

## On sizes of complete arcs in $PG(2, q)$

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### ABSTRACT

New upper bounds on the smallest size  $t_2(2, q)$  of a complete arc in the projective plane  $PG(2, q)$  are obtained for  $853 \leq q \leq 5107$  and  $q \in T_1 \cup T_2$ , where  $T_1 = \{173, 181, 193, 229, 243, 257, 271, 277, 293, 343, 373, 409, 443, 449, 457, 461, 463, 467, 479, 487, 491, 499, 529, 563, 569, 571, 577, 587, 593, 599, 601, 607, 613, 617, 619, 631, 641, 661, 673, 677, 683, 691, 709\}$ , and  $T_2 = \{5119, 5147, 5153, 5209, 5231, 5237, 5261, 5279, 5281, 5303, 5347, 5641, 5843, 6011, 8192\}$ . From these new bounds it follows that for  $q \leq 2593$  and  $q = 2693, 2753$ , the relation  $t_2(2, q) < 4.5\sqrt{q}$  holds. Also, for  $q \leq 5107$  we have  $t_2(2, q) < 4.79\sqrt{q}$ . It is shown that for  $23 \leq q \leq 5107$  and  $q \in T_2 \cup \{2^{14}, 2^{15}, 2^{18}\}$ , the inequality  $t_2(2, q) < \sqrt{q} \ln^{0.75} q$  is true. Moreover, the results obtained allow us to conjecture that this estimate holds for all  $q \geq 23$ . The new upper bounds are obtained by finding new small complete arcs with the help of a computer search using randomized greedy algorithms. Also new constructions of complete arcs are proposed. These constructions form families of  $k$ -arcs in  $PG(2, q)$  containing arcs of all sizes  $k$  in a region  $k_{\min} \leq k \leq k_{\max}$ , where  $k_{\min}$  is of order  $\frac{1}{3}q$  or  $\frac{1}{4}q$  while  $k_{\max}$  has order  $\frac{1}{2}q$ . The completeness of the arcs obtained by the new constructions is proved for  $q \leq 2063$ . There is reason to suppose that the arcs are complete for all  $q > 2063$ . New sizes of complete arcs in  $PG(2, q)$  are presented for  $169 \leq q \leq 349$  and  $q = 1013, 2003$ .

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### 1. Introduction

Let  $PG(2, q)$  be the projective plane over the Galois field  $F_q$ . A  $k$ -arc is a set of  $k$  points no three of which are collinear. A  $k$ -arc is called complete if it is not contained in a  $(k + 1)$ -arc of  $PG(2, q)$ . For an introduction in projective geometries over finite fields, see [26,47,49].

In [27,28] the close relationship between the theory of  $k$ -arcs, coding theory and mathematical statistics is presented. In particular, a complete arc in a plane  $PG(2, q)$ , points of which are treated as 3-dimensional  $q$ -ary columns, defines a parity check matrix of a  $q$ -ary linear code with codimension 3, Hamming distance 4, and covering radius 2. Arcs can be interpreted as linear maximum distance separable (MDS) codes [53,54] and they are related to optimal coverings arrays [24] and to superregular matrices [29].

One of the main problems in the study of projective planes, which is also of interest in coding theory, is finding the spectrum of possible sizes of complete arcs.

A great part of this work is devoted to upper bounds on  $t_2(2, q)$ , the smallest size of a complete arc in  $PG(2, q)$ . Also we propose new constructions of complete arcs in  $PG(2, q)$  and consider the spectrum of their possible sizes.

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Surveys of results on the sizes of plane complete arcs, methods of their construction and comprehension of the relating properties can be found in [3–5,16,25–28,36,42–45,47–53]. In particular, as it is noted in [27,28], the following idea of Segre [48] and Lombardo-Radice [36] is fruitful: the points of the arc are chosen, with some exceptions, among the points of a conic or a cubic curve. We use this idea for constructions of complete arcs and for finding the spectrum of arc sizes; see Sections 2, 5 and 6.

The maximum size  $m_2(2, q)$  of a complete arc in  $PG(2, q)$  is well known. It holds that

$$m_2(2, q) = \begin{cases} q + 1 & \text{if } q \text{ odd} \\ q + 2 & \text{if } q \text{ even.} \end{cases}$$

On the other hand, finding an estimation of the minimum size  $t_2(2, q)$  is a hard open problem.

Problems connected with small complete plane arcs are considered in [3–5,8,9,14,17,19–23,26,30,35,37–39,46–48,50–53]; see also the references therein.

We denote the aggregates of  $q$  values:

$$T_1 = \{173, 181, 193, 229, 243, 257, 271, 277, 293, 343, 373, 409, 443, 449, 457, 461, 463, 467, 479, 487, 491, 499, 529, 563, 569, 571, 577, 587, 593, 599, 601, 607, 613, 617, 619, 631, 641, 661, 673, 677, 683, 691, 709\};$$

$$T_2 = \{5119, 5147, 5153, 5209, 5231, 5237, 5261, 5279, 5281, 5303, 5347, 5641, 5843, 6011, 8192\};$$

$$T_3 = \{2^{14}, 2^{15}, 2^{18}\};$$

$$Q = \{961, 1024, 1369, 1681, 2401\} = \{31^2, 2^{10}, 37^2, 41^2, 7^4\};$$

$$N = \{601, 5^4, 661, 3^6, 29^2, 31^2, 2^{10}, 37^2, 41^2, 7^4, 2^{18}\}.$$

The exact values of  $t_2(2, q)$  are known only for  $q \leq 32$ , see [37] and the recent work [38], where the equalities  $t_2(2, 31) = t_2(2, 32) = 14$  are proved. Also, there are the following lower bounds (see [46,47]):

$$t_2(2, q) > \begin{cases} \sqrt{2q} + 1 & \text{for any } q \\ \sqrt{3q} + \frac{1}{2} & \text{for } q = p^h, \quad p \text{ prime, } h = 1, 2, 3. \end{cases}$$

Let  $\bar{t}_2(2, q)$  be the smallest *known* size of a complete arc in  $PG(2, q)$ . For  $q \leq 841$  the values of  $\bar{t}_2(2, q)$  (up to June 2009) are collected in [5, Tab. 1] whence it follows that  $\bar{t}_2(2, q) < 4\sqrt{q}$  for  $q \leq 841$ . In [19], see also [23], complete  $(4\sqrt{q} - 4)$ -arcs are obtained for  $q = p^2$  odd,  $q \leq 1681$  or  $q = 2401$ . In [9,14] complete  $(4\sqrt{q} - 4)$ -arcs are obtained for even  $q = 64, 256, 1024$ . By the results cited above, it holds that

$$t_2(2, q) < 4\sqrt{q} \quad \text{for } 2 \leq q \leq 841, \quad q \in Q. \tag{1.1}$$

For even  $q = 2^h$ ,  $11 \leq h \leq 15$ , the smallest known sizes of complete  $n$ -arcs in  $PG(2, q)$  are obtained in [9]; see also [5, p. 35]. They are as follows:  $\bar{t}_2(2, 2^{11}) = 201$ ,  $\bar{t}_2(2, 2^{12}) = 307$ ,  $\bar{t}_2(2, 2^{13}) = 461$ ,  $\bar{t}_2(2, 2^{14}) = 665$ ,  $\bar{t}_2(2, 2^{15}) = 993$ . Also,  $(6\sqrt{q} - 6)$ -arcs in  $PG(2, q)$ ,  $q = 4^{2h+1}$ , are constructed in [8]; for  $h \leq 4$  it is proved that they are complete. It gives a complete 3066-arc in  $PG(2, 2^{18})$ .

Let  $t(\mathcal{P}_q)$  be the size of the smallest complete arc in any (not necessarily Galois) projective plane  $\mathcal{P}_q$  of order  $q$ . In [30], for *sufficiently large*  $q$ , the following result is proved (we give it in the form of [28, Tab. 2.6]):

$$t(\mathcal{P}_q) \leq d\sqrt{q} \log^c q, \quad c \leq 300, \tag{1.2}$$

where  $c$  and  $d$  are constants independent of  $q$  (i.e. universal constants). The logarithm basis is not noted as the estimate is asymptotic.

In this work, by computer search using randomized greedy algorithms (see Section 2), new small complete arcs in  $PG(2, q)$  are obtained for all  $q \in T_1 \cup T_2$  and for  $853 \leq q \leq 5107$ . For  $q = 601, 661$ , the new complete arcs arose from a theoretical study on orbits of subgroups, helped by computer [22]. From the sizes of the new arcs, with the use of (1.1) and [5, Tab. 1], [8], the following theorems result (see also Theorems 3.2, 3.3 and 4.1 for more details).

**Theorem 1.1.** *In  $PG(2, q)$ , the following holds.*

$$t_2(2, q) < 4.5\sqrt{q} \quad \text{for } q \leq 2593, \quad q = 2693, 2753.$$

$$t_2(2, q) < 4.79\sqrt{q} \quad \text{for } q \leq 5107.$$

$$t_2(2, q) < 4.98\sqrt{q} \quad \text{for } q \in T_2.$$

**Theorem 1.2.** *In  $PG(2, q)$ ,*

$$t_2(2, q) < \sqrt{q} \ln^{0.75} q \quad \text{for } 23 \leq q \leq 5107, \quad q \in T_2 \cup T_3. \tag{1.3}$$

Moreover, the study of the values of  $\bar{t}_2(2, q)$  allows us to conjecture that the estimate (1.3) holds for all  $q \geq 23$  and the last estimate of Theorem 1.1 is right for all  $q \leq 8192$ .

**Conjecture 1.3.** In  $PG(2, q)$ ,

$$\begin{aligned} t_2(2, q) &< \sqrt{q} \ln^{0.75} q \quad \text{for } q \geq 23. \\ t_2(2, q) &< 5\sqrt{q} \quad \text{for } q \leq 8192. \end{aligned} \tag{1.4}$$

Regarding the spectrum of complete arc sizes, we note (going after [28, p. 209]) that in the literature, complete arcs have been constructed with sizes approximately  $\frac{1}{2}q$  (see [2,5,13,18,26,33,34,36,42,44,45,49]),  $\frac{1}{3}q$  (see [1,31,50,51,55]),  $\frac{1}{4}q$  (see [31,52]),  $2q^{0.9}$  (see [50] where such arcs are constructed for  $q > 7^{10}$ ). In particular, for even  $q \geq 8$  there is a complete  $\frac{1}{2}(q+4)$ -arc [26]. Important results on the spectrum of complete arc sizes are collected in [28, Th. 2.6] where it is noted, for example, that in  $PG(2, q)$  with  $q$  square there exists a complete  $(q - \sqrt{q} + 1)$ -arc. In [32], large complete arcs in  $PG(2, q^n)$  are defined and new infinite families of such arcs are constructed.

Much attention is given to  $\frac{1}{2}(q+5)$ -arcs and  $\frac{1}{2}(q+7)$ -arcs sharing  $\frac{1}{2}(q+3)$  points with a conic for  $q$  odd [2,4,5], [12, Rem. 3], [13,18,33,34,44]. It is proved that for all odd  $q$  there is a complete  $\frac{1}{2}(q+5)$ -arc [34]; see also [2]. Also, a complete  $\frac{1}{2}(q+7)$ -arc exists at least for the following odd  $q$ :

$$\begin{aligned} 25 \leq q \leq 167 & [3, \text{Tab. 2}], [5, \text{Sec. 2, Tab. 2}], [16, \text{Tab. 2.4}], [34, \text{Introduction}]; \\ q \equiv 2 \pmod{3}, q \leq 4523 & [5,13]; \\ q \equiv 1 \pmod{4}, q \leq 337 & [18]; \\ q = 2bt - 1, t \text{ odd prime}, b = 1, 2 & [34, \text{Introduction}]; \\ q \equiv 3 \pmod{4} \text{ and condition of [34, Cor. 4.16(i)]} & \text{ holds, see [34, Introduction]}; \\ q^2 \equiv 1 \pmod{16} \text{ and condition of [34, Cor. 4.17]} & \text{ holds, see [34, Introduction]}. \end{aligned} \tag{1.5}$$

For  $q \leq 167$  the known sizes of complete arcs in  $PG(2, q)$  are collected in [3, Tab. 2], [5, Tab. 2], [16, Tab. 2.4].

In this work new **Constructions A–C** of complete arcs are proposed; see Section 5. These constructions form families of complete  $k$ -arcs in  $PG(2, q)$  containing arcs of all sizes  $k$  in a region  $k_{\min} \leq k \leq k_{\max}$  where  $k_{\min}$  is of order  $\frac{1}{3}q$  or  $\frac{1}{4}q$  while  $k_{\max}$  has order  $\frac{1}{2}q$ . The completeness of the arcs obtained by the new constructions is proved for  $q \leq 2063$ . Moreover, there is reason to suppose that the arcs are complete for all  $q > 2063$ . From **Theorems 5.8, 5.18** and **5.25** the following theorem results.

**Theorem 1.4.** **Constructions A–C** of Section 5 form families of complete  $k$ -arcs in  $PG(2, q)$  containing arcs of all sizes  $k$  in the following regions:

$$\begin{aligned} \text{(i) Construction A : } \left\lfloor \frac{q+8}{3} \right\rfloor \leq k \leq \frac{q+5}{2}, q \text{ prime,} \\ 109 \leq q \leq 2063, q = 73, 97, 101, 103. \\ \text{(ii) Construction B : } \left\lfloor \frac{q+8}{3} \right\rfloor \leq k \leq \left\lfloor \frac{q+4}{2} \right\rfloor, q \not\equiv 3 \pmod{4} \text{ is a prime power,} \\ 128 \leq q \leq 2063, q = 89, 109, 113, 121. \\ \text{(iii) Construction C : } \frac{q+13}{4} \leq k \leq \frac{q+5}{2}, q \equiv 3 \pmod{4} \text{ is a prime power,} \\ 347 \leq q \leq 2063, \\ q = 199, 227, 239, 243, 251, 263, 271, 283, 307, 311, 331. \end{aligned}$$

For the given  $q$ , in order to show that arcs obtained by a construction are complete, we should calculate by computer some special value, say  $\bar{L}_q$  (see **Definitions 5.4, 5.14** and **5.21**) and check if  $\bar{L}_q \leq R_q$ , where  $R_q = \lfloor \frac{1}{3}(q-1) \rfloor$  for **Constructions A** and **B** and  $R_q = \frac{1}{4}(q-3)$  for **Construction C**. The calculations are relatively simple. Moreover, for  $q \leq 2063$  it holds that  $\bar{L}_q < \sqrt{q} \ln q$  and the difference  $R_q - \bar{L}_q$  has a tendency to increase when  $q$  grows; see **Theorems 5.7(iii), (iv), 5.17(iii), (iv), 5.24(iii), (iv)**. It allows us to conjecture the following; cf. **Conjecture 5.26**.

**Conjecture 1.5.** The assertions of **Theorem 1.4** hold also for all  $q > 2063$ .

A  $k$ -arc of **Constructions A** and **B** contains  $k-2$  points in common with a conic and two points lying on a tangent to the conic (**Construction A**) or on a bisecant of the conic (**Construction B**). A  $k$ -arc of **Construction C** contains  $k-3$  points in common with a conic, two points lying on a bisecant and one point on a tangent. Note that in [1],  $k$ -arcs containing  $k-2$  points of a hyperoval (the nucleus among them) and two points on its bisecant are constructed. Also, in the space  $PG(3, q)$ ,  $k$ -caps with  $k-2$  points in common with a quadric are considered in [15,40,41]. In [15,40], the cap contains two points on a tangent to the quadric while in [41] two points lie on an external line.

The complete arcs of **Constructions A–C** can be used as starting objects for the inductive constructions of [9,32]; see **Remark 5.28**. In that way, using results of [9] together with **Constructions A–C**, one can generate infinite sets of families of complete caps in projective spaces  $PG(v, 2^n)$  of growing dimensions  $v$ . Also, infinite families of large complete arcs in  $PG(2, q^n)$  with growing  $n$  can be obtained by constructions of [32] using arcs of **Constructions A–C**.

In this work, using **Constructions A–C** and randomized greedy algorithms, new complete arcs in  $PG(2, q)$  are obtained for  $169 \leq q \leq 349$  and  $q = 1013, 2003$ .

In Section 2 we describe the greedy algorithms used for obtaining new arcs. In Section 3 we collect the known and new upper bounds on  $t_2(2, q)$  for  $q \leq 5107$  and  $q \in T_2 \cup T_3$ . The bounds are represented by tables, where values of  $\bar{t}_2(2, q)$  are written, and by the corresponding relations. In Section 4 we give the upper bounds on  $t_2(2, q)$  in the form of (1.3) and substantiate **Conjecture 1.3**. In Section 5 new **Constructions A–C** of complete arcs are described. Finally, in Section 6 we present new sizes of complete arcs in  $PG(2, q)$  with  $169 \leq q \leq 349$  and  $q = 1013, 2003$ .

Some of the results of this work were briefly presented without proofs in [6]; see also [7].

## 2. An approach to computer search

In this paper for computer search we use the randomized greedy algorithms [3, Sec. 2], [11, Sec. 2] that are convenient for relatively large  $q$  and for obtaining examples of different sizes of complete arcs. At every step an algorithm minimizes or maximizes an objective function  $f$  but some steps are executed in a random manner. The number of these steps and their ordinal numbers have been taken intuitively. Also, if the same extremum of  $f$  can be obtained in distinct ways, one way is chosen randomly.

We begin to construct a complete arc by using a starting set of points  $S_0$ . At the  $i$ -th step one point is added to the set and we obtain a point set  $S_i$ . As the value of the objective function  $f$  we consider the number of points in  $PG(2, q)$  that lie on bisecants of the set obtained. For small arcs we look for the maximum of the objective function  $f$ . For the spectrum of arc sizes we use both the maximum and the minimum of  $f$ .

On every “random” steps we take  $d_q$  of randomly chosen uncovered points of  $PG(2, q)$  and compute the objective function  $f$  adding each of these  $d_q$  points to  $S_i$ . The point providing the extremum is included into  $S_i$ . The value of  $d_q$  is given intuitively depending upon  $q$ , upon the number of chosen points (i.e.  $|S_{i-1}|$ ), and upon the current task (small arcs or the spectrum of arc sizes). For example, one can put  $d_q = 1$  for finding the spectrum and  $d_q = 100\beta$  with  $\beta = 1, 2, \dots$  for small arcs.

We can use  $S_0$  as a subset of points of an arc obtained in previous stages of the search. Also, for finding the spectrum of arc sizes it is fruitful to take  $S_0$  as a part of points of a conic. A generator of random numbers is used for a random choice. To get arcs with distinct sizes, starting conditions of the generator are changed for the same set  $S_0$ . In this way the algorithm works in a convenient limited region of the search space to obtain examples improving the size of the arc from which the fixed points have been taken.

In order to obtain arcs with new sizes one should make sufficiently many attempts with the randomized greedy algorithms. For small arcs, the so called predicted sizes considered in Section 4 are useful for understanding if a good result has been obtained. If the result is not close to the predicted size, the attempts should be continued.

Note also that arcs with sizes close to  $\bar{t}_2(2, q)$  usually are obtained as a byproduct when we execute the computer search for the smallest arcs using a few attempts.

## 3. Small complete $k$ -arcs in $PG(2, q)$ , $q \leq 5107$ , $q \in T_2$

Throughout the paper, in all tables we denote  $A_q = \lfloor a_q\sqrt{q} - \bar{t}_2(2, q) \rfloor$ , where

$$a_q = \begin{cases} 4 & \text{if } q \leq 841, q \in Q \\ 4.5 & \text{if } 853 \leq q \leq 2593, q = 2693, 2753, q \notin Q \\ 5 & \text{if } 2609 \leq q \leq 8192, q \notin \{2693, 2753\}. \end{cases}$$

Also, in all tables,  $B_q$  is a superior approximation of  $\bar{t}_2(2, q)/\sqrt{q}$ .

For  $q \leq 841$ , the values of  $\bar{t}_2(2, q)$  (up to June 2009) are collected in [5, Tab. 1]. In this work we obtained small arcs with new sizes for  $q \in T_1$ . The new arcs are obtained by computer search, based on the randomized greedy algorithms. Complete 90-arcs for  $q = 601, 661$  came from a theoretical study on orbits of subgroups, helped by computer; see [22]. These arcs are announced also in [10, Tab. 1]. A complete 104-arc for  $q = 709$  is obtained by the greedy algorithm with the starting point set taking from [22]. The current values of  $\bar{t}_2(2, q)$  for  $q \leq 841$  are given in Table 1. The data for  $q \in T_1$  improving results of [5, Tab. 1] are written in Table 1 in bold font. The exact values  $\bar{t}_2(2, q) = t_2(2, q)$  are marked by the dot “.”. In particular, due to the recent result [38] we noted the values  $t_2(2, 31) = t_2(2, 32) = 14$ .

From Table 1 and the results of [19,23], on complete  $(4\sqrt{q} - 4)$ -arcs for  $q = p^2$  (see Introduction) we obtain **Theorem 3.1** improving and extending the results of [5, Th. 1].

**Theorem 3.1.** *In  $PG(2, q)$ , the following holds.*

$$\begin{aligned} t_2(2, q) &< 4\sqrt{q} \quad \text{for } 2 \leq q \leq 841, q \in Q. \\ t_2(2, q) &\leq 3\sqrt{q} \quad \text{for } 2 \leq q \leq 89, q = 101; \\ t_2(2, q) &< 3.5\sqrt{q} \quad \text{for } 2 \leq q \leq 277; \\ t_2(2, q) &< 3.8\sqrt{q} \quad \text{for } 2 \leq q \leq 509, q = 521, 523, 529, 601, 661. \end{aligned} \tag{3.1}$$

**Table 1**

The smallest known sizes  $\bar{t}_2 = \bar{t}_2(2, q) < 4\sqrt{q}$  of complete arcs in planes  $PG(2, q)$ ,  $q \leq 841$ .  $A_q = \lfloor 4\sqrt{q} - \bar{t}_2(2, q) \rfloor$ ,  $B_q \geq \bar{t}_2(2, q) / \sqrt{q}$ .

$q$	$\bar{t}_2$	$A_q$	$B_q$	$q$	$\bar{t}_2$	$A_q$	$B_q$	$q$	$\bar{t}_2$	$A_q$	$B_q$	$q$	$\bar{t}_2$	$A_q$	$B_q$
2	4.	1	2.83	128	34	11	3.01	347	67	7	3.60	<b>599</b>	<b>94</b>	<b>3</b>	<b>3.85</b>
3	4.	2	2.31	131	36	9	3.15	349	67	7	3.59	<b>601</b>	<b>90</b>	<b>8</b>	<b>3.68</b>
4	6.	2	3.00	137	37	9	3.17	353	68	7	3.62	<b>607</b>	<b>95</b>	<b>3</b>	<b>3.86</b>
5	6.	2	2.69	139	37	10	3.14	359	69	6	3.65	<b>613</b>	<b>96</b>	<b>3</b>	<b>3.88</b>
7	6.	4	2.27	149	39	9	3.20	361	69	7	3.64	<b>617</b>	<b>96</b>	<b>3</b>	<b>3.87</b>
8	6.	5	2.13	151	39	10	3.18	367	70	6	3.66	<b>619</b>	<b>96</b>	<b>3</b>	<b>3.86</b>
9	6.	6	2.00	157	40	10	3.20	<b>373</b>	<b>70</b>	<b>7</b>	<b>3.63</b>	625	96	4	3.84
11	7.	6	2.12	163	41	10	3.22	379	71	6	3.65	<b>631</b>	<b>97</b>	<b>3</b>	<b>3.87</b>
13	8.	6	2.22	167	42	9	3.26	383	71	7	3.63	<b>641</b>	<b>98</b>	<b>3</b>	<b>3.88</b>
16	9.	7	2.25	169	42	10	3.24	389	72	6	3.66	643	99	2	3.91
17	10.	6	2.43	<b>173</b>	<b>43</b>	<b>9</b>	<b>3.27</b>	397	73	6	3.67	647	99	2	3.90
19	10.	7	2.30	179	44	9	3.29	401	74	6	3.70	653	100	2	3.92
23	10.	9	2.09	<b>181</b>	<b>44</b>	<b>9</b>	<b>3.28</b>	<b>409</b>	<b>74</b>	<b>6</b>	<b>3.66</b>	659	100	2	3.90
25	12.	8	2.40	191	46	9	3.33	419	76	5	3.72	<b>661</b>	<b>90</b>	<b>12</b>	<b>3.51</b>
27	12.	8	2.31	<b>193</b>	<b>46</b>	<b>9</b>	<b>3.32</b>	421	76	6	3.71	<b>673</b>	<b>101</b>	<b>2</b>	<b>3.90</b>
29	13.	8	2.42	197	47	9	3.35	431	77	6	3.71	<b>677</b>	<b>102</b>	<b>2</b>	<b>3.93</b>
31	<b>14.</b>	8	2.52	199	47	9	3.34	433	77	6	3.71	<b>683</b>	<b>102</b>	<b>2</b>	<b>3.91</b>
32	<b>14.</b>	8	2.48	211	49	9	3.38	439	78	5	3.73	<b>691</b>	<b>103</b>	<b>2</b>	<b>3.92</b>
37	15	9	2.47	223	51	8	3.42	<b>443</b>	<b>78</b>	<b>6</b>	<b>3.71</b>	701	104	1	3.93
41	16	9	2.50	227	51	9	3.39	<b>449</b>	<b>79</b>	<b>5</b>	<b>3.73</b>	<b>709</b>	<b>104</b>	<b>2</b>	<b>3.91</b>
43	16	10	2.45	<b>229</b>	<b>51</b>	<b>9</b>	<b>3.38</b>	<b>457</b>	<b>80</b>	<b>5</b>	<b>3.75</b>	719	106	1	3.96
47	18	9	2.63	233	52	9	3.41	<b>461</b>	<b>80</b>	<b>5</b>	<b>3.73</b>	727	106	1	3.94
49	18	10	2.58	239	53	8	3.43	<b>463</b>	<b>80</b>	<b>6</b>	<b>3.72</b>	729	104	4	3.86
53	18	11	2.48	241	53	9	3.42	<b>467</b>	<b>81</b>	<b>5</b>	<b>3.75</b>	733	107	1	3.96
59	20	10	2.61	<b>243</b>	<b>53</b>	<b>9</b>	<b>3.40</b>	<b>479</b>	<b>82</b>	<b>5</b>	<b>3.75</b>	739	107	1	3.94
61	20	11	2.57	251	55	8	3.48	<b>487</b>	<b>83</b>	<b>5</b>	<b>3.77</b>	743	108	1	3.97
64	22	10	2.75	256	55	9	3.44	<b>491</b>	<b>83</b>	<b>5</b>	<b>3.75</b>	751	108	1	3.95
67	23	9	2.81	<b>257</b>	<b>55</b>	<b>9</b>	<b>3.44</b>	<b>499</b>	<b>84</b>	<b>5</b>	<b>3.77</b>	757	109	1	3.97
71	22	11	2.62	263	56	8	3.46	503	85	4	3.79	761	109	1	3.96
73	24	10	2.81	269	57	8	3.48	509	85	5	3.77	769	110	0	3.97
79	26	9	2.93	<b>271</b>	<b>57</b>	<b>8</b>	<b>3.47</b>	512	86	4	3.81	773	111	0	4.00
81	26	10	2.89	<b>277</b>	<b>58</b>	<b>8</b>	<b>3.49</b>	521	86	5	3.77	787	112	0	4.00
83	27	9	2.97	281	59	8	3.52	523	86	5	3.77	797	112	0	3.97
89	28	9	2.97	283	59	8	3.51	<b>529</b>	<b>87</b>	<b>5</b>	<b>3.79</b>	809	113	0	3.98
97	30	9	3.05	289	60	8	3.53	541	89	4	3.83	811	113	0	3.97
101	30	10	2.99	<b>293</b>	<b>60</b>	<b>8</b>	<b>3.51</b>	547	89	4	3.81	821	114	0	3.98
103	31	9	3.06	307	62	8	3.54	557	90	4	3.82	823	114	0	3.98
107	32	9	3.10	311	63	7	3.58	<b>563</b>	<b>91</b>	<b>3</b>	<b>3.84</b>	827	115	0	4.00
109	32	9	3.07	313	63	7	3.57	<b>569</b>	<b>91</b>	<b>4</b>	<b>3.82</b>	829	115	0	4.00
113	33	9	3.11	317	63	8	3.54	<b>571</b>	<b>92</b>	<b>3</b>	<b>3.86</b>	839	115	0	3.98
121	34	10	3.10	331	65	7	3.58	<b>577</b>	<b>92</b>	<b>4</b>	<b>3.84</b>	841	112	4	3.87
125	35	9	3.14	337	66	7	3.60	<b>587</b>	<b>93</b>	<b>3</b>	<b>3.84</b>				
127	35	10	3.11	<b>343</b>	<b>66</b>	<b>8</b>	<b>3.57</b>	<b>593</b>	<b>94</b>	<b>3</b>	<b>3.87</b>				

Also,

- $t_2(2, q) \leq 4\sqrt{q} - 9$  for  $37 \leq q \leq 211$ ,  $q = 23, 227, 229, 233, 241, 243, 256, 257, 661$ ;
- $t_2(2, q) \leq 4\sqrt{q} - 8$  for  $23 \leq q \leq 307$ ,  $q = 317, 343, 601, 661$ ;
- $t_2(2, q) \leq 4\sqrt{q} - 7$  for  $19 \leq q \leq 353$ ,  $q = 16, 361, 373, 383, 601, 661$ ;
- $t_2(2, q) \leq 4\sqrt{q} - 6$  for  $9 \leq q \leq 409$ ,  $q = 421, 431, 433, 443, 463, 601, 661$ ;
- $t_2(2, q) \leq 4\sqrt{q} - 5$  for  $8 \leq q \leq 499$ ,  $q = 509, 521, 523, 529, 601, 661$ ;
- $t_2(2, q) \leq 4\sqrt{q} - 4$  for  $7 \leq q \leq 557$ ,  $q = 569, 577, 601, 625, 661, 729, 841$ ,  $q \in Q$ ;
- $t_2(2, q) < 4\sqrt{q} - 3$  for  $7 \leq q \leq 641$ ,  $q = 661, 729, 841$ ,  $q \in Q$ ;
- $t_2(2, q) \leq 4\sqrt{q} - 2$  for  $3 \leq q \leq 691$ ,  $q = 709, 729, 841$ ,  $q \in Q$ ;
- $t_2(2, q) < 4\sqrt{q} - 1$  for  $2 \leq q \leq 761$ ,  $q = 841$ ,  $q \in Q$ .

In Table 2, the current values of  $\bar{t}_2(2, q)$  for  $853 \leq q \leq 2593$  are given. The data for  $q = p^2$  with  $\bar{t}_2(2, q) = 4\sqrt{q} - 4$  [19,23] are written in bold font.

From Table 2, we obtain Theorem 3.2.

**Theorem 3.2.** In  $PG(2, q)$ , the following holds.

$$t_2(2, q) < 4.5\sqrt{q} \text{ for } q \leq 2593, q = 2693, 2753. \tag{3.2}$$

**Table 2**

The smallest known sizes  $\bar{t}_2 = \bar{t}_2(2, q) < 4.5\sqrt{q}$  of complete arcs in planes  $PG(2, q)$ ,  $853 \leq q \leq 2593$ ,  $A_q = \lfloor a_q\sqrt{q} - \bar{t}_2(2, q) \rfloor$ ,  $B_q \geq \bar{t}_2(2, q)/\sqrt{q}$ .

$q$	$\bar{t}_2$	$A_q$	$B_q$	$q$	$\bar{t}_2$	$A_q$	$B_q$	$q$	$\bar{t}_2$	$A_q$	$B_q$	$q$	$\bar{t}_2$	$A_q$	$B_q$
853	117	14	4.01	1277	150	10	4.20	1693	178	7	4.33	2141	205	3	4.44
857	118	13	4.04	1279	150	10	4.20	1697	178	7	4.33	2143	205	3	4.43
859	118	13	4.03	1283	150	11	4.19	1699	178	7	4.32	2153	205	3	4.42
863	118	14	4.02	1289	151	10	4.21	1709	178	8	4.31	2161	205	4	4.41
877	119	14	4.02	1291	151	10	4.21	1721	179	7	4.32	2179	207	3	4.44
881	120	13	4.05	1297	151	11	4.20	1723	180	6	4.34	2187	207	3	4.43
883	120	13	4.04	1301	152	10	4.22	1733	180	7	4.33	2197	208	2	4.44
887	120	14	4.03	1303	151	11	4.19	1741	181	6	4.34	2203	208	3	4.44
907	122	13	4.06	1307	152	10	4.21	1747	181	7	4.34	2207	208	3	4.43
911	122	13	4.05	1319	153	10	4.22	1753	181	7	4.33	2209	208	3	4.43
919	123	13	4.06	1321	153	10	4.21	1759	182	6	4.34	2213	209	2	4.45
929	124	13	4.07	1327	153	10	4.21	1777	183	6	4.35	2221	209	3	4.44
937	124	13	4.06	1331	154	10	4.23	1783	183	7	4.34	2237	210	2	4.45
941	125	13	4.08	1361	156	10	4.23	1787	183	7	4.33	2239	210	2	4.44
947	125	13	4.07	1367	156	10	4.22	1789	184	6	4.36	2243	210	3	4.44
953	126	12	4.09	<b>1369</b>	<b>144</b>	<b>4</b>	<b>3.90</b>	1801	184	6	4.34	2251	211	2	4.45
<b>961</b>	<b>120</b>	<b>4</b>	<b>3.88</b>	1373	157	9	4.24	1811	184	7	4.33	2267	211	3	4.44
967	127	12	4.09	1381	157	10	4.23	1823	186	6	4.36	2269	212	2	4.46
971	127	13	4.08	1399	158	10	4.23	1831	186	6	4.35	2273	212	2	4.45
977	127	13	4.07	1409	159	9	4.24	1847	187	6	4.36	2281	212	2	4.44
983	128	13	4.09	1423	160	9	4.25	1849	187	6	4.35	2287	213	2	4.46
991	127	14	4.04	1427	160	9	4.24	1861	188	6	4.36	2293	213	2	4.45
997	129	13	4.09	1429	160	10	4.24	1867	188	6	4.36	2297	213	2	4.45
1009	130	12	4.10	1433	161	9	4.26	1871	189	5	4.37	2309	214	2	4.46
1013	130	13	4.09	1439	161	9	4.25	1873	189	5	4.37	2311	214	2	4.46
1019	131	12	4.11	1447	162	9	4.26	1877	189	5	4.37	2333	215	2	4.46
1021	131	12	4.10	1451	162	9	4.26	1879	189	6	4.37	2339	216	1	4.47
<b>1024</b>	<b>124</b>	<b>4</b>	<b>3.88</b>	1453	162	9	4.25	1889	190	5	4.38	2341	216	1	4.47
1031	132	12	4.12	1459	162	9	4.25	1901	190	6	4.36	2347	216	2	4.46
1033	132	12	4.11	1471	163	9	4.25	1907	191	5	4.38	2351	216	2	4.46
1039	132	13	4.10	1481	164	9	4.27	1913	191	5	4.37	2357	217	1	4.47
1049	133	12	4.11	1483	164	9	4.26	1931	192	5	4.37	2371	217	2	4.46
1051	133	12	4.11	1487	164	9	4.26	1933	192	5	4.37	2377	216	3	4.44
1061	134	12	4.12	1489	164	9	4.26	1949	193	5	4.38	2381	217	2	4.45
1063	134	12	4.11	1493	165	8	4.28	1951	193	5	4.37	2383	218	1	4.47
1069	135	12	4.13	1499	165	9	4.27	1973	195	4	4.40	2389	218	1	4.47
1087	136	12	4.13	1511	166	8	4.28	1979	195	5	4.39	2393	218	2	4.46
1091	136	12	4.12	1523	167	8	4.28	1987	196	4	4.40	2399	219	1	4.48
1093	136	12	4.12	1531	167	9	4.27	1993	196	4	4.40	<b>2401</b>	<b>192</b>	<b>4</b>	<b>3.92</b>
1097	137	12	4.14	1543	167	9	4.26	1997	196	5	4.39	2411	220	0	4.49
1103	137	12	4.13	1549	168	9	4.27	1999	196	5	4.39	2417	220	1	4.48
1109	138	11	4.15	1553	169	8	4.29	2003	197	4	4.41	2423	220	1	4.47
1117	138	12	4.13	1559	169	8	4.29	2011	197	4	4.40	2437	221	1	4.48
1123	139	11	4.15	1567	170	8	4.30	2017	197	5	4.39	2441	221	1	4.48
1129	139	12	4.14	1571	170	8	4.29	2027	198	4	4.40	2447	221	1	4.47
1151	141	11	4.16	1579	170	8	4.28	2029	198	4	4.40	2459	222	1	4.48
1153	141	11	4.16	1583	171	8	4.30	2039	199	4	4.41	2467	223	0	4.49
1163	142	11	4.17	1597	172	7	4.31	2048	199	4	4.40	2473	223	0	4.49
1171	142	11	4.15	1601	172	8	4.30	2053	199	4	4.40	2477	223	0	4.49
1181	143	11	4.17	1607	172	8	4.30	2063	200	4	4.41	2503	224	1	4.48
1187	144	11	4.18	1609	172	8	4.29	2069	200	4	4.40	2521	225	0	4.49
1193	144	11	4.17	1613	173	7	4.31	2081	201	4	4.41	2531	226	0	4.50
1201	145	10	4.19	1619	173	8	4.30	2083	201	4	4.41	2539	226	0	4.49
1213	145	11	4.17	1621	173	8	4.30	2087	201	4	4.40	2543	226	0	4.49
1217	146	10	4.19	1627	174	7	4.32	2089	201	4	4.40	2549	226	1	4.48
1223	146	11	4.18	1637	174	8	4.31	2099	202	4	4.41	2551	227	0	4.50
1229	146	11	4.17	1657	175	8	4.30	2111	203	3	4.42	2557	227	0	4.49
1231	147	10	4.19	1663	176	7	4.32	2113	203	3	4.42	2579	228	0	4.49
1237	147	11	4.18	1667	176	7	4.32	2129	204	3	4.43	2591	229	0	4.50
1249	148	11	4.19	1669	176	7	4.31	2131	204	3	4.42	2593	229	0	4.50
1259	149	10	4.20	<b>1681</b>	<b>160</b>	<b>4</b>	<b>3.91</b>	2137	204	4	4.42				

$t_2(2, q) < 4.1\sqrt{q}$  for  $q \leq 1013$ ,  $q = 1021, 1024, 1039, 1369, 1681, 2401$ ;

$t_2(2, q) < 4.2\sqrt{q}$  for  $q \leq 1283$ ,  $q = 1297, 1303, 1369, 1681, 2401$ ;

$t_2(2, q) < 4.3\sqrt{q}$  for  $q \leq 1583$ ,  $q = 1601, 1607, 1609, 1619, 1621, 1657, 1681, 2401$ ;

$t_2(2, q) < 4.4\sqrt{q}$  for  $q \leq 1999$ ,  $q = 2011, 2017, 2027, 2029, 2048, 2053, 2069, 2087, 2089, 2401$ .

Also,

$$\begin{aligned}
 t_2(2, q) &< 4.5\sqrt{q} - 12 \quad \text{for } q \leq 1103, q = 1117, 1129, 1369, 1681, 2401; \\
 t_2(2, q) &< 4.5\sqrt{q} - 11 \quad \text{for } q \leq 1193, q = 1213, 1223, 1229, 1237, 1249, 1283, 1297, 1303, 1369, 1681, 2401; \\
 t_2(2, q) &< 4.5\sqrt{q} - 10 \quad \text{for } q \leq 1369, q = 1381, 1399, 1429, 1681, 2401; \\
 t_2(2, q) &< 4.5\sqrt{q} - 9 \quad \text{for } q \leq 1489, q = 1499, 1531, 1543, 1549, 1681, 2401; \\
 t_2(2, q) &< 4.5\sqrt{q} - 8 \quad \text{for } q \leq 1583, q = 1601, 1607, 1609, 1619, 1621, 1637, 1657, 1681, 1709, 2401; \\
 t_2(2, q) &< 4.5\sqrt{q} - 7 \quad \text{for } q \leq 1721, q = 1733, 1747, 1753, 1783, 1787, 1811, 2401; \\
 t_2(2, q) &< 4.5\sqrt{q} - 6 \quad \text{for } q \leq 1867, q = 1879, 1901, 2401; \\
 t_2(2, q) &< 4.5\sqrt{q} - 5 \quad \text{for } q \leq 1951, q = 1979, 1997, 1999, 2017, 2401; \\
 t_2(2, q) &< 4.5\sqrt{q} - 4 \quad \text{for } q \leq 2099, q = 2137, 2161, 2401; \\
 t_2(2, q) &< 4.5\sqrt{q} - 3 \quad \text{for } q \leq 2187, q = 2203, 2207, 2209, 2221, 2243, 2267, 2377, 2401; \\
 t_2(2, q) &< 4.5\sqrt{q} - 2 \quad \text{for } q \leq 2333, q = 2347, 2351, 2371, 2377, 2381, 2393, 2401; \\
 t_2(2, q) &< 4.5\sqrt{q} - 1 \quad \text{for } q \leq 2401, q = 2417, 2423, 2437, 2441, 2447, 2459, 2503, 2549.
 \end{aligned}$$

In Table 3, the current values of  $\bar{t}_2(2, q)$  for  $2609 \leq q \leq 5107$  are given. The data with  $\bar{t}_2(2, q) < 4.5\sqrt{q}$  are written in bold font.

Values of  $\bar{t}_2(2, q)$  for relatively great  $q \in T_2 \cup T_3$  are given in Table 4. The notation  $\bar{D}_q\left(\frac{3}{4}\right)$  is explained in the next section.

In Table 2 for  $q \in Q$ , we use the results of [9,19,23]; see also [5, p. 35]. In Table 4, for  $q \in T_3$  we use the results of [8,9], see also [5, p. 35] and Introduction. The remaining sizes  $k$  for small complete  $k$ -arcs in Tables 2–4 are obtained in this work by computer search with the help of the randomized greedy algorithms.

Note that a complete 199-arc in  $PG(2, 2048)$  of Table 2, a complete 301-arc in  $PG(2, 4096)$  of Table 3, and a complete 450-arc in  $PG(2, 8192)$  of Table 4 improve the results of [9] for  $q = 2^{11}, 2^{12}, 2^{13}$ ; see Introduction.

From Tables 3 and 4, we obtain Theorem 3.3.

**Theorem 3.3.** *In  $PG(2, q)$ , the following holds.*

$$\begin{aligned}
 t_2(2, q) &< 4.79\sqrt{q} \quad \text{for } q \leq 5107. & (3.3) \\
 t_2(2, q) &< 4.6\sqrt{q} \quad \text{for } q \leq 3209, q = 3221, 3229, 3251, 3253, 3257, 3271, 3299, 3301, 3319, 3323, 3343, 3347; \\
 t_2(2, q) &< 4.7\sqrt{q} \quad \text{for } q \leq 4093, q = 4099, 4111, 4129, 4133, 4139, 4153, 4157, 4159, 4177, 4217, \\
 &4219, 4229, 4241, 4243, 4253, 4271, 4273, 4297, 4363, 4423.
 \end{aligned}$$

Also,

$$\begin{aligned}
 t_2(2, q) &< 5\sqrt{q} - 22 \quad \text{for } q \leq 3391, q = 3413, 3433, 3457, 3461, 3463, 3467, 3469, 3481, 3491, 3499, \\
 &3511, 3517, 3529, 3533, 3539, 3559, 3583, 3607, 3617, 3643, 3739; \\
 t_2(2, q) &< 5\sqrt{q} - 21 \quad \text{for } q \leq 3659, q = 3673, 3677, 3697, 3701, 3709, 3721, 3727, 3733, 3739, 3779, \\
 &3797, 3803, 3821, 3853, 3919, 4021, 4027, 4153; \\
 t_2(2, q) &< 5\sqrt{q} - 20 \quad \text{for } q \leq 3911, q = 3919, 3923, 3929, 3931, 3947, 3967, 4001, 4003, 4013, 4019, \\
 &4021, 4027, 4049, 4057, 4073, 4099, 4153, 4159, 4177, 4229, 4253, 4273, 4363, 4423; \\
 t_2(2, q) &< 5\sqrt{q} - 19 \quad \text{for } q \leq 4297, q = 4357, 4363, 4423, 4447, 4463, 4481, 4489, 4517; \\
 t_2(2, q) &< 5\sqrt{q} - 16 \quad \text{for } q \leq 4969, q = 4987, 4993, 5003, 5009, 5011, 5021, 5041, 5059, 5107, 5119.
 \end{aligned}$$

#### 4. Observations on $\bar{t}_2(2, q)$ values

We look for upper estimates of the collection of  $\bar{t}_2(2, q)$  values from Tables 1–4 in the form (1.2), see [30] and [28, Tab. 2.6]. For definiteness, we use the natural logarithms. Let  $c$  be a constant independent of  $q$ . We introduce  $D_q(c)$  and  $\bar{D}_q(c)$  as follows:

$$\begin{aligned}
 t_2(2, q) &= D_q(c)\sqrt{q} \ln^c q, \\
 \bar{t}_2(2, q) &= \bar{D}_q(c)\sqrt{q} \ln^c q. & (4.1)
 \end{aligned}$$

Let  $\bar{D}_{\text{aver}}(c, q_0)$  be the average value of  $\bar{D}_q(c)$  calculated in the region  $q_0 \leq q \leq 5107$  and  $q \in T_2$  under condition  $q \notin N$ . From Tables 1–4, we obtain Observation 1.





Table 3 (continued)

$q$	$\bar{t}_2$	$A_q$	$B_q$	$q$	$\bar{t}_2$	$A_q$	$B_q$	$q$	$\bar{t}_2$	$A_q$	$B_q$	$q$	$\bar{t}_2$	$A_q$	$B_q$
3163	257	24	4.57	3767	286	20	4.66	4423	312	20	4.70	5077	341	15	4.79
3167	258	23	4.59	3769	286	20	4.66	4441	315	18	4.73	5081	341	15	4.79
3169	258	23	4.59	3779	286	21	4.66	4447	314	19	4.71	5087	341	15	4.79
3181	259	23	4.60	3793	287	20	4.67	4451	316	17	4.74	5099	342	15	4.79
3187	258	24	4.58	3797	287	21	4.66	4457	315	18	4.72	5101	342	15	4.79
3191	259	23	4.59	3803	287	21	4.66	4463	315	19	4.72	5107	341	16	4.78
3203	259	23	4.58	3821	288	21	4.66	4481	315	19	4.71				
3209	260	23	4.59	3823	289	20	4.68	4483	317	17	4.74				
3217	261	22	4.61	3833	289	20	4.67	4489	316	19	4.72				

Table 4

The smallest known sizes  $\bar{t}_2 = \bar{t}_2(2, q)$  of complete arcs in planes  $PG(2, q)$  with  $q \in T_2 \cup T_3$ ,  $A_q = \lfloor 5\sqrt{q} - \bar{t}_2(2, q) \rfloor$ ,  $B_q > \bar{t}_2(2, q)/\sqrt{q}$ .

$q$	$\bar{t}_2$	$A_q$	$B_q$	$\bar{D}_q(\frac{3}{4})$	$q$	$\bar{t}_2$	$A_q$	$B_q$	$\bar{D}_q(\frac{3}{4})$	$q$	$\bar{t}_2$	$A_q$	$B_q$	$\bar{D}_q(\frac{3}{4})$
5119	341	16	4.77	0.9540	5261	348	14	4.80	0.9581	5843	370	12	4.85	0.9578
5147	343	15	4.79	0.9566	5279	349	14	4.81	0.9589	6011	377	10	4.87	0.9598
5153	344	14	4.80	0.9587	5281	349	14	4.81	0.9587	8192	450	2	4.98	0.9560
5209	346	14	4.80	0.9582	5303	349	15	4.80	0.9564	$2^{14}$	665		5.20	0.9449
5231	347	14	4.80	0.9585	5347	352	13	4.82	0.9599	$2^{15}$	993		5.49	0.9474
5237	347	14	4.80	0.9579	5641	363	12	4.84	0.9592	$2^{18}$	3066		5.99	0.9020

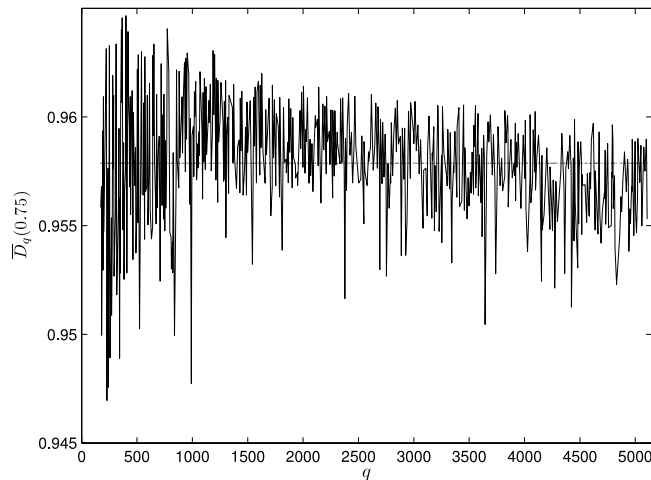


Fig. 1. The values of  $\bar{D}_q(0.75)$  for  $173 \leq q \leq 5107$ ,  $q \notin N$ .  $\bar{D}_{aver}(0.75, 173) = 0.95787$ .

**Observation 1.** Let  $173 \leq q \leq 5107$  or  $q \in T_2$ , under condition  $q \notin N$ . Then

- (i) When  $q$  grows,  $\bar{D}_q(0.8)$  has a tendency to decrease.
- (ii) When  $q$  grows,  $\bar{D}_q(0.5)$  has a tendency to increase.
- (iii) When  $q$  grows, the values of  $\bar{D}_q(0.75)$  oscillate about the average value  $\bar{D}_{aver}(0.75, 173) = 0.95787$  (see Fig. 1). Also,

$$\begin{aligned}
 &0.946 < \bar{D}_q(0.75) < 0.9647 \quad \text{if } 173 \leq q \leq 997, \\
 &0.953 < \bar{D}_q(0.75) < 0.9631 \quad \text{if } 1009 \leq q \leq 1999, \\
 &0.951 < \bar{D}_q(0.75) < 0.9615 \quad \text{if } 2003 \leq q \leq 2999, \\
 &0.950 < \bar{D}_q(0.75) < 0.9608 \quad \text{if } 3001 \leq q \leq 3989, \\
 &0.951 < \bar{D}_q(0.75) < 0.9603 \quad \text{if } 4001 \leq q.
 \end{aligned} \tag{4.2}$$

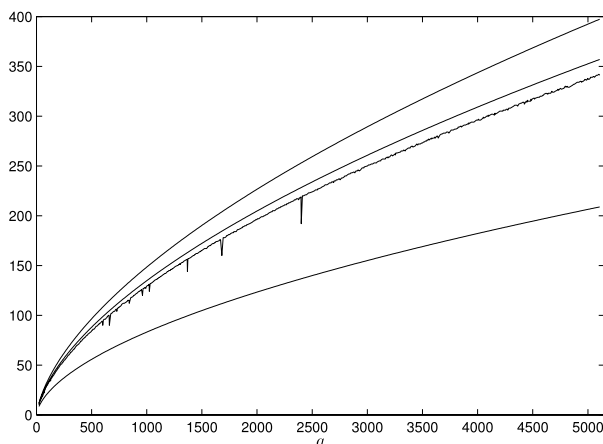
Data for relatively big  $q$ , collected in Table 4, in large confirm Observation 1.

By Observation 1 it seems that the values of  $D_q(0.75)$  and  $\bar{D}_q(0.75)$  are sufficiently convenient for estimates of  $t_2(2, q)$  and  $\bar{t}_2(2, q)$ .

From Tables 1–4, we obtain Theorem 4.1.

**Theorem 4.1.** In  $PG(2, q)$ ,

$$t_2(2, q) < 0.9987\sqrt{q} \ln^{0.75} q \quad \text{for } 23 \leq q \leq 5107, \quad q \in T_2 \cup T_3. \tag{4.3}$$



**Fig. 2.** The values of  $\sqrt{q} \ln^{0.8} q$  (the top curve),  $\sqrt{q} \ln^{0.75} q$  (the 2-nd curve),  $\bar{t}_2(2, q)$  (the 3-rd curve), and  $\sqrt{q} \ln^{0.5} q$  (the bottom curve) for  $23 \leq q \leq 5107$ .

In [Theorem 1.2](#) we slightly rounded the estimate [\(4.3\)](#).

The graphs of values of  $\sqrt{q} \ln^{0.8} q$ ,  $\sqrt{q} \ln^{0.75} q$ ,  $\bar{t}_2(2, q)$ , and  $\sqrt{q} \ln^{0.5} q$  are shown in [Fig. 2](#), where  $\sqrt{q} \ln^{0.8} q$  is the top curve and  $\sqrt{q} \ln^{0.5} q$  is the bottom one.

One can see in [Fig. 2](#) that always  $\bar{t}_2(2, q) < \sqrt{q} \ln^{0.75} q$  and, moreover, when  $q$  grows, the graphs  $\sqrt{q} \ln^{0.75} q$  and  $\bar{t}_2(2, q)$  diverge so that positive difference  $\sqrt{q} \ln^{0.75} q - \bar{t}_2(2, q)$  increases.

We denote

$$\hat{t}_2(2, q) = \bar{D}_{\text{aver}}(0.75, 173) \sqrt{q} \ln^{0.75} q, \quad \bar{\Delta}_q = \bar{t}_2(2, q) - \hat{t}_2(2, q), \quad \bar{P}_q = \frac{100 \bar{\Delta}_q}{\bar{t}_2(2, q)} \%. \tag{4.4}$$

One can treat  $\hat{t}_2(2, q)$  as a predicted value of  $t_2(2, q)$ . Then  $\bar{\Delta}_q$  is the difference between the smallest known size  $\bar{t}_2(2, q)$  of complete arcs and the predicted value. Finally,  $\bar{P}_q$  is this difference in percentage terms of the smallest known size.

**Observation 2.** Let  $173 \leq q \leq 5107$  or  $q \in T_2, q \notin N$ . Then

$$-2.18 < \bar{\Delta}_q < 0.78. \tag{4.5}$$

$$\begin{aligned} -1.16\% < \bar{P}_q < 0.71\% & \text{ if } 173 \leq q \leq 997, \\ -0.49\% < \bar{P}_q < 0.54\% & \text{ if } 1009 \leq q \leq 1999, \\ -0.66\% < \bar{P}_q < 0.37\% & \text{ if } 2003 \leq q \leq 2999, \\ -0.79\% < \bar{P}_q < 0.30\% & \text{ if } 3001 \leq q \leq 3989, \\ -0.70\% < \bar{P}_q < 0.25\% & \text{ if } 4001 \leq q. \end{aligned} \tag{4.6}$$

By [\(4.5\)](#) and [\(4.6\)](#), see also [Figs. 3](#) and [4](#), the upper bounds of  $\bar{\Delta}_q$  and  $\bar{P}_q$  are relatively small. Moreover, the upper bound of  $\bar{P}_q$  decreases when  $q$  grows. Therefore the values of  $\bar{\Delta}_q$  and  $\bar{P}_q$  are useful for computer search of small arcs.

The relations [\(4.2\)](#)–[\(4.6\)](#), [Theorems 1.2](#) and [4.1](#), and [Figs. 1–4](#) are the foundation for [Conjecture 1.3](#).

**Remark 4.2.** By above,  $\sqrt{q} \ln^{0.75} q$  seems to be a reasonable upper bound on the current collection of  $\bar{t}_2(2, q)$  values. It gives some reference points for computer search and foundations for [Conjecture 1.3](#) on the upper bound for  $t_2(2, q)$ . In principle, the constant  $c = 0.75$  can be slightly reduced to move the curve  $\sqrt{q} \ln^c q$  near to the curve of  $\bar{t}_2(2, q)$ ; see [Fig. 2](#). For example, from [Tables 1–4](#), [Theorem 4.3](#) holds.

**Theorem 4.3.** In  $PG(2, q)$ ,

$$t_2(2, q) < \sqrt{q} \ln^{0.735} q \text{ for } 32 \leq q \leq 5107, q \in T_2 \cup T_3. \tag{4.7}$$

### 5. Constructions of families of complete arcs in $PG(2, q)$

In the homogeneous coordinates of a point  $(x_0, x_1, x_2)$  we put  $x_0 \in \{0, 1\}, x_1, x_2 \in F_q$ . Let  $F_q^* = F_q \setminus \{0\}$ . Let  $\xi$  be a primitive element of  $F_q$ . Recall that indexes of powers of  $\xi$  are calculated modulo  $q - 1$ .

Throughout this section we use the conic  $\mathcal{C}$  of equation  $x_1^2 = x_0 x_2$ . We denote points of  $\mathcal{C}$  as follows:

$$A_i = (1, i, i^2), \quad i \in F_q; \quad \bar{A}_d = (1, \xi^d, \xi^{2d}), \quad d \in \{0, 1, \dots, q - 2\}; \quad A_\infty = (0, 0, 1).$$

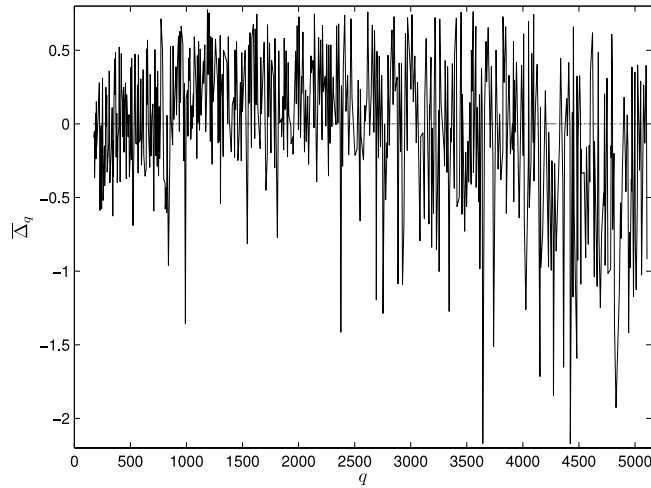


Fig. 3. The values of  $\bar{\Delta}_q = \bar{t}_2(2, q) - \widehat{t}_2(2, q)$  for  $173 \leq q \leq 5107, q \notin N$ .

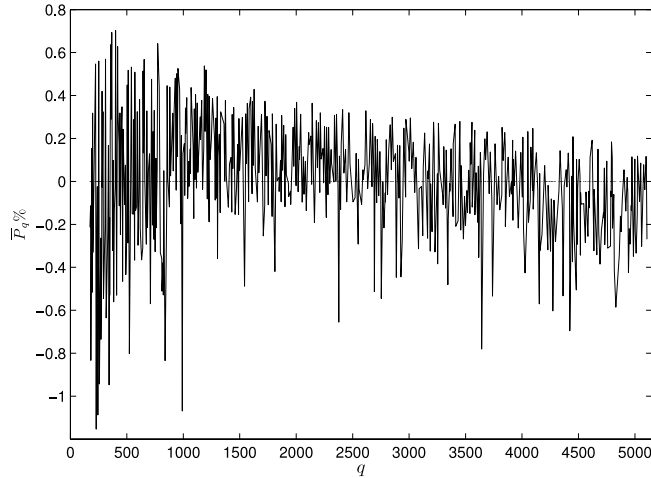


Fig. 4. The values of  $\bar{P}_q = 100\bar{\Delta}_q/\bar{t}_2(2, q)\%$  for  $173 \leq q \leq 5107, q \notin N$ .

5.1. Arcs with two points on a tangent to a conic

Throughout this subsection,  $q \geq 19$  is an odd prime. Let  $H$  be an integer in the region

$$\left\lfloor \frac{q-1}{3} \right\rfloor \leq H \leq \frac{q-1}{2}. \tag{5.1}$$

We denote by  $\mathcal{V}_H$  the following  $(H + 1)$ -subset of the conic  $\mathcal{C}$ :

$$\mathcal{V}_H = \{A_i : i = 0, 1, 2, \dots, H\} \subset \mathcal{C}. \tag{5.2}$$

We denote the points of  $PG(2, q)$ :

$$P = (0, 1, 0), \quad T_H = (0, 1, b_H), \quad b_H = \begin{cases} 2H + 1 & \text{if } H = \left\lfloor \frac{1}{3}(q-1) \right\rfloor \\ 2H & \text{if } \left\lfloor \frac{1}{3}(q-1) \right\rfloor < H \leq \frac{1}{2}(q-1). \end{cases} \tag{5.3}$$

Let  $\ell_0$  be the line of equation  $x_0 = 0$ . It is the tangent to  $\mathcal{C}$  at  $A_\infty$ . It holds that  $\{P, T_H\} \subset \ell_0$ .

**Construction A.** Let  $q$  be an odd prime. Let  $H, \mathcal{V}_H, P$  and  $T_H$  be given by (5.1)–(5.3). We construct a point  $(H + 3)$ -set  $\mathcal{K}_H$  in the plane  $PG(2, q)$  as follows:

$$\mathcal{K}_H = \mathcal{V}_H \cup \{P, T_H\}.$$

The following lemma can be proved by elementary calculations.

**Lemma 5.1.** (i) Let  $i \neq j$ . A point  $(0, 1, b)$  is collinear with points  $A_i, A_j$  if and only if

$$b = i + j. \tag{5.4}$$

(ii) Let  $i \neq j, a, b \in F_q, b \neq a^2$ . Then a point  $(1, a, b)$  is collinear with  $A_i, A_j$  if and only if  $b = a(i + j) - ij$ .

(iii) Let  $a \in F_q, a \neq i$ . Then a point  $(1, a, i^2)$  is collinear with  $P, A_i$ .

**Theorem 5.2.** The  $(H + 3)$ -set  $\mathcal{K}_H$  of Construction A is an arc in  $PG(2, q)$ .

**Proof.** By (5.1), (5.2), the sum  $i + j$  in (5.4) is running on  $\{1, 2, \dots, 2H - 1\}$ , where  $2H - 1 \leq q - 2$  if  $\lfloor \frac{1}{3}(q - 1) \rfloor < H$  and  $2H - 1 < \frac{2}{3}(q - 1)$  if  $H = \lfloor \frac{1}{3}(q - 1) \rfloor$ . So,  $\{0, b_H\} \cap \{1, 2, \dots, 2H - 1\} = \emptyset$ , see (5.3). Therefore  $P$  and  $T_H$  do not lie on bisecants of  $\mathcal{V}_H$ . In other side, any point of  $\mathcal{V}_H$  does not lie on the line  $PT_H$  as  $PT_H$  is a tangent to  $\mathcal{C}$  in  $A_\infty$ .  $\square$

**Theorem 5.3.** Let  $H$  be given by (5.1). Then all points of  $\ell_0 \cup \mathcal{C} \setminus \mathcal{V}_H$  lie on bisecants of  $\mathcal{K}_H$ .

**Proof.** All points of  $\ell_0$  are covered as two points  $P$  and  $T_H$  of this line belong to  $\mathcal{K}_H$ .

Let  $\mathcal{R}$  and  $\mathcal{S}$  be sets of integers modulo  $q$ , i.e.  $\mathcal{R} \cup \mathcal{S} \subset F_q$ .

Let  $\mathcal{R} = \{-H, -(H - 1), \dots, -1\} = \{q - H, q - (H - 1), \dots, q - 1\}$ . By Lemma 5.1(i), points  $A_j, A_{-j}, P$  are collinear. Therefore, a point  $A_j$  of  $\mathcal{C} \setminus \mathcal{V}_H$  with  $j \in \mathcal{R}$  lies on the bisecant of  $\mathcal{K}_H$  through  $P$  and  $A_{-j}$ , where  $-j \in \{H, H - 1, \dots, 1\}, A_{-j} \in \mathcal{V}_H$ .

Let  $\mathcal{S} = \{b_H - H, b_H - (H - 1), \dots, b_H - 1, b_H\}$ . By Lemma 5.1(i), a point  $A_j$  of  $\mathcal{C} \setminus \mathcal{V}_H$  with  $j \in \mathcal{S}$  lies on the bisecant  $T_H A_{b_H - j}$ , where  $b_H - j \in \{H, H - 1, \dots, 1, 0\}, A_{b_H - j} \in \mathcal{V}_H$ .

Let  $b_H = 2H + 1$ . Then  $\mathcal{S} = \{H + 1, H + 2, \dots, 2H + 1\}$ . Also, by (5.3),  $H = \lfloor \frac{1}{3}(q - 1) \rfloor$  whence  $H = \frac{1}{3}(q - v)$ , where  $v \in \{1, 2\}$  and  $v \equiv q \pmod{3}$ . Hence,  $3H = q - v, 2H + 1 = q - H - v + 1 \in \{q - H, q - H - 1\}$ .

Let  $b_H = 2H$ . Then  $\mathcal{S} = \{H, H + 1, \dots, 2H\}$ . Also, by (5.3),  $H > \lfloor \frac{1}{3}(q - 1) \rfloor$  whence  $H > \frac{1}{3}(q - v)$ , where  $v \in \{1, 2\}$  is as above. Therefore  $3H > q - v, 2H > q - H - v, 2H \geq q - H - v + 1 \in \{q - H, q - H - 1\}$ .

We proved that  $\{H + 1, H + 2, \dots, q - 1\} \subseteq \mathcal{S} \cup \mathcal{R}$ . Also we showed that the points  $A_j$  of  $\mathcal{C} \setminus \mathcal{V}_H$  with  $j \in \mathcal{S} \cup \mathcal{R}$  are covered by bisecants of  $\mathcal{K}_H$  through  $P$  (if  $j \in \mathcal{R}$ ) or through  $T_H$  (if  $j \in \mathcal{S}$ ). In the other side,  $\mathcal{C} \setminus \mathcal{V}_H = \{A_j : j = H + 1, H + 2, \dots, q - 1\} \cup \{A_\infty\}$ , where  $A_\infty \in \ell_0$ . So, all points of  $\mathcal{C} \setminus \mathcal{V}_H$  are covered.  $\square$

**Definition 5.4.** Let  $q$  be an odd prime. Let  $\bar{H}$  be an integer and let

$$\mathcal{P}_{\bar{H}} = \{P\} \cup \{A_i : i = 0, 1, 2, \dots, \bar{H}\}.$$

We call *critical value of  $\bar{H}$*  and denote by  $\bar{H}_q$  the smallest value of  $\bar{H}$  such that all points of the form  $(1, a, b), a, b \in F_q, b \neq a^2$ , lie on bisecants of  $\mathcal{P}_{\bar{H}}$ .

**Theorem 5.5.** Let  $q \geq 19$  be an odd prime. Let  $\bar{H}_q \leq \frac{1}{2}(q - 1)$  and let

$$\max \left\{ \bar{H}_q, \left\lfloor \frac{q - 1}{3} \right\rfloor \right\} \leq H \leq \frac{q - 1}{2}.$$

Then the arc  $\mathcal{K}_H$  of Construction A is complete.

**Proof.** We use Theorem 5.3 and Definition 5.4.  $\square$

In this subsection, we put  $q \geq 19$  as we checked by computer that  $\frac{1}{2}(q - 1) < \bar{H}_q$  if  $q \leq 17$ .

**Corollary 5.6.** Let  $q \geq 19$  be an odd prime. Let  $\bar{H}_q \leq \frac{1}{2}(q - 1)$ . Then Construction A forms a family of complete  $k$ -arcs in  $PG(2, q)$  containing arcs of all sizes  $k$  in the region

$$\max \left\{ \bar{H}_q, \left\lfloor \frac{q - 1}{3} \right\rfloor \right\} + 3 \leq k \leq \frac{q + 5}{2}.$$

If  $\bar{H}_q \leq \lfloor \frac{1}{3}(q - 1) \rfloor$  then the cardinality of this family is equal to  $\lceil \frac{1}{6}(q + 5) \rceil$  and the size of the smallest complete arc of the family is  $\lfloor \frac{1}{3}(q + 8) \rfloor$ .

By computer search using Lemma 5.1(ii), (iii) we obtained the following theorem.

**Theorem 5.7.** Let  $q \geq 19$  be an odd prime. Let  $\bar{H}_q$  be given by Definition 5.4. We introduce  $D_q$  and  $\Delta_q$  as follows:  $\bar{H}_q = D_q \sqrt{q} \ln q, \Delta_q = \lfloor \frac{1}{3}(q - 1) \rfloor - \bar{H}_q$ . Then the following holds.

**Table 5**

The values  $\bar{H}_q, \bar{G}_q, \bar{J}_q$  for cases  $\bar{H}_q, \bar{G}_q > \lfloor \frac{1}{3}(q-1) \rfloor, \bar{J}_q > \frac{1}{4}(q-3)$ .

$q$	$\bar{H}_q$	$\bar{G}_q$	$\bar{J}_q$	$q$	$\bar{H}_q$	$\bar{G}_q$	$\bar{J}_q$	$q$	$\bar{H}_q$	$\bar{G}_q$	$\bar{J}_q$	$q$	$\bar{H}_q$	$\bar{G}_q$	$\bar{J}_q$	$q$	$\bar{J}_q$
19	9			43	19		16	71	24		25	103			34	167	42
23	11			47	18		18	73		27		107	36		30	179	62
27			12	49		18		79	29		27	125		43		191	49
29	13			53	20	19		81		29		127			39	211	54
31	14		14	59	23		24	83	29		29	131			38	223	60
32		15		61	22	22		89	32			139			38	343	86
37	16	16		64		24		97		33		151			42		
41	16	19		67	24		23	101		35		163			41		

- (i)  $\lfloor \frac{q-1}{3} \rfloor < \bar{H}_q \leq \frac{q-1}{2}$  if  $19 \leq q \leq 71$  and  $q = 79, 83, 89, 107$ .
- (ii)  $\bar{H}_q \leq \lfloor \frac{q-1}{3} \rfloor$  if  $109 \leq q \leq 2063, q = 73, 97, 101, 103$ .
- (iii)  $\bar{H}_q < 0.84\sqrt{q} \ln q$  if  $19 \leq q \leq 2063$ . (5.5)
- (iv)  $89 \leq \Delta_q \leq 198, 0.59 < D_q < 0.82$ , if  $109 \leq q \leq 599$ ;  
 $187 \leq \Delta_q \leq 288, 0.59 < D_q < 0.78$ , if  $599 < q \leq 1049$ ;  
 $241 \leq \Delta_q \leq 464, 0.61 < D_q < 0.84$ , if  $1367 < q \leq 2063$ .

For situations  $\lfloor \frac{1}{3}(q-1) \rfloor < \bar{H}_q$ , the values of  $\bar{H}_q$  are given in Table 5.

**Theorem 5.8.** Let  $q$  be an odd prime with  $109 \leq q \leq 2063$ , or  $q = 73, 97, 101, 103$ . Then Construction A forms a family of complete  $k$ -arcs in  $PG(2, q)$  containing arcs of all sizes  $k$  in the region

$$\lfloor \frac{q+8}{3} \rfloor \leq k \leq \frac{q+5}{2}.$$

**Proof.** We use Corollary 5.6 and Theorem 5.7(ii). □

5.2. Arcs with two points on a bisecant of a conic

Throughout this subsection,  $q \geq 32$  is a prime power. Let  $G$  be an integer in the region

$$\lfloor \frac{q-1}{3} \rfloor \leq G \leq \lfloor \frac{q-3}{2} \rfloor. \tag{5.6}$$

We denote by  $\mathcal{D}_G$  the following  $(G+1)$ -subset of the conic  $\mathcal{C}$ :

$$\mathcal{D}_G = \{\bar{A}_d : d = 0, 1, 2, \dots, G\} \subset \mathcal{C}. \tag{5.7}$$

Clearly,  $A_0 \notin \mathcal{D}_G$ . Let

$$\gamma \in F_q, \quad \gamma = \begin{cases} -1 = \xi^{(q-1)/2} & \text{for } q \text{ odd} \\ 1 & \text{for } q \text{ even.} \end{cases} \tag{5.8}$$

We denote the points of  $PG(2, q)$ :

$$Z = (1, 0, \gamma\xi^0), \quad B_G = (1, 0, \gamma\xi^{\beta_G}), \tag{5.9}$$

where

$$\beta_G = \begin{cases} 2G+1 & \text{if } \lfloor \frac{1}{3}(q-1) \rfloor = G, q \equiv 0 \pmod{3} \\ 2G & \text{if } \lfloor \frac{1}{3}(q-1) \rfloor = G, q \not\equiv 0 \pmod{3} \\ 2G & \text{if } \lfloor \frac{1}{3}(q-1) \rfloor < G \leq \lfloor \frac{1}{2}(q-3) \rfloor, \text{ arbitrary } q. \end{cases} \tag{5.10}$$

Let  $\ell_1$  be the line of equation  $x_1 = 0$ . It is the bisecant  $A_\infty A_0$  of  $\mathcal{C}$ . We have  $\{Z, B_G\} \subset \ell_1$ .

Using (5.6), (5.8), by elementary calculations we obtained the following lemma.

**Lemma 5.9.** (i) Let  $d \neq t$ . A point  $(0, 1, b)$  is collinear with points  $\bar{A}_d, \bar{A}_t$  if and only if

$$b = \xi^d + \xi^t. \tag{5.11}$$

(ii) A point  $(0, 1, b)$  is collinear with points  $\bar{A}_d$  and  $(1, 0, \gamma U)$  if and only if

$$b = \frac{\xi^{2d} + U}{\xi^d}. \tag{5.12}$$

**Corollary 5.10.** (i) For all  $q$ , the point  $P = (0, 1, 0)$  does not lie on any bisecant of  $\mathcal{D}_G$ .

(ii) Let  $q$  be even. Then the points  $P, Z, \bar{A}_d$  are collinear if and only if  $d = 0$ .

(iii) Let  $q \equiv 1 \pmod{4}$ . Then the points  $P, Z, \bar{A}_d$  are collinear if and only if  $d \in \{\frac{1}{4}(q-1), \frac{3}{4}(q-1)\}$ .

(iv) Let  $q \equiv 3 \pmod{4}$  and let  $\beta_G = 2G$ . Then the points  $P, Z, \bar{A}_d$  and  $P, B_G, \bar{A}_d$  are not collinear for any  $d$ .

**Proof.** (i) In (5.11), the case  $b = 0$  implies  $\xi^d + \xi^t = 0$ . For even  $q$ , it is impossible. For odd  $q$ , we obtain  $\xi^d = -\xi^t$  whence, by (5.8),  $d = t + (q-1)/2$ . By (5.6), it is impossible.

(ii)–(iv) In (5.12), the case  $b = 0$  implies  $\xi^{2d} + U = 0$  whence  $\xi^{2d} + 1 = 0$  if  $(1, 0, \gamma U) = Z$  and  $\xi^{2d} + \xi^{\beta_G} = 0$  if  $(1, 0, \gamma U) = B_G$ . Recall that  $d \leq q-2$ .

(ii) Here  $q$  is even but  $q-1$  is odd. For  $(1, 0, \gamma U) = Z$ , we have  $\xi^{2d} = 1$  whence  $d = 0$ .

(iii) Here both  $q-1$  and  $\frac{1}{2}(q-1)$  are even. If  $(1, 0, \gamma U) = Z$ , then  $\xi^{2d} = -1 = \xi^{(q-1)/2}$  whence  $2d \equiv \frac{1}{2}(q-1) \pmod{q-1}$ . It is possible if  $d = \frac{1}{4}(q-1)$  or  $d = \frac{3}{4}(q-1)$ .

(iv) Here  $q-1$  is even whereas  $\frac{1}{2}(q-1)$  is odd. If  $(1, 0, \gamma U) = Z$ , then  $\xi^{2d} = -1 = \xi^{(q-1)/2}$  whence  $2d \equiv \frac{1}{2}(q-1) \pmod{q-1}$ . This is impossible. For  $(1, 0, \gamma U) = B_G$ , it holds that  $\xi^{2d} = -\xi^{\beta_G} = \xi^{2G+(q-1)/2}$ , which is impossible.  $\square$

**Construction B.** Let  $q$  be a prime power. Assume that  $q \not\equiv 3 \pmod{4}$ . Let  $G, \mathcal{D}_G, Z$ , and  $B_G$  be given by (5.6)–(5.10). We construct a point  $(G+3)$ -set  $\mathcal{W}_G$  in  $PG(2, q)$  as follows:

$$\mathcal{W}_G = \mathcal{D}_G \cup \{Z, B_G\}.$$

From Lemma 5.1 it follows.

**Lemma 5.11.** Let  $d \neq t$ . A point  $(1, 0, \gamma \xi^\beta)$  is collinear with  $\bar{A}_d, \bar{A}_t$  if and only if

$$\beta = d + t. \tag{5.13}$$

**Theorem 5.12.** The  $(G+3)$ -set  $\mathcal{W}_G$  of Construction B is an arc.

**Proof.** By (5.6), (5.7), the sum  $d+t$  in (5.13) is running on  $\{1, 2, \dots, 2G-1\}$ , where  $2G-1 \leq q-3$ . So,  $\{0, \beta_G\} \cap \{1, 2, \dots, 2G-1\} = \emptyset$ , see (5.10). Therefore  $Z$  and  $B_G$  do not lie on bisecants of  $\mathcal{D}_G$ . In other side, any point of  $\mathcal{D}_G$  does not lie on the line  $ZB_G$  as  $ZB_G$  is the bisecant of  $\mathcal{C}$  through  $A_\infty$  and  $A_0$ , where  $\{A_\infty, A_0\} \cap \mathcal{D}_G = \emptyset$ .  $\square$

**Theorem 5.13.** Let  $q$  be a prime power. Assume that  $q \not\equiv 3 \pmod{4}$ . Let  $G$  be given by (5.6). Then all points of  $\{P\} \cup \ell_1 \cup \mathcal{C} \setminus \mathcal{D}_G$  lie on bisecants of the arc  $\mathcal{W}_G$  of Construction B.

**Proof.** By (5.6), (5.7),  $\{\bar{A}_0, \bar{A}_{(q-1)/4}\} \subset \mathcal{D}_G$ . So, the point  $P$  is covered by Corollary 5.10(ii), (iii).

All points of  $\ell_1$  are covered as two points  $Z$  and  $B_G$  of this line belong to  $\mathcal{W}_G$ .

Throughout this proof,  $\mathcal{R}$  and  $\mathcal{S}$  are sets of integers modulo  $q-1$ . It can be said that  $\mathcal{R}$  and  $\mathcal{S}$  are sets of indexes of powers of  $\xi$ .

Let  $\mathcal{R} = \{-G, -(G-1), \dots, -1\} = \{q-1-G, q-1-(G-1), \dots, q-2\}$ . By Lemma 5.11, points  $\bar{A}_t, \bar{A}_{-t}, Z$  are collinear. Therefore, a point  $\bar{A}_t$  of  $\mathcal{C} \setminus \mathcal{D}_G$  with  $t \in \mathcal{R}$  lies on the bisecant of  $\mathcal{W}_G$  through  $\bar{A}_{-t}$  and  $Z$ , where  $-t \in \{G, G-1, \dots, 1\}, \bar{A}_{-t} \in \mathcal{D}_G$ .

Let  $\mathcal{S} = \{\beta_G - G, \beta_G - (G-1), \dots, \beta_G - 1, \beta_G\}$ . By Lemma 5.11, a point  $\bar{A}_t$  of  $\mathcal{C} \setminus \mathcal{D}_G$  with  $t \in \mathcal{S}$  lies on the bisecant  $B_G \bar{A}_{\beta_G - t}$ , where  $\beta_G - t \in \{G, G-1, G-2, \dots, 1, 0\}, \bar{A}_{\beta_G - t} \in \mathcal{D}_G$ .

Let  $\beta_G = 2G+1$ . Then  $\mathcal{S} = \{G+1, G+2, \dots, 2G+1\}$ . Also, by (5.10),  $q \equiv 0 \pmod{3}$  and  $G = \lfloor \frac{1}{3}(q-1) \rfloor = \frac{1}{3}(q-3)$  whence  $3G = q-3$  and  $2G+1 = q-G-2$ .

Let  $\beta_G = 2G$ . Then  $\mathcal{S} = \{G, G+1, \dots, 2G\}$ . Let  $v \in \{0, 1, 2\}$  and  $v \equiv q \pmod{3}$ . For  $G = \lfloor \frac{1}{3}(q-1) \rfloor$ , we have  $v \neq 0$ , see (5.10), whence  $G = \frac{1}{3}(q-v)$  and  $2G = q-G-v \geq q-G-2$ . For  $G > \lfloor \frac{1}{3}(q-1) \rfloor$ , we have  $G > \frac{1}{3}(q-3)$  whence  $2G > q-G-3$  and  $2G \geq q-G-2$ .

We proved that  $\{G+1, G+2, \dots, q-2\} \subseteq \mathcal{S} \cup \mathcal{R}$ . Also we showed that the points  $\bar{A}_t$  of  $\mathcal{C} \setminus \mathcal{D}_G$  with  $t \in \mathcal{S} \cup \mathcal{R}$  are covered by bisecants of  $\mathcal{W}_G$  either through  $Z$  (if  $t \in \mathcal{R}$ ) or through  $B_G$  (if  $t \in \mathcal{S}$ ). In the other side,  $\mathcal{C} \setminus \mathcal{D}_G = \{\bar{A}_t : t = G+1, G+2, \dots, q-2\} \cup \{A_\infty, A_0\}$ , where  $\{A_\infty, A_0\} \subset \ell_1$ . So, all points of  $\mathcal{C} \setminus \mathcal{D}_G$  are covered.  $\square$

**Definition 5.14.** Let  $q$  be a prime power. Let  $q \not\equiv 3 \pmod{4}$ . For integer  $\bar{G}$ , let

$$\mathcal{Z}_{\bar{G}} = \{Z\} \cup \{\bar{A}_d : d = 0, 1, 2, \dots, \bar{G}\}.$$

We call *critical value* of  $\bar{G}$  and denote by  $\bar{G}_q$  the *smallest value* of  $\bar{G}$  such that all points  $(1, a, b)$  with  $a \in F_q^*, b \in F_q, b \neq a^2$ , and all points  $(0, 1, b)$  with  $b \in F_q^*$ , lie on bisecants of  $\mathcal{Z}_{\bar{G}}$ .

**Theorem 5.15.** Let  $q \geq 32$  be a prime power. Let  $q \not\equiv 3 \pmod{4}$ . If  $\bar{G}_q \leq \lceil \frac{1}{2}(q - 3) \rceil$  and

$$\max \left\{ \bar{G}_q, \left\lfloor \frac{q-1}{3} \right\rfloor \right\} \leq G \leq \left\lfloor \frac{q-3}{2} \right\rfloor,$$

then the arc  $\mathcal{W}_G$  of Construction B is complete.

**Proof.** We use Theorem 5.13 and Definition 5.14.  $\square$

In this subsection, we put  $q \geq 32$  as we checked by computer that  $\lceil \frac{1}{2}(q - 3) \rceil < \bar{G}_q$  if  $q \leq 29$ .

**Corollary 5.16.** Let  $q \not\equiv 3 \pmod{4}$  be a prime power. If  $\bar{G}_q \leq \lceil \frac{1}{2}(q - 3) \rceil$ , then Construction B forms a family of complete  $k$ -arcs in  $PG(2, q)$  containing arcs of all sizes  $k$  in the region

$$\max \left\{ \bar{G}_q, \left\lfloor \frac{q-1}{3} \right\rfloor \right\} + 3 \leq k \leq \left\lfloor \frac{q-3}{2} \right\rfloor + 3 = \begin{cases} \frac{1}{2}(q+3) & \text{if } q \text{ odd} \\ \frac{1}{2}(q+4) & \text{if } q \text{ even.} \end{cases}$$

If  $\bar{G}_q \leq \lfloor \frac{1}{3}(q - 1) \rfloor$ , then size of the smallest complete arc of the family is  $\lfloor \frac{1}{3}(q + 8) \rfloor$ .

By computer search using Lemmas 5.1, 5.9 and 5.11 we obtained the following theorem.

**Theorem 5.17.** Let  $q \not\equiv 3 \pmod{4}$  be a prime power. Let  $\bar{G}_q$  be given by Definition 5.14. We introduce  $d_q$  and  $\delta_q$  as follows:  $\bar{G}_q = d_q \sqrt{q} \ln q, \delta_q = \lfloor \frac{1}{3}(q - 1) \rfloor - \bar{G}_q$ . Then the following holds.

- (i)  $\lfloor \frac{q-1}{3} \rfloor < \bar{G}_q \leq \left\lfloor \frac{q-3}{2} \right\rfloor$  if  $32 \leq q \leq 81$  and  $q = 97, 101, 125$ .
- (ii)  $\bar{G}_q \leq \left\lfloor \frac{q-1}{3} \right\rfloor$ , if  $128 \leq q \leq 2063, q = 89, 109, 113, 121$ .
- (iii)  $\bar{G}_q < 0.87 \sqrt{q} \ln q$  if  $32 \leq q \leq 2063$ . (5.14)
- (iv)  $1 \leq \delta_q \leq 93, 0.60 < d_q < 0.81,$  if  $128 \leq q \leq 593$ ;  
 $89 \leq \delta_q \leq 196, 0.64 < d_q < 0.84,$  if  $593 < q \leq 1049$ ;  
 $183 \leq \delta_q \leq 267, 0.62 < d_q < 0.80,$  if  $1049 < q \leq 1361$ ;  
 $271 \leq \delta_q \leq 464, 0.62 < d_q < 0.87,$  if  $1361 < q \leq 2063$ .

For situations  $\lfloor \frac{1}{3}(q - 1) \rfloor < \bar{G}_q$ , the values of  $\bar{G}_q$  are given in Table 5.

**Theorem 5.18.** Let  $q \not\equiv 3 \pmod{4}$  be a prime power. Let  $128 \leq q \leq 2063$ , or  $q = 89, 109, 113, 121$ . Then Construction B forms a family of complete  $k$ -arcs in  $PG(2, q)$  containing arcs of all sizes  $k$  in the region

$$\left\lfloor \frac{q+8}{3} \right\rfloor \leq k \leq \begin{cases} \frac{1}{2}(q+3) & \text{if } q \text{ odd} \\ \frac{1}{2}(q+4) & \text{if } q \text{ even.} \end{cases}$$

**Proof.** We use Corollary 5.16 and Theorem 5.17(ii).  $\square$

### 5.3. Arcs with three points outside a conic

Throughout this subsection,  $q \geq 27$  is a prime power and also  $q \equiv 3 \pmod{4}$ .

Let  $J$  be an integer in the region

$$\frac{q-3}{4} \leq J \leq \frac{q-3}{2}. \tag{5.15}$$

Let

$$\beta_J = 2J. \tag{5.16}$$

Notations  $\mathcal{D}_J$  and  $B_J$  are taken from (5.7) and (5.9) with substitution of  $G$  by  $J$  and by applying (5.16). Using (5.15) and (5.16), it is easy to see that Corollary 5.10(i), (iv), Theorem 5.12 and their proofs hold for  $\mathcal{D}_J$ ,  $B_J$ , and  $\beta_J$  as well as for  $\mathcal{D}_G$ ,  $B_G$ , and  $\beta_G$ .

**Construction C.** Let  $q \equiv 3 \pmod{4}$  be a prime power. Let  $P, J, \mathcal{D}_J, Z, B_J$ , and  $\beta_J$  be given by (5.3), (5.15), (5.7), (5.9) and (5.16). We construct a point  $(J + 4)$ -set  $\mathcal{E}_J$  in  $PG(2, q)$  as follows:

$$\mathcal{E}_J = \mathcal{D}_J \cup \{P, Z, B_J\}.$$

**Theorem 5.19.** The  $(J + 4)$ -set  $\mathcal{E}_J$  of Construction C is an arc.

**Proof.** The set  $\mathcal{D}_J \cup \{Z, B_J\}$  is an arc due to Theorem 5.12. By Corollary 5.10(i), (iv), the point  $P$  does not lie on bisecants of  $\mathcal{D}_J$  and  $\mathcal{D}_J \cup \{Z, B_J\}$ . Finally,  $P, Z, B_J$  are not collinear.  $\square$

**Theorem 5.20.** Let  $q \equiv 3 \pmod{4}$  be a prime power. Let  $J$  and  $\beta_J$  be given by (5.15) and (5.16). Then all points of  $\ell_1 \cup \mathcal{C} \setminus \mathcal{D}_J$  lie on bisecants of the arc  $\mathcal{E}_J$  of Construction C.

**Proof.** All points of  $\ell_1$  are covered as two points  $Z$  and  $B_J$  of this line belong to  $\mathcal{E}_J$ .

Throughout this proof,  $\mathcal{R}, \mathcal{S}$ , and  $\mathcal{T}$  are sets of integers modulo  $q - 1$ . It can be said that  $\mathcal{R}, \mathcal{S}$ , and  $\mathcal{T}$  are sets of indexes of powers of  $\xi$ . We act similarly to the proof of Theorem 5.13.

Let  $\mathcal{R} = \{-J, -(J - 1), \dots, -1\} = \{q - 1 - J, q - 1 - (J - 1), \dots, q - 2\}$ . By Lemma 5.11, a point  $\bar{A}_t$  of  $\mathcal{C} \setminus \mathcal{D}_J$  with  $t \in \mathcal{R}$  lies on the bisecant of  $\mathcal{E}_J$  through  $\bar{A}_{-t}$  and  $Z$ , where  $-t \in \{J, J - 1, \dots, 1\}$ ,  $\bar{A}_{-t} \in \mathcal{D}_J$ .

Let  $\mathcal{S} = \{\beta_J - J, \beta_J - (J - 1), \dots, \beta_J - 1, \beta_J\}$ . By Lemma 5.11, a point  $\bar{A}_t$  of  $\mathcal{C} \setminus \mathcal{D}_J$  with  $t \in \mathcal{S}$  lies on the bisecant  $B_J \bar{A}_{\beta_J - t}$ , where  $\beta_J - t \in \{J, J - 1, J - 2, \dots, 1, 0\}$ ,  $\bar{A}_{\beta_J - t} \in \mathcal{D}_J$ .

Let  $\mathcal{T} = \{\frac{1}{2}(q - 1), \frac{1}{2}(q - 1) + 1, \dots, \frac{1}{2}(q - 1) + J\}$ . By (5.8), (5.11), points  $P, \bar{A}_t$ , and  $\bar{A}_{t+(q-1)/2}$  are collinear. Therefore, a point  $\bar{A}_t$  of  $\mathcal{C} \setminus \mathcal{D}_J$  with  $t \in \mathcal{T}$  lies on the bisecant  $P \bar{A}_{t+(q-1)/2}$ , where  $t + \frac{1}{2}(q - 1) \in \{q - 1, q, \dots, q - 1 + J\} = \{0, 1, \dots, J\}$ ,  $\bar{A}_{t+(q-1)/2} \in \mathcal{D}_J$ .

As  $\beta_J = 2J$ , we have  $\mathcal{S} = \{J, J + 1, \dots, 2J\}$ , where  $2J \geq \frac{1}{2}(q - 1) - 1$ ; see (5.15). Also, by (5.15),  $\frac{1}{2}(q - 1) + J \geq \frac{1}{4}(3q - 5)$  while  $q - 1 - J \leq \frac{1}{4}(3q - 5) + 1$ .

We proved that  $\{J + 1, J + 2, \dots, q - 2\} \subseteq \mathcal{S} \cup \mathcal{R} \cup \mathcal{T}$ . Also we showed that the points  $\bar{A}_t$  of  $\mathcal{C} \setminus \mathcal{D}_J$  with  $t \in \mathcal{S} \cup \mathcal{R} \cup \mathcal{T}$  are covered by bisecants of  $\mathcal{E}_J$  either through  $Z$  (if  $t \in \mathcal{R}$ ) or through  $B_J$  (if  $t \in \mathcal{S}$ ) or, finally, through  $P$  (if  $t \in \mathcal{T}$ ). In the other side,  $\mathcal{C} \setminus \mathcal{D}_J = \{\bar{A}_t : t = J + 1, J + 2, \dots, q - 2\} \cup \{A_\infty, A_0\}$ , where  $\{A_\infty, A_0\} \subset \ell_1$ . So, all points of  $\mathcal{C} \setminus \mathcal{D}_J$  are covered.  $\square$

**Definition 5.21.** Let  $q \equiv 3 \pmod{4}$  be a prime power. For integer  $\bar{J}$ , let

$$\mathcal{Q}_{\bar{J}} = \{P, Z\} \cup \{\bar{A}_d : d = 0, 1, 2, \dots, \bar{J}\}.$$

We call *critical value* of  $\bar{J}$  and denote by  $\bar{J}_q$  the *smallest* value of  $\bar{J}$  such that all points  $(1, a, b)$  with  $a \in F_q^*, b \in F_q, b \neq a^2$ , and all points  $(0, 1, b)$  with  $b \in F_q^*$ , lie on bisecants of  $\mathcal{Q}_{\bar{J}}$ .

**Theorem 5.22.** Let  $q \geq 27$  be a prime power. Let  $q \equiv 3 \pmod{4}$ . If  $\bar{J}_q \leq \frac{1}{2}(q - 3)$  and

$$\max \left\{ \bar{J}_q, \frac{q - 3}{4} \right\} \leq J \leq \frac{q - 3}{2},$$

then the arc  $\mathcal{E}_J$  of Construction C is complete.

**Proof.** We use Theorem 5.20 and Definition 5.21.  $\square$

In this subsection, we put  $q \geq 27$  as we checked by computer that  $\frac{1}{2}(q - 3) < \bar{J}_q$  if  $q \leq 23$ .

**Corollary 5.23.** Let  $q \equiv 3 \pmod{4}$  be a prime power. If  $\bar{J}_q \leq \frac{1}{2}(q - 3)$ , then Construction C forms a family of complete  $k$ -arcs in  $PG(2, q)$  containing arcs of all sizes  $k$  in the region

$$\max \left\{ \bar{J}_q, \frac{q - 3}{4} \right\} + 4 \leq k \leq \frac{q + 5}{2}.$$

If  $\bar{J}_q \leq \frac{1}{4}(q - 3)$ , then the cardinality of this family is equal to  $\frac{1}{4}(q - 3)$  and the size of the smallest complete arc of the family is  $\frac{1}{4}(q + 13)$ .

By computer search using Lemmas 5.1, 5.9 and 5.11 we obtained the following theorem.

**Theorem 5.24.** Let  $q \equiv 3 \pmod{4}$  be a prime power. Let  $\bar{J}_q$  be given by Definition 5.21. We introduce  $r_q$  and  $\theta_q$  as follows:  $\bar{J}_q = r_q \sqrt{q} \ln q, \theta_q = \frac{1}{4}(q - 3) - \bar{J}_q$ . Then it holds that



$$\begin{aligned}
 & \text{(i) } \frac{q-3}{4} < \bar{J}_q \leq \frac{q-3}{2} \quad \text{if } 27 \leq q \leq 191 \text{ and } q = 211, 223, 343; \\
 & \text{(ii) } \bar{J}_q \leq \frac{q-3}{4}, \quad \text{if } 347 \leq q \leq 2063, \quad q = 199, 227, 239, 243, 251, 263, 271, 283, 307, 311, 331; \\
 & \text{(iii) } \bar{J}_q < 0.85\sqrt{q} \ln q \quad \text{if } 27 \leq q \leq 2063; \\
 & \text{(iv) } \begin{array}{lll} 3 \leq \theta_q \leq 44, & 0.61 < r_q < 0.78, & \text{if } 347 \leq q \leq 599; \\ 18 \leq \theta_q \leq 111, & 0.61 < r_q < 0.85, & \text{if } 599 < q \leq 991; \\ 89 \leq \theta_q \leq 167, & 0.61 < r_q < 0.76, & \text{if } 991 < q \leq 1367; \\ 163 \leq \theta_q \leq 296, & 0.59 < r_q < 0.73, & \text{if } 1367 < q \leq 2063. \end{array}
 \end{aligned}
 \tag{5.17}$$

For situations  $\frac{1}{4}(q-3) < \bar{J}_q$ , the values of  $\bar{J}_q$  are given in Table 5.

**Theorem 5.25.** *Let  $q \equiv 3 \pmod{4}$  be a prime power. Let  $347 \leq q \leq 2063$ , or  $q = 199, 227, 239, 243, 251, 263, 271, 283, 307, 311, 331$ . Then Construction C forms a family of complete  $k$ -arcs in  $PG(2, q)$  containing arcs of all sizes  $k$  in the region*

$$\frac{q+13}{4} \leq k \leq \frac{q+5}{2}.$$

**Proof.** We use Corollary 5.23 and Theorem 5.24(ii).  $\square$

Based on Theorems 5.7, 5.17 and 5.24, we conjecture the following; cf. Conjecture 1.5.

**Conjecture 5.26.** *Let  $\bar{H}_q, \bar{G}_q, \bar{J}_q$  be given by Definitions 5.4, 5.14 and 5.21. Let for  $\bar{H}_q, q$  be prime while for  $\bar{G}_q$  and  $\bar{J}_q$  it holds that  $q$  is a prime power. Finally, let  $q \not\equiv 3 \pmod{4}$  for  $\bar{G}_q$  and  $q \equiv 3 \pmod{4}$  for  $\bar{J}_q$ . Then the following holds.*

$$\begin{aligned}
 \bar{H}_q &\leq \left\lfloor \frac{q-1}{3} \right\rfloor \quad \text{if } q \geq 109; & \bar{G}_q &\leq \left\lfloor \frac{q-1}{3} \right\rfloor \quad \text{if } q \geq 128; & \bar{J}_q &\leq \frac{q-3}{4} \quad \text{if } q \geq 347. \\
 \bar{H}_q &< \sqrt{q} \ln q \quad \text{if } q \geq 19; & \bar{G}_q &< \sqrt{q} \ln q \quad \text{if } q \geq 32; & \bar{J}_q &< \sqrt{q} \ln q \quad \text{if } q \geq 27.
 \end{aligned}
 \tag{5.18}$$

**Remark 5.27.** It is interesting to compare the relations (5.5), (5.14), (5.17) and (5.18) with Theorems 1.2, 4.1 and 4.3 and Conjecture 1.3 and to compare also computer results provided by Tables 1 and 2 with Theorems 5.7, 5.17 and 5.24. One can see that the upper estimates of  $t_2(2, q), \bar{H}_q, \bar{G}_q$ , and  $\bar{J}_q$  have the same structure and the values of  $\bar{t}_2(2, q), \bar{H}_q, \bar{G}_q$ , and  $\bar{J}_q$  have a close order. This seems to be natural as almost all points of  $PG(2, q)$  lie on bisecants of  $\mathcal{P}_{\bar{H}_q}, \mathcal{Z}_{\bar{G}_q}$ , and  $\mathcal{Q}_{\bar{J}_q}$ ; see Definitions 5.4, 5.14 and 5.21.

**Remark 5.28.** The complete arcs of Constructions A–C can be used as starting objects in inductive constructions. For example, for even  $q$ , arcs of Construction B can be used in constructions of [9, Ths. 1.1, 3.14–3.17, 4.6–4.8]. In that way, one can generate infinite sets of families of complete caps in projective spaces  $PG(v, 2^n)$  of growing dimensions  $v$ . For every  $v$ , constructions of [9] can obtain a complete cap from every complete arc of Construction B. Also, it can be shown that in Constructions A–C all points not on conic are external. So, the arcs of Constructions A and C for  $q \equiv 3 \pmod{4}$  can be used as starting objects in constructions of [32]; see [32, Th. 23]. Thereby, infinite families of large complete arcs in  $PG(2, q^n)$  with growing  $n$  can be obtained.

### 6. On the spectrum of possible sizes of complete arcs in $PG(2, q)$

The main known results on the spectrum of possible sizes of complete arcs in  $PG(2, q)$  are given in Introduction with the corresponding references. Taking into account the results cited in Introduction, we denote

$$M_q = \begin{cases} \frac{1}{2}(q+4) & \text{for even } q \\ \frac{1}{2}(q+7) & \text{for odd } q \text{ included to (1.5)} \\ \frac{1}{2}(q+5) & \text{for odd } q \text{ not included to (1.5)}. \end{cases}$$

We suppose that the smallest known sizes  $\bar{t}_2(2, q)$  are given in Tables 1–4 of this paper.

**Theorem 6.1.** *In  $PG(2, q)$  with  $25 \leq q \leq 349$ , and  $q = 1013, 2003$ , there are complete  $k$ -arcs of all the sizes in the region*

$$\bar{t}_2(2, q) \leq k \leq M_q.$$

**Proof.** For  $25 \leq q \leq 167$  the assertion of the theorem follows from [3, Tab. 2] and [5, Tab. 2]. For  $169 \leq q \leq 349$  and  $q = 1013, 2003$ , we used **Constructions A–C** of Section 5. The sizes not following from the constructions are obtained in this work by the randomized greedy algorithms.  $\square$

An experience obtained in computer search for the proof of **Theorem 6.1** allows us to do **Conjecture 6.2**. Here we took into account sizes that can be got by **Constructions A–C** and the remark on sizes close to  $\bar{t}_2(2, q)$  in the end of Section 2. Note also that the remaining sizes for all  $169 \leq q \leq 349$  and  $q = 1013, 2003$  were relatively easily obtained by the greedy algorithms with point subset of a conic taken as the starting set  $S_0$ . For this we used consequently subsets of cardinality approximately 15%, 20%, 25%, 30% of the conic cardinality  $q + 1$ .

**Conjecture 6.2.** *Let  $353 \leq q \leq 5107$ ,  $q \in T_2 \cup T_3$ , be a prime power. Then in  $PG(2, q)$  there are complete  $k$ -arcs of all the sizes in the region  $\bar{t}_2(2, q) \leq k \leq M_q$ . Moreover, complete  $k$ -arcs with  $\bar{t}_2(2, q) \leq k \leq \frac{1}{2}(q + 5)$  can be obtained either by **Constructions A–C** or by the randomized greedy algorithms.*

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