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On Measures of "Useful" Information*

BHU DEV SHARMA,† JAGDISH MITTER,‡ AND MAN MOHAN††

† *Faculty of Mathematics, University of Delhi, Delhi-110007, India,*
‡ *Department of Electrical Engineering, Mosul College of Engineering, Mosul, Iraq, and*
†† *Department of Mathematics, Ramjas College, University of Delhi, Delhi-110007, India*

The quantitative–qualitative measure of information as given by Belis and Guiaşu is additive, the additivity being a modification of the additivity of Shannon's measure with due place for utilities of the scheme in this property. A characterization of Belis and Guiaşu's measure depending upon the additivity postulate has been provided. The additivity can be relaxed, and there can be several ways of choosing a nonadditive law in place of additivity. Starting from a particular type of nonadditivity relation we characterize a measure of nonadditive "useful" information, which may be considered as a quantitative-qualitative measure corresponding to the Havrda–Charvat–Vajda–Daróczy entropy of degree β.

1. INTRODUCTION

Considering transmitted signals of a communication system as random abstract events, the quantitative aspect of information is developed based upon the probabilities of occurrence of these events. For such a discrete system, an *information scheme* may be represented in the form

$$S = \begin{bmatrix} E_1, E_2, \ldots, E_n \\ p_1, p_2, \ldots, p_n \end{bmatrix} = \begin{bmatrix} \mathscr{E} \\ \mathscr{P} \end{bmatrix}, \tag{1.1}$$

where $\mathscr{E} = (E_1, E_2, \ldots, E_n)$ is the family of events with respect to some random experiment and $\mathscr{P} = (p_1, p_2, \ldots, p_n)$, $\sum_{i=1}^{n} p_i = 1$, is the probability distribution defined over it.

Shannon's measure of information (Shannon and Weaver, 1949) for the information scheme (1.1) is given by

$$I(\mathscr{P}) = -\sum_{i=1}^{n} p_i \log p_i, \qquad \sum_{i=1}^{n} p_i = 1. \tag{1.2}$$

This quantity, in some sense, measures the amount of information contained

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in the scheme. However, this measure does not take into account the effectiveness or importance of the events. This is so because (1.2) depends only on the probabilities associated with the events of the information scheme. In a practical situation of probabilistic nature, quite often subjective considerations get involved with the study. These considerations take into account the effectiveness of the outcomes. Motivated by this idea, Belis and Guiaşu (1968) introduced a "utility distribution" $\mathscr{U} = (u_1, u_2, \ldots, u_n)$, where each u_i is a nonnegative real number accounting for the utility of the occurrence of the ith event, with the probability scheme $\mathscr{P}$.

As a consequence, studies may be based on what may be called the "Utility Information Scheme" given by

$$S^* = \begin{bmatrix} E_1, E_2, \ldots, E_n \\ u_1, u_2, \ldots, u_n \\ p_1, p_2, \ldots, p_n \end{bmatrix} = \begin{bmatrix} \mathscr{E} \\ \mathscr{U} \\ \mathscr{P} \end{bmatrix}. \tag{1.3}$$

It is worthwhile to mention here that the utility u_i of an event is in general independent of its probability of occurrence p_i.

The "quantitative–qualitative" measure of information (called "useful information" by Guiaşu and Picard, 1971) for the utility information scheme (1.3), as obtained by Belis and Guiaşu (1968), is given by

$$I(\mathscr{U}; \mathscr{P}) = -\sum_{i=1}^{n} u_i p_i \log p_i, \qquad u_i \geqslant 0, \sum_{i=1}^{n} p_i = 1. \tag{1.4}$$

It may be observed that $I(\mathscr{U}; \mathscr{P})$ reduces to (1.2) when the utilities are all ignored (which can be achieved by taking $u_i = 1$ for each i).

In a recent work Sharma, Mohan, and Mitter (1977b) have suggested some applications of (1.4) in the analysis of business and accounting data, by assigning utilities to the items in a balance sheet, so as to represent the user's preference pattern associated with the accounting names. It has been discussed there in some detail that this approach seems to be largely free from one of the deficiencies that led to criticism of Lev's approach (Lev, 1968).[1] Recently, Sharma, Mohan, and Mitter (1977a) and Sharma and Mohan (1978, 1978a) obtained moments of a maximum likelihood estimate of (1.4). Applications of measure (1.4) have also been made recently in the Theory of Questionnaires by Guiaşu and Picard (1971), Picard (1972), and Bouchon (1976). In support of the role that "useful average code length" in the work of Guiaşu and Picard (1971) may find, Longo

[1] Lev has suggested the use of Shannon's measure of information (1.2) in measuring the loss of information due to aggregation in financial statements. One of the basic criticisms (see Bernhardt and Copeland, 1970; Horowitz and Horowitz, 1976) in applying Lev's approach is that all items in a balance sheet are given equal importance irrespective of their accounting names.

(1976) has given a variety of interesting interpretations, for some well-suited practical situations. The utility information scheme imbued with operational insight is likely to prove more applicable to other areas also. Interestingly, the first applications of information theoretic concepts in the Theory of Games have come about using measure (1.4) only (see Guiaşu, 1970).

Coming back to (1.2), the Shannon measure is additive in the sense that

$$I(\mathscr{P}*\mathscr{Q}) = I(\mathscr{P}) + I(\mathscr{Q}), \tag{1.5}$$

where $\mathscr{P} = (p_1, p_2, ..., p_n)$, $0 < p_i \leqslant 1, \sum_{i=1}^{n} p_i = 1$; $\mathscr{Q} = (q_1, q_2, ..., q_m)$, $0 < q_j \leqslant 1, \sum_{j=1}^{m} q_j = 1$; and

$$\mathscr{P}*\mathscr{Q} = \{p_i q_j \mid p_i \in \mathscr{P} \wedge q_j \in \mathscr{Q}\}.$$

A modification of (1.5) for utility schemes would require defining the utility distribution associated with the product distribution $\mathscr{P}*\mathscr{Q}$ in terms of the individual utility distributions associated with $\mathscr{P}$ and $\mathscr{Q}$, respectively. Let $\mathscr{U} = (u_1, u_2, ..., u_n)$ and $\mathscr{V} = (v_1, v_2, ..., v_m)$ be the utility distributions associated with $\mathscr{P}$ and $\mathscr{Q}$, respectively. Then, if

$$\mathscr{U}*\mathscr{V} = \{u_i v_j \mid u_i \in \mathscr{U} \wedge v_j \in \mathscr{V}\}$$

is the utility distribution associated with $\mathscr{P}*\mathscr{Q}$, it can easily be seen that (1.4) satisfies

$$I(\mathscr{U}*\mathscr{V}; \mathscr{P}*\mathscr{Q}) = \bar{\mathscr{V}} I(\mathscr{U}; \mathscr{P}) + \bar{\mathscr{U}} I(\mathscr{V}; \mathscr{Q}), \tag{1.6}$$

where

$$\bar{\mathscr{U}} = \sum_{i=1}^{n} u_i p_i, \qquad \text{and} \qquad \bar{\mathscr{V}} = \sum_{j=1}^{m} v_j q_j.$$

With ever increasing applications of the informational approach, subadditivity rather than additivity is becoming an acceptable basis. In social, psychological, chemical, and biological systems additivity does not quite prevail (this is probably not true for rational, logical, and mechanical systems). For instance, in chemical systems a catalytic agent plays a role that cannot be accounted for by considering the additivity of the effects of individual reactions only. In biological systems the interactions between various drugs call for nonadditivity of the individual effects. The same is true of the interactions between treatments at various levels in the design of experiments. Our decision-making process has therefore to consider a suitable model based on a suitable modification of additivity. We consider one such simple model.

Various generalizations of (1.2) satisfying the nonadditivity relation

$$I(\mathscr{P}*\mathscr{Q}) = I(\mathscr{P}) + I(\mathscr{Q}) + K \cdot I(\mathscr{P}) \cdot I(\mathscr{Q}),$$

where K is some constant, have been studied by Havrda and Charvat (1967), Vajda (1968), Daróczy (1970), Sharma and Mittal (1975), and Picard (1976). A study of the nonadditive generalization of (1.6) has been made recently by Emptoz (1976).

The purpose of this paper is to give an axiomatic characterization of the "useful information" depending upon property (1.6) and a generalization of this property, to which we have referred as the nonadditivity property. In Section 2, we characterize (1.4) for the generalized form of utility information scheme (1.3), for which $\sum_{i=1}^{n} p_i \leqslant 1$. In Section 3, a nonadditivity property is formulated, and a measure of "useful" information satisfying that property has been characterized. The nonadditive measure is the counterpart of the Havrda–Charvat–Vajda–Daróczy nonadditive generalization of Shannon measure, and contains a parameter.

2. Measure of Useful Information

The occurrence of a single event removes a double uncertainty, the quantitative one related to its probability of occurrence p and the qualitative one related to its utility u with respect to some goal. Thus the useful information provided by the occurrence of the event must be a function of both variables p and u. To indicate this, we denote useful information, conveyed by an event whose probability of occurrence is p and utility is u, by $I(u; p)$.

We now determine the form for $I(u; p)$ under the following two postulates:

p-1 (Additivity). Given two independent events E_1, E_2 with probabilities p_1, p_2 and utilities u_1, u_2, their joint occurrence with probability $p_1 p_2$ and utility $u_1 u_2$ provides useful information which is a sum of two parts, each part being the useful information conveyed by one event multiplied by the utility of the other, i.e.,

$$I(u_1 u_2 ; p_1 p_2) = u_2 I(u_1 ; p_1) + u_1 I(u_2 , p_2), \tag{2.1}$$

where $u_i > 0$, $0 < p_i \leqslant 1$ for $i = 1, 2$.

p-2 (Nonnegativity). The measure of useful information with positive utility is nonnegative, and no useful information is conveyed by the ocurrence of an event which is sure to occur whatever its utility be, i.e.,

$$I(u; p) \geqslant 0 \qquad \text{for all} \quad u > 0, \qquad 0 < p \leqslant 1,$$

and

$$I(u; p) = 0 \qquad \text{whenever} \quad p = 1. \tag{2.2}$$

Theorem 2.1. *The useful information $I(u; p)$ conveyed by the occurrence of* *a single event with probability p and utility u ($u > 0$, $0 < p \leqslant 1$) satisfying Postulates* p-1 *and* p-2 *can be only of the form*

$$I(u; p) = -ku \log p, \tag{2.3}$$

where k is an arbitrary nonnegative constant.

Proof. Since $u_1 u_2 \neq 0$, from (2.1) we get

$$\frac{I(u_1 u_2\,;\,p_1 p_2)}{u_1 u_2} = \frac{I(u_1\,;\,p_1)}{u_1} + \frac{I(u_2\,;\,p_2)}{u_2}$$

or

$$F(u_1 u_2\,;\,p_1 p_2) = F(u_1\,;\,p_1) + F(u_2\,;\,p_2), \tag{2.4}$$

where

$$F(u_i\,;\,p_i) = I(u_i\,;\,p_i)/u_i \qquad \text{for all} \quad i = 1, 2.$$

Now letting $F(u_i\,;\,p_i) = G(\log u_i\,;\,\log p_i)$ for $i = 1, 2$, we can rewrite (2.4) as

$$G(\log u_1 u_2\,;\,\log p_1 p_2) = G(\log u_1\,;\,\log p_1) + G(\log u_2\,;\,\log p_2)$$

or

$$G(\log u_1 + \log u_2\,;\log p_1 + \log p_2) = G(\log u_1\,;\log p_1) + G(\log u_2\,;\log p_2). \tag{2.5}$$

The solution of (2.5) under the nonnegativity postulate (see Aczél, 1966, p. 215) is given by

$$G(\log u_1\,;\,\log p_1) = c_1 \log u_1 + c_2 \log p_1\,,\ p_1 > 0,\ u_1 > 0,$$

where c_1 and c_2 are arbitrary constants such that G is nonnegative.

Thus

$$F(u_1\,;\,p_1) = c_1 \log u_1 + c_2 \log p_1\,,\ p_1 > 0,\ u_1 > 0,$$

and hence

$$I(u_1\,;\,p_1) = c_1 u_1 \log u_1 + c_2 u_1 \log p_1\,,\ p_1 > 0,\ u_1 > 0, \tag{2.6}$$

where c_1 and c_2 are constants as specified earlier.

Finally, invoking p-2, it follows from (2.6) that $c_1 = 0$ and c_2 is a nonpositive arbitrary real number, say $c_2 = -k$, where k is an arbitrary nonnegative constant.

Thus we get

$$I(u_1\,,\,p_1) = -ku_1 \log p_1$$

or

$$I(u, p) = -ku \log p, \text{ since } u_1\,,\,p_1 \text{ are arbitrary.} \quad \blacksquare$$

We refer to (2.3) as *useful self-information* conveyed by an event whose probability of occurrence is p and utility is u ($p, u \neq 0$).

Remarks. In deriving the above result, we have taken $u_i > 0$. This was to avoid the obvious mathematical complexities because of the presence of $\log u$ in the proof. However, if (2.1) is supposed to hold also for u_1 or u_2 equal to zero, then (2.3) remains valid for $u = 0$, which conforms with the information theoretical interpretation that an event with zero utility has no useful information.

In Theorem 2.1, we obtained a measure of useful information conveyed by the occurrence of a single event E of a given utility information scheme. Now, we derive an expression for $I(\mathscr{U}; \mathscr{P})$, the useful information conveyed by the utility information scheme (1.3). The measure of information is, however, assumed to depend upon the probability and utility of the events and not upon the events as such. We derive the expression for $I(\mathscr{U}; \mathscr{P})$ under the following set of postulates for the generalized utility information schemes, viz.,

$$S = \begin{bmatrix} E_1, E_2, ..., E_n \\ u_1, u_2, ..., u_n \\ p_1, p_2, ..., p_n \end{bmatrix} = \begin{bmatrix} \mathscr{E} \\ \mathscr{U} \\ \mathscr{P} \end{bmatrix}, \qquad u_i > 0,\ 0 < p_i \leqslant 1,\ \sum_{i=1}^{n} p_i \leqslant 1, \tag{2.7}$$

and

$$T = \begin{bmatrix} F_1, F_2, ..., F_m \\ v_1, v_2, ..., v_m \\ q_1, q_2, ..., q_m \end{bmatrix} = \begin{bmatrix} \mathscr{F} \\ \mathscr{V} \\ \mathscr{Q} \end{bmatrix}, \qquad v_j > 0,\ 0 < q_j \leqslant 1,\ \sum_{j=1}^{m} q_j \leqslant 1, \tag{2.8}$$

where

$$\sum_{i=1}^{m} p_i + \sum_{j=1}^{m} q_j \leqslant 1.$$

P-1′ (Additivity). Given the utility information schemes S and T, the following holds

$$I(\mathscr{U}*\mathscr{V}; \mathscr{P}*\mathscr{Q}) = \bar{\mathscr{V}} I(\mathscr{U}; \mathscr{P}) + \bar{\mathscr{U}} I(\mathscr{V}; \mathscr{Q}), \tag{2.9}$$

where

$$\mathscr{U}*\mathscr{V} = \{u_i v_j \mid u_i \in \mathscr{U} \wedge v_j \in \mathscr{V}\},$$
$$\mathscr{P}*\mathscr{Q} = \{p_i q_j \mid p_i \in \mathscr{P} \wedge q_j \in \mathscr{Q}\},$$

and

$$\bar{\mathscr{U}} = \sum_{i=1}^{n} u_i p_i \Big/ \sum_{i=1}^{n} p_i, \qquad \bar{\mathscr{V}} = \sum_{j=1}^{m} v_j q_j \Big/ \sum_{j=1}^{m} q_j .$$

P-2′ (Nonnegativity). $I(\mathscr{U}; \mathscr{P})$ is nonnegative and vanishes whenever $\mathscr{P}$ degenerates, i.e., it is of the type (0, 0,..., 0, 1, 0,..., 0).

P-3′ (MEAN VALUE). For the utility information schemes S and T, the useful information conveyed by the information scheme $S \cup T$ is given by

$$I(S \cup T) = \frac{(\sum_{i=1}^{n} p_i)\, I(S) + (\sum_{j=1}^{m} q_j)\, I(T)}{\sum_{i=1}^{n} p_i + \sum_{j=1}^{m} q_j}, \tag{2.10}$$

where $I(S) = I(\mathscr{U}; \mathscr{P})$, $I(T) = I(\mathscr{V}; \mathscr{Q})$, and

$$S \cup T = \begin{bmatrix} E_1, E_2, \ldots, E_n, F_1, F_2, \ldots, F_m \\ u_1, u_2, \ldots, u_n, v_1, v_2, \ldots, v_m \\ p_1, p_2, \ldots, p_n, q_1, q_2, \ldots, q_m \end{bmatrix}.$$

THEOREM 2.2. *The useful information conveyed by the utility information scheme S satisfying Postulates* P-1′ *to* P-3′ *is given by*

$$I(\mathscr{U}; \mathscr{P}) = -k \sum_{i=1}^{n} u_i p_i \log p_i \Big/ \sum_{i=1}^{n} p_i, \qquad u_i > 0,\ \sum_{i=1}^{n} p_i \leqslant 1, \tag{2.11}$$

where k is an arbitrary nonnegative constant.

Proof. We may consider

$$S = S_1 \cup S_2 \cup S_3 \cdots \cup S_n,$$

where

$$S_i = \begin{bmatrix} E_i \\ u_i \\ p_i \end{bmatrix}, \qquad p_i > 0, \quad i = 1, 2, \ldots, n. \tag{2.12}$$

The Postulates P-1′ and P-2′ for S_i reduce to Postulates p-1 and p-2, respectively, and therefore from Theorem 2.1 we have

$$I(S_i) = -k u_i \log p_i, \qquad i = 1, \ldots, n,$$

where k is an arbitrary nonnegative constant. It now follows from Postulate P-3′ that

$$I(\mathscr{U}; \mathscr{P}) = I(S_1 \cup S_2 \cdots \cup S_n)$$
$$= -k \sum_{i=1}^{n} u_i p_i \log p_i \Big/ \sum_{i=1}^{n} p_i. \quad \blacksquare$$

Remarks. If $\sum_{i=1}^{n} p_i = 1$, i.e., if the information scheme is complete, then (2.11) reduces to the Belis and Guiaşu measure (1.4). Moreover, by fixing the units of useful information, the constant k can be determined. Thus if we take $I(1; \frac{1}{2}) = 1$, then $k = 1(\log_2 2)$, and

$$I(\mathscr{U}; \mathscr{P}) = -\sum_{i=1}^{n} u_i p_i \log_2 p_i. \tag{2.13}$$

As is usual, we define $0 \log 0 \triangleq 0$, and with this convention, measure (2.11) is defined even when some probabilities in a distribution $\mathscr{P}$ take the value zero.

3. Generalized Useful Information

In the previous section, we characterized the measure of useful information, based upon the additivity relation (1.6). A generalized additive useful information measure has also been suggested and used recently by Gurdial and Pessoa (1977). One may be interested in investigating the measure of useful information based upon some nonadditivity relation. Various generalizations of (1.6) can be called nonadditivity relations, but we consider the one given by

$$I(\mathscr{U}*\mathscr{V}; \mathscr{P}*\mathscr{Q}) = \mathscr{V}I(\mathscr{U}; \mathscr{P}) + \mathscr{U}I(\mathscr{V}; \mathscr{Q}) + KI(\mathscr{U}; \mathscr{P}) \cdot I(\mathscr{V}; Q), \quad (3.1)$$

where K is an arbitrary constant. Obviously, if $K = 0$, then (3.1) reduces to (1.6).

In order to determine the form of "generalized useful information" based upon the nonadditivity relation (3.1), the additivity postulate p-1 is replaced by the following nonadditivity postulate.

P-I (Nonadditivity). Given two independent events E_1, E_2 with probabilities p_1, p_2 and utilities u_1, u_2, their joint occurrence with probability $p_1 p_2$ and utility $u_1 u_2$ provides useful information, given by

$$I(u_1 u_2; p_1 p_2) = u_2 I(u_1; p_1) + u_1 I(u_2; p_2) + KI(u_1; p_1) \cdot I(u_2; p_2), \quad (3.2)$$

where $u_i > 0$, $0 < p_i \leqslant 1$, and $K \neq 0$ is an arbitrary constant.

Theorem 3.1. *The useful information $I(u; p)$ conveyed by the occurrence of an event with probability p and utility u $(0 < p \leqslant 1,\ u > 0)$ satisfying continuity w.r.t. its arguments, Postulates* P-I, p-2, *and the condition*

$$I(1; \tfrac{1}{2}) = 1, \quad (3.3)$$

can be only of the form

$$I(u; p) = u \left\{ \frac{p^c - 1}{2^{-c} - 1} \right\}, \quad (3.4)$$

where $c \neq 0$ is an arbitrary constant.

Proof. Since $u_1 u_2 \neq 0$, we have from (3.2),

$$F(u_1 u_2; p_1 p_2) = F(u_1; p_1) + F(u_2; p_2) + KF(u_1; p_1) F(u_2; p_2), \qquad K \neq 0, \quad (3.5)$$

where

$$F(u; p) = I(u; p)/u. \tag{3.6}$$

We rewrite (3.5) as

$$H(x_1 + x_2 ; y_1 + y_2) = H(x_1 ; y_1) \cdot H(x_2 ; y_2), \tag{3.7}$$

where $x_i = \log u_i$, $y_j = \log p_j$, for $i, j = 1, 2$, and

$$H(x; y) = 1 + Ke^{-x}I(e^x; e^y), \qquad K \neq 0 \ (x, y \in R). \tag{3.8}$$

Then it follows from the continuity assumption, (3.7), and (3.8) itself that $H(x; y)$ is positive. Indeed, by (3.7) we have

$$H(x; y) = [H(x/2; y/2)]^2 \geqslant 0 \qquad (x \geqslant 0, y \geqslant 0). \tag{3.9}$$

It is well known that every solution of (3.9) is either everywhere or nowhere 0 (for nonnegative arguments). The first possibility is excluded by (3.8), so $H(x; y) > 0$ for $x \geqslant 0$, $y \geqslant 0$. This can easily be extended to $x < 0$ and $y < 0$ by $H(0; y) = H(x; y) \cdot H(-x; 0)$ and $H(x; 0) = H(x; y) \cdot H(0; -y)$, respectively.

Thus from (3.7) we have

$$L(x_1 + x_2 ; y_1 + y_2) = L(x_1 ; y_1) + L(x_2 ; y_2), \tag{3.10}$$

where

$$L(x; y) = \log H(x; y), \qquad x, y \in R. \tag{3.11}$$

The continuous solution of (3.10) (see Aczél, 1966) is

$$L(x_1 ; y_1) = c_1 x_1 + c_2 y_1 , \tag{3.12}$$

where c_1, c_2 are arbitrary constants.

Now it follows from (3.12), (3.11), and (3.8) that the continuous solution to (3.2) is

$$I(u_1 ; p_1) = u_1 \left\{ \frac{u_1^{c_1} p_1^{c_2} - 1}{K} \right\}, \qquad K \neq 0, \tag{3.13}$$

where c_1, c_2 are arbitrary constants. (In fact c_2 and K are of opposite sign as follows from the normality used below.)

Now using Postulate p-2 in (3.13) we get $c_1 = 0$, and thus we have

$$I(u_1 ; p_1) = u_1 \left\{ \frac{p_1^{c_2} - 1}{K} \right\}, \qquad K \neq 0. \tag{3.14}$$

Invoking condition (3.3), (3.14) gives $K = 2^{-c_2} - 1$, and thus

$$I(u_1\,;\,p_1) = u_1 \left\{\frac{p_1^{c_2} - 1}{2^{-c_2} - 1}\right\}.$$

Clearly, since $K \neq 0$, c_2 must be different from zero. As u_1, p_1, c_1 are arbitrary, we have thus proved that

$$I(u; p) = u \left\{\frac{p^c - 1}{2^{-c} - 1}\right\}, \qquad u > 0, 0 < p \leqslant 1,$$

where $c \neq 0$ is an arbitrary constant. ∎

Measure (3.4) is expected to give the additive self-information (2.4) when the constant K of (3.2) tends to zero, i.e., when $c \to 0$. To maintain the convention in the existing literature, it is desirable to take $c = \alpha - 1$, so that additivity can follow when $\alpha \to 1$. Thus we have

$$I(u; p) = u \left\{\frac{p^{\alpha-1} - 1}{2^{1-\alpha} - 1}\right\}, \qquad u > 0, p > 0, \alpha \neq 1, \tag{3.15}$$

and the nonadditivity postulate (3.2) then takes the form

$$I(u_1u_2\,;\,p_1p_2) = u_2I(u_1\,;\,p_1) + u_1I(u_2\,;\,p_2) + (2^{1-\alpha} - 1)\,I(u_1\,;\,p_1)\cdot I(u_2\,;\,p_2),$$
$$u_i > 0, p_i > 0, \alpha \neq 1. \tag{3.16}$$

We refer to (3.15) as the *generalized useful self-information of degree* α, conveyed by an event whose probability of occurrence is p and utility is u ($p, u > 0$).

To indicate the presence of a parameter α, we denote this expression by $I^\alpha(u; p)$. Further, by the same arguments as given in the remarks following Theorem 2.1, we may include the case $u_i = 0$ also, taking $I^\alpha(0, p) = 0$.

Now we determine the nonadditive form of useful information for the generalized utility information scheme (1.3). We do it by taking Postulates P-2′, P-3′, and the following two postulates, of which the first is nonadditivity and the second defines the unit.

P-I′ (NONADDITIVITY). For the generalized utility information schemes S and T, the following holds:

$$I(\mathscr{U}*\mathscr{V}; \mathscr{P}*\mathscr{Q}) = \mathscr{V}I(\mathscr{U}; \mathscr{P}) + \mathscr{U}I(\mathscr{V}; \mathscr{Q}) + (2^{1-\alpha} - 1)\,I(\mathscr{U}; \mathscr{P})\cdot I(\mathscr{V}; \mathscr{Q});$$
$$\alpha \neq 1, \tag{3.17}$$

where

$$\mathscr{U}*\mathscr{V} = \{u_iv_j \mid u_i \in \mathscr{U} \wedge v_j \in \mathscr{V}\},$$
$$\mathscr{P}*\mathscr{Q} = \{p_iq_j \mid p_i \in \mathscr{P} \wedge q_j \in \mathscr{Q}\},$$

and

$$\bar{\mathscr{U}} = \sum_{i=1}^{n} u_i p_i \Big/ \sum_i p_i , \qquad \bar{\mathscr{V}} = \sum_{j=1}^{m} v_j q_j \Big/ \sum_j q_j .$$

P-II′ (UNIT). Two equally likely events both having unit utility convey a unit amount of useful information, i.e.,

$$I(1, 1; \tfrac{1}{2}, \tfrac{1}{2}) = 1. \tag{3.18}$$

THEOREM 3.2. *The nonadditive useful information conveyed by the utility information scheme S satisfying the Postulates* P-I′, P-II′, P-2′, *and* P-3′ *can be only of the form*

$$I^\alpha(\mathscr{U}; \mathscr{P}) = \frac{\sum_{i=1}^n u_i p_i (p_i^{\alpha-1} - 1)}{(2^{1-\alpha} - 1) \sum_{i=1}^n p_i}, \qquad \alpha \neq 1, \quad \sum_{i=1}^{n} p_i \leqslant 1. \tag{3.19}$$

Proof. Considering as earlier

$$S = S_1 \cup \cdots \cup S_n ,$$

where

$$S_1 = \begin{bmatrix} E_i \\ U_i \\ P_i \end{bmatrix}, \qquad p_i > 0, i = 1, 2,..., n,$$

Postulates P-I′ and P-2′ for S_i reduce to Postulates P-I and p-2, respectively. Then from result (3.14) of Theorem 3.1, we have

$$I^\alpha(S_i) = u_i \left\{ \frac{p_i^c - 1}{K} \right\}, \qquad K \neq 0, \qquad \text{for} \quad i = 1, 2,..., n, \tag{3.20}$$

where c is an arbitrary constant.

Thus it follows from Postulate P-3′ that

$$I^\alpha(\mathscr{U}; \mathscr{P}) = \frac{\sum_{i=1}^n u_i p_i (p_i^c - 1)}{K(\sum_{i=1}^n p_i)}, \qquad \sum_{i=1}^{n} p_i \leqslant 1, K \neq 0, \tag{3.21}$$

where c is an arbitrary constant.

Invoking Postulate P-II′, (3.21) gives $K = 2^{-c} - 1$. Clearly since $K \neq 0$, c must be different from zero. Thus we have

$$I^\alpha(\mathscr{U}; \mathscr{P}) = \frac{\sum_{i=1}^n u_i p_i (p_i^c - 1)}{(\sum_{i=1}^n p_i)(2^{-c} - 1)}, \qquad \sum_{i=1}^{n} p_i \leqslant 1, c \neq 0. \tag{3.22}$$

Following arguments analogous to those for obtaining (3.15) we must choose $c = \alpha - 1$, and thus we finally obtain

$$I^{\alpha}(\mathscr{U}; \mathscr{P}) = \frac{\sum_{i=1}^{n} u_i p_i (p_i^{\alpha-1} - 1)}{(2^{1-\alpha} - 1)(\sum_{i=1}^{n} p_i)}, \qquad \sum_{i=1}^{n} p_i \leqslant 1,\ \alpha \neq 1. \quad \blacksquare \tag{3.23}$$

If $\sum_{i=1}^{n} p_i = 1$, i.e., when $\mathscr{P}$ is complete, then (3.23) reduces to

$$I^{\alpha}(\mathscr{U}; \mathscr{P}) = \frac{\sum_{i=1}^{n} u_i p_i (p_i^{\alpha-1} - 1)}{(2^{1-\alpha} - 1)}, \ \alpha \neq 1, \tag{3.24}$$

which is the useful nonadditive information corresponding to the entropy of degree α, studied by Havrda and Charvat (1967), Vajda (1968) and Daróczy (1970). We call (3.24) the *generalized useful information of degree* α.

It will be noted that

$$\lim_{\alpha \to 1} I^{\alpha}(\mathscr{U}; \mathscr{P}) = -\sum_{i=1}^{n} u_i p_i \log p_i ,$$

which is measure (2.13).

It will also be noted that $I^{\alpha}(\mathscr{U}; \mathscr{P})$ vanishes if at least one of u_i or p_i is zero for every i. This means that no useful information is supplied by a utility information scheme in which possible events are useless. Thus in order to have a nonvanishing measure of useful information, some possible events must be useful in the scheme. The convexity and branching properties of $I^{\alpha}(\mathscr{U}; \mathscr{P})$ were established recently by Sharma, Mohan, and Mitter (1978). In a recent work, Sharma and Mohan (1978b) obtained expressions for the mean and variance of an m.l.e. of $I^{\alpha}(\mathscr{U}; \mathscr{P})$. Also, recently, Mohan and Mitter (1978) derived some bounds for $I(\mathscr{U}; \mathscr{P})$ and $I^{\alpha}(\mathscr{U}; \mathscr{P})$.

The additive form (1.6) is a limiting case of the nonadditive form (3.17), and the parameter α controls the contribution of the nonadditive factor in Postulate P-I′. Various other interpretations to α can be given. The following is interesting from applicative point of view.

If we consider the ensemble of events E_i, probabilities p_i, and utilities u_i as a cybernetic system $E_i \mid u_i \mid p_i$ (see Belis and Guiaşu, 1968), then one can interpret the parameter α as a flexibility parameter or as a preassigned number associated with different cybernetic systems. For instance, two cybernetic systems with the same set of E_i, u_i, p_i may have different useful information (with respect to the same goal) for two different values of α. The parameter α may represent an environment factor, such as temperature, humidity. In a recent work Sharma, Mohan, and Dev (1976) have interpreted the parameter α as the parameter of "information consciousness" in aggregating financial accounts. Some applications of useful information have been pointed out in the beginning of the paper. Applications of generalized useful information in information theory need to be explored further.

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