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NOTE

On Regressive Ramsey Numbers

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FOR MY MENTORS DON BONAR AND GERALD THOMPSON

We prove the following relation between regressive and classical Ramsey numbers $R^n_{\text{reg}}(n+2) = R_{n-1}(n) + 2$. This is used to compute $R^3_{\text{reg}}(5) = 8$, $R^4_{\text{reg}}(6) = 15$, and $R^5_{\text{reg}}(7) \geqslant 36$. We prove that $R^2_{x+k}(4) \leqslant 2^{k+1}(3+k) - (k+1)$, and use this to compute $R^2_{\text{reg}}(5) = 15$. Finally, we provide the bounds $195 \leqslant R^2_{\text{reg}}(6) \leqslant 5 \cdot 2^{42} + 2^{39} - 2$. © 2002 Elsevier Science (USA)

0. INTRODUCTION

This paper is concerned with the topic of *regressive* functions and sets which are *min-homogeneous* for those functions. Let $[n]$ denote the first n positive integers: $[n] = \{1, 2, \ldots, n\}$. If X is a set of positive integers, let $[X]^k$ denote the collection of subsets of X having cardinality k. We write $[n]^k$ for the collection of cardinality k subsets of $[n]$. In what follows, elements of a set are always listed in order.

Let X be a set of positive integers. A function $f : [X]^k \to \mathbf{N}$ to the natural numbers is said to be *regressive* if $f(\mathbf{s}) < \min(\mathbf{s})$ for all $\mathbf{s} \in [X]^k$. We write $f(s_1, s_2)$ for $f(\{s_1, s_2\})$ with the assumption that $s_1 < s_2$, and we extend this convention in a natural way for $k > 2$. If $f : [n]^k \to \mathbf{N}$ is regressive, a subset $H \subseteq [n]^k$ is said to be *min-homogeneous for f* if for all $\mathbf{s}, \mathbf{t} \in [H]^k$, $\min(\mathbf{s}) = \min(\mathbf{t})$ implies $f(\mathbf{s}) = f(\mathbf{t})$.

For fixed positive integers k and n, let $R^n_{\text{reg}}(k)$ denote the least positive integer m such that for any regressive function $f : [m]^n \to \mathbf{N}$, there exists a set $H \subseteq [m]$, such that $|H| \geqslant k$ and H is min-homogeneous for f. Kanamori and McAloon [3] prove that for positive integers k and n, there is a such a number, and hence there is a least such number. It is not hard to see that $R^2_{\text{reg}}(3) = 3$ and $R^2_{\text{reg}}(4) = 5$.

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In [4], Kojman and Shelah give an elementary proof of a fact proved by Kanamori and McAloon [3], that regressive Ramsey numbers are Ackermannian. Kojman and Shelah prove that the function $v(k) = R^2_{\text{reg}}(k)$ eventually dominates every primitive recursive function. They also state as an open problem the computation of small regressive Ramsey numbers.

In this paper, we prove a relation between regressive Ramsey numbers and classical two-color Ramsey numbers and use this relation to compute several regressive Ramsey numbers. We also compute some bounds on more general g-regressive Ramsey numbers and use these bounds in the computation $R^2_{\text{reg}}(5) = 15$.

1. $R^N_{\text{REG}}(N+2)$

Let $R_k(l)$ denote the classical 2-color Ramsey number. That is, $R_k(l)$ is the least positive integer n such that for any 2-coloring $f : [n]^k \to \{0, 1\}$ there exists a set $T \subseteq [n]$ with $|T| = l$ and f constant on $[T]^k$. Such a set T is called *monochromatic* or *homogeneous for f.*

In the following theorem, we relate classical Ramsey numbers to regressive Ramsey numbers.

THEOREM 1.1. $R^n_{\text{reg}}(n+2) = R_{n-1}(n) + 2$.

Proof. Let $r = R_{n-1}(n) + 2$ and suppose $f : [r]^n \to N$ is regressive. Let $A = \{3, \ldots, r\}$ and define a two-coloring g on $[A]^{n-1}$ by $g(\mathbf{x}) = f(2, \mathbf{x})$. Here we are writing $f(a, \mathbf{x})$ for $f(a, x_1, x_2, \ldots, x_{n-1})$. Since $|A| = R_{n-1}(n)$, there exists a set $T \subseteq A$ with $|T| = n$ and $[T]^{n-1}$ monochromatic for g. Defining $H = \{1, 2\} \cup T$, we see that $f(1, \mathbf{x}) = 0$ for all $\mathbf{x} \in \{H - \{1\}\}^{n-1}$, and $f(2, \mathbf{x}) = g(\mathbf{x})$ is constant for $\mathbf{x} \in \{H - \{1, 2\}\}^{n-1}$. It follows that H is min-homogeneous for f and has cardinality $n+2$.

On the other hand, since $|\{3, 4, \ldots, r-1\}| = R_{n-1}(n) - 1$, there exists a *Ramsey* two-coloring $g : \{3, 4, \ldots, r-1\}^{n-1} \to \{0, 1\}$, i.e. $\{3, 4, \ldots, r-1\}$ has no cardinality n subset which is monochromatic for g. If $\mathbf{x} = \{x_1, x_2, \ldots, x_n\} \in [1, r-1]^n$, write $\mathbf{x}' = \{x_2, x_3, \ldots, x_n\} \in [2, r-1]^{n-1}$. Now define

$$f(\mathbf{x}) = \begin{cases} 0 & \text{if } \min(\mathbf{x}) = 1, \\ g(\mathbf{x}') & \text{if } \min(\mathbf{x}) \geqslant 2. \end{cases} \tag{1}$$

We show that f has no min-homogeneous set of cardinality $n+2$, hence $R^n_{\text{reg}}(n+2) > R_{n-1}(n) + 1$. Suppose $H = \{h_1, h_2, h_3, \ldots, h_{n+2}\} \subseteq [1, r-1]$

with $|H| = n+2$. Let $H' = \{h_3, h_4, \ldots, h_{n+2}\} \subseteq [3, r-1]$, so $|H'| = n$. If H were min-homogeneous for f, then $f(h_2, *) = g(*)$ would be constant on $[H']^{n-1}$, i.e. H' would be monochromatic for g. This contradicts the choice of g as a Ramsey coloring of $[3, r-1]^{n-1}$, so H cannot be min-homogeneous for f. ∎

It is worth remarking that coloring f defined above is a regressive *two*-coloring.

It is known that $R_2(3) = 6$, $R_3(4) = 13$ (see [2, 5]), and $R_4(5) \geqslant 34$ (unpublished, see [7]), so we have the following:

Corollary 1.1. $R^3_{\mathrm{reg}}(5) = 8$, $R^4_{\mathrm{reg}}(6) = 15$, *and* $R^5_{\mathrm{reg}}(7) \geqslant 36$.

2. $R^2_{\mathrm{REG}}(5)$

In this section we build up to the computation of $R^2_{\mathrm{reg}}(5)$. If $g : \mathbf{N} \to \mathbf{N}$ and $f : [m]^n \to \mathbf{N}$, f is said to be g-regressive (as defined in [3]) if $f(\mathbf{x}) < g(\min(\mathbf{x}))$ for all $\mathbf{x} \in [m]^n$. Here we encounter only the situation in which $g(x) = x + k$ for some fixed k. It is not hard to see that the regressive Ramsey numbers $R^n_{x+k}(l)$ always exist.

Lemma 2.1. *For all* $l \geqslant 3$ *and* $n \geqslant 2$, $R^n_{\mathrm{reg}}(l) = R^n_{x+1}(l-1) + 1$.

Proof. Fix a positive integer r. For any function $f : [r]^n \to \mathbf{N}$, define $f^{(1)} : [r-1]^n \to \mathbf{N}$ as $f^{(1)}(x_1, x_2 \ldots, x_n) = f(x_1 + 1, x_2 + 1, \ldots, x_n + 1)$.

Clearly, $f(\mathbf{x}) < \min(\mathbf{x})$ for all $\mathbf{x} \in [2, r]^n$ if and only if $f^{(1)}(\mathbf{x}) < \min(\mathbf{x}) + 1$ for all $\mathbf{x} \in [1, r-1]^n$. Thus f regressive implies $f^{(1)}$ is $(x+1)$-regressive, and $f^{(1)}$ $(x+1)$-regressive implies f is regressive provided also that $f(\mathbf{x}) = 0$ when $\min(\mathbf{x}) = 1$.

For any set $H \subset [r]$, define $H^{(1)} = \{x - 1 | x \in H, \ x \neq 1\}$. Note that $H^{(1)} \subset [r-1]$ and $|H^{(1)}| \geqslant |H| - 1$.

Assuming f is regressive, if $\mathbf{x}, \mathbf{y} \in [H]^n$ with $\min(\mathbf{x}) = \min(\mathbf{y}) \neq 1$, $f(\mathbf{x}) = f(\mathbf{y})$ if and only if $f^{(1)}(\mathbf{x}) = f^{(1)}(\mathbf{y})$. It follows immediately that H is min-homogeneous for f if and only if $H^{(1)}$ is $(x+1)$ min-homogeneous for $f^{(1)}$. ∎

Next, we provide an exponential upper bound for the numbers $R^2_{x+k}(4)$, $k \in \mathbf{N}$.

Lemma 2.2. *For* $k \in \mathbf{N}$, $R^2_{x+k}(4) \leqslant 2^{k+1}(3+k) - (k+1)$.

Proof. Given $k \in \mathbf{N}$, let $n = 2^{k+1}(3+k) - (k+1)$. For $i \in [0,k]$, define $A_i = \{x \in [2,n] | f(1,x) = i\}$, and $a_i = \min(A_i)$. Note that $[2,n]$ is a disjoint union of the A_i's. By re-ordering, we may assume $a_i < a_j$ whenever $i < j$, i.e. the a_i's appear in order. We claim that for some $i \in [0,k]$, $|A_i| > 2^i(3+k)$. If not, then $|A_i| \leqslant 2^i(3+k)$ for each i, so $|\bigcup_i A_i| \leqslant (3+k) \cdot \sum_{i=0}^{k} 2^i = (2^{k+1}-1)(3+k)$. But $|[2,n]| = n-1 = (2^{k+1}-1)(3+k)+1$, contradicting the fact that $[2,n] = \bigcup_i A_i$.

Now let l be the least value such that $|A_l| > 2^l(3+k)$. Since $|A_i| \leqslant 2^i(3+k)$ for each $i \in [0, l-1]$, we have $a_l \leqslant |\{1\} \cup (\bigcup_{i=0}^{l-1} A_i)| + 1 \leqslant 2 + (3+k)(2^l - 1) = 2^l(3+k) - k - 1$. Write $A'_l = A_l - \{a_l\}$. Since f is $(x+k)$-regressive, $f(a_l, *)$ maps A'_l to $[0, a_l - 1 + k]$. But $|[0, a_l - 1 + k]| = a_l + k \leqslant 2^l(3+k) - 1$ and $|A'_l| \geqslant 2^l(3+k)$ implies $f(a_l, *)$ cannot be injective on A'_l. There must exist $b, c \in A'_l$ with $f(a_l, b) = f(a_l, c)$, so $\{1, a_l, b, c\}$ is min-homogeneous for f. ∎

We use what was proved above to compute $R^2_{\text{reg}}(5)$.

THEOREM 2.1. $R^2_{\text{reg}}(5) = 15$.

Proof. By Lemmas 2.2 and 2.1, $R^2_{\text{reg}}(5) = R^2_{x+1}(4) + 1 \leqslant 14 + 1 = 15$. To complete the proof we show that $R^2_{\text{reg}}(5) > 14$ by constructing a regressive map $f : [14]^2 \to \mathbf{N}$ having no min-homogeneous set of cardinality 5. Define the regressive function $f : [14]^2 \to \mathbf{N}$ by

$$f(i,j) = \begin{cases} f(1,j) = 0 & \text{for } j \in [2,14], \\ f(2,j) = 0 & \text{for } j \in [3,6], \\ f(2,j) = 1 & \text{for } j \in [7,14], \\ f(i,j) \ = j - i \,(\mathrm{mod}\, i) & \text{otherwise.} \end{cases} \tag{2}$$

First note that if H is min-homogeneous for f, then so is $\{1\} \cup H$, so we may assume $1 \in H$. Write $H = \{1, a, x, y, z\} \subset [1, 14]$. f is defined so that for $i \geqslant 7$ the values $f(i,x), x \in [i+1, 14]$ are all distinct. That is, $f(x,y) = f(x,z)$ implies $x \leqslant 6$. Moreover, because of the periodicity of the functions $f(i, *)$ for $i \geqslant 3$, there are no triples x, y, z with $f(a,x) = f(a,y) = f(a,z)$ except for $a \leqslant 4$. We may now assume $2 \leqslant a \leqslant 4$ and $3 \leqslant x \leqslant 6$. Consider a set of the form $H = \{1, 2, x, y, z\}$. Min-homogeneity would require $f(2,x) = f(2,y) = f(2,z)$ so either $x, y, z \geqslant 7$, in which case $f(x,y) \neq f(x,z)$, or $\{x,y,z\} \subset [3,6]$. In the latter case $x = 3$ or $x = 4$, but the definition of f admits no pairs $\{y,z\} \subset [4,6]$ with $f(3,y) = f(3,z)$ or $f(4,y) = f(4,z)$.

Now consider a set of the form $H = \{1, 3, x, y, z\}$. If H is min-homogeneous then we have a monochromatic triple $f(3,x) = f(3,y) =$

$f(3,z)$. Because the function $f(3,*)$ cycles with period 3, this occurs only if $x \equiv y \equiv z \pmod 3$. Moreover, $f(x,y) = f(x,z)$ would require that $y \equiv z \pmod x$. Since 4 and 5 are prime to 3, $x = 4$ or $x = 5$ would require $y \equiv z \pmod{12}$ or $y \equiv z \pmod{15}$, but this cannot happen for $y, z \in [5,14]$. In case $x = 6$ the (mod 3) requirement forces $y = 9$ and $z = 12$, but $f(6,9) \neq f(6,12)$. We have eliminated all possibility of a min-homogeneous set of the form $\{1,3,x,y,z\}$.

The case $H = \{1,4,x,y,z\}$ is dismissed in the same way, but more quickly. Here $x = 5$ or $x = 6$, and the periodicity of the functions $f(5,*)$ and $f(6,*)$ would require that y and z are equivalent modulo 20 or 12, which cannot happen for $y, z \in [6,14]$. ∎

3. A BOUND FOR $R^2_{\mathrm{REG}}(6)$

In this section, we extend the ideas from Lemma 2.1 to establish a bound for $R^2_{\mathrm{reg}}(6)$. We have

LEMMA 3.1. $R^2_{\mathrm{reg}}(6) \leqslant 5 \cdot 2^{42} + 2^{39} - 1$.

Proof. By Lemma 2.1, this follows immediately after we have proved that $R^2_{x+1}(5) \leqslant 5 \cdot 2^{42} + 2^{39} - 2$. For any $(x+1)$-regressive function $f : [1,n]^2 \to \mathbf{N}$, define $A_i = \{x \in [2,n] | f(1,x) = i\}$ and $a_i = \min(A_i)$ for $i = 0,1$. By re-ordering if necessary, we may assume $a_0 < a_1$, and so $a_0 = 2$.

Suppose that $a_1 > 38$, so that $f(1,x) = 0$ for $x \in [2,38]$. For $j \in [0,2]$, define $A_{0j} = \{x \in [3,38] | f(2,x) = j\}$. Let s be the least value such that $|A_{0s}| > 5 \cdot 2^s$. Such an s must exist, for if not, $36 = |[3,38]| = |A_{00} \cup A_{01} \cup A_{02}| \leqslant 5 + 10 + 20 = 35$.

We have $a_{0s} = \min(A_{0s}) \leqslant 2 + (\sum_{j=0}^{s-1} |A_{0j}|) + 1 \leqslant 3 + 5\sum_{j=0}^{s-1} 2^s = 5 \cdot 2^s - 2$.

Write $A'_{0s} = A_{0s} - \{a_{0s}\}$. Since f is $(x+1)$-regressive, $f(a_{0s},*)$ maps A'_{0s} to $[0,a_{0s}]$. But now $|[0,a_{0s}]| \leqslant 5 \cdot 2^s - 1$ and $|A'_{0s}| \geqslant 5 \cdot 2^s$, so $f(a_{0s},*)$ cannot be injective on A'_{0s}. That is, there exist $b, c \in A'_{0s}$ with $f(a_{0s},b) = f(a_{0s},c)$. The set $\{1,2,a_{0s},b,c\}$ is then min-homogeneous for f.

Now suppose $a_1 \leqslant 38$ and consider the collection of sets of the form $A_{ij} = \{x \in [3,n] | f(1,x) = i, f(a_i,x) = j\}$, where for each $i \in [0,1]$, $j \in [0,a_i]$. Let q be the number of such sets which are non-empty. Given the assumptions on a_0 and a_1, we have $q \leqslant (a_0+1) + (a_1+1) \leqslant 42$. Rename the A_{ij}'s as C_s's indexed by increasing minimum value, i.e. so that $c_l = \min(C_l)$ for $l \in [0,q-1]$, and for $s,t \in [0,q-1]$, $s < t$ implies $c_s < c_t$. Note that $a_1 \notin C_l$ for any l. Since $a_1 < a_{1j}$ for all j, and since A_{0j} is defined only for $j = 0,1,2$, there is a least $0 \leqslant k \leqslant 3$ with $a_1 < c_k$. Ultimately, our bound will be $5 \cdot 2^q + 2^{q-k} - 2$. Using reasoning similar to the $a_1 > 38$ case above, it is not hard to see that $k = 0, 1, 2, 3$ (resp.) limits a_1 to 3, 8, 18 or 38 (resp.), or yields a min-homogeneous set of cardinality 5. All cases follow from the worst case: $k = 3$ and $q = 38$.

Now suppose $f : [1, n] \to \mathbf{N}$ is regressive and with $n = 5 \cdot 2^{42} + 2^{39} - 2$. We claim there is some $s < k$ with $|C_s| > 5 \cdot 2^s$ or some $s \geqslant k$ with $|C_s| > 5 \cdot 2^s + 2^{s-k}$. If not, $n = |[1, n]| = |\{1, a_0, a_1\}| + \sum_{i=0}^{k-1} |C_s| + \sum_{i=k}^{q} |C_s| \leqslant 3 + \sum_{i=0}^{k-1} 5 \cdot 2^i + \sum_{i=k}^{q} (5 \cdot 2^i + 2^{i-k}) = 3 + 5(2^k - 1) + 5 \cdot 2^k(2^{q-k+1} - 1) + (2^{q-k+1} - 1) = 5 \cdot 2^{q+1} + 2^{q-k+1} - 3 < n$. Assume s is the least value satisfying this claim.

If $s < k$, $c_s < a_1$, so C_s is of the form A_{0i} for some i. In the worst case, the value $c_s = \min(C_s)$ may be preceded by 1, a_0, and all the elements of earlier C_i's. That is $c_s \leqslant |\{1, a_0\}| + \sum_{i=0}^{s-1} |C_i| + 1 \leqslant 3 + \sum_{i=0}^{s-1} 5 \cdot 2^i = 5 \cdot 2^s - 2$. Write $C'_s = C_s - \{c_s\}$. We have $|[0, c_s]| \leqslant 5 \cdot 2^2 - 1$ and since $|C'_s| \geqslant 5 \cdot 2^s$, the map $f(c_s, *) : C'_s \to [0, c_s]$ cannot be injective. There exist $d, e \in C'_s$ with $f(c_s, d) = f(c_s, e)$ and it follows that $\{1, a_0, c_s, d, e\} = \{1, 2, c_s, d, e\}$ is min-homogeneous.

Similarly, if $s \geqslant k$, then $a_1 < c_s$ and $c_s = \leqslant 3 + \sum_{i=0}^{k-1} 5 \cdot 2^i + \sum_{i=k}^{s-1} (5 \cdot 2^i + 2^{i-k}) + 1 = 5 \cdot 2^s + 2^{s-k} - 2$. As above, $|[0, c_s]| \leqslant 5 \cdot 2^s + 2^{s-k} - 1$ and $|C'_s| \geqslant 5 \cdot 2^s - 2^{s-k}$, so the map $f(c_s, *) : C'_s \to [0, c_s]$ cannot be injective. There must exist $d, e \in C'_s$ with $f(c_s, d) = f(c_s, e)$ and $\{1, a_i, c_s, d, e\}$ is min-homogeneous, where $i = f(1, c_s)$. ∎

Using techniques similar to those described in [1, 6], the author was able to construct a regressive function $f : [1, 195]^2 \to \mathbf{N}$ which has no min-homogeneous 6-set. We state without further proof:

PROPOSITION 3.1. $195 \leqslant R^2_{\text{reg}}(6) \leqslant 5 \cdot 2^{42} + 2^{39} - 1$.

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