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Journal of Algebra 301 (2006) 703–727

JOURNAL OF Algebra

www.elsevier.com/locate/jalgebra

Multigraded regularity: Coarsenings and resolutions

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Received 19 May 2005

Available online 9 November 2005

Communicated by Luchezar L. Avramov

Abstract

Let $S = k[x_1, \ldots, x_n]$ be a $\mathbb{Z}^r$-graded ring with $\deg(x_i) = \mathbf{a}_i \in \mathbb{Z}^r$ for each i and suppose that M is a finitely generated $\mathbb{Z}^r$-graded S-module. In this paper we describe how to find finite subsets of $\mathbb{Z}^r$ containing the multidegrees of the minimal multigraded syzygies of M. To find such a set, we first coarsen the grading of M so that we can view M as a $\mathbb{Z}$-graded S-module. We use a generalized notion of Castelnuovo–Mumford regularity, which was introduced by D. Maclagan and G. Smith, to associate to M a number which we call the regularity number of M. The minimal degrees of the multigraded minimal syzygies are bounded in terms of this invariant.

1. Introduction

Let $S = k[x_1, \ldots, x_n]$ with $\deg(x_i) = 1$ and M be a finitely generated graded S-module. The Castelnuovo–Mumford regularity of M, denoted $\operatorname{reg}(M)$, is a cohomological invariant that bounds the "size" of its minimal free resolution. The module M is d-regular if

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0021-8693/$ – see front matter
doi:10.1016/j.jalgebra.2005.09.032

$H^i_{\mathfrak{m}}(M)_p = 0$ for all $i \geqslant 0$ and all $p \geqslant d - i + 1$, where $\mathfrak{m} = \langle x_1, \dots, x_n\rangle$. The *regularity* of M is the smallest integer d for which M is d-regular. If

$$0 \to F_r \to \cdots \to F_i \to \cdots \to F_0 \to M \to 0$$

is a minimal free graded resolution of M, then the degrees of the generators of F_i are bounded above by $\operatorname{reg}(M) + i$. Hence, $\operatorname{reg}(M)$ gives a finite set which contains all of the possible degrees of minimal generators of the modules F_i.

The goal of this paper is to find a cohomological theory of regularity which will complement that of [11] and give finite bounds on the degrees of the minimal syzygies of M in a $\mathbb{Z}^r$-graded setting. More precisely, suppose that $S = k[x_1, \dots, x_n]$ is $\mathbb{Z}^r$-graded with $\deg(x_i) = \mathbf{a}_i \in \mathbb{Z}^r$ for each i, and let M be a finitely generated $\mathbb{Z}^r$-graded S-module. Under suitable hypotheses on the grading, M has a finite $\mathbb{Z}^r$-graded minimal free resolution (cf. Definition 2.1). We say that a finite subset D of $\mathbb{Z}^r$ such that the multidegree of every minimal generator of M belongs to D is a *finite bound* on the degrees of the generators of M.

Partial solutions to this problem have appeared in previous investigations into the regularity of multigraded modules (cf. [1,8,10,11,14]). We summarize these approaches here.

Maclagan and Smith [11] developed a multigraded theory of regularity for sheaves on a simplicial toric variety X with an algebraic variant defined in terms of the vanishing of graded pieces of $H^i_B(M)$, the ith local cohomology module of M. Here B is the irrelevant ideal of the homogeneous coordinate ring of X which is multigraded by a finitely generated abelian group A. Multigraded regularity, which we denote by $\operatorname{reg}_B(M)$ and refer to as the *B-regularity* of M, is a subset of A, and the degrees of the minimal syzygies of M lie in the complement of certain shifts of $\operatorname{reg}_B(M)$. As we see in Example 1.1, these complements may not be bounded if $\operatorname{rank}(A) \geqslant 2$, so B-regularity does not necessarily give a finite bound on the degrees of the minimal syzygies.

Example 1.1. Consider the bigraded coordinate ring of $\mathbb{P}^1 \times \mathbb{P}^1$, that is, $S = k[x_0, x_1, y_0, y_1]$ with $\deg(x_i) = (1, 0)$ and $\deg(y_i) = (0, 1)$. Let I be the defining ideal of the set of points

$$X = \big\{[1:0] \times [1:0], [1:0] \times [0:1], [0:1] \times [1:0], [0:1] \times [0:1]\big\}.$$

The bigraded Hilbert function of S/I is

$$H_{S/I} = \begin{bmatrix} \vdots & \vdots & \vdots & \\ 2 & 4 & 4 & \dots \\ 2 & 4 & 4 & \dots \\ 1 & 2 & 2 & \dots \end{bmatrix}$$

where we identify the value at the (i, j)th integer Cartesian coordinate with the value of $H_{S/I}(i, j)$. Applying Proposition 6.7 of [11] gives

$$\operatorname{reg}_B(S/I) = \big\{(i, j) \in \mathbb{N}^2 \mid H_{S/I}(i, j) = \deg X = 4\big\} = (1, 1) + \mathbb{N}^2.$$

The degrees of the minimal generators are contained in the unbounded set

$$\mathbb{N}^2 \backslash \big(\big((1,2) + \mathbb{N}^2\big) \cup \big((2,1) + \mathbb{N}^2\big)\big).$$

More generally, the degrees of the minimal ith syzygies are contained in

$$\mathbb{N}^2 \backslash \bigcup_{m,n \geqslant 0,\, m+n=i} \big((1,1) + (m,n)\big) + \mathbb{N}^2,$$

which is not finite. To foreshadow the approach we shall use in this paper, note that the ideal $I = \langle x_0 x_1, y_0 y_1 \rangle$ is a $\mathbb{Z}$-graded complete intersection with $\operatorname{reg}(S/I) = 2$. So, if F is any minimal ith syzygy with $\deg F = (a,b)$, then $a + b \leqslant 2 + i$. There are only a finite number of $(a,b) \in \mathbb{N}^2$ with this property.

Hoffman and the third author [10] introduced a notion of strong regularity for bigraded modules over the homogeneous coordinate ring of $\mathbb{P}^n \times \mathbb{P}^m$, $S = k[x_0, \ldots, x_n, y_0, \ldots, y_m]$, with $\deg(x_i) = (1,0)$ and $\deg(y_i) = (0,1)$. Strong regularity, which is defined in terms of the vanishing of graded pieces of the local cohomology modules $H^i_{\mathbf{x}}(M)$, $H^i_{\mathbf{y}}(M)$, and $H^i_{\mathbf{x}+\mathbf{y}}(M)$ where $\mathbf{x} = \langle x_1, \ldots, x_n \rangle$ and $\mathbf{y} = \langle y_1, \ldots, y_m \rangle$, gives a finite set that bounds the degrees of the minimal syzygies of M. This approach relies on the Mayer–Vietoris sequence to relate the various local cohomology modules. Generalizations to this approach may be possible using a more complicated spectral sequence, but we take a simpler approach here.

Extending earlier work of Aramova, Crona, and De Negri [1] (see also Römer [13]), the first two authors [14] studied the resolutions of finitely generated modules over the coordinate ring of $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_r}$. If M is such a module, one can assign to M a vector $\underline{r}(M) = (d_1, \ldots, d_r) \in \mathbb{N}^r$ called the resolution regularity vector of M. This vector provides a finite bound on the degrees of the minimal syzygies of M; more precisely, if F is a minimal multigraded ith syzygy of M, then $\underline{0} \leqslant \deg F \leqslant (d_1 + i, \ldots, d_r + i)$. Work of Hà in [8] and results in [14] investigate the relationship between the resolution regularity vector and the definition of Maclagan and Smith. Hà also introduced a multigraded analog of the a^*-invariant to study both $\underline{r}(M)$ and $\operatorname{reg}_B(M)$.

Our point of view is shaped by the observation that in the work of Hoffman and the third author a finite bound on the degrees of syzygies may be obtained by considering only vanishings of $H^i_{\mathbf{x}+\mathbf{y}}(M)$, and this may be extended to nonstandard multigraded rings. We also exploit the fact that Maclagan and Smith's definition of regularity does provide a finite bound on the degrees of syzygies if M is a finitely generated module over the coordinate ring of a weighted projective space, that is, the nonstandard $\mathbb{Z}$-graded ring $S = k[x_1, \ldots, x_n]$ with $\deg(x_i) = a_i \in \mathbb{Z}$ for each i.

Our method can be summarized as follows: Let $M = \bigoplus_{\mathbf{a} \in \mathbb{Z}^r} M_{\mathbf{a}}$ be a finitely generated $\mathbb{Z}^r$-graded module over a positively $\mathbb{Z}^r$-graded ring (see Definition 2.1) $S = k[x_1, \ldots, x_n]$. We can coarsen the grading of M to form a $\mathbb{Z}$-graded module by picking a suitable vector $\mathbf{v}$ and setting $M^{[\mathbf{v}]} = \bigoplus_{m \in \mathbb{Z}} (\bigoplus_{\mathbf{a} \cdot \mathbf{v} = m} M_{\mathbf{a}})$. The module $M^{[\mathbf{v}]}$ is an $S^{[\mathbf{v}]}$-module where $S^{[\mathbf{v}]}$ is the nonstandard graded polynomial ring with $\deg(x_i) = \mathbf{a}_i \cdot \mathbf{v}$. We associate to M a *regularity number*, denoted $\text{reg-num}_{\mathbf{v}}(M)$, where $\text{reg-num}_{\mathbf{v}}(M)$ is defined using the regularity of $M^{[\mathbf{v}]}$ as defined by Maclagan and Smith. Our main result (cf. Corollary 3.11) uses reg-num$_{\mathbf{v}}(M)$ to obtain finite bounds on which multidegrees may appear in a minimal $\mathbb{Z}^r$-graded free resolution of M.

Different vectors $\mathbf{w}$ result in different coarsenings, and hence, reg-num$_{\mathbf{w}}(M)$ may give us different bounds on the multidegrees. We examine how we can use different coarsenings of M to improve our bounds on the multidegrees. Even though there are an infinite number of possible coarsening vectors $\mathbf{w}$ (and hence, different degree bounds), we show (cf. Theorem 3.12) that all of the information on degree bounds that can be obtained from coarsenings can be obtained from a finite number of such vectors. We call a finite set of vectors with this property a minimal set of positive coarsening vectors.

This paper is structured as follows. In Section 2 we recall important notions related to multigraded rings and positive coarsenings of the grading. In Section 3 we introduce the idea of the regularity number of a $\mathbb{Z}^r$-graded module M under a given positive coarsening, and we show how this number can be used to bound degrees appearing in a minimal free resolution of M. In Section 4 we show that scalar multiples of a positive coarsening vector all give the same degree bounds. In Section 5 we relate reg-num$_{\mathbf{v}}(M)$ to the notion of the multigraded regularity in [11] and the notion of regularity based on resolutions in [14]. In Section 6 we illustrate the theory that has come before with examples.

2. Preliminaries

We briefly summarize the relevant definitions and properties of multigraded rings and multigraded resolutions in Section 2.1. We refer the reader to [12, Chapter 8] for a comprehensive introduction to multigraded rings and modules. We describe the notion of coarsening a multigrading in Section 2.2. (See also [12, Chapters 7 and 8].)

2.1. *Multigraded polynomial rings and resolutions*

2.1.1. *Multigraded rings and modules*

Let $S = k[x_1, \dots, x_n]$ with a $\mathbb{Z}^r$-grading given by a group homomorphism $\deg : \mathbb{Z}^n \to \mathbb{Z}^r$ where $\deg(x_i) = \mathbf{a}_i$ for some $\mathbf{a}_i \in \mathbb{Z}^r$. Let $Q = \deg(\mathbb{N}^n)$ denote the subsemigroup of $\mathbb{Z}^r$ generated by $\mathbf{a}_1, \dots, \mathbf{a}_n$.

If S has a $\mathbb{Z}^r$-grading, then we say $F \in S$ is homogeneous of degree $\mathbf{a}$ if all the terms of F have degree $\mathbf{a}$. We let $S_{\mathbf{a}}$ denote the k-vector space consisting of all the homogeneous forms of degree $\mathbf{a}$ for each $\mathbf{a} \in \mathbb{Z}^r$. A finitely generated S-module M is $\mathbb{Z}^r$-graded if

$$M = \bigoplus_{\mathbf{a} \in \mathbb{Z}^r} M_{\mathbf{a}} \quad \text{and} \quad S_{\mathbf{a}} M_{\mathbf{b}} \subseteq M_{\mathbf{a}+\mathbf{b}} \quad \text{for all } \mathbf{a}, \mathbf{b} \in \mathbb{Z}^r.$$

For any $\mathbf{a} \in \mathbb{Z}^r$, define $M(\mathbf{a})$ to be the finitely generated $\mathbb{Z}^r$-graded S-module where $M(\mathbf{a})_{\mathbf{p}} = M_{\mathbf{p}+\mathbf{a}}$ for all $\mathbf{p}$.

2.1.2. *Multigraded resolutions*

Given a finitely generated $\mathbb{Z}^r$-graded S-module M, we shall be interested in the $\mathbb{Z}^r$-graded minimal free resolution of M. However, this notion may not be well defined because the notion of minimality breaks down without additional hypotheses. To obtain the desired behavior, we will impose the additional constraints defined below on the grading of S that guarantee a version of Nakayama's lemma. (See [12, Chapter 8.2].)

Definition 2.1. [12, Definition 8.7] The polynomial ring S is *positively multigraded* by $\mathbb{Z}^r$ if $\deg(x_i) \neq 0$ for all i and the semigroup Q defined above has no nonzero invertible elements.

The Cox homogeneous coordinate ring of a complete smooth toric variety (see [5]) is an important example of a positively multigraded ring. We will say a bit more about this case in Section 5.1. Example 2.2 is a special case of such a ring.

Example 2.2. We say that the polynomial ring S is a *standard multigraded ring* if S is the homogeneous coordinate ring of the product of projective spaces $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_r}$. Equivalently, $S = k[x_{1,0}, \ldots, x_{1,n_1}, \ldots, x_{r,0}, \ldots, x_{r,n_r}]$ is $\mathbb{Z}^r$-graded by setting $\deg(x_{i,j}) = e_i$, the ith standard basis vector.

We assume throughout this paper that S is positively multigraded. Consequently, if M is a finitely generated $\mathbb{Z}^r$-graded S-module, then $\dim_k M_{\mathbf{a}} < \infty$ for all $\mathbf{a} \in \mathbb{Z}^r$. (See [12, Theorem 8.6] for a proof and other equivalent properties.)

In this setting there is a well-defined notion of a minimal free resolution of a $\mathbb{Z}^r$-graded module M (cf. [12, Chapter 8.3]). That is, there exists an exact complex of the form

$$\mathbf{F}_\bullet : 0 \to F_\ell \to F_{\ell-1} \to \cdots \to F_1 \to F_0 \to M \to 0 \tag{1}$$

such that $\ell \leqslant n$, and $F_i = \bigoplus_{\mathbf{a} \in \mathbb{Z}^r} S(-\mathbf{a})^{\beta_{i,\mathbf{a}}(M)}$ is finitely generated. The number

$$\beta_{i,\mathbf{a}}(M) = \dim_k \operatorname{Tor}_i^S(M, k)_{\mathbf{a}}$$

is the ith graded Betti number of M of degree $\mathbf{a}$. This invariant of M counts the number of minimal generators of degree $\mathbf{a}$ in the ith syzygy module of M. Our goal is to find a mechanism for finding a finite set $\mathcal{D}_i(M) \subset \mathbb{Z}^r$ such that $\beta_{i,\mathbf{a}}(M)$ is always zero for $\mathbf{a} \in \mathbb{Z}^r \backslash \mathcal{D}_i(M)$. In other words, we wish to provide a finite set of possible values for the degrees of the minimal ith syzygies.

Although the degrees of elements of S lie in Q, the degrees of elements of M may not. However, by [12, Theorem 8.20], the multigraded Hilbert series of M, $H(M; \mathbf{t}) = \sum_{\mathbf{a} \in \mathbb{Z}^r} \dim_k(M_{\mathbf{a}}) \mathbf{t}^{\mathbf{a}}$ lies in $\mathbb{Z}[[Q]][\mathbb{Z}^r] = \mathbb{Z}[[Q]] \otimes_{\mathbb{Z}[Q]} \mathbb{Z}[\mathbb{Z}^r]$, the ring of Laurent series supported on *finitely many translates of Q* (see [12, p. 155]). This just means that the degrees of elements of M lie in $(\mathbf{b}_1 + Q) \cup \cdots \cup (\mathbf{b}_k + Q)$ for some finite collection of $\mathbf{b}_i$'s in $\mathbb{Z}^r$.

2.2. Coarsening gradings

If M is a finitely generated $\mathbb{Z}^r$-graded S-module, we can *coarsen* the $\mathbb{Z}^r$-grading, thus allowing us to view M as a $\mathbb{Z}$-graded module (see [12, Corollary 7.23]). We pass to a

$\mathbb{Z}$-grading by choosing a vector $\mathbf{v} \in \mathbb{Z}^r$ and defining $\deg_{\mathbf{v}}(m) := \deg(m) \cdot \mathbf{v}$ for $m \in M$ using the dot product in $\mathbb{Z}^r$. Note that $\deg_{\mathbf{v}}(x_i m) = \deg_{\mathbf{v}}(x_i) + \deg_{\mathbf{v}}(m)$ for all variables x_i and elements $m \in M$.

We write $S^{[\mathbf{v}]}$ to denote the $\mathbb{Z}$-graded polynomial ring $S^{[\mathbf{v}]}$ with $(S^{[\mathbf{v}]})_m = \bigoplus_{\mathbf{a}\cdot\mathbf{v}=m} S_{\mathbf{a}}$ and $M^{[\mathbf{v}]}$ for the $\mathbb{Z}$-graded $S^{[\mathbf{v}]}$-module $M^{[\mathbf{v}]} = \bigoplus_{m\in\mathbb{Z}}(\bigoplus_{\mathbf{a}\cdot\mathbf{v}=m} M_{\mathbf{a}})$. If the coarsening vector is clear, we will sometimes drop the superscript.

We want the $\mathbb{Z}$-graded ring $S^{[\mathbf{v}]}$, which can be viewed as the coordinate ring of a weighted projective space, to be positively graded.

Definition 2.3. A vector $\mathbf{v} \in \mathbb{Z}^r$ is a *positive coarsening vector* for the $\mathbb{Z}^r$-graded ring S if $\deg_{\mathbf{v}}(x_i) > 0$ for all i.

Lemma 2.4. *If S is positively $\mathbb{Z}^r$-graded, then a positive coarsening vector exists.*

Proof. The set

$$\operatorname{pos}(Q) = \left\{ \sum \lambda_i \mathbf{a}_i \;\middle|\; \lambda_i \in \mathbb{R}_{\geqslant 0} \right\}$$

is the convex cone in $\mathbb{R}^r$ generated by the vectors in Q. Our assumption that S is positively multigraded implies that $\operatorname{pos}(Q)$ is a pointed cone, i.e., $\operatorname{pos}(Q)$ does not contain any non-trivial linear subspaces. If $\operatorname{pos}(Q)$ is pointed, then $\mathbf{0}$ is a face, so there exist hyperplanes $H \subset \mathbb{R}^r$ so that $\operatorname{pos}(Q) - \{\mathbf{0}\}$ lies strictly on one side of H. If we take $\mathbf{v}$ to be a normal vector to H on the same side of H as $\operatorname{pos}(Q)$, then $\deg_{\mathbf{v}}(x_i) > 0$ for each i. $\square$

Remark 2.5. Let S be positively multigraded by $\mathbb{Z}^r$ with degrees in the subsemigroup $Q \subset \mathbb{Z}^r$, and let M be a finitely generated $\mathbb{Z}^r$-graded S-module. If $\mathbf{v} \in \mathbb{Z}^r$ is a positive coarsening vector, then $\dim_k(M^{[\mathbf{v}]})_m < \infty$ for all $m \in \mathbb{Z}$ since the ring $S^{[\mathbf{v}]}$ is positively graded.

Suppose $\mathbf{v}$ is a positive coarsening vector for S, and that we have a multigraded minimal free resolution of the $\mathbb{Z}^r$-graded S-module M of the form (1). Using $\mathbf{v}$ we can pass to a $\mathbb{Z}$-graded minimal free resolution of $M^{[\mathbf{v}]}$ as an $S^{[\mathbf{v}]}$-module:

$$0 \to F_\ell^{[\mathbf{v}]} \to F_{\ell-1}^{[\mathbf{v}]} \to \cdots \to F_1^{[\mathbf{v}]} \to F_0^{[\mathbf{v}]} \to M^{[\mathbf{v}]} \to 0 \tag{2}$$

where

$$F_i^{[\mathbf{v}]} = \bigoplus_{\mathbf{a}\in\mathbb{Z}^r} S^{[\mathbf{v}]}(-\mathbf{a}\cdot\mathbf{v})^{\beta_{i,\mathbf{a}}(M)} = \bigoplus_{m\in\mathbb{Z}} S^{[\mathbf{v}]}(-m)^{\sum_{\mathbf{a}\cdot\mathbf{v}=m} \beta_{i,\mathbf{a}}(M)}.$$

The above expression allows us to deduce that

$$\sum_{\mathbf{a}\cdot\mathbf{v}=m} \beta_{i,\mathbf{a}}(M) = \beta_{i,m}\big(M^{[\mathbf{v}]}\big).$$

The heart of our method is to find bounds on m such that $\beta_{i,m}(M^{[\mathbf{v}]}) \neq 0$ by investigating the regularity of $M^{[\mathbf{v}]}$ as a $\mathbb{Z}$-graded $S^{[\mathbf{v}]}$-module. This, in turn, gives us information on which $\mathbf{a} \in \mathbb{Z}^r$ have the property that $\beta_{i,\mathbf{a}}(M) \neq 0$.

3. Regularity of $\mathbb{Z}$-graded modules

Our main tool will be a special case of the multigraded regularity introduced in [11] which is also related to the commutative version of the notion of regularity introduced in [2]. Throughout this section we assume that M is a finitely generated $\mathbb{Z}^r$-graded S-module and that $\mathbf{v} \in \mathbb{Z}^r$ is a positive coarsening vector.

Specializing [11, Definition 4.1] gives us:

Definition 3.1. Let M be a finitely generated $\mathbb{Z}^r$-graded S-module M and let $\mathbf{v}$ be a positive coarsening vector. Let $\mathfrak{m} = \langle x_1, \ldots, x_n \rangle$ and $c_{\mathbf{v}} = \operatorname{lcm}(\deg_{\mathbf{v}}(x_i))_{i=1}^n$. Define

$$\operatorname{reg}_{\mathbf{v}}(M) = \left\{ p \in \mathbb{Z} \mid H^i_{\mathfrak{m}}(M)_q = 0,\ \forall i \geqslant 0,\ \forall q \in p + \mathbb{N}c_{\mathbf{v}}[1-i] \right\}$$

where

$$\mathbb{N}c_{\mathbf{v}}[i] = ic_{\mathbf{v}} + \mathbb{N}c_{\mathbf{v}} = \left\{ c_{\mathbf{v}}(i+d) \mid d \in \mathbb{N} \right\}.$$

Note that $\operatorname{reg}_{\mathbf{v}}(M)$ is a special case of the definition of the regularity of $M^{[\mathbf{v}]}$ given in [11].

3.1. The regularity number

The vanishing conditions required by the definition of $\operatorname{reg}_{\mathbf{v}}(M)$ are only required for shifted multiples of $c_{\mathbf{v}}$. So, if $c_{\mathbf{v}} > 1$, then $p \in \operatorname{reg}_{\mathbf{v}}(M)$ may not imply that $q \in \operatorname{reg}_{\mathbf{v}}(M)$ for all $q \geqslant p$ as demonstrated in the following example.

Example 3.2. Suppose that $S = K[x_1, x_2]$ with $\deg(x_i) = 4$ for $i = 1, 2$. Then since $H^0_{\mathfrak{m}}(S) = H^1_{\mathfrak{m}}(S) = 0$,

$$\begin{aligned}\operatorname{reg}(S) &= \left\{ u \in \mathbb{Z} \mid H^2_{\mathfrak{m}}(S)_p = 0\ \forall p \in u + 4\mathbb{N}[1-2] \right\} \\ &= \left\{ u \in \mathbb{Z} \mid H^2_{\mathfrak{m}}(S)_p = 0\ \forall p \in u - 4 + 4\mathbb{N} \right\}.\end{aligned}$$

If we compute $H^2_{\mathfrak{m}}(S)$ using a Čech complex, then $H^2_{\mathfrak{m}}(S)$ is a quotient of $S_{x_1x_2}$ by all elements of $S_{x_1x_2}$ that do not have both x_1 and x_2 in the denominator when written in lowest terms. Since all of the elements of $S_{x_1x_2}$ have degrees that are multiples of 4, $H^2_{\mathfrak{m}}(S)_p = 0$ for all p that are not multiples of 4.

Of course it is also true that $H^2_{\mathfrak{m}}(S)_p = 0$ for all $p > -8$ since any monomial written in lowest terms that has degree > -8 cannot have both x_1 and x_2 in the denominator. So in this example, if $q > -4$, then

$$H^2_{\mathfrak{m}}(S)_{q-4+4m} = 0 \quad \text{for all } m \in \mathbb{N}$$

since $q-4>-8$. Therefore, $q\in\operatorname{reg}(S)$, if $q>-4$, and as well, $-5\in\operatorname{reg}(S)$. However $-4\notin\operatorname{reg}(S)$ since $H^2_{\mathfrak m}(S)_{-4}\neq 0$.

Despite the behavior exhibited in Example 3.2, we can guarantee that there is an integer $p\in\operatorname{reg}_{\mathbf v}(M)$ such that $q\in\operatorname{reg}_{\mathbf v}(M)$ for all $q\geqslant p$.

Theorem 3.3. *Let M be a finitely generated $\mathbb{Z}^r$-graded S-module. Then there exists a $p\in\operatorname{reg}_{\mathbf v}(M)$ such that $q\geqslant p$ implies that $q\in\operatorname{reg}_{\mathbf v}(M)$.*

In light of the previous result we have:

Definition 3.4. The *regularity number* of M (with respect to a positive coarsening vector $\mathbf v$), denoted $\operatorname{reg-num}_{\mathbf v}(M)$, is

$$\operatorname{reg-num}_{\mathbf v}(M):=\inf\bigl\{p\in\operatorname{reg}_{\mathbf v}(M)\mid q\in\operatorname{reg}_{\mathbf v}(M)\text{ for all }q\geqslant p\bigr\}.$$

We need the following lemma to prove Theorem 3.3.

Lemma 3.5. *There exists a $p\in\operatorname{reg}_{\mathbf v}(S)$ such that $q\geqslant p$ implies that $q\in\operatorname{reg}_{\mathbf v}(S)$. In particular*

$$\operatorname{reg-num}_{\mathbf v}(S)=(n-1)c_{\mathbf v}+1-\sum\deg_{\mathbf v}(x_i).$$

Proof. Let $\mathfrak m=\langle x_1,\dots,x_n\rangle$. The set

$$\begin{aligned}\operatorname{reg}_{\mathbf v}(S)&=\bigl\{u\in\mathbb Z\mid H^i_{\mathfrak m}(S)_w=0\ \forall i\geqslant 0\text{ and }\forall w\in u+\mathbb N c_{\mathbf v}[1-i]\bigr\}\\&=\bigl\{u\in\mathbb Z\mid H^n_{\mathfrak m}(S)_w=0\ \forall w\in u+\mathbb N c_{\mathbf v}[1-n]\bigr\}\\&=\bigl\{u\in\mathbb Z\mid H^n_{\mathfrak m}(S)_w=0\ \forall w\in u+(1-n)c_{\mathbf v}+\mathbb N c_{\mathbf v}\bigr\}.\end{aligned}$$

The module $H^n_{\mathfrak m}(S)$ has a $\mathbb Z^n$-grading and $H^n_{\mathfrak m}(S)_{\mathbf q}\neq 0$ for $\mathbf q\in\mathbb Z^n$ if and only if all the coordinates of $\mathbf q$ are negative (see [6]). Therefore, it is certainly true that $H^n_{\mathfrak m}(S)_w=0$ if $w>-\sum\deg_{\mathbf v}(x_i)$. Hence

$$\Bigl\{u\in\mathbb Z\Bigm| u>(n-1)c_{\mathbf v}-\sum\deg_{\mathbf v}(x_i)\Bigr\}\subseteq\operatorname{reg}_{\mathbf v}(S).$$

Note that the inclusion may be strict as in Example 3.2.

We see that $\operatorname{reg-num}_{\mathbf v}(S)$ cannot be equal to $(n-1)c_{\mathbf v}-\sum\deg_{\mathbf v}(x_i)$ since then we would have $H^n_{\mathfrak m}(S)_w=0$ where

$$w=(n-1)c_{\mathbf v}-\sum\deg_{\mathbf v}(x_i)+(1-n)c_{\mathbf v}=-\sum\deg_{\mathbf v}(x_i).$$

Since $-\sum\deg_{\mathbf v}(x_i)$ is the total degree of a Laurent monomial in which all of the exponents are negative, we know that $H^n_{\mathfrak m}(S)_w\neq 0$. □

Corollary 3.6. *Let S be as above and let $\mathbf{d}_i \in \mathbb{Z}^r$. Then*

$$\text{reg-num}_{\mathbf{v}}\big(S(-\mathbf{d}_1) \oplus \cdots \oplus S(-\mathbf{d}_n)\big) = \max\big\{\text{reg-num}_{\mathbf{v}}(S) + \mathbf{d}_i \cdot \mathbf{v}\big\}_{i=1}^{n}.$$

Proof. This follows from the fact that

$$H^i_{\mathfrak{m}}\big(S(-\mathbf{d}_1) \oplus \cdots \oplus S(-\mathbf{d}_n)\big)_w \cong H^i_{\mathfrak{m}}(S)_{w-\mathbf{d}_1\cdot\mathbf{v}} \oplus \cdots \oplus H^i_{\mathfrak{m}}(S)_{w-\mathbf{d}_n\cdot\mathbf{v}}. \qquad \square$$

Proof of Theorem 3.3. We use induction on the projective dimension of M. The result is trivial if $M = 0$, and if M has projective dimension zero, then the result holds by the previous lemma since M is free in this case.

Suppose now that the projective dimension of M is greater than 0. Then there exists a short exact sequence

$$0 \to M' \to F \to M \to 0$$

where F is a free module. Since M' has projective dimension strictly less than the projective dimension of M, the result holds for M'. Therefore, there exist p_0, p_1 such that $p_0 \in \text{reg}_{\mathbf{v}}(M')$ implies $q \in \text{reg}_{\mathbf{v}}(M')$ for all $q \geqslant p_0$ and $p_1 \in \text{reg}_{\mathbf{v}}(F)$ implies $q \in \text{reg}_{\mathbf{v}}(F)$ for all $q \geqslant p_1$. Using [11, Lemma 7.1], we see that

$$\big(\text{reg}_{\mathbf{v}}(M') - c_{\mathbf{v}}\big) \cap \text{reg}_{\mathbf{v}}(F) \subseteq \text{reg}_{\mathbf{v}}(M).$$

Hence $q \in \text{reg}_{\mathbf{v}}(M)$ for all $q \geqslant \max\{p_0 - c_{\mathbf{v}}, p_1\}$. $\square$

3.2. Bounds on the degrees of the syzygies

The following theorem and its corollary are the main results of this paper. Both results give bounds on the degrees in $\mathbb{Z}^r$ that may appear in a free resolution of M. These results are similar to those in [2, §5] and [11, §7] (see Section 5).

Theorem 3.7. *Let M be a finitely generated $\mathbb{Z}^r$-graded S-module, and*

$$s_{\mathbf{v}} := \max\Big\{nc_{\mathbf{v}} - \sum \deg_{\mathbf{v}}(x_i), c_{\mathbf{v}}\Big\}.$$

If F is a minimal generator of the ith syzygy module of $M^{[\mathbf{v}]}$, then

$$\deg F \leqslant \text{reg-num}_{\mathbf{v}}(M) + is_{\mathbf{v}} + c_{\mathbf{v}} - 1.$$

Remark 3.8. Note that if $\deg_{\mathbf{v}}(x_i) = 1$ for all i, then $c_{\mathbf{v}} = s_{\mathbf{v}} = 1$ and the statement of the above theorem specializes to the statement that F_i is generated by elements of degree at most $\text{reg-num}_{\mathbf{v}}(M) + is_{\mathbf{v}} + 1 - 1 = \text{reg-num}_{\mathbf{v}}(M) + i$. So we recover the usual result in the standard graded case.

We need two familiar results about $\text{reg-num}_{\mathbf{v}}(M)$ that follow from results in [11].

Lemma 3.9. *If*

$$0 \to M' \to M \to M'' \to 0$$

is a short exact sequence of finitely generated $\mathbb{Z}^r$*-graded modules, then*

(1) $\operatorname{reg-num}_{\mathbf{v}}(M'') \leqslant \max\{\operatorname{reg-num}_{\mathbf{v}}(M), \operatorname{reg-num}_{\mathbf{v}}(M') - c_{\mathbf{v}}\}$.
(2) $\operatorname{reg-num}_{\mathbf{v}}(M) \leqslant \max\{\operatorname{reg-num}_{\mathbf{v}}(M'), \operatorname{reg-num}_{\mathbf{v}}(M'')\}$.
(3) $\operatorname{reg-num}_{\mathbf{v}}(M') \leqslant \max\{\operatorname{reg-num}_{\mathbf{v}}(M), \operatorname{reg-num}_{\mathbf{v}}(M'') + c_{\mathbf{v}}\}$.

Proof. The result follows from Lemma 7.1 of [11] and the definition of $\operatorname{reg-num}_{\mathbf{v}}(M)$. □

Lemma 3.10. *Let M be a finitely generated* $\mathbb{Z}^r$*-graded module. Then the minimal generators of* $M^{[\mathbf{v}]}$ *have degrees at most* $\operatorname{reg-num}_{\mathbf{v}}(M) + c_{\mathbf{v}} - 1$.

Proof. The result follows from Theorem 5.4 in [11] since we have taken

$$c_{\mathbf{v}} = \operatorname{lcm}\bigl(\deg_{\mathbf{v}}(x_i)\bigr)_{i=1}^{n}. \qquad \square$$

Proof of Theorem 3.7. We construct a minimal free resolution $\mathbf{F}_\bullet$ satisfying the claim as follows. By Lemma 3.10 we know that the minimal generators of M have degree at most $\operatorname{reg-num}_{\mathbf{v}}(M) + c_{\mathbf{v}} - 1$. So we have

$$0 \to M_1 \to F_0 \to M \to 0$$

and we see that

$$\begin{aligned}\operatorname{reg-num}_{\mathbf{v}}(M_1) &\leqslant \max\Bigl\{(n-1)c_{\mathbf{v}} - \sum \deg_{\mathbf{v}}(x_i) + 1 + \bigl(\operatorname{reg-num}_{\mathbf{v}}(M) + c_{\mathbf{v}} - 1\bigr), \\ &\qquad \operatorname{reg-num}_{\mathbf{v}}(M) + c_{\mathbf{v}}\Bigr\} \\ &= \max\Bigl\{nc_{\mathbf{v}} - \sum \deg_{\mathbf{v}}(x_i) + \operatorname{reg-num}_{\mathbf{v}}(M), \operatorname{reg-num}_{\mathbf{v}}(M) + c_{\mathbf{v}}\Bigr\} \\ &= s_{\mathbf{v}} + \operatorname{reg-num}_{\mathbf{v}}(M).\end{aligned}$$

Now we can proceed by induction on the projective dimension of M. There exists a short exact sequence

$$0 \to M_{i+1} \to F_i \to M_i \to 0$$

where $\operatorname{reg-num}_{\mathbf{v}}(M_i) \leqslant i s_{\mathbf{v}} + \operatorname{reg-num}_{\mathbf{v}}(M)$. Then F_i is generated by elements of degree at most $i s_{\mathbf{v}} + \operatorname{reg-num}_{\mathbf{v}}(M) + c_{\mathbf{v}} - 1$. Furthermore,

$$\begin{aligned}\text{reg-num}_{\mathbf{v}}(M_{i+1}) &\leqslant \max\Big\{(n-1)c_{\mathbf{v}} - \sum \deg_{\mathbf{v}}(x_i) + 1 + \big(is_{\mathbf{v}} + \text{reg-num}_{\mathbf{v}}(M) + c_{\mathbf{v}} - 1\big),\\ &\qquad is_{\mathbf{v}} + \text{reg-num}_{\mathbf{v}}(M) + c_{\mathbf{v}}\Big\}\\ &= \max\Big\{nc_{\mathbf{v}} - \sum \deg_{\mathbf{v}}(x_i) + is_{\mathbf{v}} + \text{reg-num}_{\mathbf{v}}(M),\\ &\qquad is_{\mathbf{v}} + \text{reg-num}_{\mathbf{v}}(M) + c_{\mathbf{v}}\Big\}\\ &= (i+1)s_{\mathbf{v}} + \text{reg-num}_{\mathbf{v}}(M).\end{aligned}$$

We conclude that F_{i+1} is generated by elements of degree at most

$$(i+1)s_{\mathbf{v}} + \text{reg-num}_{\mathbf{v}}(M) + c_{\mathbf{v}} - 1$$

by Lemma 3.10. □

Corollary 3.11. *Let* $\mathbf{F}_\bullet$ *be a minimal free* $\mathbb{Z}^r$*-graded resolution of* M*. Assume that the Hilbert series of* M *is supported on* $\bigcup(\mathbf{b}_i + Q)$ *where* $\mathbf{b}_1, \ldots, \mathbf{b}_k \in \mathbb{Z}^r$*. Then the minimal generators of* F_i *have multidegrees contained in the finite set*

$$\mathcal{D}_{i,\mathbf{v}}(M) := \Big\{\mathbf{a} \in \bigcup(\mathbf{b}_i + Q) \;\Big|\; \mathbf{a}\cdot\mathbf{v} \leqslant \text{reg-num}_{\mathbf{v}}(M) + is_{\mathbf{v}} + c_{\mathbf{v}} - 1\Big\}.$$

3.3. Minimal sets of positive coarsening vectors

If $\mathbf{v}$ is a positive coarsening vector, then the set $\mathcal{D}_{i,\mathbf{v}}(M)$ of Corollary 3.11 bounds the multidegrees of the generators of F_i, the ith syzygy module of M. In fact, since Corollary 3.11 holds for every positive coarsening vector $\mathbf{v}$, we see that we may strengthen the result as follows: the minimal generators of F_i have multidegrees contained in

$$\mathcal{D}_i(M) := \bigcap_{\mathbf{v}} \mathcal{D}_{i,\mathbf{v}}(M) = \bigcap_{\mathbf{v}} \Big\{\mathbf{a} \in \bigcup(\mathbf{b}_k + Q) \;\Big|\; \mathbf{a}\cdot\mathbf{v} \leqslant \text{reg-num}_{\mathbf{v}}(M) + is_{\mathbf{v}} + c_{\mathbf{v}} - 1\Big\}$$

where the intersection is taken over all vectors $\mathbf{v}$ that are positive coarsening vectors.

Although there are an infinite number of positive coarsening vectors, the region $\mathcal{D}_i(M)$ may be computed from only finitely many positive coarsening vectors $\mathbf{v}$.

Theorem 3.12. *Let* M *be a finitely generated* $\mathbb{Z}^r$*-graded* S*-module. Then there exists a finite set of positive coarsening vectors* $\mathcal{V} = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ *such that*

$$\mathcal{D}_i(M) = \bigcap_{\mathbf{v}_j \in \mathcal{V}} \Big\{\mathbf{a} \in \bigcup(\mathbf{b}_k + Q) \;\Big|\; \mathbf{a}\cdot\mathbf{v}_j \leqslant \text{reg-num}_{\mathbf{v}_j}(M) + is_{\mathbf{v}_j} + c_{\mathbf{v}_j} - 1\Big\}.$$

Proof. Let $\mathbf{v}$ be any positive coarsening vector for M. Then $\mathcal{D}_{i,\mathbf{v}}(M)$ contains only finitely many integral vectors. We know that $\mathcal{D}_i(M) \subseteq \mathcal{D}_{i,\mathbf{v}}(M)$ by definition. Since $\mathcal{D}_{i,\mathbf{v}}(M)$ is finite, we see that the set $\mathcal{D}_{i,\mathbf{v}}(M) - \mathcal{D}_i(M)$ must also be finite. For each $\mathbf{p} \in \mathcal{D}_{i,\mathbf{v}}(M) - \mathcal{D}_i(M)$ there must be some positive coarsening vector $\mathbf{v_p}$ such that

$$\mathbf{p} \cdot \mathbf{v_p} > \text{reg-num}_{\mathbf{v_p}}(M) + i s_{\mathbf{v_p}} + c_{\mathbf{v_p}} - 1.$$

The set of all $\mathbf{v_p}$ together with $\mathbf{v}$ is finite and $\mathcal{D}_i(M)$ may be computed from these vectors. □

In light of the previous result, it is natural to introduce the following definition:

Definition 3.13. Let M be a finitely generated $\mathbb{Z}^r$-graded S-module. We shall say a set $\mathcal{V} = \mathcal{V}(M) = \{\mathbf{v}_1, \dots, \mathbf{v}_n\} \subseteq \mathbb{Z}^r$ is a *minimal set of positive coarsening vectors* for M if

(1) each $\mathbf{v}_i$ is a positive coarsening vector of S,
(2) $\mathcal{D}_i(M) = \bigcap_{\mathbf{v}_j \in \mathcal{V}} \mathcal{D}_{i,\mathbf{v}_j}(M)$,
(3) if $\mathcal{V}'$ is a proper subset of $\mathcal{V}$, then $\mathcal{D}_i(M) \neq \bigcap_{\mathbf{v} \in \mathcal{V}'} \mathcal{D}_{i,\mathbf{v}}(M)$.

Remark 3.14. As we shall see in Section 4, it appears quite difficult to find a minimal set of positive coarsening vectors (see Remark 4.5 for more details). We therefore propose some natural questions about this minimal set: is there a method to identify $\mathcal{V}$ from the $\mathbb{Z}^r$-grading? is the size of $\mathcal{V}$ an invariant? is $\mathcal{V}$ unique?

3.4. *Further properties of the regularity number*

It is clear from the above discussion that $\text{reg-num}_{\mathbf{v}}(M)$ gives information about the $\mathbb{Z}$-graded resolution of $M^{[\mathbf{v}]}$. In the standard graded case, the process can be reversed, that is, the regularity of M can be read off of the resolution. As the following result and example show, we can obtain a lower bound on $\text{reg-num}_{\mathbf{v}}(M)$ from the $\mathbb{Z}$-graded resolution of $M^{[\mathbf{v}]}$, but we cannot determine an upper bound on $\text{reg-num}_{\mathbf{v}}(M)$ from the invariants of the resolution.

Theorem 3.15. *Let M be a finitely generated positively $\mathbb{Z}^r$-graded S-module, and let* $\mathbf{v}$ *be a positive coarsening vector. If*

$$0 \to F_\ell^{[\mathbf{v}]} \to F_{\ell-1}^{[\mathbf{v}]} \to \cdots \to F_1^{[\mathbf{v}]} \to F_0^{[\mathbf{v}]} \to M^{[\mathbf{v}]} \to 0$$

where $F_i^{[\mathbf{v}]} = \bigoplus_{j \in \mathbb{Z}} S^{[\mathbf{v}]}(-j)^{\beta_{i,j}(M^{[\mathbf{v}]})}$ is a minimal $\mathbb{Z}$-graded resolution of $M^{[\mathbf{v}]}$, then

$$\max_{i,j}\bigl\{j - i s_{\mathbf{v}} - c_{\mathbf{v}} + 1 \mid \beta_{i,j}\bigl(M^{[\mathbf{v}]}\bigr) \neq 0\bigr\} \leqslant \text{reg-num}_{\mathbf{v}}(M).$$

Proof. Suppose $\beta_{i,j}(M^{[\mathbf{v}]}) \neq 0$. So, there exists a minimal generator of the ith syzygy module of degree j. By Theorem 3.7 we must have

$$j \leqslant \text{reg-num}_{\mathbf{v}}(M) + i s_{\mathbf{v}} + c_{\mathbf{v}} - 1 \quad \Leftrightarrow \quad j - i s_{\mathbf{v}} - c_{\mathbf{v}} + 1 \leqslant \text{reg-num}_{\mathbf{v}}(M).$$

The above inequality is true for all i, j with $\beta_{i,j}(M^{[\mathbf{v}]}) \neq 0$, so it also holds for the maximum such value. □

When $\mathbf{v} = (1, \dots, 1)$, then $\text{reg-num}_{\mathbf{v}}(M)$ is simply the Castelnuovo–Mumford regularity of M, and in this case, $\text{reg-num}_{\mathbf{v}}(M)$ equals $\max_{i,j}\{j - is_{\mathbf{v}} - c_{\mathbf{v}} + 1 \mid \beta_{i,j}(M^{[\mathbf{v}]}) \neq 0\}$. In fact, this is often taken as a definition. In general, $\text{reg-num}_{\mathbf{v}}(M)$ may be bigger than this lower bound as exhibited in the following example.

Example 3.16. Let $S = k[x, y]$ with $\deg(x) = (1, 0)$ and $\deg(y) = (0, 1)$. Let M be the free S-module $M = S(-1, 0) \oplus S(0, -1)$. Now M has projective dimension equal to zero, so a resolution of M is simply

$$0 \to S(-1, 0) \oplus S(0, -1) \to M \to 0.$$

A positive coarsening vector $\mathbf{v}$ of S must have the form $\mathbf{v} = (v_1, v_2) \geqslant (1, 1)$. Thus, a resolution of $M^{[\mathbf{v}]}$ as a $\mathbb{Z}$-graded module is

$$0 \to S^{[\mathbf{v}]}(-v_1) \oplus S^{[\mathbf{v}]}(-v_2) \to M^{[\mathbf{v}]} \to 0.$$

Then

$$\max\bigl\{j - is_{\mathbf{v}} - c_{\mathbf{v}} + 1 \mid \beta_{i,j}\bigl(M^{[\mathbf{v}]}\bigr) \neq 0\bigr\} = \max\{v_1 - c_{\mathbf{v}} + 1, v_2 - c_{\mathbf{v}} + 1\}.$$

On the other hand, by Corollary 3.6 we have

$$\text{reg-num}_{\mathbf{v}}(M) = \text{reg-num}_{\mathbf{v}}(S) + \max\{v_1, v_2\} = c_{\mathbf{v}} + 1 - v_1 - v_2 + \max\{v_1, v_2\}.$$

Suppose that $\mathbf{v} = (v_1, v_2)$ has been picked so that $\gcd(v_1, v_2) = 1$ and $v_1 > v_2 > 1$ (such pairs exist, e.g., $\mathbf{v} = (5, 3)$). Then $c_{\mathbf{v}} = v_1 v_2$ and

$$\begin{aligned}\max\bigl\{j - is_{\mathbf{v}} - c_{\mathbf{v}} + 1 \mid \beta_{i,j}\bigl(M^{[\mathbf{v}]}\bigr) \neq 0\bigr\} &= v_1 - v_1 v_2 + 1 = v_1(1 - v_2) + 1 \\ &< v_1 v_2 - v_2 + 1 = \text{reg-num}_{\mathbf{v}}(M).\end{aligned}$$

Note that we may also define the familiar a-invariants of $M^{[\mathbf{v}]}$ in this context as well:

Definition 3.17. Let M be a finitely generated positively $\mathbb{Z}^r$-graded S-module and let $\mathbf{v}$ be a positive coarsening vector. For each $i \geqslant 0$, let

$$a^i\bigl(M^{[\mathbf{v}]}\bigr) := \max\bigl\{p \mid H^i_{\mathfrak{m}}\bigl(M^{[\mathbf{v}]}\bigr)_p \neq 0\bigr\}$$

where $a^i(M^{[\mathbf{v}]}) = -\infty$ if $H^i_{\mathfrak{m}}(M^{[\mathbf{v}]}) = 0$.

Remark 3.18. Benson [2] defines a notion of regularity for a module M over a graded commutative ring H with nonstandard $\mathbb{Z}$-grading to be $\max\{a^i(M)+i\}$. However, to link a notion of regularity defined in terms of local cohomology with degrees of generators as in Lemma 3.10, the standard proof proceeds by induction on the dimension of M via a ring element x that is a nonzero divisor (or almost a nonzerodivisor) on M (see [2,3,11]). Since $c_{\mathbf{v}}$ is the least common multiple of the degrees of the variables x_i, we know we can find an element that is almost a nonzerodivisor in degree $c_{\mathbf{v}}$ from [11, Proposition 3.1]. This is why we require vanishings in local cohomology shifted by $c_{\mathbf{v}}$. (See also [11, Theorem 5.4].) Benson links his notion of regularity to free resolutions by passing to a Noether normalization R of H generated by elements which form a filter-regular (almost regular) sequence on M. Under suitable hypotheses there are nice links between Benson's regularity and free resolutions of M over R. However, since we are interested in resolutions of M over S, we have chosen to follow along the lines of [11].

The quantity $\text{reg-num}_{\mathbf{v}}(M)$ interacts with the a-invariants as follows:

Theorem 3.19. *Keeping notation and hypotheses as above,*

$$\text{reg-num}_{\mathbf{v}}(M)=\max_i\left\{a^i\left(M^{[\mathbf{v}]}\right)-c_{\mathbf{v}}(1-i)+1\right\}.$$

Proof. First, we show that $a^i(M^{[\mathbf{v}]})-c_{\mathbf{v}}(1-i)+1\leqslant \text{reg-num}_{\mathbf{v}}(M)$ for all i. If $r=\text{reg-num}_{\mathbf{v}}(M)$, then $H^i_{\mathfrak{m}}(M)_p=0$ for all $p\in r+j+c_{\mathbf{v}}(1-i)+\mathbb{N}c_{\mathbf{v}}$, for all $j\in\mathbb{N}$ and in particular, for all $p\geqslant r+c_{\mathbf{v}}(1-i)$. Therefore,

$$a^i\left(M^{[\mathbf{v}]}\right)\leqslant r+c_{\mathbf{v}}(1-i)-1$$

or $a^i(M^{[\mathbf{v}]})-c_{\mathbf{v}}(1-i)+1\leqslant \text{reg-num}_{\mathbf{v}}(M)$ for all i.

Now we show that $\text{reg-num}_{\mathbf{v}}(M)\leqslant \max_i\{a^i(M^{[\mathbf{v}]})-c_{\mathbf{v}}(1-i)+1\}$. Suppose that $p\geqslant \max_i\{a^i(M^{[\mathbf{v}]})-c_{\mathbf{v}}(1-i)+1\}$. It suffices to show that $p\in \text{reg}_{\mathbf{v}}(M)$, which holds if and only if $H^i_{\mathfrak{m}}(M)_q=0$ for all $i\geqslant 0$ and for all

$$q\in p+c_{\mathbf{v}}(1-i)+\mathbb{N}c_{\mathbf{v}}.$$

Since $q>a^i(M^{[\mathbf{v}]})$ we have $\text{reg-num}_{\mathbf{v}}(M)\leqslant \max_i\{a^i(M^{[\mathbf{v}]})-c_{\mathbf{v}}(1-i)+1\}$. □

4. Positive coarsening vectors and their scalar multiples

In this section we show that if $\mathbf{v}$ is a positive coarsening vector, then no new information on the degrees of the generators is obtained by using the positive coarsening vector $d\mathbf{v}$ with $d\in\mathbb{N}_{>0}$. As a consequence, if $\mathbf{v}=(v_1,\dots,v_r)$ and $\ell=\gcd(v_1,\dots,v_r)$, then $\mathbf{v}$ can be replaced with $\mathbf{v}'=(v_1/\ell,\dots,v_r/\ell)$.

If M is a finitely generated $\mathbb{Z}^r$-graded S-module, and if $\mathbf{v}$ is a positive coarsening vector, then

$$M^{[\mathbf{v}]} := \bigoplus_{m\in\mathbb{Z}} \Bigl(\bigoplus_{\mathbf{a}\cdot\mathbf{v}=m} M_{\mathbf{a}} \Bigr) \quad \text{and} \quad M^{[d\mathbf{v}]} := \bigoplus_{m\in\mathbb{Z}} \Bigl(\bigoplus_{\mathbf{a}\cdot d\mathbf{v}=m} M_{\mathbf{a}} \Bigr).$$

As a $\mathbb{Z}$-graded module, the degree j piece of $M^{[d\mathbf{v}]}$ is given by

$$\bigl(M^{[d\mathbf{v}]}\bigr)_j = \begin{cases} 0, & \text{if } j \not\equiv 0 \pmod{d}, \\ M^{[\mathbf{v}]}_{j/d}, & \text{if } j \equiv 0 \pmod{d}. \end{cases}$$

Lemma 4.1. *Let M be a finitely generated $\mathbb{Z}^r$-graded S-module. For all $j \in \mathbb{Z}$*

$$H^i_{\mathfrak{m}}\bigl(M^{[d\mathbf{v}]}\bigr)_j = \begin{cases} 0, & \text{if } j \not\equiv 0 \pmod{d}, \\ H^i_{\mathfrak{m}}(M^{[\mathbf{v}]})_{j/d}, & \text{if } j \equiv 0 \pmod{d}. \end{cases}$$

Proof. The isomorphism $\phi : M^{[\mathbf{v}]} \to M^{[d\mathbf{v}]}$ where $\phi(f) = f$ satisfies $\phi((M^{[\mathbf{v}]})_j) = (M^{[d\mathbf{v}]})_{dj}$ and gives rise to the corresponding isomorphism of local cohomology modules. □

Theorem 4.2. *Let M be a finitely generated $\mathbb{Z}^r$-graded S-module. If $p \in \operatorname{reg}_{\mathbf{v}}(M^{[\mathbf{v}]})$, then*

$$dp - \ell \in \operatorname{reg}_{d\mathbf{v}}\bigl(M^{[d\mathbf{v}]}\bigr) \quad \textit{for } \ell = 0, \dots, d-1.$$

Proof. We will consider the case $\ell = 0$ and $\ell \neq 0$ separately. Suppose $\ell = 0$. We want to show that $dp \in \operatorname{reg}_{d\mathbf{v}}(M^{[d\mathbf{v}]})$. Fix an $i \geqslant 0$ and let

$$q \in dp + \mathbb{N}\, dc_{\mathbf{v}}[1-i].$$

So, $q = dp + dc_{\mathbf{v}}(1 - i + j)$ for some $j \in \mathbb{N}$. Since $q \equiv 0 \pmod{d}$ for all i, j, we have by Lemma 4.1

$$H^i_{\mathfrak{m}}\bigl(M^{[d\mathbf{v}]}\bigr)_q = H^i_{\mathfrak{m}}\bigl(M^{[\mathbf{v}]}\bigr)_{q/d} = H^i_{\mathfrak{m}}\bigl(M^{[\mathbf{v}]}\bigr)_{p+c_{\mathbf{v}}(1-i+j)}.$$

But $p \in \operatorname{reg}_{\mathbf{v}}(M^{[\mathbf{v}]})$ so $H^i_{\mathfrak{m}}(M^{[\mathbf{v}]})_{p+c_{\mathbf{v}}(1-i+j)} = 0$. Thus $dp \in \operatorname{reg}_{d\mathbf{v}}(M^{[d\mathbf{v}]})$.

Suppose now that $\ell \neq 0$ and fix an $i \geqslant 0$. Let

$$q \in (dp - \ell) + \mathbb{N} dc_{\mathbf{v}}[1-i].$$

But then $q = dp - \ell + dc_{\mathbf{v}}(1 - i + j)$ for some $j \in \mathbb{N}$, and hence $q \not\equiv 0 \pmod{d}$. By Lemma 4.1 we get $H^i_{\mathfrak{m}}(M^{[d\mathbf{v}]})_q = 0$, which implies that $dp - \ell \in \operatorname{reg}_{d\mathbf{v}}(M^{[d\mathbf{v}]})$. □

Corollary 4.3. *Let M be a finitely generated $\mathbb{Z}^r$-graded S-module. Then*

$$\operatorname{reg-num}_{d\mathbf{v}}\bigl(M^{[d\mathbf{v}]}\bigr) = d \operatorname{reg-num}_{\mathbf{v}}\bigl(M^{[\mathbf{v}]}\bigr) - d + 1.$$

Proof. Let $r = \text{reg-num}_{\mathbf{v}}(M^{[\mathbf{v}]})$, and thus $r' \in \text{reg}_{\mathbf{v}}(M^{[\mathbf{v}]})$ for all $r' \geqslant r$. The previous theorem implies that for all $s \geqslant dr - d + 1$, $s \in \text{reg}_{d\mathbf{v}}(M^{[d\mathbf{v}]})$. So

$$dr - d + 1 \geqslant \text{reg-num}_{d\mathbf{v}}\left(M^{[d\mathbf{v}]}\right).$$

On the other hand, we know that $r - 1 \notin \text{reg}_{\mathbf{v}}(M^{[\mathbf{v}]})$. If $d(r-1) \in \text{reg}_{d\mathbf{v}}(M^{[d\mathbf{v}]})$ this would imply that

$$H^i_{\mathfrak{m}}\left(M^{[d\mathbf{v}]}\right)_q = 0 \quad \forall i \geqslant 0,\ \forall q \in d(r-1) + \mathbb{N}dc_{\mathbf{v}}[1-i].$$

But by Lemma 4.1, this would imply

$$H^i_{\mathfrak{m}}\left(M^{[\mathbf{v}]}\right)_{q'} = 0 \quad \forall i \geqslant 0,\ \forall q' \in (r-1) + \mathbb{N}c[1-i],$$

contradicting the fact that $r - 1 \notin \text{reg}_{\mathbf{v}}(M^{[\mathbf{v}]})$. Combining the inequality

$$\text{reg-num}_{d\mathbf{v}}\left(M^{[d\mathbf{v}]}\right) > d(r-1) = dr - d$$

with our previous inequality gives the desired conclusion. □

We can now show that if we have a bound on the multidegrees of the generators using $\mathbf{v}$, we will not obtain new bounds on the multidegrees if we use a scalar multiple of $\mathbf{v}$.

Theorem 4.4. *Let* $\mathbf{v}$ *be a positive coarsening vector and let* $d \in \mathbb{N}_{>0}$*. Then*

$$\mathcal{D}_{i,d\mathbf{v}}(M) = \mathcal{D}_{i,\mathbf{v}}(M).$$

Proof. A simple calculation will show that $c_{d\mathbf{v}} = dc_{\mathbf{v}}$ and $s_{d\mathbf{v}} = ds_{\mathbf{v}}$. Suppose that $\mathbf{a} \in \mathcal{D}_{i,d\mathbf{v}}(M)$. So

$$\begin{aligned}\mathbf{a} \cdot d\mathbf{v} &\leqslant \text{reg-num}_{d\mathbf{v}}\left(M^{[d\mathbf{v}]}\right) + is_{d\mathbf{v}} + c_{d\mathbf{v}} - 1 \\ &= d\,\text{reg-num}_{\mathbf{v}}\left(M^{[\mathbf{v}]}\right) - d + 1 + i\,ds_{\mathbf{v}} + dc_{\mathbf{v}} - 1.\end{aligned}$$

Dividing both sides of the inequality by d gives

$$\mathbf{a} \cdot \mathbf{v} \leqslant \text{reg-num}_{\mathbf{v}}\left(M^{[\mathbf{v}]}\right) + is_{\mathbf{v}} + c_{\mathbf{v}} - 1,$$

that is, $\mathbf{a} \in \mathcal{D}_{i,\mathbf{v}}(M)$. By reversing the argument, we also can show that $\mathcal{D}_{i,\mathbf{v}}(M) \subseteq \mathcal{D}_{i,d\mathbf{v}}(M)$. □

Remark 4.5. The result above tells us that a minimal set of positive coarsening vectors will not contain parallel vectors. However, to proceed further, we need a mechanism to compare the sets $\mathcal{D}_{i,\mathbf{v}}(M)$ and $\mathcal{D}_{i,\mathbf{w}}(M)$ for two arbitrary positive coarsening vectors $\mathbf{v}$ and $\mathbf{w}$. The primary obstacle in describing a comparison is relating the vanishing of $H^i_{\mathfrak{m}}(M^{[\mathbf{v}]})$ with that of $H^i_{\mathfrak{m}}(M^{[\mathbf{w}]})$, as we did in Lemma 4.1 for $\mathbf{v}$ and $d\mathbf{v}$. In particular, we want a relationship like Corollary 4.3 for $\text{reg-num}_{\mathbf{v}}(M)$ and $\text{reg-num}_{\mathbf{w}}(M)$.

5. Connections with other notions of regularity

In this section we relate $\text{reg-num}_{\mathbf{v}}(M)$ to two recent notions of multigraded regularity. In particular, we show how to use $\text{reg-num}_{\mathbf{v}}(M)$ to determine bounds on the multigraded regularity of M as defined in [11]. As well, we relate $\text{reg-num}_{\mathbf{v}}(M)$ to the resolution regularity vector of M as defined by [1,14].

5.1. *B*-regularity

We assume that our multigraded ring S is the homogeneous coordinate ring of a smooth complete toric variety X of dimension $d = n - r$ whose Chow group $A_{d-1}(X) \cong \mathbb{Z}^r$ for some r. The ring S comes equipped with a square-free "irrelevant" monomial ideal B. The group in which the degrees of S lie is the Chow group $A_{d-1}(X)$.

Alternatively, we may work in the more general setting described in Section 2 of [11] by imposing the additional condition that S is positively multigraded by $\mathbb{Z}^r$ and that B is chosen in a way that is compatible with the grading.

Assume that an isomorphism $A_{d-1}(X) \cong \mathbb{Z}^r$ and a set $\mathcal{C} = \{\mathbf{c}_1, \ldots, \mathbf{c}_\ell\} \subset \mathbb{Z}^r$ have been chosen. We let $\mathbb{N}\mathcal{C}$ denote the subsemigroup of $\mathbb{Z}^r$ generated by $\mathcal{C}$.

Definition 5.1. [11, Definition 4.1] Let M be a finitely generated $\mathbb{Z}^r$-graded S-module. If $\mathbf{m} \in \mathbb{Z}^r$, then $\mathbf{m}$ is in the *B-regularity* of M from level i with respect to $\mathcal{C}$ if

$$H_B^j(M)_{\mathbf{p}} = 0 \quad \text{for all } j \geqslant i \quad \text{and} \quad \mathbf{p} \in \mathbf{m} + \mathbb{N}\mathcal{C}[1-j]$$

where

$$\mathbb{N}\mathcal{C}[j] = \bigcup_{\mathbf{w} \in \mathbb{N}^\ell,\ \sum w_i = |j|} \operatorname{sign}(j)(w_1\mathbf{c}_1 + \cdots + w_\ell \mathbf{c}_\ell) + \mathbb{N}\mathcal{C}.$$

The set of all such $\mathbf{m}$ is denoted $\operatorname{reg}_B^i(M)$ and $\operatorname{reg}_B(M) = \operatorname{reg}_B^0(M)$.

The B-regularity of the polynomial ring S can always be computed topologically via Proposition 3.2 of [11]. Once the sets $\operatorname{reg}_B^i(S)$ are known, we can compute a lower bound on $\operatorname{reg}_B(M)$ for any finitely generated $\mathbb{Z}^r$-graded S-module from the coarsely graded minimal free resolution of M, and hence, from $\text{reg-num}_{\mathbf{v}}(M)$.

Theorem 5.2. *Let M be a finitely generated $\mathbb{Z}^r$-graded S-module. If $\mathbf{v}$ is a positive coarsening vector, and if*

$$\mathbf{F}_\bullet : \cdots \to F_i \to \cdots \to F_0 \to M^{[\mathbf{v}]} \to 0$$

is a minimal free $\mathbb{Z}$-graded resolution of $M^{[\mathbf{v}]}$ with $F_i = \bigoplus_j S^{[\mathbf{v}]}(-d_{i,j})$, then

$$D_0 \cap \left(\bigcup_{1 \leqslant j \leqslant \ell} -\mathbf{c}_j + \bigcup_{\phi:[d+1]\to[\ell]} \left(\bigcap_{1 \leqslant i \leqslant d+1} (-\mathbf{c}_{\phi(2)} - \cdots - \mathbf{c}_{\phi(i)} + D_i) \right) \right) \subseteq \operatorname{reg}_B(M)$$

where

$$D_i = \bigcap_{\mathbf{a}\cdot\mathbf{v}=d_{i,j}\ \text{for some}\ j} \big(\mathbf{a} + \operatorname{reg}_B^i(S)\big).$$

Proof. We know that M has a minimal free $\mathbb{Z}^r$-graded resolution

$$0 \to \cdots \to G_i \to \cdots \to G_0 \to M \to 0$$

where $G_i = \bigoplus_k S(-\mathbf{a}_{i,k})$. Coarsening this resolution with the vector $\mathbf{v}$ gives a minimal free $\mathbb{Z}$-graded resolution of $M^{[\mathbf{v}]}$ which must be isomorphic to $\mathbf{F}_\bullet$. One can easily see that [14, Lemma A.1 and Theorem A.2] (variants of [11, Lemma 7.1 and Theorem 7.2]) hold in this slightly more general situation, so we know that

$$D_0 \cap \Bigg(\bigcup_{1\leqslant j\leqslant \ell} -\mathbf{c}_j + \bigcup_{\phi:[d+1]\to[\ell]} \Bigg(\bigcap_{1\leqslant i\leqslant d+1} \big(-\mathbf{c}_{\phi(2)} - \cdots - \mathbf{c}_{\phi(i)} + \operatorname{reg}_B^i(G_i)\big)\Bigg)\Bigg) \subseteq \operatorname{reg}_B(M).$$

So it is enough to show that $D_i \subseteq \operatorname{reg}_B^i(G_i)$. But

$$\operatorname{reg}_B^i(G_i) = \bigcap_k \big(\mathbf{a}_{i,k} + \operatorname{reg}_B^i(S)\big),$$

which contains D_i since for each k such that $\mathbf{a}_{i,k} \neq 0$, there exists a $d_{i,j}$ in the $\mathbb{Z}$-graded resolution of $M^{[\mathbf{v}]}$ with $\mathbf{a}_{i,k} \cdot \mathbf{v} = d_{i,j}$. □

Corollary 5.3. *Under the same hypotheses as Theorem* 5.2*,*

$$\bigcup_{\phi:[d+1]\to[\ell]} D_0 \cap \Bigg(\bigcup_{1\leqslant j\leqslant \ell} -\mathbf{c}_j + \bigcup_{\phi:[d+1]\to[\ell]} \Bigg(\bigcap_{1\leqslant i\leqslant d+1} \big(-\mathbf{c}_{\phi(2)} - \cdots - \mathbf{c}_{\phi(i)} + D_i\big)\Bigg)\Bigg) \subseteq \operatorname{reg}_B(M)$$

where

$$D_i = \bigcap_{\mathbf{a}\cdot\mathbf{v}\leqslant \text{reg-num}_{\mathbf{v}}(M)+is_{\mathbf{v}}+\mathbf{c}_{\mathbf{v}}-1} \big(\mathbf{a} + \operatorname{reg}_B^i(S)\big).$$

Proof. Keeping the setup of the proof of Theorem 5.2,

$$\big\{\mathbf{a} \mid \mathbf{a}\cdot\mathbf{v} \leqslant \text{reg-num}_{\mathbf{v}}(M) + is_{\mathbf{v}} + c_{\mathbf{v}} - 1\big\} \supseteq \{\mathbf{a} \mid \mathbf{a}\cdot\mathbf{v} = d_{i,j} \text{ for some } j\}. \qquad \square$$

5.2. Resolution regularity vector

We shall now assume that S is the standard multigraded homogeneous coordinate ring of $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_r}$. Generalizing the bigraded case found in [1], the first two authors introduced the following definition in [14]:

Definition 5.4. Let M be a finitely generated $\mathbb{Z}^r$-graded S-module. For $\ell = 1, \ldots, r$, let

$$d_\ell := \max\left\{a_\ell \mid \operatorname{Tor}_i^S(M,k)_{(a_1,\ldots,a_\ell+i,\ldots,a_r)} \neq 0\right\}$$

for some i and for some $a_1, \ldots, a_{\ell-1}, a_{\ell+1}, \ldots, a_r$. We call $\underline{r}(M) := (d_1, \ldots, d_r)$ the *resolution regularity vector* of M.

It follows that if $\underline{r}(M) = (d_1, \ldots, d_r)$ and if F if is a minimal generator of the ith syzygy module of M, then the ℓth coordinate of $\deg F$ is bounded above $d_\ell + i$.

Theorem 5.5. *Let M be a finitely generated $\mathbb{Z}^r$-graded S-module whose Hilbert series is supported on $\mathbb{N}^r$, and let*

$$\mathbf{F}_\bullet : \cdots \to F_i \to \cdots \to F_0 \to M \to 0$$

be a minimal free $\mathbb{Z}^r$-graded resolution of M. Let $d = \text{reg-num}_{\mathbf{v}}(M)$ when $\mathbf{v} = (1, \ldots, 1)$.

(1) *The free module F_i is generated by elements whose degrees are contained in the set of all $\mathbf{a} \in \mathbb{N}^r$ satisfying $a_1 + \cdots + a_r \leqslant d + i$.*
(2) *Set $m = \min\{\text{proj-dim}(M), n_1 + \cdots + n_r + 1\}$. If $m > 0$, then*

$$\underline{r}(M) \leqslant (d, \ldots, d) \quad \textit{and} \quad (d+m, \ldots, d+m) + \mathbb{N}^r\left[-(m-1)\right] \subseteq \operatorname{reg}_B(M).$$

Proof. Statement (1) follows from Corollary 3.11. For (2), the first part is clear, and the second follows from Corollary 2.3 of [14]. □

6. Illustrative examples

We end this paper with some examples that illustrate the strengths and weaknesses of using coarsening vectors to find bounds on the degrees of the multigraded generators of the syzygies in the $\mathbb{Z}^r$-graded free resolutions of finitely generated $\mathbb{Z}^r$-graded S-modules.

6.1. Points in multi-projective spaces

For our first set of examples, we consider the ideals of sets of points in $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_r}$. The benefit of studying this class of ideals is that we already have some information on the regularity of these ideals (see [9,11,14] for more on this).

Let X be a finite set of reduced points in $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_r}$, that is, the defining ideal of X equals its radical. The defining ideal of X, denoted I_X, is an $\mathbb{N}^r$-graded homogeneous ideal of the $\mathbb{N}^r$-graded polynomial ring $S = k[x_{1,0}, \dots, x_{1,n_1}, \dots, x_{r,0}, \dots, x_{r,n_r}]$ where $\deg(x_{i,j}) = e_i$, the ith standard basis vector of $\mathbb{N}^r$. The *Hilbert function* of X is the numerical function $H_X : \mathbb{N}^r \to \mathbb{N}$ defined by $\mathbf{i} \mapsto \dim_k(S/I_X)_{\mathbf{i}}$.

As shown in [11], the B-regularity of S/I_X is linked to H_X. Specifically,

Theorem 6.1. *Let X be a set of reduced points in $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_r}$. Then*

$$\operatorname{reg}_B(S/I_X) = \{\mathbf{i} \in \mathbb{N}^r \mid H_X(\mathbf{i}) = \deg X = |X|\}.$$

The resolution regularity vector of S/I_X can also be read off of the Hilbert function of S/I_X. The following theorem is a special case of Theorem 4.2 of [14].

Theorem 6.2. *Let X be a set of reduced points in $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_r}$. Then*

$$\underline{r}(S/I_X) = (t_1, \dots, t_r)$$

where $t_i = \min\{t \in \mathbb{N} \mid H_X(te_i) = |\pi_i(X)|\}$ and $\pi_i : \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_r} \to \mathbb{P}^{n_i}$ is the ith projection morphism.

The following two examples compare these notions of regularity with the notion developed in this paper.

Example 6.3. In $\mathbb{P}^1 \times \mathbb{P}^1$, let $P_{ij} = [1 : i] \times [1 : j]$. Consider the set of points $X = \{P_{11}, P_{12}, P_{13}, P_{14}, P_{21}, P_{31}, P_{41}, P_{51}\}$. The Hilbert function of X is

$$\begin{bmatrix} \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\ 4 & 5 & 6 & 7 & 8 & 8 & \dots \\ 4 & 5 & 6 & 7 & 8 & 8 & \dots \\ 3 & 4 & 5 & 6 & 7 & 7 & \dots \\ 2 & 3 & 4 & 5 & 6 & 6 & \dots \\ 1 & 2 & 3 & 4 & 5 & 5 & \dots \end{bmatrix},$$

where the number in the location (i, j) corresponds to the value of $H_X(i, j)$. By Theorem 6.1, $\operatorname{reg}_B(S/I_X) = (4, 3) + \mathbb{N}^2$. The degrees of the minimal ith syzygies then belong to the unbounded set

$$\mathcal{A}_i = \mathbb{N}^2 \setminus \bigcup_{m,n \geqslant 0,\ m+n=i} \big((4+m, 3+n) + \mathbb{N}^2\big).$$

Theorem 6.2 tells us that $\underline{r}(S/I_X) = (4, 3)$ since $4 = \min\{t \mid H_X(t, 0) = |\pi_1(X)| = 5\}$ and $3 = \min\{t \mid H_X(0, t) = |\pi_2(X)| = 4\}$. The resolution regularity vector tells us that the degree of a minimal ith syzygy belongs to the finite set

$$\mathcal{B}_i = \{(m, n) \in \mathbb{N}^2 \mid (0, 0) \leqslant (m, n) \leqslant (4+i, 3+i)\}.$$

Observe that $\mathcal{A}_i \not\subset \mathcal{B}_i$ and $\mathcal{B}_i \not\subset \mathcal{A}_i$ since if $i = 2$, $(6,5) \in \mathcal{B}_2$ but is not in $\mathcal{A}_2$, while $(7,0) \in \mathcal{A}_2$ but not in $\mathcal{B}_2$.

Let us now consider the coarsening vector $\mathbf{v} = (1,1)$. Under this coarsening vector $\text{reg-num}_{\mathbf{v}}(S/I_X)$ agrees with the standard notion of regularity of a $\mathbb{Z}$-graded module. Thus $\text{reg-num}_{\mathbf{v}}(S/I_X) = 4$ since the $\mathbb{Z}$-graded resolution of $(S/I_X)^{[\mathbf{v}]}$ has the form

$$0 \to S^{[\mathbf{v}]}(-5) \oplus S^{[\mathbf{v}]}(-6) \to S^{[\mathbf{v}]}(-5) \oplus S^{[\mathbf{v}]}(-4) \oplus S^{[\mathbf{v}]}(-2) \to S^{[\mathbf{v}]} \to (S/I_X)^{[\mathbf{v}]} \to 0.$$

The generators of the ith syzygy module of S/I_X must therefore have degrees belonging to the set

$$\mathcal{C}_i = \mathcal{D}_{i,\mathbf{v}}(S/I_X) = \{(m,n) \in \mathbb{N}^2 \mid m + n \leqslant 4 + i\}.$$

Note that $\mathcal{B}_i \not\subset \mathcal{C}_i$ and $\mathcal{C}_i \not\subset \mathcal{B}_i$ since $(0,5) \in \mathcal{C}_1$ but $(0,5) \notin \mathcal{B}_1$, while $(4,5) \in \mathcal{B}_1$ but not in $\mathcal{C}_1$.

On the other hand, $\mathcal{C}_i \subseteq \mathcal{A}_i$ for all i. So, for this example, the coarse grading yields new information on degrees of the minimal syzygies when compared to the B-regularity of [11].

The previous example showed that a coarse grading can reveal new information about the degrees of the minimal syzygies. However, this may not always be the case, as Example 6.5 will show. We first recall some relevant results about points in generic position.

A set of points $X \subseteq \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_r}$ is said to be in *generic position* if

$$H_X(\mathbf{i}) = \min\{\dim_k S_{\mathbf{i}}, |X|\} \quad \text{for all } \mathbf{i} \in \mathbb{N}^r.$$

The regularity of S/I_X as a $\mathbb{Z}$-graded ring was computed in [9] when X is in generic position:

Theorem 6.4. *Let X be a set of reduced points in $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_r}$ in generic position. Then*

$$\operatorname{reg}(S/I_X) = \max\{d_1, \ldots, d_r\}$$

where $d_i = \min\{d \mid \binom{d+n_i}{d} \geqslant |X|\}$ for $i = 1, \ldots, r$.

Example 6.5. Let $X \subseteq \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_r}$ be a set of points in generic position. By Theorem 6.1,

$$\operatorname{reg}_B(S/I_X) = \left\{ \mathbf{i} \in \mathbb{N}^r \;\middle|\; \binom{n_1 + i_1}{n_1} \cdots \binom{n_r + i_r}{n_r} \geqslant |X| \right\}.$$

The degrees of the minimal ith syzygies must therefore be elements of

$$\mathcal{A}_i = \mathbb{N}^r \setminus \bigcup_{i_1,\ldots,i_r \geqslant 0,\ i_1+\cdots+i_r = i} \big(\operatorname{reg}_B(S/I_X) + (i_1, \ldots, i_r)\big) + \mathbb{N}^r.$$

For points in generic position, the sets $\mathcal{A}_i$ are finite.

We compare this to the information we obtained from the coarsely graded ring. We use $\mathbf{v} = (1, \dots, 1)$ as our positive coarsening vector. Then $\text{reg-num}_{\mathbf{v}}(S/I_X) = \max\{d_1, \dots, d_r\}$ where $d_i = \min\{d \mid \binom{d+n_i}{d} \geqslant |X|\}$ by Theorem 6.4. Thus

$$\mathcal{C}_i = \mathcal{D}_{i,\mathbf{v}}(S/I_X) = \{\mathbf{a} \in \mathbb{N}^r \mid \mathbf{a} \cdot \mathbf{v} \leqslant \text{reg-num}_{\mathbf{v}}(R/I_X) + i\}.$$

But for all i, $\mathcal{A}_i \subseteq \mathcal{C}_i$, so no new information about the multidegrees of ith syzygies is obtained by using the positive coarsening vector $\mathbf{v} = (1, \dots, 1)$.

Example 6.6. We find a minimal set of positive coarsening vectors for $X = \{[1:0] \times [1:0], [1:0] \times [0:1], [0:1] \times [1:0], [0:1] \times [0:1]\}$, the set of points of Example 1.1. The defining ideal $I_X = \langle x_0x_1, y_0y_1 \rangle$ is the Stanley–Reisner ideal of the simplicial complex Δ equal to the 4-cycle with edges $\{x_0, y_0\}, \{x_0, y_1\}, \{x_1, y_0\}, \{x_1, y_1\}$ (where we label the vertices with the variables from the ring).

Because I_X is a complete intersection, S/I_X is a Cohen–Macaulay ring of dimension 2. Thus, $H^i_{\mathfrak{m}}(S/I_X) = 0$ for all $i \neq 2$. When $i = 2$, we can compute the fine $\mathbb{N}^4$-graded Hilbert series of $H^2_{\mathfrak{m}}(S/I_X)$ using Hochster's theorem (cf. [12, Theorem 13.13]). In particular, the Hilbert series $H(H^2_{\mathfrak{m}}(S/I_X); x_0, x_1, y_0, y_1)$ is

$$\begin{aligned}
&= 1 + \frac{x_0^{-1}}{1 - x_0^{-1}} + \frac{x_1^{-1}}{1 - x_1^{-1}} + \frac{y_0^{-1}}{1 - y_0^{-1}} + \frac{y_1^{-1}}{1 - y_1^{-1}} + \frac{x_0^{-1}}{1 - x_0^{-1}} \cdot \frac{y_0^{-1}}{1 - y_0^{-1}} \\
&\quad + \frac{x_0^{-1}}{1 - x_0^{-1}} \cdot \frac{y_1^{-1}}{1 - y_1^{-1}} + \frac{x_1^{-1}}{1 - x_1^{-1}} \cdot \frac{y_0^{-1}}{1 - y_0^{-1}} + \frac{x_1^{-1}}{1 - x_1^{-1}} \cdot \frac{y_1^{-1}}{1 - y_1^{-1}} \\
&= 1 + \left(\frac{1}{x_0} + \frac{1}{x_0^2} + \cdots\right) + \left(\frac{1}{x_1} + \frac{1}{x_1^2} + \cdots\right) + \left(\frac{1}{y_0} + \frac{1}{y_0^2} + \cdots\right) \\
&\quad + \left(\frac{1}{y_1} + \frac{1}{y_1^2} + \cdots\right) + \left(\frac{1}{x_0y_0} + \frac{1}{x_0^2y_0} + \cdots\right) + \left(\frac{1}{x_0y_1} + \frac{1}{x_0^2y_1} + \cdots\right) \\
&\quad + \left(\frac{1}{x_1y_0} + \frac{1}{x_1y_0^2} + \cdots\right) + \left(\frac{1}{x_1y_1} + \frac{1}{x_1y_1^2} + \cdots\right).
\end{aligned}$$

Let $\mathbf{v} = (a, b) \in \mathbb{N}^2$ be a positive coarsening vector for S/I_X. This implies that $a, b \geqslant 1$. We can also assume that $\gcd(a, b) = 1$ (from Section 4) and that $a \leqslant b$. Since $\deg(x_i) = a$ and $\deg(y_i) = b$, the $\mathbb{Z}$-graded Hilbert series of $H^2_{\mathfrak{m}}((S/I_X)^{[\mathbf{v}]})$ is

$$\begin{aligned}
H(H^2_{\mathfrak{m}}((S/I_X)^{[\mathbf{v}]}); t) &= 1 + 2(t^{-a} + t^{-2a} + \cdots) + 2(t^{-b} + t^{-2b} + \cdots) \\
&\quad + 4(t^{-a-b} + t^{-2a-b} + \cdots).
\end{aligned}$$

Therefore, we see that $a^2((S/I_X)^{[\mathbf{v}]}) = 0$ and $a^i((S/I_X)^{[\mathbf{v}]}) = -\infty$ for all $i \neq 2$. By Theorem 3.19

$$\text{reg-num}_{\mathbf{v}}(S/I_X) = 0 - ab(1 - 2) + 1 = ab + 1$$

since $c_{\mathbf{v}} = ab$.

We can now show that for any positive coarsening vector $\mathbf{v} = (a, b)$ and for each $i \geqslant 0$,

$$\mathcal{D}_{i,\mathbf{1}}(S/I_X) \subseteq \mathcal{D}_{i,\mathbf{v}}(S/I_X)$$

where $\mathbf{1} = (1, 1)$, and consequently, $\mathcal{V} = \{\mathbf{1}\}$ is a minimal set of positive coarsening vectors for S/I_X. Without loss of generality, we can assume $\gcd(a, b) = 1$ and $a \leqslant b$. If $(x, y) \in \mathcal{D}_{i,\mathbf{1}}(S/I_X) \subset \mathbb{N}^2$, then $x + y \leqslant \text{reg-num}_{\mathbf{1}}(S/I_X) + is_{\mathbf{1}} + c_{\mathbf{1}} - 1$. But since $s_{\mathbf{1}} = c_{\mathbf{1}} = 1$ and $\text{reg-num}_{\mathbf{1}}(S/I_X) = 2$ (because I_X is a complete intersection), we deduce that $x + y \leqslant 2 + i$. Thus

$$\begin{aligned}(x, y) \cdot (a, b) &= ax + by \leqslant bx + by \leqslant 2b + bi \\ &\leqslant 2ab + abi = (ab + 1) + abi + ab - 1.\end{aligned}$$

Because $c_{\mathbf{v}} = ab$, $s_{\mathbf{v}} \geqslant c_{\mathbf{v}}$, and since $\text{reg-num}_{\mathbf{v}}(S/I_X) = ab + 1$, we have

$$(x, y) \cdot (a, b) \leqslant \text{reg-num}_{\mathbf{v}}(S/I_X) + is_{\mathbf{v}} + c_{\mathbf{v}} - 1,$$

that is, $(x, y) \in \mathcal{D}_{i,\mathbf{v}}(S/I_X)$, as desired.

Remark 6.7. The previous examples suggest that if one is interested in computing bounds on the degrees of the minimal syzygies, one might wish to use the notions of B-regularity, the resolution regularity vector, and the regularity number of a positive coarsening vector together, as one approach is not always better than another.

6.2. *Hirzebruch surface*

Consider the case where $S = k[x_1, x_2, x_3, x_4]$ is the coordinate ring of the Hirzebruch surface $\mathbb{F}_s$, $s \geqslant 1$. The ring S is multigraded by $A = \mathbb{Z}^2$ with $\deg(x_1) = (1, 0)$, $\deg(x_2) = (-s, 1)$, $\deg(x_3) = (1, 0)$, and $\deg(x_4) = (0, 1)$.

The solid dots in Fig. 1(i) represent the degrees of elements of S. The positive coarsening vectors of S are the interior points (represented as solid dots) of the cone in Fig. 1(ii).

Example 6.8. If we choose coarsening vector $\mathbf{v} = (1, s+1)$, then $\deg_{\mathbf{v}}(x_1) = \deg_{\mathbf{v}}(x_3) = 1$, $\deg_{\mathbf{v}}(x_2) = 1$ and $\deg_{\mathbf{v}}(x_4) = s + 1$. Therefore, $c_{\mathbf{v}} = s + 1$, $\text{reg-num}_{\mathbf{v}}(S) = 2s$, and $s_{\mathbf{v}} = 3s$.

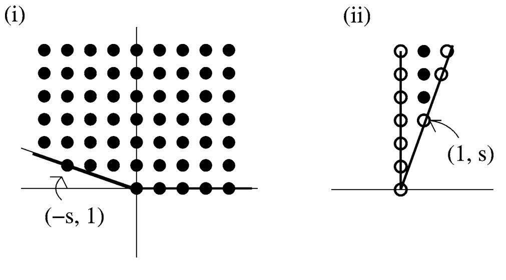

Fig. 1.

If we choose coarsening vector $\mathbf{u} = (1, 2s)$, then $\deg_{\mathbf{u}}(x_1) = \deg_{\mathbf{u}}(x_3) = 1$, $\deg_{\mathbf{u}}(x_2) = s$ and $\deg_{\mathbf{u}}(x_4) = 2s$. Therefore, $c_{\mathbf{u}} = 2s$, $\text{reg-num}_{\mathbf{u}}(S) = 3s - 1$, and $s_{\mathbf{u}} = 5s - 2$. In particular, when $s = 2$, then $\text{reg-num}_{\mathbf{v}}(S) = 4$, and $\text{reg-num}_{\mathbf{u}}(S) = 8$.

Example 6.9. Let S be the coordinate ring of the Hirzebruch surface $\mathbb{F}_s$, and consider the S-module $M = S/\langle x_1x_2, x_3x_4\rangle$. This is the same ring as the ring of Example 6.6 except that we have renamed our indeterminates and we have changed our grading. So $H^i_{\mathfrak{m}}(M) = 0$ for all $i \neq 2$, and for $i = 2$, the fine $\mathbb{N}^4$-graded Hilbert series of $H^2_{\mathfrak{m}}(M)$ is the same as the one given in Example 6.6. If we choose the positive coarsening vector $\mathbf{v} = (1, s+1)$, then the $\mathbb{Z}$-graded Hilbert series of $H^2_{\mathfrak{m}}(M^{[\mathbf{v}]})$ is

$$H\big(H^2_{\mathfrak{m}}\big(M^{[\mathbf{v}]}\big); t\big) = 1 + 3\big(t^{-1} + t^{-2} + \cdots\big) + \big(t^{-s-1} + t^{-2s-2} + \cdots\big) + \cdots;$$

in other words, $H^2_{\mathfrak{m}}(M)_p = 0$ for all $p \geqslant 1$ but $H^2_{\mathfrak{m}}(M)_0 \neq 0$. So, $a^2(M^{[\mathbf{v}]}) = 0$ and $a^i(M^{[\mathbf{v}]}) = -\infty$ for $i \neq 2$. Since $c_{\mathbf{v}} = (s+1)$ by Theorem 3.19 we have

$$\text{reg-num}_{\mathbf{v}}(M) = 0 - (s+1)(1-2) + 1 = s + 2.$$

Acknowledgments

We thank David Cox for suggesting the collaboration and for his encouragement, and Tài Hà, Robert Lazarsfeld, Diane Maclagan and Greg Smith for their comments and suggestions. We computed many examples using the computer algebra packages *CoCoA* [4] and *Macaulay 2* [7]. The first author was partially supported by an NSF postdoctoral fellowship at the University of Massachusetts in Amherst in 2004–2005 on Grant No. 0201607 and also thanks the Clare Boothe Luce Foundation for support. The second author thanks NSERC for financial support while working on this project and the hospitality of Mt. Holyoke College while visiting the first author. The third author thanks NSF-AWM and GRFC for travel support.

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