the leaf defined by $\tilde{C}_z$, as the covering map $\beta_{M|L} : L \to \mathbb{CP}^1$ is not finite (otherwise L is compact and C_z is algebraic), L must cut infinitely many times F_0 and C_z is not isomorphic to $\mathbb{C}^*$, which is not possible. □

After Proposition 2 we can assume that any irreducible component of $D \cup E$ is invariant by $\tilde{\mathcal{F}}$. Otherwise Theorem 1 follows by the results of Section 2.

3.3. Existence of a second integral

Let us come back to the beginning of Section 3, and consider (4).

Lemma 9. *It holds $Y^2F = 0$. In particular $\bar{X}$ is complete on the Zariski open set W' of W defined by $W \setminus (\{F = 0\} \cup \{F = \infty\} \cup P)$.*

Proof. Let $\mathscr{R}_0$ be a connected component of $h^{-1}(\tilde{C}_z)$. As $\mathscr{R}_0$ does not meet the exceptional divisor of $s : M \to \hat{M}$ then $h_{|\mathscr{R}_0} : \mathscr{R}_0 \to \tilde{C}_z$ is a non-ramified finite covering map. Hence $h^*_{|\mathscr{R}_0}(\tilde{X}_{|\tilde{C}_z}) = \bar{X}_{|\mathscr{R}_0}$ is complete. On the other hand $\bar{X}$ and Y are tangent on $\mathscr{R}_0$ according to (4). Thus $\mathscr{R}_0$ is a Riemann surface contained in a trajectory R_z of Y. Let $\varphi_z : \Omega_z \to R_z$ be the corresponding solution. We have two possibilities from 3.- of Remark 5:

(i) $R_z \notin \{R_{z_i}\}_{i=1}^s$. Since $Y_{|R_z}$ is complete $\Omega_z = \mathbb{C}$.
(ii) $R_z \in \{R_{z_i}\}_{i=1}^s$. Let us suppose that $R_z = R_{z_j}$. We take the solution $f_{z_j} : \mathbb{C} \to R_{z_j} \cup \{\bar{\theta}_{jl}\}_{l=0}^h$ of (5) and the discrete subset $\Delta = \{{f_{z_j}}^{-1}(\bar{\theta}_{jl})\}_{l=0}^h$ of $\mathbb{C}$. Since ${f_{z_j}}_{|\mathbb{C}\setminus\Delta} = \varphi_z$ then $\Omega_z = \mathbb{C} \setminus \Delta$.

It follows from (i) and (ii) that φ_z is a *univaluated* holomorphic map. Let us note that $\bar{X}_{|R_z}$ must be also complete because $\bar{X}_{|\mathscr{R}_0}$ is complete. Using these two facts and that $T \in \Omega_z \mapsto \varphi_z(T) \in R_z$ is a covering map then

$$\varphi_z^*(\bar{X}_{|R_z}) = \varphi_z^*(\bar{X})(T) = (F \circ \varphi_z(T)) \cdot \varphi_z^*(Y) = (F \circ \varphi_z(T)) \frac{\partial}{\partial T} \tag{9}$$

is a complete vector field on Ω_z. This is only possible if $\Omega_z = \mathbb{C}$ or $\mathbb{C}^*$ and $(F \circ \varphi_z)(T) = aT + b$, for $a, b \in \mathbb{C}$. We conclude that $Y(F)(\varphi_z(T)) = (F \circ \varphi_z)'(T)$ is constant and hence Y^2F vanishes along R_z, which can be assumed to be non-algebraic since C_z is by hypothesis. Hence $Y^2F = 0$.

Let us take a point $z \in W' = W \setminus (\{F = 0\} \cup \{F = \infty\} \cup P)$. If S_z is the trajectory of $\bar{X}$ through z, as Y is holomorphic on W' (1.- of Remark 5) and tangent to $\bar{X}$ on S_z by (4) then S_z defines a trajectory R_z of Y. Since it holds (9), the fact that $Y^2F = 0$ implies that $\bar{X}_{|S_z}$ is complete. □

After Lemma 9, $\bar{X}$ is complete on $W' = W \setminus (\{F = 0\} \cup \{F = \infty\} \cup P)$. According to 1.- and 2.- of Remark 5, $\tilde{X}$ is complete on $M \setminus h(W \setminus W')$. By Propositions 1 and 2, $\tilde{X}$ is complete on $M \setminus (h(W \setminus W') \cup E \cup D)$. If we project by π we see that X is complete on a Zariski open set of $\mathbb{C}^2$ and it can be extended to $\mathbb{C}^2$ as a complete vector field. Therefore X is complete.

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