A piezoelectric screw dislocation interacting with an imperfect piezoelectric bimaterial interface

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Abstract

A general method is presented for the analytical solution of a piezoelectric screw dislocation located within one of two joined piezoelectric half-planes. The bonding along the half-plane is considered to be imperfect with the assumption that the imperfect interface is mechanically compliant and dielectrically weakly (or highly) conducting. For a mechanically compliant interface tractions are continuous but displacements are discontinuous across the imperfect interface. In this context, jumps in the displacement components are assumed to be proportional to their respective interface traction components. Similarly, for a dielectrically weakly conducting interface the normal electric displacement is continuous but the electric potential is discontinuous across the interface. The jump in electric potential is proportional to the normal electric displacement. In contrast, for a dielectrically highly conducting interface the electric potential is continuous across the interface whereas the normal electric displacement has a discontinuity across the interface which is proportional to a certain differential expression of the electric potential. Explicit expressions are derived for the complex field potentials. The results show that there are two dimensionless parameters measuring the interface “rigidity” as compared to one for the purely elastic case. When the imperfect interface is compliant and weakly conducting, the two dimensionless parameters can be positive real values or complex conjugates with positive real parts. When the imperfect interface is compliant and highly conducting the two dimensionless parameters can only be positive real values. An expression for the image force acting on the screw dislocation due to its interaction with a compliant and weakly conducting interface is also given. It is found that the image force is only dependent on two dimensionless generalized Dundurs constants as well as two dimensionless parameters measuring the interface “rigidity”.

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1. Introduction

In the last few decades piezoelectric materials have evolved into a significant branch of modern engineering materials due to their role in many modern devices and composite structures. The favorable behavior of these
materials is a result of a unique electromechanical coupling effect which is not present in conventional elastic materials. The electroelastic interaction between a material interface and a defect (such as dislocations, cracks and inclusions) plays a significant role in studying the electromechanical coupling phenomenon; consequently, a considerable amount of study has been undertaken in this area (see, for example, Pak, 1990; Suo et al., 1992; Liu et al., 1999; Lee et al., 2000; Liu et al., 2004 among others).

In all the aforementioned works the interface is treated as perfectly bonded with tractions, displacements, normal electric displacement and electric potential assumed to be continuous across the interface. However, in reality, the perfect bonding condition is a convenient idealization of a more complex situation. In fact, this condition ignores the presence of interfacial damage such as imperfect adhesions and defects. To respond to this deficiency an analytical model of imperfect bonding has been developed. (for details see, for example, Benveniste and Miloh, 1986; Achenbach and Zhu, 1989; Hashin, 1991; Benveniste and Miloh, 1999; Sudak et al., 1999; Benveniste and Miloh, 2001 and references therein).

In the present work, we are interested in modeling the interaction between an imperfect interface separating two transversely isotropic piezoelectric half-planes and a screw dislocation located within one of the two half-planes. Two types of imperfect interface will be discussed. The first one is a mechanically compliant, dielectrically weakly conducting interface and the second one is also mechanically compliant but dielectrically highly conducting. For a mechanically compliant imperfect interface we adopt one of the more useful models that being the linear spring model (see, for example, Achenbach and Zhu, 1989; Sudak et al., 1999; Fan and Wang, 2003; Wang et al., in press). In this model, the tractions are continuous but the displacements are discontinuous across the interface. In this context, the displacement jumps are proportional, in terms of ‘spring-factor’ type parameters, to their respective tractions components. In the case of a dielectrically weakly conducting interface (Benveniste and Miloh, 1986; Chen, 2001; Fan and Sze, 2001) the normal electric displacement is continuous but electric potential is discontinuous across the interface. Thus, the jump in electric potential is proportional to the normal electric displacement. On the other hand, for a dielectrically highly conducting interface (Benveniste and Miloh, 1999; Chen, 2001) the electric potential is continuous across the interface but the normal electric displacement has a discontinuity across the interface which is proportional to a certain differential expression for the electric potential.

In Section 2 the foundations of the theory are discussed. By employing the complex variable formulation for the two different kinds of imperfect interface conditions, the original boundary value problem for the two different conditions can be discussed within the same framework. In Section 3 we derive a set of coupled first-order partial differential equations for the analytic function vector in the lower half-plane which is free of the action of the screw dislocation. By considering the eigenvalue problem certain orthogonal relationships for the eigenvectors are derived. These relations allow us to decouple the set of coupled partial differential equations into two independent first-order partial differential equations for two newly defined analytic functions whose solutions can be conveniently expressed in terms of the exponential integral. Explicit expressions are derived for the complex field potentials both in the upper and lower piezoelectric half-planes. In Section 4 the image force on the screw dislocation is calculated and several examples illustrating the imperfect interfaces are discussed.

2. Governing equations and boundary conditions

2.1. Preliminaries

Consider a screw dislocation located at a point \( z = i\delta, \ delta > 0 \) in the upper piezoelectric half-plane of a piezoelectric bimaterial. Both the upper and the lower half-planes, denoted by \( S_1 \) and \( S_2 \), respectively, are transversely isotropic with the poling direction parallel to the \( z \)-axis (Fig. 1). The screw dislocation is assumed to be straight and infinitely long in the \( z \)-direction suffering a displacement jump \( b \) and an electric potential jump \( \Delta \phi \) across the slip plane. The dislocation also has a line force \( p \) and line charge \( q \) along its core. Throughout the paper, unless otherwise specified, the subscripts 1 and 2 (or the superscripts (1) and (2)) are used to identify the respective quantities in \( S_1 \) and \( S_2 \).

For the antiplane problem considered, the governing field equations and constitutive equations can be simplified considerably and are given as follows.
where the comma followed by $x$ (or $y$) denotes the partial derivative with respect to $x$ (or $y$), $\sigma_{2x}$, $\sigma_{2y}$ are the shear stress components, $D_{x}$, $D_{y}$ are the electric displacement components, $E_{x}$, $E_{y}$ are the electric fields, $w$ is the out-of-plane displacement, $\phi$ is the electric potential, $c_{44}$, $e_{15}$ and $\varepsilon_{11}$ are, respectively, the longitudinal elastic modulus measured in a constant electric field, the piezoelectric modulus and the electric permittivity measured at a constant strain.

Let us assume that the interface between the two piezoelectric half-planes is imperfect. Then, as outlined in the introduction, the two different kinds of imperfect interfaces that are considered in this investigation are (1) a mechanically compliant, dielectrically weakly conducting interface and (2) a mechanically compliant, dielectrically highly conducting interface. Then the boundary condition for a compliant, weakly conducting imperfect interface is given by

\[
\begin{align*}
\sigma_{2x}^{(1)} + \sigma_{2y}^{(1)} &= 0, & D_{x}^{(1)} + D_{y}^{(2)} &= 0, \\
E_{x} &= -\phi_{x}, & E_{y} &= -\phi_{y}, \\
\begin{bmatrix}
\sigma_{2x}^{(1)} \\
D_{y}^{(1)} \\
\sigma_{2x}^{(2)} \\
D_{y}^{(2)}
\end{bmatrix}
&= 
\begin{bmatrix}
c_{44} & -e_{15} \\
e_{15} & \varepsilon_{11}
\end{bmatrix}
\begin{bmatrix}
w_{y}^{(1)} \\
E_{y}^{(2)}
\end{bmatrix},
\end{align*}
\]

where the two interface parameters $\alpha$ and $\beta$ are uniform and nonnegative constants. For instance, if $\alpha = \beta = 0$ Eq. (4) corresponds to a perfectly bonded interface; if $\alpha, \beta \to \infty$ Eq. (4) describes a completely debonded and charge-free (insulating) interface.

Similarly, the boundary conditions for a compliant, highly conducting interface are given by (Chen, 2001)

\[
\begin{align*}
\sigma_{2x}^{(1)} &= \sigma_{2y}^{(2)}, & E_{x}^{(1)} &= E_{x}^{(2)}, \\
\begin{bmatrix}
w_{y}^{(1)} - w_{y}^{(2)} = \alpha \sigma_{2y}^{(2)}, & \phi_{1}^{(1)} - \phi_{2}^{(2)} = -\beta D_{y}^{(2)}, & y = 0,
\end{align*}
\]

\[
\sigma_{2y}^{(1)} = \sigma_{2y}^{(2)}, & E_{x}^{(1)} = E_{x}^{(2)}; \\
D_{y}^{(1)} - D_{y}^{(2)} = \gamma \frac{\partial w_{y}^{(2)}}{\partial x}, & y = 0,
\]

Fig. 1. A screw dislocation interacting with an imperfect interface between two piezoelectric half-planes.
where the two interface parameters $\chi$ and $\gamma$ are uniform and nonnegative constants. As in the previous case, if $\chi = \gamma = 0$ Eq. (5) represents a perfectly bonded interface or if $\chi, \gamma \rightarrow \infty$ Eq. (5) describes a completely debonded and equipotential interface.

### 2.2. Complex variable formulation for compliant, weakly conducting interface

Let us now introduce the generalized displacement vector $\mathbf{U} = [w \phi]^T$ and the generalized stress function vector $\Phi$ which is related to the stresses and electric displacements through the following

$$\begin{bmatrix} \sigma_{zy} \\ D_y \end{bmatrix} = \Phi_x, \quad \begin{bmatrix} \sigma_{zx} \\ D_x \end{bmatrix} = -\Phi_y.$$

(6)

Let us also introduce the generalized stiffness matrix $C$ as

$$C = \begin{bmatrix} c_{44} & e_{15} \\ e_{15} & -e_{11} \end{bmatrix},$$

(7)

which is real and symmetric but not positive definite. Then the generalized displacement and stress function vectors can be concisely expressed in terms of a 2D analytic function vector $f(z)$ of a single complex variable $z = x + iy$ as

$$\mathbf{U} = \text{Im}\{f(z)\}, \quad \Phi = \text{Re}\{Cf(z)\}.$$

(8)

In addition, the strains, stresses, electric fields and electric displacements can also be concisely expressed in terms of $f(z)$ as follows

$$\begin{bmatrix} \gamma_{zy} + i\gamma_{zy} \\ -E_y - iE_x \end{bmatrix} = f'(z), \quad \begin{bmatrix} \sigma_{zy} + i\sigma_{zy} \\ D_y + iD_x \end{bmatrix} = Cf'(z),$$

(9)

where the strains $\gamma_{zx}$ and $\gamma_{zy}$ are related to the out-of-plane displacement $w$ through

$$\gamma_{zx} = w_{,x}, \quad \gamma_{zy} = w_{,y}.$$

(10)

### 2.3. Complex variable formulation for compliant, highly conducting interface

In order to be consistent with formulation given in Section 2.2 and have the ability to treat both boundary value problems simultaneously in the same setting let us define a function $\phi$ which is related to the electric displacements through the following relations

$$D_y = \phi_{,x}, \quad D_x = -\phi_{,y}.$$

(11)

Having this definition in place, we can now introduce the generalized displacement vector $\mathbf{U} = [w \phi]^T$ and the generalized stress function vector $\Phi$ which is related to the stresses and electric fields through the following form

$$\begin{bmatrix} \sigma_{zy} \\ -E_x \end{bmatrix} = \Phi_x, \quad \begin{bmatrix} \sigma_{zx} \\ E_y \end{bmatrix} = -\Phi_y.$$

(12)

In view of Eq. (2) it is clear that the second component of $\Phi$, in Eq. (25), is in fact the electric potential $\phi$.

Let us also define the following complex generalized stiffness matrix $C$

$$C = \begin{bmatrix} \tilde{c}_{44} & i\tilde{e}_{15} \\ -i\tilde{e}_{15} & \tilde{e}_{11} \end{bmatrix}$$

(13)

which is a $2 \times 2$ positive definite Hermitian matrix (i.e., $C^T = C$). In the above expression note that $\tilde{c}_{44} = c_{44} + e_{15}^2/\varepsilon_{11} \geq c_{44}$ is the piezoelectrically stiffened elastic constant.

As in Section 2.2, the generalized displacement and stress function vectors can now be concisely expressed in terms of a 2D analytic function vector $f(z)$ of a single complex variable $z = x + iy$ as
\[ U = \text{Im}\{f(z)\}, \quad \Phi = \text{Re}\{Cf(z)\}. \quad (14) \]

In addition, the strains, stresses, electric fields and electric displacements can also be concisely expressed in terms of \( f(z) \) as follows
\[
\begin{bmatrix}
\gamma_{xy} + i\gamma_{yz} \\
-D_x + iD_y
\end{bmatrix} = f'(z), 
\begin{bmatrix}
\sigma_{xy} + i\sigma_{xz} \\
-E_x + iE_y
\end{bmatrix} = Cf'(z). \quad (15)
\]

**Remark 1.** Eqs. (8) and (14) for the generalized displacement vector \( U \) and the generalized stress function vector \( \Phi \), for the two different interface conditions, are identical.

### 2.4. Expression for the boundary conditions in terms of \( U \) and \( \Phi \)

In view of Eq. (11) and the electric field–electric potential relations (Eq. (2)), the boundary conditions given by Eq. (5) for a compliant, highly conducting interface can be equivalently written as
\[
\begin{align*}
\sigma_{xy}^{(1)} &= \sigma_{xy}^{(2)}, \\
E_x^{(1)} &= E_x^{(2)}, \\
\nu^{(1)} - \nu^{(2)} &= \gamma\sigma_{xy}^{(2)}, \\
\phi^{(1)} - \phi^{(2)} &= -\gamma E_x^{(2)}, \\
y &= 0.
\end{align*} \quad (16)
\]

Thus, the boundary conditions given by Eq. (4) for a compliant, weakly conducting interface and Eq. (16) for a compliant, highly conducting interface can both be commonly written in terms of the generalized displacement and stress function vectors \( U \) and \( \Phi \), respectively, as follows
\[
\Phi_1 = \Phi_2, \quad U_1 - U_2 = \Lambda \frac{\partial \Phi_2}{\partial x}, \quad y = 0, \quad (17)
\]
where \( \Lambda = \text{diag}[\varpi, -\beta] \) for a compliant, weakly conducting interface and \( \Lambda = \text{diag}[\varpi, \gamma] \) for a compliant, highly conducting interface.

### 3. Field potentials

The boundary conditions, given by Eq. (17) for a compliant, weakly (or highly) conducting interface, can be equivalently expressed in terms of the two analytic function vectors \( f_1(z) \) and \( f_2(z) \) as follows
\[
\begin{align*}
C_1f_1^*(x) + \overline{C}_1\overline{f}_1^*(x) &= C_2f_2^*(x) + \overline{C}_2\overline{f}_2^*(x), \\
f_1^*(x) - f_1^*(x) - f_2^*(x) + f_2^*(x) &= i\Lambda[C_2f_2^*(x) + \overline{C}_2\overline{f}_2^*(x)], \\
y &= 0. \quad (18)
\end{align*}
\]

It immediately follows from Eq. (18) that
\[
\begin{align*}
f_1(z) &= C_1^{-1}\overline{C}_2\overline{f}_2(z) + f_0(z) - C_1^{-1}\overline{C}_1\overline{f}_1(z), \\
f_1(z) &= C_1^{-1}C_2f_2(z) - \overline{C}_1^{-1}C_1f_0(z) + \overline{f}_0(z), \quad (19)
\end{align*}
\]
where \( f_0(z) = \frac{\text{Im}z-i\eta}{\text{Im}z-i\eta} (C_1 + \overline{C}_1)^{-1}(C_1\overline{b} - i\overline{f}) \) is the complex potential for a piezoelectric screw dislocation located at the point \( z = i\eta \) in a homogeneous piezoelectric body. Note that for a compliant, weakly conducting interface \( b = [b^T \Delta \phi]^T \) and \( \overline{f} = [\overline{p} - q]^T \), whereas for a compliant, highly conducting interface \( b = [b^T \Delta \phi]^T \) and \( \overline{f} = [\overline{p} - q]^T \). Substituting Eq. (19) into Eq. (18) yields
\[
\begin{align*}
(I + C_1^{-1}\overline{C}_2)\overline{f}_2^*(x) - i\Lambda\overline{C}_2\overline{f}_2^*(x) - (I + C_1^{-1}\overline{C}_1)\overline{f}_1(x) \\
= (I + C_1^{-1}\overline{C}_2)f_2^*(x) + i\Lambda C_2f_2^*(x) - (I + C_1^{-1}C_1)f_0(x), \\
y &= 0. \quad (20)
\end{align*}
\]

Clearly, the left hand side of Eq. (20) is analytic in the upper half-plane while the right hand side of Eq. (20) is analytic in the lower half-plane. Thus, the left and right sides of Eq. (20) are identically zero in the upper and lower half-planes, respectively. Consequently, it follows that
\[
HC_2f_2(z) + i\Lambda C_2f_2(z) = (I + C_1^{-1}C_1)f_0(z), \quad y \leq 0, \quad (21)
\]
where $H = \mathbf{C}_1^{-1} + \mathbf{C}_2^{-1}$ is a $2 \times 2$ Hermitian matrix. Note that the above equation is a set of coupled first-order partial differential equations for the analytic function vector $f_2(z)$ defined in the lower half-plane which is free of the singularity (the screw dislocation).

If the imperfect interface is compliant and weakly conducting then it can be easily verified that

$$
H = \begin{bmatrix} H_{11} & H_{12} \\ H_{12} & -H_{22} \end{bmatrix},
$$

(22)

where the three real components $H_{11}(H_{11} > 0), H_{22}(H_{22} > 0), H_{12}$ are explicitly given by

$$
H_{11} = \frac{1}{c_{44}^{(1)}} + \frac{1}{c_{44}^{(2)}}, \quad H_{22} = \frac{c_{44}^{(1)} c_{11}^{(1)}}{c_{44}^{(2)} c_{11}^{(2)}}, \quad H_{12} = \frac{c_{44}^{(1)} c_{11}^{(2)}}{c_{44}^{(2)} c_{11}^{(1)}}.
$$

(23)

On the other hand, if the imperfect interface is compliant and highly conducting then it can also be verified that

$$
H = \begin{bmatrix} \tilde{H}_{11} & i\tilde{H}_{12} \\ -i\tilde{H}_{12} & \tilde{H}_{22} \end{bmatrix},
$$

(24)

where the three real components $\tilde{H}_{11}(\tilde{H}_{11} > 0), \tilde{H}_{22}(\tilde{H}_{22} > 0), \tilde{H}_{12}$ are explicitly given by

$$
\tilde{H}_{11} = \frac{1}{c_{44}^{(1)}} + \frac{1}{c_{44}^{(2)}}, \quad \tilde{H}_{22} = \frac{c_{44}^{(1)} c_{11}^{(1)}}{c_{44}^{(2)} c_{11}^{(2)}}, \quad \tilde{H}_{12} = \frac{c_{44}^{(1)} c_{11}^{(2)}}{c_{44}^{(2)} c_{11}^{(1)}}.
$$

(25)

Clearly, since the inequality $\tilde{H}_{11}\tilde{H}_{22} > \tilde{H}_{12}^2$ holds then if the imperfect interface is compliant and highly conducting the Hermitian matrix $H$ must be positive definite.

In order to be able to solve the coupled partial differential equation given by (21), let consider the following eigenvalue problem

$$(H - \lambda \mathbf{1}) \mathbf{z} = 0.$$  

(26)

The solutions (eigenvalues, eigenvectors and orthogonal relationships) to the above eigenvalue problem for the two kinds of imperfect interface will be discussed separately in following sections.

3.1. When the interface is compliant, weakly conducting

When the two interface parameters $\alpha$ and $\beta$ satisfy the following inequality

$$
\sqrt{\frac{\alpha}{\beta}} \leq \sqrt{\frac{H_{11}H_{22} + H_{12}^2}{H_{22}} - |H_{12}|} \quad \text{or} \quad \sqrt{\frac{\alpha}{\beta}} \geq \sqrt{\frac{H_{11}H_{22} + H_{12}^2 + |H_{12}|}{H_{22}}},
$$

(27)

or equivalently $(\alpha H_{11} - \beta H_{22})^2 \geq 4\alpha \beta H_{12}^2$ then Eq. (26) has two positive real eigenvalues given by

$$
\lambda_1 = \frac{\beta H_{11} + \beta H_{22} + \sqrt{(-\beta H_{11} - \beta H_{22})^2 - 4\alpha \beta H_{12}^2}}{2\alpha \beta},
$$

$$
\lambda_2 = \frac{\beta H_{11} + \beta H_{22} - \sqrt{(-\beta H_{11} - \beta H_{22})^2 - 4\alpha \beta H_{12}^2}}{2\alpha \beta},
$$

(28)

where $\lambda_1 \geq \lambda_2 > 0$.

On the other hand, when the two interface parameters $\alpha$ and $\beta$ satisfy the following inequality

$$
\sqrt{\frac{\alpha}{\beta}} \geq \sqrt{\frac{H_{11}H_{22} + H_{12}^2 - |H_{12}|}{H_{22}}}
$$

(29)

or equivalently $(\alpha H_{11} - \beta H_{22})^2 < 4\alpha \beta H_{12}^2$ then Eq. (26) has two complex eigenvalues given by
\[
\lambda_1 = \frac{\beta H_{11} + \alpha H_{22} + \sqrt{4\alpha^2 H_{12}^2 - \beta^2 H_{11}^2}}{2\beta}, \\
\lambda_2 = \frac{\beta H_{11} + \alpha H_{22} - \sqrt{4\alpha^2 H_{12}^2 - \beta^2 H_{11}^2}}{2\beta},
\]
(30)

where \( \text{Re}\{\lambda_1\} = \text{Re}\{\lambda_2\} > 0. \)

Furthermore, the two eigenvectors of Eq. (26) associated with the two eigenvalues are determined to be
\[
\xi_1 = \begin{bmatrix} H_{12}/(\lambda_1 x - H_{11}) \\ 1 \end{bmatrix}, \quad \xi_2 = \begin{bmatrix} 1/(\lambda_2 x - H_{11})/H_{12} \end{bmatrix}.
\]
(31)

It can be verified that when the two eigenvalues are real and distinct then the following orthogonal relationships with respect to \( \Lambda \) and \( \mathbf{H} \) holds
\[
\begin{bmatrix} \xi_1^T \\ \xi_2^T \end{bmatrix} \Lambda \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = \begin{bmatrix} \frac{2H_{12}^2 - \beta(\lambda_1 x - H_{11})^2}{(\lambda_1 x - H_{11})^2} & 0 \\ 0 & \frac{2H_{12}^2 - \beta(\lambda_2 x - H_{11})^2}{(\lambda_2 x - H_{11})^2} \end{bmatrix},
\]
(32)
\[
\begin{bmatrix} \xi_1^T \\ \xi_2^T \end{bmatrix} \mathbf{H} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = \begin{bmatrix} \frac{\lambda_1 2H_{12}^2 - \beta(\lambda_1 x - H_{11})^2}{(\lambda_1 x - H_{11})^2} & 0 \\ 0 & \frac{\lambda_2 2H_{12}^2 - \beta(\lambda_2 x - H_{11})^2}{(\lambda_2 x - H_{11})^2} \end{bmatrix}.
\]
(33)

In addition, when the two eigenvalues of Eq. (26) are identical or equivalently when \( \sqrt{\frac{2}{\beta}} = \frac{\sqrt{H_{11} H_{22} + H_{12}^2}}{H_{12}} \) or \( \sqrt{\frac{2}{\beta}} = \frac{\sqrt{H_{11} H_{22} + H_{12}^2}}{H_{12}} \) then only one independent eigenvector associated with the two identical eigenvalues exists. For this special case, the above orthogonal relationships given by Eqs. (32) and (33) are invalid and a small perturbation technique is required to make the two eigenvalues different.

When the two eigenvalues are complex conjugates then the following orthogonal relationships with respect to \( \Lambda \) and \( \mathbf{H} \) are established
\[
\begin{bmatrix} \bar{\xi}_1^T \\ \bar{\xi}_1^T \end{bmatrix} \Lambda \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = \begin{bmatrix} \frac{2H_{12}^2 - \beta(\lambda_1 x - H_{11})^2}{H_{12}(\lambda_1 x - H_{11})} & 0 \\ 0 & \frac{2H_{12}^2 - \beta(\lambda_2 x - H_{11})^2}{H_{12}(\lambda_2 x - H_{11})} \end{bmatrix},
\]
(34)
\[
\begin{bmatrix} \bar{\xi}_1^T \\ \bar{\xi}_1^T \end{bmatrix} \mathbf{H} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = \begin{bmatrix} \frac{\lambda_1 2H_{12}^2 - \beta(\lambda_1 x - H_{11})^2}{H_{12}(\lambda_1 x - H_{11})} & 0 \\ 0 & \frac{\lambda_2 2H_{12}^2 - \beta(\lambda_2 x - H_{11})^2}{H_{12}(\lambda_2 x - H_{11})} \end{bmatrix}.
\]
(35)

3.2. When the interface is compliant, highly conducting

In this case, when the two eigenvalues of (26) are strictly positive and real the eigenvalues can be explicitly determined to be
\[
\lambda_1 = \frac{\gamma H_{11} + \tilde{\beta} H_{22} + \sqrt{(\gamma H_{11} + \tilde{\beta} H_{22})^2 + 4\tilde{\alpha} \tilde{H}_{12}^2}}{2\tilde{\gamma}} > 0,
\]
\[
\lambda_2 = \frac{\gamma H_{11} + \tilde{\beta} H_{22} - \sqrt{(\gamma H_{11} + \tilde{\beta} H_{22})^2 + 4\tilde{\alpha} \tilde{H}_{12}^2}}{2\tilde{\gamma}} > 0.
\]
(36)

The two eigenvectors of (26) associated with the two eigenvalues are determined to be
\[
\xi_1 = \begin{bmatrix} \tilde{H}_{12}/(\lambda_1 x - \tilde{H}_{11}) \\ -i \end{bmatrix}, \quad \xi_2 = \begin{bmatrix} i/(\lambda_2 x - \tilde{H}_{11})/\tilde{H}_{12} \end{bmatrix}.
\]
(37)

It can then be verified that the orthogonal relationships with respect to \( \Lambda \) and \( \mathbf{H} \) is given by
[\begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} \begin{bmatrix} A \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = \begin{bmatrix} \tilde{H}_1^{1} + \gamma (\lambda_1 x - H_{11})^2 \\ 0 \\ 0 \\ \tilde{H}_2^{1} + \gamma (\lambda_2 x - H_{11})^2 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ \tilde{H}_2^{1} + \gamma (\lambda_2 x - H_{11})^2 \end{bmatrix},

(38)

\begin{bmatrix} \tilde{H}_1^{1} + \gamma (\lambda_1 x - H_{11})^2 \\ 0 \\ 0 \\ \tilde{H}_2^{1} + \gamma (\lambda_2 x - H_{11})^2 \end{bmatrix}.

(39)

3.3. Decoupling and the field potentials

In order to decouple Eq. (21) and subsequently solve the partial differential equation let us introduce the following transformation

\[ C_2 \mathbf{f}_2(z) = \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} \begin{bmatrix} h_1(z) \\ h_2(z) \end{bmatrix}, \quad \begin{bmatrix} 0 \\ \tilde{H}_2^{1} + \gamma (\lambda_2 x - H_{11})^2 \end{bmatrix},

\]

where \( h_1(z) \) and \( h_2(z) \) are two newly introduced analytic functions. In what follows, we will discuss the decoupling of the original partial differential equation and provide solutions to the field potentials according to the two different kinds of imperfect interface conditions.

3.3.1. Field potentials for a compliant and weakly conducting interface

In view of Eq. (40) and the orthogonal relations given by Eqs. (32)–(35), it is readily found that Eq. (21), irrespective of whether the eigenvalues are real or complex conjugates, can always be decoupled into the following two independent first-order partial differential equations for \( h_1(z) \) and \( h_2(z) \) as follows:

\[
\begin{aligned}
-i \lambda_1 h_1(z) + h_1'(z) &= -K_1 (\lambda_1 x - H_{11}) \ln(z - i \delta), \\
-i \lambda_2 h_2(z) + h_2'(z) &= -K_2 H_{12} \ln(z - i \delta),
\end{aligned}
\]

where the two constants \( K_1 \) and \( K_2 \) are given by

\[
K_1 = \frac{\epsilon^{(0)}_{11} H_{12} + \epsilon^{(0)}_{11} (\lambda_2 x - H_{11}) |p + \epsilon^{(0)}_{44} (\lambda_1 x - H_{11})|q + \epsilon^{(0)}_{44} \epsilon^{(0)}_{11} |H_{12} b + (\lambda_1 x - H_{11})| \Delta \phi}{\epsilon^{(0)}_{44} \epsilon^{(0)}_{11} |H_{12} b + (\lambda_1 x - H_{11})| \Delta \phi},
\]

\[
K_2 = \frac{\epsilon^{(0)}_{11} H_{12} + \epsilon^{(0)}_{11} (\lambda_2 x - H_{11}) |p + \epsilon^{(0)}_{44} (\lambda_2 x - H_{11})|q + \epsilon^{(0)}_{44} \epsilon^{(0)}_{11} |H_{12} b + (\lambda_2 x - H_{11})| \Delta \phi}{\epsilon^{(0)}_{44} \epsilon^{(0)}_{11} |H_{12} b + (\lambda_2 x - H_{11})| \Delta \phi},
\]

The solution to the above can be expressed in terms of the exponential integral as (for more details, see Wang et al., in press) and Sudak and Wang (2006))

\[
\begin{aligned}
h_1'(z) &= K_1 (\lambda_1 x - H_{11}) \exp[i \lambda_1 (z - i \delta)] E_1[i \lambda_1 (z - i \delta)], \\
h_2'(z) &= K_2 H_{12} \exp[i \lambda_2 (z - i \delta)] E_1[i \lambda_2 (z - i \delta)],
\end{aligned}
\]

where the exponential integral is defined by (Abramovitz and Stegun, 1972)

\[
E_1(z) = - \int_{\infty}^{z} \frac{e^{-t}}{t} \, dt.
\]

(43)

Having solved for \( h_1(z) \) and \( h_2(z) \) the original analytic function vectors \( \mathbf{f}_1(z) \) and \( \mathbf{f}_2(z) \) can then be easily determined to be
Observe that the last term in the expression of fields and electric displacements) for the two piezoelectric half-planes Eqs. (45) and (46) are simply substituted

Having determined analytic functions (21) can be decoupled into the following two independent first-order partial differential equations as for the

In view of Eq. (40) and the orthogonal relations given by Eqs. (38) and (39) it can be readily shown that Eq. (47) and (49) can be decoupled into the following two independent first-order partial differential equations as for the analytic functions $h_1(z)$ and $h_2(z)$ as follows:

where the two constants $T_1$ and $T_2$ are given by

Observe that the last term in the expression of $f'_1(z)$ is the singular part due to the piezoelectric screw dislocation. It can be shown that $T_1 = K_1, T_2 = K_2$ when $\lambda_1$ and $\lambda_2$ are real and $T_1 = K_2, T_2 = K_1$ when $\lambda_1$ and $\lambda_2$ are complex conjugates, (i.e., $\lambda_1 = \lambda_2$). To determine the field quantities (such as strains, stresses, electric fields and electric displacements) for the two piezoelectric half-planes Eqs. (45) and (46) are simply substituted into Eq. (9).

3.3.2. Field potentials for a compliant and highly conducting interface

In view of Eq. (40) and the orthogonal relations given by Eqs. (38) and (39) it can be readily shown that Eq. (21) can be decoupled into the following two independent first-order partial differential equations as for the analytic functions $h_1(z)$ and $h_2(z)$ as follows:

where the two constants $K_1$ and $K_2$ are given by

The solution to the above can also be expressed in terms of the exponential integral as

Having determined $h_1(z)$ and $h_2(z)$ the original analytic function vectors $f'_1(z)$ and $f'_2(z)$ can then be easily determined to be
\[ f'_1(z) = \frac{1}{c_{44}} \left[ \tilde{H}_{12} + e_{15}^{(1)}(\lambda_1 \chi - \tilde{H}_{11}) - i\left[\tilde{H}_{12} + e_{15}^{(1)}(\lambda_2 \chi - \tilde{H}_{11}) \right] \right] \]

\[ \times \left[ K_1 \exp[-i\lambda_1(z + i\delta)]E_1[-i\lambda_1(z + i\delta)] \right. \]

\[ \left. \left[ iK_2 \exp[-i\lambda_2(z + i\delta)]E_1[-i\lambda_2(z + i\delta)] \right. \right. \]

\[ - \frac{1}{2\pi} \left[ \frac{\epsilon_{11}^{(1)}(\epsilon_{11}^{(1)} + 2\epsilon_{11}^{(2)} \phi')}{\epsilon_{44}^{(1)} \epsilon_{11}^{(1)}} \right] \frac{1}{z + i\delta}, \]

\[ y \geq 0 \]

\[ f'_2(z) = \frac{1}{c_{44}} \left[ \tilde{H}_{12} - e_{15}^{(2)}(\lambda_1 \chi - \tilde{H}_{11}) - i\left[\tilde{H}_{12} - e_{15}^{(2)}(\lambda_2 \chi - \tilde{H}_{11}) \right] \right] \]

\[ \times \left[ K_1 \exp[i\lambda_1(z - i\delta)]E_1[i\lambda_1(z - i\delta)] \right. \]

\[ \left. \left[ -iK_2 \exp[i\lambda_2(z - i\delta)]E_1[i\lambda_2(z - i\delta)] \right. \right. \]

\[ y \leq 0 \]  

(51)

(52)

Note that the last term in the expression for \( f'_1(z) \) is the singular part due to the piezoelectric screw dislocation. Substituting Eqs. (51) and (52) into Eq. (15) will yield the field quantities (such as strain, stress, electric field and electric displacement) for the two piezoelectric half-planes.

It is observed from the above derivations that two dimensionless parameters \( \lambda_1 \delta \) and \( \lambda_2 \delta \) are needed to measure the interface “rigidity” as contrary to the purely elastic case in which only one parameter is enough to measure the interface “rigidity” (Fan and Wang, 2003). When the imperfect interface is compliant and weakly conducting the two dimensionless parameters can be either positive real values or complex conjugates with positive real parts. When the imperfect interface is compliant and highly conducting the two dimensionless parameters can only be positive real values.

4. Image force on the screw dislocation

Let us now consider the image force (or the generalized Peach–Koehler force) acting on the screw dislocation. In view of the work of Lee et al., 2000), the image force can be calculated by employing the field potentials given by Eqs. (45) and (51) derived in the previous section. For example, if the dislocation only suffers a nonzero displacement jump \( b \neq 0 \) with \( p = q = \Delta \phi = 0 \) and, in addition, the interface is compliant and weakly conducting, then the image force on the screw dislocation is given by

\[ F_y = -\frac{c_{44}^2}{4\alpha^2} \left( 1 - \rho \left( f(\eta_1) + f(\eta_2) \right) + \kappa \mp \sqrt{1 + \kappa \left[ \frac{\eta_1 - \eta_2}{\eta_1 + \eta_2} \right]^2} \right), \]

\[ F_x = 0, \]  

(53)

where \( F_x \) and \( F_y \) are, respectively, the \( x \) and \( y \) components of the image force, \( \eta_1 = \lambda_1 \delta, \eta_2 = \lambda_2 \delta, \kappa = \frac{H_{12}}{H_{11}H_{22}} \) and \( \rho = \frac{H_{22}}{c_{44}^2[H_{11}H_{22} + H_{12}]} = \frac{1}{c_{44}^2[H_{11}(1 + \kappa)]} \) can be considered as two dimensionless generalized Dundurs constants for the piezoelectric bimaterial. In addition, the expression \( f(\eta_i) = 2\eta_i \exp(2\eta_i)E_1(2\eta_i), i = 1, 2 \) is utilized from Fan and Wang (2003). Furthermore, the choice of the sign ‘+’ is selected in such a way that when \( \alpha H_{12} \gg \beta H_{11} \), the ‘−’ sign is selected. When \( \alpha H_{12} < \beta H_{11} \) the ‘+’ sign is selected. Note that when the two dimensionless
parameters \( \eta_1 \) and \( \eta_2 \) are complex conjugates the inequality \( \text{Im}\{\eta_1\}/\text{Re}\{\eta_1\} \leq \sqrt{\kappa} \) must hold. In fact, Eq. (53) demonstrates that the image force on the screw dislocation is only dependent on the two generalized Dundurs constants \( \kappa, \rho \) and the two constants \( \eta_1, \eta_2 \) measuring the interface “rigidity”. Due to the fact the following inequality always holds

\[
f(\eta_1) + f(\eta_2) + \left[ \kappa \mp 1 + \kappa \sqrt{\frac{(\eta_1 - \eta_2)^2}{(\eta_1 + \eta_2)^2} + \kappa} \right] \frac{(\eta_1 + \eta_2)[f(\eta_1) - f(\eta_2)]}{\eta_1 - \eta_2} > 0,
\]

then it is possible to find an equilibrium position for the screw dislocation in which \( F_y = 0 \) provided the following condition is met

\[
f(\eta_1) + f(\eta_2) + \left[ \kappa \mp 1 + \kappa \sqrt{\frac{(\eta_1 - \eta_2)^2}{(\eta_1 + \eta_2)^2} + \kappa} \right] \frac{(\eta_1 + \eta_2)[f(\eta_1) - f(\eta_2)]}{\eta_1 - \eta_2} = \frac{1}{\rho}.
\]

In what follows, let us examine several special cases to verify the correctness of the solution and to show the versatility of the above expression for the image force on the screw dislocation.

### 4.1. An imperfect interface between two purely elastic half-planes

When the bimaterial is purely elastic we have that \( H_{12} = 0 \) or \( \kappa = 0 \) since \( e_{15}^{(1)} = e_{15}^{(2)} = 0 \). In this case Eq. (53) reduces to

\[
F_y = -\frac{c_{44}^{(1)} b^2}{4\pi \delta} \left[ 1 - \frac{2c_{44}^{(2)}}{c_{44}^{(1)} + c_{44}^{(2)}} f \left( \frac{\delta c_{44}^{(1)} + c_{44}^{(2)}}{\delta c_{44}^{(1)} + c_{44}^{(2)}} \right) \right],
\]

which is the result derived by Fan and Wang (2003) for a screw dislocation interacting with an imperfect interface between two purely elastic half-planes.

### 4.2. A perfect bonding interface between two piezoelectric half-planes

When the interface is perfect (i.e., \( \alpha, \beta \to 0 \) or \( \eta_1, \eta_2 \to \infty \)), Eq. (53) reduces to

\[
F_y = -\frac{c_{44}^{(1)} b^2 (1 - 2\rho)}{4\pi \delta},
\]

which is the result derived by Liu et al. (1999). In deriving Eq. (57) we utilized the following asymptotic expansion for \( f(\eta) \)

\[
1 - f(\eta) \approx \frac{1}{2\eta} - \frac{1}{2\eta^2} + o\left(\frac{1}{\eta^2}\right), \text{ when } \eta \to \infty.
\]

### 4.3. A fully debonded and insulating interface between two piezoelectric half-planes

When the interface is completely debonded and charge-free (i.e., \( \alpha, \beta \to \infty \) or \( \eta_1, \eta_2 \to 0 \)), Eq. (53) reduces to

\[
F_y = -\frac{c_{44}^{(1)} b^2}{4\pi \delta},
\]

which is the result derived by Pak (1990) for a piezoelectric screw dislocation near a traction-free and charge-free surface. Note that in deriving Eq. (59) we have utilized \( f(0) = 0 \) and

\[
\lim_{\eta_1, \eta_2 \to 0} \frac{f(\eta_1) - f(\eta_2)}{\eta_1 - \eta_2} = \lim_{\eta_1 \to 0} f'(\eta_1) \approx \lim_{\eta_1 \to 0} \frac{f(\eta_1)}{\eta_1}.
\]
4.4. A mechanically compliant and dielectrically perfect interface

When the interface is mechanically compliant and dielectrically perfect ($\beta \to 0$) it follows from Eq. (28) that

$$\lambda_1 \to \infty, \quad \lambda_2 = \frac{H_{11}H_{22} + H_{12}^2}{\beta H_{11}}.$$  \hspace{1cm} (61)

Consequently, Eq. (53) reduces to

$$F_y = -\frac{c_{41}^{(1)}b^2\{1 - 2\rho f(\eta_1)\}}{4\pi\delta}. \hspace{1cm} (62)$$

Furthermore, if the two half-planes are purely elastic ($H_{12} = 0$) then Eq. (62) will further reduce to Eq. (56).

4.5. A mechanically perfect and dielectrically weakly conducting interface

When the interface is mechanically perfect ($\gamma \to 0$) and dielectrically weakly conducting it follows from Eq. (28) that

$$\lambda_1 \to \infty, \quad \lambda_2 = \frac{H_{11}H_{22} + H_{12}^2}{\beta H_{11}}.$$  \hspace{1cm} (63)

In this case Eq. (53) reduces to

$$F_y = -\frac{c_{41}^{(1)}b^2\{1 - 2\rho[1 + \kappa[1 - f(\eta_2)]\}}{4\pi\delta}. \hspace{1cm} (64)$$

Furthermore, if the interface is also dielectrically perfect ($\beta \to 0$ or $\eta_2 \to \infty$) Eq. (64) can further reduce to Eq. (57). On the other hand, if the interface is insulating ($\beta \to \infty$ or $\eta_2 \to 0$) Eq. (64) reduces to

$$F_y = -\frac{c_{41}^{(1)}b^2\{1 - 2\rho(1 + \kappa)\}}{4\pi\delta}. \hspace{1cm} (65)$$

4.6. The degenerate case where $\lambda_1 = \lambda_2$ or $\eta_1 = \eta_2 = \eta$

Here it is interesting to note that the result for the degenerate case where $\lambda_1 = \lambda_2$ or $\eta_1 = \eta_2 = \eta$ can also be conveniently derived from Eq. (53) as

$$F_y = -\frac{c_{41}^{(1)}b^2\{1 - 2\rho\left[f(\eta) - \frac{\sqrt{\kappa}\eta f'(\eta)}{\sqrt{\kappa} \pm \sqrt{1 + \kappa}}\right]\}}{4\pi\delta}, \hspace{1cm} (66)$$

where the sign ‘±’ is chosen in such a way that when $\alpha H_{22} \geq \beta H_{11}$ (or more precisely $\sqrt{\beta} = \frac{\sqrt{H_{11}H_{22} + H_{12}^2}}{H_{22}} + |H_{12}|$) the ‘+’ sign is selected. When $\alpha H_{22} < \beta H_{11}$ (or more precisely $\sqrt{\beta} = \frac{\sqrt{H_{11}H_{22} + H_{12}^2} - |H_{12}|}{H_{22}}$) the ‘–’ is selected and $f'(\eta)$ can be expressed in terms of $f(\eta)$ as

$$f'(\eta) = \lim_{\eta_1, \eta_2 \to \eta} \frac{f(\eta_1) - f(\eta_2)}{\eta_1 - \eta_2} = \frac{f(\eta)}{\eta} + 2f(\eta) - 2.$$  \hspace{1cm} (67)

In view of Eq. (67), Eq. (66) can also be expressed as

$$F_y = -\frac{c_{41}^{(1)}b^2\{1 - 2\rho\left[\frac{\sqrt{\kappa}\eta g(\eta) \pm \sqrt{1 + \kappa}f(\eta)}{\sqrt{\kappa} \pm \sqrt{1 + \kappa}}\right]\}}{4\pi\delta}, \hspace{1cm} (68)$$

where $g(\eta) = 2\eta[1 - f(\eta)] \leq f(\eta), 0 \leq f(\eta), g(\eta) \leq 1$.

In this special degenerate case an equilibrium position for the screw dislocation in which $F_y = 0$ can be found when
Let us examine in more detail the influence of the parameter $g$ for various possibilities of the generalized Dundurs parameters $\eta$ and $\rho$. Due to the fact that the electromechanical coupling factor

$$\sqrt{\rho}g(\eta) \pm \sqrt{1 + \kappa f(\eta)} = \frac{\sqrt{\kappa} \pm \sqrt{1 + \kappa}}{2\rho}.$$  \hfill (69)
\[ k_e^{(i)} = \sqrt{\frac{\varepsilon_i^{(0)}}{\varepsilon_i^{(0)} + \frac{4\varepsilon_i^{(0)}}{\varepsilon_i^{(0)} + \varepsilon_i^{(0)}}}}, \quad (i = 1, 2) \] for the two piezoelectric half-planes are usually less than \( 1/\sqrt{2} \) (Wang et al., 2003) then \( \varepsilon_i^{(0)} \leq 1 \) and as such the parameter \( \kappa = \frac{H_2}{H_1} \) is varied from zero to 1. Fig. 2a illustrates the results for \( zH_{22} \geq \beta H_{11} \) (when the ‘+’ sign is selected in Eq. (69)) while Fig. 2b illustrates the results for \( zH_{22} < \beta H_{11} \) (when the ‘−’ sign is selected in Eq. (69)). Our calculations clearly demonstrate that there is no possible choice for the parameter \( \eta \) satisfying Eq. (69) when \( \rho < 1/2 \) no matter what value of \( \kappa \) is selected. In other words, it is only possible to find an equilibrium position for the screw dislocation when \( \rho \geq 1/2 \). This suggests that when the two half-planes are both purely elastic, the condition \( \rho \geq 1/2 \) implies that the lower half-plane is stiffer than the upper half-plane. In addition, note that the equilibrium position is clearly unstable. Furthermore, it is observed that for a fixed value of \( \kappa \) the parameter \( \eta \) is always a monotonically decreasing function of \( \rho \). In fact, \( \eta \) becomes very small as \( \rho \) approaches infinity and \( \eta \) becomes extremely large as \( \rho \) approaches \( 1/2 \). When \( zH_{22} \geq \beta H_{11} \) (see Fig. 2a) the parameter \( \eta \) is a monotonically increasing function of \( \kappa \) for a fixed value of \( \rho \). On the other hand, when \( zH_{22} < \beta H_{11} \) (see Fig. 2b) the parameter \( \eta \) is a monotonically decreasing function of \( \kappa \) for a fixed value of \( \rho \). By comparing Fig. 2a and b it is found that for a certain value of \( \rho \) the minimum value of \( \eta \) for \( zH_{22} \geq \beta H_{11} \), which is attained at \( \kappa = 0 \), is always equal to the maximum value of \( \eta \) for \( zH_{22} < \beta H_{11} \) which is also attained at \( \kappa = 0 \). In fact, it follows from Eq. (69) that \( 2\rho \delta(\eta) = 1 \) when \( \kappa = 0 \) for both of the cases \( zH_{22} \geq \beta H_{11} \) and \( zH_{22} < \beta H_{11} \).

5. Conclusions

We have derived closed form solutions in terms of exponential integrals for a piezoelectric screw dislocation interacting with two kinds of imperfect interfaces, namely a compliant, weakly conducting interface and a compliant, highly conducting interface. The generalized Peach–Koehler force acting on the screw dislocation is also given. It is observed that when \( b \neq 0 \) and \( p = q = \Delta \phi = 0 \) for the screw dislocation and the imperfect interface is compliant and weakly conducting, it is possible to find an equilibrium position for the screw dislocation. Furthermore, since Eq. (53) approaches Eq. (57) when \( \delta \to \infty \) and Eq. (53) approaches Eq. (59) when \( \delta \to 0 \) an equilibrium position, which is unstable, exists only if the condition \( \rho \geq 1/2 \) is satisfied. It is expected that the derived solutions can be conveniently applied to investigate a crack interacting with the imperfect piezoelectric bimaterial interface. In addition, the method presented in this research can be easily extended to investigate the case in which the imperfect interface is mechanically stiff and dielectrically weakly (or highly) conducting.

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