Completeness is determined by any non-algebraic trajectory

Alvaro Bustinduy\textsuperscript{a,}\textsuperscript{*}, Luis Giraldo\textsuperscript{b}

\textsuperscript{a} Departamento de Ingeniería Industrial, Escuela Politécnica Superior, Universidad Antonio de Nebrija, C/ Pirineos 55, 28040 Madrid, Spain

\textsuperscript{b} Departamento de Geometría y Topología, Facultad de Ciencias Matemáticas, Universidad Complutense de Madrid, Plaza de Ciencias 3, 28040 Madrid, Spain

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Abstract

It is proved that any polynomial vector field in two complex variables which is complete on a non-algebraic trajectory is complete.

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1. Introduction and statement of results

Let $X$ be a holomorphic vector field on $\mathbb{C}^2$. For any $z \in \mathbb{C}^2$, the local solution $\varphi_z(T)$ of the associated ordinary differential equation $dz/dT = X(z(T))$ with the initial condition $z(0) = z \in \mathbb{C}^2$ can be extended by analytic continuation along paths in $\mathbb{C}$, to a maximal domain $\Omega_z$, which may not be an open set of $\mathbb{C}$, but rather a Riemann domain over $\mathbb{C}$. The map
ϕ
z
: Ω
z
→ \mathbb{C}^2 is the solution of X through z, and its image ϕ\(_{z}(\Omega_z)\), that will be denoted by C\(_z\) (or L\(_z\), R\(_z\), S\(_z\)), is the trajectory of X through z.

The vector field X is complete on C\(_z\) if Ω\(_z\) = C, and X is complete if it is complete on C\(_z\), for every z ∈ \mathbb{C}^2. Each trajectory C\(_z\) on which X is complete (complete trajectory) is defined by an abstract Riemann surface uniformized by C, and by the maximum principle, analytically isomorphic to C or C∗.

Extrinsically, the topology of a trajectory can be very complicated. The simplest trajectories from this point of view are the analytic ones. One says that the trajectory C\(_z\) is analytic if it is contained in an analytic curve in \mathbb{C}^2 (but not necessarily equal to it, due to the possible presence of singularities). Otherwise C\(_z\) is a non-analytic trajectory.

An interesting remark (due to R. Moussu) is that two vector fields with a common non-analytic trajectory have to be collinear in any point. In this sense a non-analytic trajectory determines the vector field up to multiplication by a non-vanishing holomorphic function.

In this work we will consider polynomial vector fields with at most isolated zeros. The above remark for two polynomial vector fields can be restated. For a trajectory, it is enough not to be contained in an algebraic curve (that is, to be a non-algebraic trajectory) to determine the vector field up to multiplication by a nonzero constant.

In [1], Brunella studied foliations in \mathbb{C}^2 given by polynomial vector fields with a trajectory containing a planar isolated end (proper Riemann sub-surface isomorphic to \{z : r < \|z\| ≤ 1\}, where r ∈ [0, 1)), properly embedded in \mathbb{C}^2 and whose closure in \mathbb{CP}^2 contains the line at infinity. He proved that these foliations can be determined in terms of a polynomial whose generic fiber is of type C or C∗ and transversal to the foliation. As a remarkable corollary, he obtained that if the trajectory is a non-algebraic analytic plane, the foliation is given by the constant vector field after an analytic automorphism. Therefore, the trajectory in this case is determining the completeness of the vector field up to multiplication by a nowhere vanishing function.

Then if one attends to the completeness of a non-algebraic trajectory (not necessarily analytic) the following natural question arises [6, Question 5.1]:

**Question 1.** If X is a polynomial vector field in \mathbb{C}^2 with the property of being complete on a single non-algebraic trajectory, is it complete?

The main result of this work says that Question 1 has an affirmative answer:

**Theorem 1.** Let us consider a polynomial vector field X on \mathbb{C}^2 which is complete on a non-algebraic trajectory. Then X is complete.

Note that our theorem implies that any entire solution of a polynomial vector field can be determined up to an algebraic automorphism of \mathbb{C}^2. As the vector field is complete, the solution must correspond to one of the vector fields of Brunella’s classification in [4] after a polynomial automorphism.

It could be very interesting to study if a non-analytic trajectory of a (non-polynomial) holomorphic vector field determines the completeness of the vector field.

**1.1. About the proof of Theorem 1**

For the sake of completeness, throughout the paper we include some definitions and results taken from Brunella’s papers [2–4]. Let us begin by recalling some definitions. Consider the foliation \(\mathcal{F}\) generated by X on \mathbb{C}^2 extended to \mathbb{CP}^2. According to Seidenberg’s theorem, the
minimal resolution of $\mathcal{F}$ is a new foliation $\tilde{\mathcal{F}}$ defined on a rational surface $M$ after pulling back $\mathcal{F}$ by a birational morphism $\pi : M \to \mathbb{C}P^2$, that is a finite composition of blowing ups. Along with this resolution one has: (1) the Zariski open set $U = \pi^{-1}(\mathbb{C}^2)$ of $M$, over which $X$ can be lifted to a holomorphic vector field $\tilde{X}$; (2) the exceptional divisor $E$ of $U$, and (3) the divisor at infinity

$$D = M \setminus U = \pi^{-1}(\mathbb{C}P^2 \setminus \mathbb{C}^2) = \pi^{-1}(L_{\infty}),$$

that is a tree of smooth rational curves. The vector field $\tilde{X}$ can be extended to $M$, although it may have poles along one or more components of $D$. Let us still denote this extension by $\tilde{X}$. As only singularities of the foliation in $\mathbb{C}^2$ are blown up, and they are in the zero set of $X$, the vector field $\tilde{X}$ is holomorphic on the full $U$ and it has the complete trajectory $C_\zeta$ defined by $\pi^{-1}(C_\zeta)$.

Therefore the reduced foliation $\tilde{\mathcal{F}}$ has at least one tangent entire curve: the one defined by $C_\zeta$, which is Zariski dense in $M$. It implies that the Kodaira dimension $\kod(\tilde{\mathcal{F}})$ of $\tilde{\mathcal{F}}$ is 1 or 0 [13, Section IV] (see also [2, p. 131]).

In the case $\kod(\tilde{\mathcal{F}}) = 1$, [4] allows one to conclude that $\tilde{\mathcal{F}}$ is a Riccati foliation adapted to a fibration $g : M \to \mathbb{P}^1$, whose projection to $\mathbb{C}^2$ by $\pi$ defines a rational function $R$ of type $C$ or $C^*$. We can apply the study of [6] although $R$ is not a polynomial (see also [8]) and deduce the completeness of $X$. We will analyze this case in Section 2.

In the case $\kod(\tilde{\mathcal{F}}) = 0$, we know that $\tilde{\mathcal{F}}$ is generated by a vector field on a smooth compact projective surface $S$, up to contractions of $\tilde{\mathcal{F}}$-invariant curves and covering maps [4]. However, we need to go a bit further to know if these models restrict to our open $U$ a complete vector field. This is accomplished via the description of the irreducible components of $D \cup E$ that are not $\tilde{\mathcal{F}}$-invariant. When $S$ is rational, we show that in fact $D \cup E$ must be invariant if $\tilde{\mathcal{F}}$ is not Riccati with respect to a fibration $g : M \to \mathbb{P}^1$ that is projected to $\mathbb{C}^2$ by $\pi$ as a rational function $R$ of type $C$ or $C^*$. For the remaining cases, i.e. when $S$ is a $\mathbb{C}P^1$-bundle over an elliptic curve or a complex 2-torus, we prove that $D \cup E$ is always invariant by $\tilde{\mathcal{F}}$. For the proof of this last fact we will consider $S$ as a differential manifold with a certain Riemannian metric. It will enable us to compute the distance from the complete trajectory to a compact set containing the components of $D \cup E$ that are not $\tilde{\mathcal{F}}$-invariant. As a consequence of the discussion above one obtains that the lifting of $\tilde{X}$ by a certain covering map can be decomposed in the product of a complete rational vector field by a second integral of it. It allows us to conclude that the projection $\pi_*\tilde{X}$ restricted to $U$ i.e. $X$ must be complete. We will analyze this case in Section 3.

Finally, we point out that [6,7] imply that Question 1 has an affirmative answer for a non-algebraic analytic trajectory. In those works, Brunella’s results [1] are used as the main tool. The proof of our theorem is mainly based on Brunella’s approach to the classification complete polynomial vector fields in the plane [4], since they can be applied to the foliation $\mathcal{F}$ although $X$ could be in principle not complete. Theorem 1 is not only the generalization of the previous results mentioned above ([6,7]), but its proof also implies them.

2. $\kod(\mathcal{F}) = 1$

According to [13, Section IV] the absence of a first integral implies that $\tilde{\mathcal{F}}$ is a Ricatti or a Turbulent foliation, that is to say, the existence of a fibration

$$g : M \to B$$

whose generic fibre is a rational curve or an elliptic curve transverse to $\tilde{\mathcal{F}}$, respectively. Remark that $B$ is $\mathbb{C}P^1$ since $M$ is a rational surface.
2.1. Nef models and canonical models [13, Section III], [3, Section 4], [4, Section 3]

Existence of a nef model. As \( \hat{\mathcal{F}} \) is not a rational fibration it has a model \( \hat{\mathcal{F}} \) which is reduced and nef. More concretely, after a contraction \( s : M \to \hat{M} \) of the \( \hat{\mathcal{F}} \)-invariant rational curves on \( M \) over which the canonical bundle \( K_{\hat{\mathcal{F}}} \) has a negative degree one obtains (see [3, Section 4], [4, Section 3]):

1. a new surface \( \hat{M} \), maybe with cyclic quotient singularities; and
2. a reduced foliation \( \hat{\mathcal{F}} = s_*\hat{\mathcal{F}} \) on \( \hat{M} \) such that its canonical \( \mathbb{Q} \)-bundle \( K_{\hat{\mathcal{F}}} \) is nef (i.e. \( K_{\hat{\mathcal{F}}} \cdot C \geq 0 \) for any curve \( C \subset \hat{M} \)).

Recall that a cyclic quotient singularity \( p \) of \( \hat{M} \) is locally defined by \( \mathbb{B}^2 / \Gamma_{k,h} \) where \( \mathbb{B}^2 \subset \mathbb{C}^2 \) is the unit ball and \( \Gamma_{k,h} \) is the cyclic group generated by a map of the form \( (z, w) \to (e^{\frac{2\pi i}{k}} z, e^{\frac{2\pi i h}{k}} w) \) with \( k, h \) positive coprime integers such that \( 0 < h < k \). These singularities of \( \hat{M} \) are not singularities of \( \hat{\mathcal{F}} \). That is, the foliation can be lifted locally to \( \mathbb{B}^2 \setminus \{(0,0)\} \) and extended to a foliation on \( \mathbb{B}^2 \) with a non-vanishing associated vector field.

**Remark 1.** The possible cyclic singularities of \( \hat{M} \) are in the image of the exceptional divisor of \( s \). Any rational curve \( C_0 \) of that divisor is \( \hat{\mathcal{F}} \)-invariant, it has a unique singularity \( p \) of the foliation of type \( d(x^n y^m) \) with \( n, m \in \mathbb{N}^+ \), where \( C_0 = \{ y = 0 \} \), and it may also contain one cyclic quotient singularity \( q \) of order \( m \) (regular point if \( m = 1 \)). After contracting \( C_0 \) by \( s \) (since \( C_0^2 = -n/m \)) we obtain a new quotient singularity of order \( n \) (regular if \( n = 1 \)) [4, pp. 443–444].

Existence of a minimal model. After possibly additional contractions on \( \hat{M} \) of rational curves, \( q : \hat{M} \to N \), one obtains a reduced foliation \( \mathcal{H} = q_*\hat{\mathcal{F}} \) (birational to \( \hat{\mathcal{F}} \)) on a surface \( N \) regular on the (cyclic quotient) singularities of \( N \) whose canonical bundle \( K_\mathcal{H} \) is nef and such that it verifies this property: if \( K_\mathcal{H} \cdot C = 0 \Rightarrow C^2 \geq 0 \) for any curve \( C \subset N \). It is important to note that we can assume that \( q \) is given by contractions of curves which are invariant by the foliation: if \( C \) is not \( \hat{\mathcal{F}} \)-invariant it follows from the formula \( (K_{\hat{\mathcal{F}}} + C) \cdot C \geq 0 \) [3, Section 2] that \( K_{\hat{\mathcal{F}}} \cdot C = 0 \Rightarrow C^2 \geq 0 \). This model is the minimal model of \( \hat{\mathcal{F}} \).

\[
\begin{array}{ccc}
M & \xrightarrow{\pi} & \mathbb{C} \mathbb{P}^2 \\
\downarrow s & & \\
\hat{M} & \xrightarrow{q} & N
\end{array}
\]

**Remark 2.** In general the minimal model of \( \hat{\mathcal{F}} \) is not unique. However if we have another minimal model \( \mathcal{H}' \) of \( \hat{\mathcal{F}} \) defined on \( N' \) and \( p : N \to N' \) is an algebraic map defined everywhere with \( p_*\mathcal{H} = \mathcal{H}' \) then \( p \) is an isomorphism [13, Lemma III.3.1].

**Remark 3.** As \( s \) and \( q \) are given by contractions of rational curves which are invariant by the foliation neither \( \hat{C}_z \) meets the exceptional divisor of \( s \) nor \( s(\hat{C}_z) \) meets the exceptional divisor of \( q \). It implies that there must be a parabolic leaf of \( \mathcal{H} \): the leaf that contains the Riemann surface \( q(s(\hat{C}_z)) \) that supports the complete vector field \( q_s(s_*(\hat{X}_{\hat{C}_z})) \).

2.2. Turbulent case

When \( X \) is complete the case of a Turbulent \( \hat{\mathcal{F}} \) can be excluded as it is proved in [4, Lemma 1]. We now prove that it continues being valid in a more general situation.
Lemma 1. $\tilde{F}$ is not a Turbulent foliation.

Proof. Suppose that $\tilde{F}$ is Turbulent. The description of models around each fibre of $g$ after a birational morphism $\alpha : M \to M^*$ is known [3, Section 7]. The resulting foliation $\tilde{G} = \alpha_*\tilde{F}$ on $M^*$ is regular on the (cyclic quotient) singularities of $M^*$, it is Turbulent with respect to $\tilde{g} = g \circ \alpha^{-1}$, and each fiber of $\tilde{g}$ is of one of the following classes:

(a) (resp. (d)): the fibre is smooth elliptic, transversal (resp. tangent) to $\tilde{G}$ and may be multiple. 
(b) (resp. (e)): the fibre is rational with three quotient singularities of orders $k_1$, $k_2$ and $k_3$ satisfying $\frac{1}{k_1} + \frac{1}{k_2} + \frac{1}{k_3} = 1$, transversal (resp. tangent) to $G$ and of multiplicity 3, 4 or 6. 
(c) (resp. (f)): the fibre is rational with four quotient singularities of order 2; transversal (resp. tangent) to $G$ and of multiplicity 2.

We will call classes (a), (b) and (c) (resp. (d), (e) and (f)) transversal fibres (resp. tangent fibres) of $\tilde{g}$.

For any leaf $L$ of $G$ outside tangent fibres of $\tilde{g}$, $\tilde{g}|_L : L \to B_0$, with $B_0$ defined as $B$ minus the points over tangent fibres of $\tilde{g}$, is a regular covering (in orbifold’s sense). The orbifold structure in $B_0$ is the natural structure inherited from the orbifold structure on $B$ induced by (the local models of) $\tilde{g}$ [3, Section 7].

Claim 1. There must be at least one tangent fibre $G_0$ of $\tilde{g}$.

We suppose that all the fibres are transversal and obtain a contradiction. Since $B_0 = B = \mathbb{CP}^1$, the orbifold universal covering of any leaf $L, \tilde{L}$, is equal to the one of $B, \tilde{B}$.

Let us suppose that $\tilde{B}$ is $\mathbb{C}$ or $\mathbb{CP}^1$. By pulling back sections of $K_B$ under $\tilde{g}$ we obtain sections of $K_G$. We can in this way compute $K_G$ and obtain that $deg(\tilde{g}_*K_G) = -\chi_{orb}(B)$ (see, [3, Section 7]). On the other hand since $\text{kod}(\tilde{F}) = \text{kod}(\tilde{G}) = 1$ then $deg(\tilde{g}_*K_G) > 0$. It follows that $\chi_{orb}(B) < 0$, which is impossible if $B$ is parabolic (see Appendix E, Lemma E.4, in [14]). Thus $\tilde{B}$ is a disk.

As all the leaves of $G$ are hyperbolic and the singularities are isolated (in fact $G$ is regular), $K_G$ is nef [5, Remark 8.8]. Moreover, it is clear that $K_G : C = 0 \Rightarrow C^2 \geq 0$; if $C$ is not $G$-invariant it follows from the formula $(K_G + C) : C \geq 0$ [3, Section 2]. If $C$ is $G$-invariant the Camacho–Sad formula [3, Section 2] implies that $C^2 = 0$ because $G$ is regular on $C$. Therefore $G$ is a minimal model of $\tilde{F}$. But then it has necessarily a parabolic leaf (Remarks 2 and 3), which is a contradiction.

Claim 2. If there is an irreducible component $D_1$ of $D \cup E$ that is not $\tilde{F}$-invariant and which is not contained in any fiber of $g$, then $D_1 \subset \{\tilde{X} = 0\}$.

It is important to note that the strict transform of $D_1$ by $\alpha$, that we also denote by $D_1$, is a rational curve. Otherwise it is a point with infinitely many punctured disks invariant by $\tilde{G}$ through it and then a singularity of $G$, which is not possible. Hence $D_1 \cap G_0 \neq \emptyset$. Let us denote by $J$ the leaf of $\tilde{G}$ that defines the non algebraic component of $\alpha(\tilde{C}_z)$. There is at least one accumulation point of $J$ on $G_0$ because $\tilde{g}(J) = g(\tilde{C}_z)$ is $\mathbb{C}$ or $\mathbb{C}^*$. It must be a regular point of the foliation by the absence of singularities of the foliation on tangent fibers. Thus $J$ must accumulate on $G_0$. It implies that $\tilde{C}_z \cap D_1 \neq \emptyset$ and then $D_1 \subset \{\tilde{X} = 0\}$ by the completeness of $\tilde{X}|_{\tilde{C}_z}$.

Let us take the generic fiber $G$ of $g$, which is transverse to $\tilde{F}$. Obviously, $D \cap G \neq \emptyset$. In the contrary case we have an elliptic curve contained in $\mathbb{C}^2$, which is impossible (maximum principle). Among the irreducible components of $D$ cutting $G$ at least one, say $D_2$, is $\tilde{F}$-invariant. Otherwise $\tilde{X}$ would be holomorphic in a neighborhood of $G$ and it vanishes on at least one
component of $D$ transversal to $G$, which implies that $\tilde{X}$ is identically zero by Claim 2. The existence of $D_2$ is enough to construct a rational integral for a Turbulent $\tilde{F}$ as can be seen in [4, p. 438]. □

2.3. Ricatti case

Lemma 2. $g|_U$ is projected by $\pi$ as a rational function $R$ of type $\mathbb{C}$ or $\mathbb{C}^*$. Moreover, $\tilde{F}$ is $R$-complete.

Proof. Up to contraction of rational curves inside fibers of $g$, which can produce cyclic quotient singularities of the surface but on which the foliation is always regular, one has that there are five possible models for the fibers of $g$ [3, Section 7], [4, p. 439]. Let $L_0$ be the leaf of the foliation defined by $\tilde{C}_z$. One can conclude that the orbifold universal covering $\tilde{L}_0$ of $L_0$ is equal to the one of $B_0$, $\tilde{B}_0$, where $B_0$ is defined as $\mathbb{C}\mathbb{P}^1$ minus the points over tangent fibres of $g$ with the natural orbifold structure inherited from the orbifold structure on $\mathbb{C}\mathbb{P}^1$ induced by (the local models of) $g$. Since $X$ is complete on $C_z$, $\tilde{L}_0$ is biholomorphic to $\mathbb{C}$ and then $L_0$ is parabolic. This fact along with $\text{cod}(\tilde{F}) = 1$ implies by Brunella [4, Lemma 2] that there must be at least one fibre $G_0$ tangent to the foliation of class:

(d): the fibre is rational with two saddle-nodes of the same multiplicity $m$, with strong separatrices inside the fibre, or of class

(e): the fibre is rational with two quotient singularities of order 2, and a saddle-node of multiplicity $l$, with a strong separatrix inside the fibre.

Firstly one observes that there are irreducible components of $D \cup E$ that are not contained in any fiber of $g$. Let us take the generic fiber $G$ of $g$, which is transverse to $\tilde{F}$. Obviously, $D \cap G \neq \emptyset$. In the contrary case we have a rational curve contained in $\mathbb{C}^2$, which is impossible (maximum principle).

Let $D_1$ be one of these components. Then $D_1 \cap G_0 \neq \emptyset$ and there is at least one accumulation point of $\tilde{C}_z$ on $G_0$, say $p$, because $g(\tilde{C}_z)$ is $\mathbb{C}$ or $\mathbb{C}^*$. If $p$ is a regular point of the foliation, $\tilde{C}_z$ must accumulate on $G_0$. If $p$ is singular, it is a saddle-node with a strong separatrix defined by $G_0$, and therefore $\tilde{C}_z$ must also accumulate on all $G_0$ [12], in particular in the other saddle-node if it exists. There are two possibilities:

(i) If $D_1$ is $\tilde{F}$-invariant, $D_1$ is not in the divisor of poles of $\tilde{X}$. Otherwise, $D_1 \cap G_0 \neq \emptyset$ is a saddle-node $q$. Let us take the rational section $\omega$ of $K_{\tilde{F}}$ dual to $\tilde{X}$ that restricts to $\tilde{C}_z$ as the differential of times given by the flow of $\tilde{X}$ on $\tilde{C}_z$. One can construct a path $\gamma : (0, \epsilon] \to \tilde{C}_z$, with $\epsilon \in \mathbb{R}^+$ and $\gamma(t) \to q$ as $t \to 0$, such that $\int_\gamma \omega$ is finite (see [4, proof of Lemma 3]), which contradicts the completeness of $\tilde{X}$ on $\tilde{C}_z$.

(ii) If $D_1$ is not $\tilde{F}$-invariant, necessarily $\tilde{C}_z \cap D_1 \neq \emptyset$ and $D_1 \subset \{\tilde{X} = 0\}$. Otherwise, as $\tilde{C}_z \cap D_1 \neq \emptyset$ one has that $D_1 \cap G_0 \neq \emptyset$ is a saddle-node with $D_1$ defining its weak separatrix, which is $\tilde{F}$-invariant [1, Lemma 11].

It follows from (i) and (ii) that $\tilde{X}$ is holomorphic in a neighborhood of $G$, which implies as in the above lemma that (ii) does not really occur. Thus $D_1$ is $\tilde{F}$-invariant.

Therefore $D$ must cut $G$ at one or two points, and the projection $R$ of $g|_U$ via $\pi$ is of type $\mathbb{C}$ or $\mathbb{C}^*$. Moreover, the invariancy of the components of $D \cup E$ which are not contained in fibers of $g$ implies that generically $R$ is a fibration trivialized by the leaves of $\tilde{F}$, and then $\tilde{F}$ is $R$-complete. □
We will study the two possibilities after the previous lemma.

2.4. \( R \) of type \( \mathbb{C} \)

By Suzuki (see [17]) we may assume that \( R = x \), up to a polynomial automorphism. Hence \( \mathcal{F} \) is a Riccati foliation adapted to \( x \) and \( X \) is a complete vector field of the form

\[
Cx^{N} \frac{\partial}{\partial x} + [A(x)y + B(x)] \frac{\partial}{\partial y},
\]

with \( C \in \mathbb{C}, N = 0, 1 \) and \( A, B \in \mathbb{C}[x] \) (see [6, Proposition 4.2]).

2.5. \( R \) of type \( \mathbb{C}^* \)

By Suzuki (see [18]) we may assume that

\[
R = x^m (x^\ell y + p(x))^n,
\]

where \( m \in \mathbb{N}^*, n \in \mathbb{Z}^*, \) with \( (m, n) = 1, \ell \in \mathbb{N}, \) \( p \in \mathbb{C}[x] \) of degree \( < \ell \) with \( p(0) \neq 0 \) if \( \ell > 0 \) or \( p(0) \equiv 0 \) if \( \ell = 0 \), up to a polynomial automorphism.

**New coordinates.** According to relations \( x = u^n \) and \( x^\ell y + p(x) = v u^{-m} \) it is enough to take the rational map \( H \) from \( u \neq 0 \) to \( x \neq 0 \) defined by

\[
(u, v) \mapsto (x, y) = (u^n, u^{-(m+n\ell)} [v - u^m p(u^n)])
\]

in order to get \( R \circ H(u, v) = v^n \).

Although \( R \) is not necessarily a polynomial \( (n \in \mathbb{Z}) \), it is a consequence of the proof of [6, Proposition 3.2] that \( H^* \mathcal{F} \) is a Riccati foliation adapted to \( v^n \) having \( u = 0 \) as an invariant line. Thus

\[
H^* X = u^k \cdot Z
\]

\[
= u^k \cdot \left\{ a(v) u \frac{\partial}{\partial u} + c(v) \frac{\partial}{\partial v} \right\},
\]

where \( k \in \mathbb{Z} \), and \( a, c \in \mathbb{C}[v] \).

At this point one could apply the techniques of [7] to analyze the possible global 1-forms of times associated to \( X \) in order to prove the existence of an invariant line. However, applying directly the local models of [4], it follows from [8, Lemma 2] that at least one of the irreducible components of \( R \) over 0 must be a \( \mathcal{F} \)-invariant line. Hence the polynomial \( c(v) \) of (3) is in fact a monomial, and thus of the form \( c v^N \) with \( c \in \mathbb{C} \) and \( N \in \mathbb{N} \).

Finally, according to [6, pp. 661–662] we know that \( X|_{C_z} \) complete implies \( k = 0 \) and \( N = 0, 1 \). Hence \( X \) is complete.

### 3. \( \text{cod}(\mathcal{F}) = 0 \)

According to [13, Section IV] we can contract \( \tilde{\mathcal{F}} \)-invariant rational curves on \( M \) via a contraction \( s \) to obtain a new surface \( \tilde{M} \) (maybe singular with cyclic quotient singularities), a reduced foliation \( \tilde{\mathcal{F}} \) on this surface, and a finite covering map \( r \) from a smooth compact projective surface \( S \) to \( \tilde{M} \) such that: (1) \( r \) ramifies only over cyclic (quotient) singularities of \( \tilde{M} \) and (2)
the foliation $r^*(\bar{\mathcal{F}})$ is generated by a complete holomorphic vector field $Z_0$ on $S$ with isolated zeros [4, p. 443].

$$\mathbb{CP}^2 \xleftarrow{s} M \xrightarrow{r} S$$

Remark 4. Note that $\tilde{C}_z$ does not meet the exceptional divisor of the contraction $s$. Let us set $\tilde{C}_z$ as $s(\tilde{C}_z)$. Since $\tilde{C}_z$ does not contain singularities of $\tilde{M}$ then $\tilde{C}_z$ is a Riemann surface, $s_0,\tilde{C}_z : \tilde{C}_z \rightarrow \tilde{C}_z$ is a biholomorphism and $r_{|r^{-1}(\tilde{C}_z)} : r^{-1}(\tilde{C}_z) \rightarrow \tilde{C}_z$ is a non-ramified finite covering map. Thus $s_*(\tilde{X}|_{\tilde{C}_z})$ is complete on $\tilde{C}_z$ and $r^*(s_*(\tilde{X}|_{\tilde{C}_z}))$ is complete on the connected components $\mathcal{M}_i$ of $r^{-1}(\tilde{C}_z) = \bigcup_{i=0}^l \mathcal{M}_i$. Hence each $\mathcal{M}_i$ is a Riemann surface contained in a complete trajectory $T_z$ of $Z_0$ that supports the complete vector field $r^*(s_*(\tilde{X}|_{\tilde{C}_z}))|_{\mathcal{M}_i}$, which does not necessarily coincide with $Z_0|_{\mathcal{M}_i}$. It is convenient to observe that if $T_z$ is isomorphic to $\mathbb{C}^*$ then, necessarily $\mathcal{M}_i = T_z$, and the vector field $r^*(s_*(\tilde{X}|_{\tilde{C}_z}))|_{\mathcal{M}_i}$ coincides with $Z_0$ on $T_z$, up to a multiplicative constant. The discrepancy between the two complete vector fields can occur only if $T_z$ is isomorphic to $\mathbb{C}$, in which case $r^*(s_*(\tilde{X}|_{\tilde{C}_z}))|_{\mathcal{M}_i}$ could have one (and only one) zero at some point $p = T_z \setminus \mathcal{M}_i$.

It follows from [4, p. 443] that the covering $r$ can be lifted to $M$ via a birational morphism $g : W \rightarrow S$ and a ramified covering $h : W \rightarrow M$ such that $s \circ h = r \circ g$.

$$M \xleftarrow{h} W \xrightarrow{r} S$$

Let $Y$ be the lift of $Z_0$ on $W$ via $g$. Then $Y$ must be a rational vector field on $W$ generating the foliation $\bar{\mathcal{F}}$ given by $g^*(r^*(\bar{\mathcal{F}})) = h^*\bar{\mathcal{F}}$. On the other hand, $\bar{\mathcal{F}}$ is also generated by the rational vector field $\bar{X}$ on $W$ given by $h^*\bar{X}$. Hence there is a rational function $F$ on $W$ such that

$$\bar{X} = F \cdot Y.$$ (4)

Remark 5. We remark from the above construction:

1. The map $g$ is a composition of blowing-ups over a finite set $\Theta = \{\theta_i\}_{i=1}^r \subset S$ of regular points of $Z_0$. The poles of $Y$ are in $g^{-1}(\Theta)$ and they define a divisor $P \subset W$ invariant by $\bar{\mathcal{F}}$. Hence $Y$ is holomorphic on $W \setminus P$. Note that in $W \setminus P$, $Y$ has only isolated zeros.

2. Since $P$ is the exceptional divisor of $g$, $h(P)$ is the exceptional divisor of $s$ and is $\bar{\mathcal{F}}$-invariant. Then

$$h|_{W \setminus P} : W \setminus P \rightarrow M \setminus h(P)$$

is a regular covering map.

3. Let $C_{\theta_i}$ be the trajectory of $Z_0$ through $\theta_i$. $Y$ is a complete holomorphic vector field on $W \setminus \{g^{-1}(C_{\theta_i})\}_{i=1}^r$. Each $g^{-1}(C_{\theta_i}) \setminus P$ is contained in a trajectory $R_{z_i}$ of $Y$. Let us fix one of them, say $C_{\theta_j}$. Let us set $\Theta \cap C_{\theta_j} = \{\theta_j\}_{j=0}^h$ taking $j_0 = j$. Note that $R_{z_{j_0}} = R_{z_j}$ for
any \( l \). For every \( \theta_j \) there is a point \( \bar{\theta}_j \in P \) such that \( R_{\bar{\theta}_j} \cup \{ \theta_j \} \) defines a separatrix of \( \bar{F} \) through \( \bar{\theta}_j \). Note that \( \bar{\theta}_j \) is the unique singular point of \( \bar{F} \) in \( P \) such that \( g(\bar{\theta}_j) = \theta_j \). We can take around \( \bar{\theta}_j \), a neighbourhood \( U \) and coordinates \((z, w)\) such that \( \bar{F} \) is generated by \( z\partial / \partial z - w\partial / \partial w \) where \((R_{\bar{\theta}_j} \cup \{ \bar{\theta}_j \}) \cap U = \{ w = 0 \} \) and \( g^{-1}(\bar{\theta}_j) \cap U = \{ z = 0 \} \). As \( Y \) has a pole of order one along \( \{ z = 0 \} \), it follows that

\[
Y_{|\bar{F}} = \frac{\partial}{\partial z} - \frac{w}{z} \frac{\partial}{\partial w}
\]

is not complete. However, it extends on \( R_{\bar{\theta}_j} \cup \{ \bar{\theta}_j \}_l=0 \) as a complete vector field because \( g \) restricted to \( R_{\bar{\theta}_j} \) extends to \( R_{\bar{\theta}_j} \cup \{ \bar{\theta}_j \}_l=0 \) as a biholomorphism onto \( C_{\theta_j} \) and

\[
\left( g_{|\bar{F} \cup \{ \bar{\theta}_j \}_l=0} \right)^* Z_0|_{C_{\theta_j}} = Y_{|\bar{F} \cup \{ \bar{\theta}_j \}_l=0}.
\]

**Global holomorphic vector fields** [2]. The list of holomorphic vector fields with isolated singularities on compact complex surfaces is well known. In [2, Chapter 6] we can find the details when the surface is projective. In particular, for \( Z_0 \) on \( S \) we have one of the following possibilities:

(I) \( S \) has an elliptic fibration \( f : S \to B \), and \( Z_0 \) is a nontrivial holomorphic vector field on \( S \) tangent to the fibres of \( f \). Each fibre of \( f \) is a smooth elliptic curve which can be multiple, and outside multiple fibers \( f \) is a locally trivial fibration. Moreover \( Z_0 \) has an empty zero set.

(II) \( S = \mathbb{C}^2 / \Lambda \) is a 2-torus and \( Z_0 \) is a linear vector field on it, that is, the quotient of a constant vector field on \( \mathbb{C}^2 \).

(III) \( S \) is a \( \mathbb{CP}^1 \)-bundle over an elliptic curve \( \mathcal{E} \), and \( Z_0 \) is transverse to the fibers and projects on \( \mathcal{E} \) to a constant vector field. In this case \( Z_0 \) is the suspension of \( \rho \) via the representation \( \rho : \pi_1(\mathcal{E}) \to \text{Aut}(\mathbb{CP}^1) \) associated to the bundle structure, and it generates a Riccati foliation without invariant fibres and whose monodromy map is \( \rho \).

(IV) \( S \) is a rational surface, and up to a birational map we have \( Y = \mathbb{CP}^1 \times \mathbb{CP}^1 \) and \( Z_0 = v_1 \oplus v_2 \), where \( v_1 \) and \( v_2 \) are holomorphic vector fields on \( \mathbb{CP}^1 \).

In the course of the proof we will consider \( S \) in some cases as a differentiable manifold with a given Riemannian metric \( g \). If \((N, g)\) is a Riemannian manifold, we denote by \( d \) the distance given by the metric, and by \( B^d(p) \) the open ball centered at \( p \). For the basic notions of Riemannian geometry that will be used in the rest of the paper, see [16].

We will analyze the possible cases for \( Z_0 \) and \( S \). First note that Case (I) does not really occur since \( \mathcal{F} \) does not have a rational first integral.

### 3.1. Cases (II) and (III)

**Proposition 1.** If \( Z_0 \) and \( S \) are as in (II) or (III) any irreducible component of \( D \cup E \) is invariant by \( \mathcal{F} \).

**Proof.** Let \( D_0 \) be an irreducible component of \( D \cup E \) that is not invariant by \( \mathcal{F} \). There is a compact curve \( Q_0 \) (possibly singular) in \( S \) generically transversal to \( Z_0 \). It is enough to define \( Q_0 \) as one of the connected components of \( r^{-1}(s(D_0)) \). Note that \( s(D_0) \) is not a point.

**Case** (II). Let us take \( S \) as the quotient manifold \( \mathbb{C}^2 / \Lambda \). We identify \( \mathbb{C}^2 \) with \( \mathbb{R}^4 \), and \( \Lambda \) is an integral lattice of rank four.
Moreover, we can assume that and \( \mu(\mathbb{C}^2) = \mathbb{C}^2 / \Lambda \) denote the canonical submersion map. If we consider \( \mathbb{R}^4 \) with the usual euclidean \( g \), taking \( g' = \mu_* g \) as the metric on \( \mathbb{C}^2 / \Lambda \), the map \( \mu \) becomes a Riemannian covering map. We will denote by \( d \) and \( d' \) the distances in \( (\mathbb{R}^4, g) \) and \( (\mathbb{R}^4, g') \), respectively.

The vector field \( Z_0 \) is the projection by \( \mu \) of a constant vector field on \( \mathbb{C}^2 \), and thus its trajectories must be of the form \( \mu(L_t) \) where \( \{L_t\}_{t \in \mathbb{C}} \) is the family of lines parallel to a given direction. Note that \( Z_0 \) does not have singularities.

**Lemma 3.** There is a compact \( K \subset S \) such that \( Q_0 \subset K \).

**Proof.** Since \( Z_0 \) is complete and without singularities we can define

\[
K = \{ \varphi(T, z) | |T| \leq 1, \, z \in Q_0 \}
\]

(6)

where \( \varphi : \mathbb{C} \times S \to S \) is the complex flow of \( Z_0 \). If we apply the Flow Box Theorem to the points of \( Q_0 \) we easily deduce that \( Q_0 \subset K \).

We define the following function

\[
\alpha : \mathbb{C} \to [0, +\infty)
\]

\[
t \mapsto d(L_t, \mu^{-1}(Q_0)).
\]

(7)

**Remark 7.** \( \alpha \) is continuous. For any sequence \( \{t_n\}_{n \in \mathbb{N}} \subset \mathbb{C} \) converging to \( t_* \) as \( n \to \infty \), one sees that \( \alpha(t_n) \leq d(L_{t_n}, L_{t_*}) + \alpha(t_*) \) and \( \alpha(t_n) \leq d(L_{t_n}, L_{t_*}) + \alpha(t_n) \). Then, \( \lim_{n \to \infty} \alpha(t_n) = \alpha(t_*) \).

We will use that \( \alpha \) has the following property with respect to \( K \).

**Lemma 4.** \( \alpha(t) \neq 0 \) if and only if \( \mu(L_t) \cap K = \emptyset \).

**Proof.** If \( \alpha(t) \neq 0 \), it is clear from (6) that \( \mu(L_t) \cap K = \emptyset \). Otherwise \( \mu(L_t) \cap Q_0 \neq \emptyset \), which is not possible with our assumptions.

If \( \alpha(t) = 0 \), we suppose \( \mu(L_t) \cap K = \emptyset \) and obtain a contradiction.

**Fact 1.** There is \( \delta \in \mathbb{R}^+ \) such that \( d'(\mu(L_t), Q_0) \geq \delta \).

Otherwise we can define a sequence \( \{x_n\}_{n \in \mathbb{N}} \subset Q_0 \) converging to \( x_* \in Q_0 \) such that \( d'(\mu(L_t), x_n) < 1/n \) because \( Q_0 \) is compact and \( (\mathbb{R}^4, g') \) is complete. But it implies that for any ball \( \mathbb{B}_r'(x_n) \) there exists \( n(r) \in \mathbb{N}^+ \) such that \( \mathbb{B}_{1/n(r)}'(x_{n(r)}) \subset \mathbb{B}_r'(x_n) \), and hence \( \mathbb{B}_r'(x_*) \cap \mu(L_t) \neq \emptyset \), which contradicts our assumption \( \mu(L_t) \cap Q_0 = \emptyset \).

**Fact 2.** \( d(L_t, \mu^{-1}(Q_0)) \geq \delta \).

By contradiction, suppose that \( d(L_t, \mu^{-1}(Q_0)) < \delta \). Then there are \( z \in L_t \) and \( \bar{z} \in \mu^{-1}(Q_0) \) with \( d(z, \bar{z}) < \delta \). Note that \( \mu^{-1}(Q_0) \) is an analytic variety (non necessarily compact) of \( \mathbb{C}^2 \) and that \( \mu(z) \neq \mu(\bar{z}) \) by Fact 1. Let \( c \) be a segment from \( z \) to \( \bar{z} \). As \( \mu \) defines a local isometry from \( (\mathbb{R}^4, g) \) to \( (\mathbb{R}^4, g') \), we can take \( \mathbb{B}_r'(z_i) \subset \mathbb{R}^4, 0 \leq i \leq s \), centered at \( z_i \in c \), where \( z_0 = z \) and \( z_s = \bar{z} \), and in such a way that \( \mu \) restricted to each \( \mathbb{B}_{r_i}'(z_i) \) defines an isometry over its image. Moreover, we can assume that \( \mathbb{B}_{r_i}'(z_i) \cap \mathbb{B}_{r_j}'(z_j) \neq \emptyset \) if and only if \( j = i + 1 \), and thus fix \( s - 1 \) points \( z_{i, i+1} \) in these intersections. As the isometries preserve intrinsic distance, Fact 1 and the triangle inequality imply the following contradiction

\[
\delta > d(z, \bar{z}) = \sum_{i=1}^{s-1} d(z_i, z_{i,i+1}) + d(z_{i,i+1}, z_{i+1})
\]
Lemma 5. \( \mu(L_i) \cap Q_0 \neq \emptyset \) for any \( t \in \mathbb{C} \), and then \( \alpha \equiv 0 \).

**Proof.** Suppose, by contradiction, that \( \mu(L_i) \cap Q_0 = \emptyset \). It implies that \( \mu(L_i) \cap \hat{K} = \emptyset \). On the other hand, by Lemma 4, \( \alpha(t) \neq 0 \). Then \( \alpha^{-1}(0) \) is a closed set strictly contained in \( \mathbb{C} \), and if we take \( \hat{t} \) on its boundary we can fix a sequence \( \{t_n\}_{n \in \mathbb{N}} \) with \( \alpha(t_n) \neq 0 \) converging to \( \hat{t} \in \mathbb{C} \) with \( \alpha(\hat{t}) = 0 \). Note that \( \mu(L_i) \cap Q_0 \neq \emptyset \) due to \( \mu(L_i) \cap \hat{K} \neq \emptyset \) since \( \alpha(\hat{t}) = 0 \) (Lemma 4).

Let us take \( \tilde{x} \in \mu(L_i) \cap Q_0 \) with \( \mu(\tilde{x}) = \tilde{x} \), and set \( \{z_n\}_{n \in \mathbb{N}} \) converging to \( \tilde{z} \) with \( z_n \in L_{t_n} \). By continuity, \( \{\mu(z_n)\}_{n \in \mathbb{N}} \) must converge to \( \tilde{x} \). However, as \( z_n \in L_{t_n} \) for any \( n \), it holds \( \mu(z_n) \neq \mu(L_{t_n}) \cap \hat{K} \) since \( \alpha(t_n) \neq 0 \) (Lemma 4), what is a contradiction. Then \( \mu(L_i) \cap Q_0 \neq \emptyset \). \( \square \)

It follows from Remark 4 that \( \mathcal{M}_i \) is contained in a trajectory of \( Z_0 \). Hence there is \( L_{S_i} \) such that \( \mu(L_{S_i}) \supset \mathcal{M}_i \).

Lemma 6. \( \mu(L_{S_i}) \cap Q_0 = \{p_i\} \), where \( p_i \) is the unique point in \( \mu(L_{S_i}) \setminus \mathcal{M}_i \). In particular, \( \mu(L_{S_i}) \) and \( \mathcal{M}_i \) are respectively biholomorphic to \( \mathbb{C} \) and \( \mathbb{C}^* \).

**Proof.** Lemma 5 implies that \( \mu(L_{S_i}) \cap Q_0 \neq \emptyset \). Moreover it is clear that \( \mu(L_{S_i}) \cap Q_0 \subset \mu(L_{S_i}) \setminus \mathcal{M}_i \). It follows from Remark 4 that \( r^*(s_x(\tilde{X}_{\tilde{C}_z}))|_{\mathcal{M}_i} \) is complete and then it extends holomorphically by zeros on \( \mu(L_{S_i}) \setminus \mathcal{M}_i \). Since \( C_z \) is not algebraic, \( \mathcal{M}_i \) is biholomorphic to \( \mathbb{C}^* \) and \( \mu(L_{S_i}) \cap Q_0 = \mu(L_{S_i}) \setminus \mathcal{M}_i \) is a unique point \( p_i \). Thus \( \mu(L_{S_i}) \) must be biholomorphic to \( \mathbb{C} \). \( \square \)

Since the foliation defined by \( Z_0 \) on \( S \) has codimension 1 and it does not have singularities, the closure of \( \mu(L_{S_i}) \) in the open set \( U' \subset S \) of non-compact leaves, that we will denote by \( L' \), is a subvariety of real codimension 0, 1 or 2 [9, Théorème 1.4]. It holds \( U' = S \) and then \( L' \) is the closure of \( \mu(L_{S_i}) \) in \( S \). If there were one compact leaf \( J \), [9, Théorème 1.4] also assures that any non-compact leaf must accumulate \( J \). In particular \( \mu(L_{S_i}) \) accumulates \( J \). On the other side, as \( Q_0 \) cuts any leaf (Lemma 5), it must cut \( J \), and \( \mu(L_{S_i}) \) accumulates the points of \( J \cap Q_0 \), which is not possible since \( p_i \) is the unique point in \( \mu(L_{S_i}) \setminus \mathcal{M}_i \) (Lemma 6). Note that \( L' \) is \( S \) or a real compact subvariety of dimension three. If \( L' \) had real codimension 2, it would define a real compact subvariety of dimension two of \( S \) (\( L' \) is closed in \( S \)) containing \( \mu(L_{S_i}) \), which is a non-algebraic leaf. One concludes that \( \mu(L_{S_i}) \) must intersect infinitely many times \( Q_0 \), and then one obtains again a contradiction with Lemma 6.

**Case (III).** Let us consider \( S \) as a \( \mathbb{CP}^1 \)-bundle over an elliptic curve \( E \) with bundle projection \( p : S \to E \). The structure of \( S \) can be lifted as \( \mathbb{CP}^1 \)-bundle \( \tilde{S} \) over \( \mathbb{C} \) via the universal covering map \( \Gamma : \mathbb{C} \to E \); we can determine a complex surface \( \tilde{S} \), a holomorphic covering \( F : \tilde{S} \to S \) and a bundle projection \( \tilde{p} : \tilde{S} \to \mathbb{C} \) such that \( p \circ F = \Gamma \circ \tilde{p} \).

\[ \begin{array}{ccc} \tilde{S} & \xrightarrow{F} & S \\ \tilde{p} \downarrow & & \downarrow p \\ \mathbb{C} & \xrightarrow{\Gamma} & E \end{array} \]
Moreover as \( \mathbb{C} \) is contractible this \( \mathbb{CP}^1 \)-bundle is trivial. Thus \( \tilde{S} = \mathbb{C} \times \mathbb{CP}^1 \) and \( \tilde{p}(x, y) = x \) is the projection over the first factor.

**Lemma 7.** There is a holomorphic automorphism \( \sigma \) of \( \mathbb{C} \times \mathbb{CP}^1 \) such that \( \sigma^*(F^*Z_0) \) is the horizontal vector field.

**Proof.** It is clear that \( F^*Z_0 \) generates a Riccati foliation adapted to \( \tilde{p} \) and without invariant fibres. If \( \sigma(t, y) = \tilde{\phi}(t, 0, y) \), with \( \tilde{\phi} \) the complex flow of \( F^*Z_0 \), \( \sigma \) is bijective, since each trajectory of \( F^*Z_0 \) intersects each fibre of \( p \) in only one point, and \( \sigma(\mathbb{C} \times \{y\}) \) are the trajectories of \( F^*Z_0 \).

After **Lemma 7**, the trajectories of \( Z_0 \) are of the form \((F \circ \sigma)(L_t)\) where \( \{L_t\}_{t \in \mathbb{CP}^1} \) is now the family of lines \( L_t = \mathbb{C} \times \{t\} \).

**Remark 8.** As \( S \) is compact, \( S \) (as real manifold) admits a Riemannian metric \( g' \). Let us set \( \tilde{\mu} = F \circ \sigma \). The map \( \tilde{\mu} \) from \((\mathbb{R}^2 \times \mathbb{S}^2, \tilde{\mu}^* g') \) to \((S, g') \) is a local Riemannian isometry. But still more, \( F \) is a covering map and \( \sigma \) is a biholomorphism; hence \( F \circ \sigma \) is also a covering map and \( \tilde{\mu} \) is a Riemannian covering map. As \((S, g') \) is compact, it is complete, and \((\mathbb{R}^2 \times \mathbb{S}^2, \tilde{\mu}^* g') \) is complete. We will denote by \( d \) and \( d' \) the distances in \((\mathbb{R}^2 \times \mathbb{S}^2, \tilde{\mu}^* g') \) and \((S, g') \), respectively.

**The vector field** \( Z_0 \) **is complete and without zeros.** We will consider as in case (II) an irreducible component \( D_0 \) of \( D \cup E \) that is not invariant by \( \tilde{F} \), and the compact curve \( Q_0 \) (possibly singular) in \( S \) generically transversal to \( Z_0 \), defined by one of the connected components of \( r^{-1}(s(D_0)) \). As in **Lemma 3** we can determine a compact set \( K \subseteq S \) such that \( Q_0 \subset \tilde{K} \).

We will consider the continuous map (it follows as in **Remark 7**)

\[
\tilde{\alpha} : \mathbb{CP}^1 \to [0, +\infty),
\]

\[
t \mapsto d(L_t, \tilde{\mu}^{-1}(Q_0)).
\]  

(8)

Once we have fixed \( \tilde{\mu} = F \circ \sigma \), the complete metrics in **Remark 8**, the compact set \( K \) and the map \( \tilde{\alpha} \) as (8), we can prove similar Lemmas to Lemmas 3–6 of Case (II), where \( \mu \) and \( \alpha \) must be substituted by \( \tilde{\mu} \) and \( \tilde{\alpha} \) in their statements.

Let \( L_{s_i} \) be such that \( \tilde{\mu}(L_{s_i}) \supset \mathcal{M}_i \). Since \( p_1 \tilde{\mu}(L_{s_i}) : \tilde{\mu}(L_{s_i}) \to \mathcal{E} \) is a covering map, and \( \tilde{\mu}(L_{s_i}) \) is biholomorphic to \( \mathbb{C} \) (Lemma 6), \( \tilde{\mu}(L_{s_i}) \) must cover almost all the fibres of \( p \) infinitely many times. Let \( \kappa \in \mathcal{E} \) such that \( p^{-1}(\kappa) \) contains an infinite sequence of different points in \( p^{-1}(\kappa) \cap \tilde{\mu}(L_{s_i}) \). By compactness of \( p^{-1}(\kappa) \), the above sequence converges to \( q_1 \in p^{-1}(\kappa) \). Note that \( q_1 \) is a regular point of \( Z_0 \). If \( \tilde{\mu}(L_{s_i}) \) is the trajectory of \( Z_0 \) through \( q_1 \), \( \tilde{\mu}(L_{s_i}) \) must accumulate \( \tilde{\mu}(L_{s_i}) \) (flow-box theorem). On the other hand, \( \tilde{\mu}(L_{s_i}) \cap Q_{0} \neq \emptyset \) (**Lemma 5**) implies a contradiction with the fact that \( p_1 \) is the unique point in \( \tilde{\mu}(L_{s_i}) \\setminus \mathcal{M}_i \) (**Lemma 6**).

**Remark 9.** One can also obtain a contradiction by distinguishing several cases, according to the (abelian) monodromy \( \Gamma' \subset Aut(\mathbb{CP}^1) \). If \( \Gamma' \) has rank 1, then the non-algebraic leaves of \( Z_0 \) are isomorphic to \( \mathbb{C}^* \), and one gets a contradiction by using **Remark 5**, and the fact that the intersection with algebraic curves is nonempty. If \( \Gamma' \) has rank 2 then \( \Gamma' = \langle f, g \rangle \) with \( f(z) = \alpha z, g(z) = \beta z \) or \( f(z) = z + 1, g(z) = z + w \). In the first case (where, moreover, \( \alpha^n \beta^m \neq 1 \) for every \((m, n) \neq (0, 0) \)) the non-algebraic leaves are sufficiently dense to apply the same argument as in case (II). In the second case one can prove that every algebraic curve \( C \subset S \) different from the elliptic curve \( E = \{z = \infty\} \) must intersect \( E \), and from this fact it follows again that every non-algebraic trajectory of \( Z_0 \) intersects \( C \) infinitely many times. \( \square \)
3.2. Case (IV)

There is a birational transformation \( G : S \to \mathbb{CP}^1 \times \mathbb{CP}^1 \) sending \( Z_0 \) to \( G_*Z_0 = v_1 \oplus v_2 \), where \( v_1 \) and \( v_2 \) are holomorphic vector fields on \( \mathbb{CP}^1 \). The description of \( G \) can be found in [2, p. 87]. In particular, \( G \) is a finite sequence of birational transformations which are contractions of curves invariant by \( Z_0 \) or blowing-ups at zeros of \( Z_0 \). Hence the exceptional divisor of \( G \) does not meet \( \mathcal{M}_i \), and as a consequence \( G(\mathcal{M}_i) \) is biholomorphic to \( \mathcal{M}_i \). But still more, as \( \mathcal{M}_i \) supports a complete vector field according to Remark 4, we can define an entire curve \( f : \mathbb{C} \to G(\mathcal{M}_i) \).

In the absence of rational first integrals, we may assume that \( G \) is contained in \( B \).

Proof. Let \( A_0 \) be a component of \( D \cup E \) not invariant by \( \tilde{F} \), and \( Q_0 \) be the curve in \( S \) defined as in Proposition 1. If \( G_*Z_0 \) is as in (b) let us suppose that either there is one component \( D_j \) of \( D \cup E \) not invariant by \( \tilde{F} \) and different from \( D_0 \) or there is another component of \( r^{-1}(s(D_0)) \) different from \( Q_0 \).

Lemma 8. There exist an open set \( B \subset \mathbb{CP}^1 \times \mathbb{CP}^1 \) biholomorphic to \( \mathbb{C}^2 \) and an entire curve \( \tilde{f} : \mathbb{C} \to G(\mathcal{M}_i) \cap B \) tangent to \( G_*Z_0|_B \) whose image avoids at least three algebraic curves contained in \( B \).

Proof. We analyze the two cases:

- \( G_*Z_0 \) as in (a): Let \( B \) be \( \mathbb{CP}^1 \times \mathbb{CP}^1 \) minus \( \{ z = \infty \} \cup \{ w = \infty \} \). As \( \{ z = \infty \} \) and \( \{ w = \infty \} \) are invariant by \( G_*Z_0 \), \( G(\mathcal{M}_i) \subset B \). Note that \( G_*Z_0 \) on \( B \) is complete. If \( \tilde{f} = f, \tilde{f} : \mathbb{C} \to G(\mathcal{M}_i) \subset B \) is an entire map whose image avoids at least \( \{ z = 0 \}, \{ w = 0 \} \) and \( A_0 \cap B \).

- \( G_*Z_0 \) as in (b): Let \( B \) be \( \mathbb{CP}^1 \times \mathbb{CP}^1 \) minus \( \{ z = 0 \} \cup \{ w = 0 \} \). In this case \( \{ z = 0 \} \) is invariant by \( G_*Z_0 \) but \( \{ w = 0 \} \) is not. Remark that any non-algebraic trajectory of \( G_*Z_0 \) is of type \( \mathbb{C} \) and intersects \( \{ w = c \} \), with \( c \neq \infty \), in a unique point. More still, one can suppose that \( A_0 \neq \{ \tilde{w} = \tilde{0} \} \). Otherwise one defines \( Q_0 \) as any other component of \( r^{-1}(s(D_0)) \) or \( r^{-1}(s(D_j)) \), where \( D_j \) is a component of \( D \cup E \) not invariant by \( \tilde{F} \) and \( D_j \neq D_0 \).

(b.1) If \( L_z \setminus G(\mathcal{M}_i) = \emptyset \), we take \( G(\mathcal{M}_i) \cap \{ w = 0 \} = p \) and the trajectory \( G(\mathcal{M}_i) \cap B = L_z \setminus \{ p \} \simeq \mathbb{C}^* \) of \( G_*Z_0 \) on \( B \). As the universal covering of \( G(\mathcal{M}_i) \cap B \) is \( \mathbb{C} \), there exists \( \tilde{f} : \mathbb{C} \to G(\mathcal{M}_i) \cap B \) whose image avoids at least the algebraic curves: \( \{ z = \infty \}, \{ w = \infty \} \) and \( A_0 \cap B \).

(b.2) If \( L_z \setminus G(\mathcal{M}_i) = q \in \{ w = 0 \} \), the argumentation is similar to (b.1) since \( G(\mathcal{M}_i) \cap B = G(\mathcal{M}_i) = L_z \setminus \{ q \} \simeq \mathbb{C}^* \) is a trajectory of \( G_*Z_0 \) on \( B \).
(b.3) If \( L_\cdot \setminus G(\mathcal{M}_i) = q \notin \{ w = 0 \} \), we take the automorphism of \( \mathbb{CP}^1 \times \mathbb{CP}^1 \), \( (z, w) \mapsto \delta(z, w) = (z, w - q_2) \), where \( q = (q_1, q_2) \). As \( \delta \) leaves invariant \( G_*Z_0 \) since \( \delta_*G_*Z_0 = G_*Z_0 \) and \( \delta(q) \in \{ w = 0 \} \), it is enough to apply (b.2) to \( L_\cdot \setminus \delta(G(\mathcal{M}_i)) = \{ \delta(q) \} \). \( \square \)

Let \( \mathbb{CP}^2 \) be the compactification of \( B \). The image of \( \tilde{f} : \mathcal{C} \rightarrow G(\mathcal{M}_i) \cap B \) is contained in \( \mathbb{CP}^2 \) minus at least four hypersurface sections, that is, three sections defined by the algebraic curves of Lemma 8 along with the line at infinity \( \mathbb{CP}^2 \setminus B \). According to Green’s Theorem [11, p.199], \( \tilde{f}(\mathcal{C}) \) must be contained in some algebraic curve, which contradicts our assumptions. Hence (1), (2) and (3) of the statement of Proposition follows.

Note that \( \mathcal{C}_z, \mathcal{C}_z, \mathcal{C}_z, G(\mathcal{M}_i) \) and \( \mathcal{M}_i \) are biholomorphic to \( \mathbb{C}^* \), and that \( L_\cdot \simeq \mathbb{C} \) and \( G(\mathcal{M}_i) = L_\cdot \setminus \{ q \} \simeq \mathbb{C}^* \) with \( q \in \{ w = 0 \} \setminus \{ (0, 0), (\infty, 0) \} \). \( G(\mathcal{M}_i) \) has two parabolic ends, which are properly embedded in the complementary set of \( \{ z = \infty \} \cup \{ w = \infty \} \) in \( \mathbb{CP}^1 \times \mathbb{CP}^1 \): one \( \Sigma_1 \) defined by a punctured disk centered at \( q \), that is algebraic; and the other \( \Sigma_2 \) defined by \( G(\mathcal{M}_i) \setminus \Sigma_1 \), that is transcendental and accumulates \( \{ w = \infty \} \). Note that \( G_*Z_0 \) has two saddle-nodes as singularities: one at \( (0, \infty) \), with a strong separatrix inside \( \{ w = \infty \} \) and a weak separatrix inside \( \{ z = 0 \} \); and the other one at \( (\infty, \infty) \), with a strong separatrix inside \( \{ w = \infty \} \) and a weak separatrix inside \( \{ z = \infty \} \). On the other hand \( G_*Z_0 \) defines a Riccati foliation adapted to \( \beta(z, w) = w \). One may assume (maybe after blowing-up reduced singularities) that \( \tilde{F} \) is Riccati with respect to \( \beta_W = \beta \circ G \circ g \) and that \( G \circ g \) is the contraction of curves inside fibers of \( \beta_W \) that produces the local models of [4, p.439]. In this case all the fibers \( \{ w = c \} \), with \( c \neq \infty \), are transversal minus one that is tangent, \( \{ w = \infty \} \), and of class (d).

Since \( h \) is an algebraic covering map from \( W \) to \( M \), the proper mapping theorem allows one to define the trace of \( \beta_W \) as a rational function \( \beta_M \) on \( M \) [10]. Moreover, one can assume that \( \beta_M \) is a fibration after eliminating its base points. Recall that the property of being reduced is stable by blowing ups. Moreover, the possible dicritical components of the resolution of the pencil given by \( \beta_M \) must be transversal to the corresponding foliation.

By construction, the generic fiber \( F \) of \( \beta_M \) is a curve transverse to \( \tilde{F} \). Note that \( D_0 \) must be contained in a fiber \( F_0 \) of \( \beta_M \) as a consequence of (3) in the statement of this Proposition. Let us consider the following cases according to the genus of \( F \).

- If \( F \) is of genus \( \geq 2 \), it follows from [15, Theorem III.6.1] that \( \tilde{F} \) has a rational first integral, which is not possible.
- If \( F \) is of genus 1, \( \tilde{F} \) is a Turbulent foliation. Let us see that this case neither occurs because it would imply the existence of a rational first integral as before. Indeed, note that \( F \) does not cut \( F_0 \) since \( \beta_M \) is a fibration. On the other hand, \( D \cap F \neq \emptyset \) by the maximum principle. As \( D_0 \) is the unique irreducible component of \( D \cup E \) that is not invariant by \( \tilde{F} \), there must be one \( \tilde{F} \)-invariant component \( D_2 \) of \( D \) such that \( D_2 \cap F \neq \emptyset \). The existence of \( D_2 \) implies that \( \tilde{F} \) has a first integral (Lemma 1).
- If \( F \) is of genus 0, \( \tilde{F} \) is a Riccati foliation. Let us see that \( \tilde{F} \) satisfies (4) of the statement of Proposition 2. After contraction of rational curves each fiber of \( \beta_M \) admits one of the five possible models in [3, Section 7], [4, p.439]. If there is one fiber \( F_1 \) tangent to \( \tilde{F} \), as \( D_0 \) is the unique irreducible component of \( D \cup E \) which is not invariant by \( \tilde{F} \) and it is contained in \( F_0 \), then \( F \) must cut \( D \) in only one or two points near \( F_0 \). Then we can conclude as in Lemma 2 that \( \beta_M \) projects by \( \pi \) as a rational function \( R \) of type \( \mathbb{C} \) or \( \mathbb{C}^* \). Finally, one shows that all the fibers of \( \beta_M \) are not transverse to \( \tilde{F} \). In the contrary case, if \( L_0 \) is the leaf defined by \( \mathcal{C}_z \), as the covering map \( \beta_M|_L : L \to \mathbb{CP}^1 \) is not finite (otherwise \( L \) is compact and \( C_z \) is algebraic), \( L \) must cut infinitely many times \( F_0 \) and \( C_z \) is not isomorphic to \( \mathbb{C}^* \), which is not possible. \( \square \)
After Proposition 2 we can assume that any irreducible component of \( D \cup E \) is invariant by \( \tilde{\mathcal{F}} \). Otherwise Theorem 1 follows by the results of Section 2.

3.3. Existence of a second integral

Let us come back to the beginning of Section 3, and consider (4).

Lemma 9. It holds \( Y^2F = 0 \). In particular \( \bar{X} \) is complete on the Zariski open set \( W' \) of \( W \) defined by \( W \setminus ((F = 0) \cup \{F = \infty\} \cup P) \).

Proof. Let \( \mathcal{R}_0 \) be a connected component of \( h^{-1}(\bar{C}_z) \). As \( \mathcal{R}_0 \) does not meet the exceptional divisor of \( s : M \to \hat{M} \) then \( h_{|\mathcal{R}_0} : \mathcal{R}_0 \to \bar{C}_z \) is a non-ramified finite covering map. Hence \( h^*_0(\bar{X}_{|\mathcal{R}_0}) = \bar{X}_{|\mathcal{R}_0} \) is complete. On the other hand \( \bar{X} \) and \( Y \) are tangent on \( \mathcal{R}_0 \) according to (4). Thus \( \mathcal{R}_0 \) is a Riemann surface contained in a trajectory \( R_z \) of \( Y \). Let \( \varphi_z : \Omega_z \to R_z \) be the corresponding solution. We have two possibilities from 3.- of Remark 5:

(i) \( R_z \notin \{R_{z_j}\}_{j=1}^l \). Since \( Y_{|R_z} \) is complete \( \Omega_z = \mathbb{C} \).

(ii) \( R_z \in \{R_{z_j}\}_{j=1}^l \). Let us suppose that \( R_z = R_{z_j} \). We take the solution \( f_{z_j} : \mathbb{C} \to R_{z_j} \cup \{ar{\theta}_h\}_{h=0}^{h_f} \) of (5) and the discrete subset \( \Delta = \{ f_{z_j}^{-1}(\bar{\theta}_h) \}_{h=0}^{h_f} \) of \( \mathbb{C} \). Since \( f_{z_j}|\mathbb{C}\setminus\Delta = \varphi_z \) then \( \Omega_z = \mathbb{C} \setminus \Delta \).

It follows from (i) and (ii) that \( \varphi_z \) is a univaluated holomorphic map. Let us note that \( \bar{X}_{|R_z} \) must be also complete because \( \bar{X}_{|\mathcal{R}_0} \) is complete. Using these two facts and that \( T \in \Omega_z \mapsto \varphi_z(T) \in R_z \) is a covering map then

\[
\varphi_z^*(\bar{X}_{|R_z}) = \varphi_z^*(\bar{X})(T) = (F \circ \varphi_z(T)) \cdot \varphi_z^*(Y) = (F \circ \varphi_z(T)) \frac{\partial}{\partial T}
\]

is a complete vector field on \( \Omega_z \). This is only possible if \( \Omega_z = \mathbb{C} \) or \( \mathbb{C}^* \) and \( (F \circ \varphi_z)(T) = aT + b \), for \( a, b \in \mathbb{C} \). We conclude that \( Y(F(\varphi_z(T))) = (F \circ \varphi_z)'(T) \) is constant and hence \( Y^2F \) vanishes along \( R_z \), which can be assumed to be non-algebraic since \( C_z \) is by hypothesis. Hence \( Y^2F = 0 \).

Let us take a point \( z \in W' = W \setminus ((F = 0) \cup \{F = \infty\} \cup P) \). If \( S_z \) is the trajectory of \( \bar{X} \) through \( z \), as \( Y \) is holomorphic on \( W' \) (1.- of Remark 5) and tangent to \( \bar{X} \) on \( S_z \) by (4) then \( S_z \) defines a trajectory \( R_z \) of \( Y \). Since it holds (9), the fact that \( Y^2F = 0 \) implies that \( \bar{X}|S_z \) is complete. \( \square \)

After Lemma 9, \( \bar{X} \) is complete on \( W' = W \setminus ((F = 0) \cup \{F = \infty\} \cup P) \). According to 1.- and 2.- of Remark 5, \( \bar{X} \) is complete on \( M \setminus h(W \setminus W') \). By Propositions 1 and 2, \( \bar{X} \) is complete on \( M \setminus (h(W \setminus W') \cup E \cup D) \). If we project by \( \pi \) we see that \( X \) is complete on a Zariski open set of \( \mathbb{C}^2 \) and it can be extended to \( \mathbb{C}^2 \) as a complete vector field. Therefore \( X \) is complete.

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