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On a Boundary Point Repelling Automorphism Orbits*

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1. INTRODUCTION

Let Ω be a bounded domain in $\mathbf{C}^n$, $n \geqslant 1$. By an *automorphism* of Ω, we mean a one-to-one and onto holomorphic (or, *biholomorphic*) selfmap of Ω. Denote by $\mathrm{Aut}(\Omega)$ the set of all automorphisms of Ω, which is usually called *the automorphism group* of Ω. It is well known that $\mathrm{Aut}(\Omega)$ equipped with the law of composition and the topology of uniform convergence on compact subsets of Ω is a finite dimensional Lie group [6].

It follows by a standard normal family argument that *the automorphism group of a bounded domain Ω is noncompact if and only if the orbit of a point $q \in \Omega$ by the automorphism group* $\mathrm{Aut}(\Omega)$ *is noncompact* ([21, 26] for instance). On the other hand, there are many bounded domains with noncompact automorphism groups that are not homogeneous ([25, 26] e.g.). Such domains in general have boundary points at which the automorphism orbits do not accumulate. The theme of this paper is, therefore, to present obstructions coming from the geometry of the boundary near such boundary points that "repel" automorphism orbits. In fact, if one considers the bounded domains with entirely smooth boundary, this program is well summarized in the following conjecture of Greene and Krantz [12]. (Compare with [2, 3, 5, 9–11, 20].)

Conjecture (Greene–Krantz). Let p be a boundary point of infinite type in the sense of D'Angelo ([7, 14]) of a bounded domain Ω in $\mathbf{C}^n$ ($n \geqslant 2$) whose boundary is entirely C^∞ smooth. Then there does not exists an automorphism orbit in Ω which accumulates at p.

Green and Krantz [12] gave examples of domains that are complex two dimensional, circular, and variety-free in its boundary at p for which the conjecture above is true.

In this paper, we first show that the above conjecture is true if the

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domain Ω is bounded convex and if the boundary point p admits a neighborhood in which the boundary is *Levi flat*. More precisely, let

$$\Omega = \{(z_1, ..., z_n) \in \mathbf{C}^n \mid \psi(z_1, ..., z_n) < 0\},$$

where $\psi : \mathbf{C}^n \to \mathbf{R}$ is a C^∞ smooth function with

$$\nabla\psi(z_1, ..., z_n) \neq 0, \qquad \text{if} \quad \psi(z_1, ..., z_n) = 0.$$

Let p be a boundary point of Ω. We say there exists a neighborhood U of p in which the boundary $\partial\Omega$ is *Levi flat*, if, for every $z = (z_1, ..., z_n) \in \partial\Omega \cap U$,

$$\sum_{j,k=1}^{n} \frac{\partial^2\psi}{\partial z_j \, \partial \bar{z}_k}(z)\, w_j \bar{w}_k = 0,$$

for all $(w_1, ..., w_n) \in \mathbf{C}^n$ satisfying

$$\sum_{j=1}^{n} \frac{\partial\psi}{\partial z_j}(z)\, w_j = 0.$$

It is well-known that in such a case $\partial\Omega \cap U$ admits a C^∞ smooth foliation by analytic varieties.

Now we state the first two main results of this article. In what follows, we denote by Δ the open unit disk in $\mathbf{C}$, and by Δ^n the product of n copies of Δ.

THEOREM 1. *Let Ω be a bounded convex domain in $\mathbf{C}^n$. Suppose that the boundary $\partial\Omega$ is C^∞ smooth in a neighborhood of p in $\mathbf{C}^n$ and that there* exists *a neighborhood of p in which the boundary is Levi flat. If there exist a point $q \in \Omega$ and a sequence $\{\varphi_j\}_{j=1,2,...}$ of automorphisms of Ω such that $\lim_{j\to\infty} \varphi_j(q) = p$, then Ω is biholomorphic to product domain $\Delta \times \Omega'$ for some convex domain $\Omega' \in \mathbf{C}^{n-1}$.*

An immediate consequence is the following:

COROLLARY. *Let Ω be a bounded convex domain in $\mathbf{C}^2$. Suppose that the boundary $\partial\Omega$ is C^∞ smooth in a neighborhood of p in $\mathbf{C}^2$ and that there* exists *a neighborhood of p in which the boundary is Levi flat. If there exist a point $q \in \Omega$ and a sequence $\{\varphi_j\}_{j=1,2,...}$ of automorphisms of Ω such that* $\lim_{j\to\infty} \varphi_j(q) = p$, *then Ω is biholomorphic to the bidisk $\Delta \times \Delta$.*

Since the product domains cannot be biholomorphic to any bounded domain with its boundary entirely smooth ([1, 22, 24], e.g.), it implies the following result that supports in part the aforementioned Conjecture of Greene–Krantz.

THEOREM 2. *Let Ω be a bounded convex domain in $\mathbf{C}^n$, $n \geq 2$, with entirely C^∞ smooth a boundary. Let p be a boundary point that admits a neighborhood U in which $\partial\Omega$ is Levi flat. Then, there does not exist any automorphism orbit which accumulates at p.*

In Sections 5 and 6, we also present further results treating the case of variety-free boundary points of infinite type. It turns out that, from our viewpoint, such case is much harder to analyze. Our result in this direction is therefore special and can be summarized as in the following:

DEFINITION. Call a boundary point p of a bounded domain $\Omega \subset \mathbf{C}^n$ *of convex exponential type*, if there exist an open neighborhood U of p in $\mathbf{C}^n$ and a C^∞ smooth strictly convex positive real-valued function $\Phi: U \to \mathbf{R}$ defining $\Omega \cap U$, up to a linear change of complex coordinates of U, satisfying

(1) $p = (0, 0)$,

(2) $\Omega \cap U = \{(z_1, z_2) \in U \mid \operatorname{Im} z_1 > \Phi(z_2) + O(|z_1 z_2| + |z_1|^2)\}$, and

(3) $\lim_{z \to 0} \dfrac{\Phi(\lambda z)}{\Phi(z)} = \infty$ for any $\lambda > 1$.

The functions $\exp(-f(z) \cdot |z|^{-2m})$, for any positive integer m and any positive smooth function f in z with $f(0) \neq 0$ are typical examples of such Φ. Note that a boundary point p of convex exponential type cannot admit a codimension one subvariety in the boundary of Ω through p, due to the limit condition above and the fact that all the codimension one varieties contained in a convex hypersurface is Euclidean flat.

THEOREM 3. *Let Ω be a bounded domain in $\mathbf{C}^2$ whose boundary is C^∞ smooth near a boundary point $p \in \partial\Omega$ is of convex exponential type. Then there is no automorphism orbit accumulating at p nontangentially to $\partial\Omega$.*

If one considers noncompact automorphism orbits that approach the boundary nontangentially, then the convex scaling method adopts a very special form as in Proposition 5 of Section 5. In such a case, we also have

THEOREM 4. *Let Ω be a bounded convex domain in $\mathbf{C}^2$ with a C^∞ smooth boundary. Let $\partial\Omega$ admit a nontrivial analytic subset passing through $p \in \partial\Omega$, say. Then there is no automorphism orbit accumulating at p nontangentially to $\partial\Omega$.*

Nonstandard terms used in this section without precise definitions are defined in the later sections when the proofs are given.

2. Outline of Proofs

For the sake of clarity of the exposition, we describe very roughly how the proofs are organized in this paper. First of all, we present the convex-scaling technique initiated by Frankel [8] and further modified results [14, 15], which says in various cases that for a bounded convex domain any noncompact sequence automorphisms must yield a linear scaling that "stretches" the domain to an unbounded domain biholomorphic to the original domain. (Propositions 1–4 in Section 3, Lemma 4 of Section 4, Proposition 5 of Section 6).

For the proof of Theorem 1, one observes that the convex-scaling sequence stretches the domain to infinity in the complex direction normal to the boundary of the domain, and that all the other directions scales to finite speed keeping linear independence during the convex-scaling process. After some elementary plane geometry arguments, the scaled domain can be seen to be a product of a half plane and some other convex domain as desired. Detailed arguments are given in Sections 3 and 4.

The domain described in Theorem 1 is biholomorphic to a product of bounded domains, because any convex Kobayashi hyperbolic domain is biholomorphic to a bounded domain. Since no bounded domain in $\mathbf{C}^n$ with entirely smooth boundary can be biholomorphic to any product domain, the conclusion of Theorem 2 must follow in the light the convex scaling theorems mentioned above.

Theorem 3 makes use of the full power of convex scaling method. In Section 6, we show that no convex scaling by a nontangential automorphism orbit can be possible at the boundary point of exponential type, which is rather a broad subcollection of the boundary points of infinite type. However, at the same time, the convex-scaling technique asserts that some convex scaling must be possible due to convexity as long as there is a noncompact automorphism orbit accumulting at a boundary point. This contradiction enables us to conclude the nonexistence of the automorphism orbit accumulating nontangentially to the boundary point of exponential type. Theorem 4 uses the similar line reasoning, but the proof reduces to the arguments to the proof of Theorems 1 and 2.

3. Convex-Scaling Technique

In this section, we introduce a modification of theorem of Frankel (Theorem 5.6 of p. 123 in [8]), which provides a basic technique for our proof of the main theorems of this paper. We begin with

Proposition 1 (Frankel [8]). *Let D be a bounded convex domain in $\mathbf{C}^n$,*

$n \geqslant 1$. *Let* $q \in D$ *and* $\{\varphi_j\}_{j=1,2,\ldots} \subset \mathrm{Aut}(D)$. *Then, every subsequence of the sequence* $\{\omega_j : D \to \mathbf{C}^n\}_{j=1,2,\ldots}$ *defined by*

$$\omega_j(z) := [\partial\varphi_j(q)]^{-1}(\varphi_j(z) - \varphi_j(q))$$

has a subsequence that converges to a one-to-one holomorphic mapping from D *into* $\mathbf{C}^n$.

In practice, most interesting application of the above statement is when $\varphi_j(q)$ tends to a boundary point of D. So *we only consider throughout the rest of the paper the case that*

$$\lim_{j \to \infty} \varphi_j(q) = p \in \partial D, \tag{1}$$

unless specifically mentioned otherwise.

Now we consider the set-convergence. Let $\mathscr{F}$ denote the collection of all nonempty closed subsets of $\mathbf{C}^n$. The Hausdorff distance function $\rho : \mathscr{F} \times \mathscr{F} \to \mathbf{R}$ is defined by

$$\rho(S, T) = \sup\{\sup_{s \in S} \inf_{t \in T} \|s - t\|, \sup_{t \in T} \inf_{s \in S} \|s - t\|\}.$$

For the concept of *local Hausdorff set-convergence, we define as usual* $\rho_R : \mathscr{F} \times \mathscr{F} \to \mathbf{R}$ for each $R > 0$ by

$$\rho_R(S, T) = \rho(S \cap \overline{B_R(0)}, T \cap \overline{B_R(0)}),$$

where $\overline{B_R(0)}$ is the closed ball of radius R centered at the origin. As usual, we reserve the notation $B_r(x)$ for the open ball in $\mathbf{C}^n$ for appropriate n centered at x with radius r. We call a sequence $\{S_j\}$ of closed subsets of $\mathbf{C}^n$ *converges to a set* $S \in \mathscr{F}$ *in the sense of local Hausdorff set-convergence* if $\rho_R(S_j, S) \to 0$ as $j \to \infty$.

Since $\omega_j(D)$, for each j, is convex, the Blaschke selection and Proposition 1 above imply.

PROPOSITION 2. *With the same hypotheses and notations as Proposition* 1 *above, every sequence* $\{\omega_j\}$ *admits a subsequence* $\{\omega_{j_k}\}$ *which converges to a biholomorphic mapping* $\omega : D \to \omega(D)$, *and* $\omega(D)$ *co-incides with the interior of the limit set of the sequence of closure of* $\omega_{j_k}(D)$ *in the sense of local Hausdorff set-convergence.*

Since al the sets we are dealing with in our case are convex, from now on, we will simply state that $\omega(D)$ is the local Hausdorff limit of the sequence $\{\omega_j(D)\}$ for the sake of simplicity.

By changing indices, denote by $\{\omega_j\}$ the subsequence $\{\omega_{j_k}\}$ in Proposition 2 above, which converges to the biholomorphic mapping ω. Consider now the domain $\omega(D)$. It is a convex domain which is biholomorphic to the bounded convex domain D. In particular, the boundary of $\omega(D)$ has to be non-empty. Hence, Proposition 2 implies that there exists a sequence of points $s_j \in \partial(\omega_j(D))$ such that a subsequence of $\{[\partial\varphi_j(q)]^{-1}(s_j - f_j(q))\}_j$ is bounded. For, otherwise, the sequence of convex sets

$$\begin{aligned}\omega_j(D) &= [\partial\varphi_j(q)]^{-1}(\varphi_j(D) - f_j(q)) \\ &= [\partial\varphi_j(q)]^{-1}(D - f_j(q))\end{aligned}$$

converges to the convex subset of $\mathbf{C}^n$ with empty boundary in local Hausdorff set-convergence, where the minus signs between sets and points above represent the vector sum such as

$$T - x = \{y - x \mid y \in T\}.$$

Hence we arrive at the following modified version of Proposition 2, imitating the arguments above.

PROPOSITION 3. *With the same hypotheses and notations as Proposition 1 above, there exists a sequence $\{s_j\}_j$ of boundary points of D such that every subsequence of the sequence $\{\sigma_j : D \to \mathbf{C}^n\}$ of holomorphic mappings defined by*

$$\sigma_j(z) := [\partial\varphi_j(q)]^{-1}(\varphi_j(z) - s_j)$$

admits a subsequence that converges to a one-to-one holomorphic mapping $\sigma : D \to \mathbf{C}^n$ with the sequence $\{\sigma_j(D)\}_j$ converging to $\sigma(D)$ in local Hausdorff set-convergence.

If, in addition to the hypotheses of Proposition 1, $\|\partial\varphi_j(q)\|$ tends to 0 as $j \to \infty$, then the sequences $\{s_j\}$ and $\{\varphi_j(q)\}$ must converge to the same boundary point p, since the sequence $\{[\partial\varphi_j(q)]^{-1}(s_j - f_j(q))\}_j$ is bounded, as pointed out in the above. Moreover, in this case, for any $r > 0$ and $R > 0$, there exists $j_0 > 0$ such that

$$[\partial\varphi_j(q)]^{-1}(D - f_j(q)) \cap B_R(0) = [\partial\varphi_j(q)]^{-1}(D \cap B_\varepsilon(p) - f_j(q)) \cap B_R(0)$$

and

$$[\partial\varphi_j(q)]^{-1}(D - s_j) \cap B_R(0) = [\partial\varphi_j(q)]^{-1}(D \cap B_\varepsilon(p) - s_j) \cap B_R(0)$$

for all $j \geqslant j_0$. Hence, we obtain the following localized result (see [14]) which says roughly that the result of convex scaling by $\{\varphi_j\}$ is then governed by the local shape of the boundary of D near p.

PROPOSITION 4. *Keeping the notations from Propositions 1–3, if the sequence of norms* $\|\partial\varphi_j(q)\|$ *tends to zero as* $j \to \infty$ *in addition to the hypotheses of Proposition 1, then the set-limits* $\omega(D)$ *and* $\sigma(D)$ *are completely determined by the sequences* $\{[\partial\varphi_j(q)]^{-1}(D \cap B_\varepsilon(p) - f_j(q))\}$ *and* $\{[\partial\varphi_j(q)]^{-1}(D \cap B_\varepsilon(p) - s_j)\}$, *respectively.*

4. PROOF OF THEOREM 1 IN COMPLEX DIMENSION TWO

Let $\Omega \subset \mathbf{C}^2$, p, q and φ_j be as in the statement of Theorem 1. First, we analyze how the eigenvalues of $\partial\varphi_j(q)$ behaves in the following Lemmas 0 to 3.

LEMMA 0. $\|\partial\varphi_j(q)\|$ *does not tend to* 0 *as* $j \to \infty$.

Proof. Choose a nontrivial analytic set, say V, in $\partial\Omega$ passing through p, then it is Euclidean flat by the Maximum Principle. Replace V by the analytic disk

$$\zeta \mapsto \zeta v + p : \Delta(\delta) \to V \subset \partial\Omega,$$

where $\Delta(\delta)$ denotes the open disk in $\mathbf{C}$ centered at the origin with radius $\delta > 0$, and where $v \in \mathbf{C}^n$ is a nonzero vector parallel to V. By convexity and Levi flatness of $\partial\Omega$ at p, we may assume without loss of generality that $V + \varphi_j(q) \in \Omega$ for all j. Then,

$$\varphi_j^{-1} \circ h_j : \Delta(\delta) \to \Omega$$

satisfies $\varphi_j \circ h_j(0) = q$, where $h_j(\xi) := \zeta v + \varphi_j(q)$ for each j. By a standard normal family argument, $(\varphi_j \circ h_j)'(0)$ must be a bounded sequence, which implies the desired result (cf. [27]).

Before we proceed any further, we point out that we will taking subsequence frequently, owing to the fact that if there exists a subsequence of $\omega_j(\Omega)$ [or, of $\sigma_j(\Omega)$, resp.] which converges to a domain that is biholomorphic to the bidisk, we get the desired conclusion by Propositions 2 and 3. Thus, throughout the rest of the argument of this case, we simply use the notation $\{\varphi_j\}_j$ for each subsequence of $\{\varphi_j\}_j$ we choose in the following arguments. And each step of arguments in the following starts with the newly chosen subsequence $\{\varphi_j\}_j$ which is extracted from the preceding subsequence $\{\varphi_j\}_j$, to avoid complicated stacks of subscripts.

Denote by λ_j, Λ_j, for each j, the eigenvalues of the complex 2×2 matrix $\partial\varphi_j(q)$ arranged so that $|\lambda_j| \leqslant |\Lambda_j|$.

LEMMA 1. *There exists a constant $K>0$ independent of j such that $K^{-1} \leqslant \|\partial\varphi_j(q)\| \leqslant K$ for all j. Furthermore, $\lambda_j \to 0$ as $j \to \infty$.*

Proof. The lower bound estimate is a consequence of Lemma 0. The upper bound is obtained from the fact that φ_j, for each j, is an automorphism of the bounded domain Ω. Namely, $\{\varphi_j\}$ is a precompact normal family of holomorphic mappings, and hence has a uniform bound independent of j for $\|\partial\varphi_j(q)\|$.

For the remaining claim, choose any subsequence of $\{\varphi_j\}_j$ that converges to $\varphi: \Omega \to \mathbf{C}^2$. Then clearly $\varphi(\Omega) \subset \bar{\Omega}$. On the other hand, $\varphi(q) = \lim_{j\to\infty} \varphi_j(q) = p \in \partial\Omega$. Hence, $\varphi(\Omega) \subset \partial\Omega$ due to the pseudo-convexity of Ω. So, $\det \partial\varphi = 0$ which in turn implies that $\det \partial\varphi_j(q) \to 0$ as $j \to \infty$. Now, the second assertion of the lemma follows immediately.

Now we need refine the content of Lemma 0. Consider a convex function $\Psi: \mathbf{C}^2 \to \mathbf{R}$ defining Ω in the following sense:

$$\Omega = \{(z, w) \in \mathbf{C}^2 \mid \Psi(z, w) < 0\}$$

$$\partial\Omega = \{(z, w) \in \mathbf{C}^2 \mid \Psi(z, w) = 0\}$$

$$\mathbf{C}^2 \backslash \Omega = \{(z, w) \in \mathbf{C}^2 \mid \Psi(z, w) > 0\}.$$

Without loss of generality, we further assume that

$$\Psi(p) = \Psi(0, 0) = 0, \quad \text{and} \quad \Omega \subset \{(z, w) \in \mathbf{C}^2 \mid \operatorname{Im} z > 0\}. \tag{1}$$

Recall that $\partial\Omega$ is C^∞ smooth near $p = (0, 0)$. So, in particular, Im z axis is parallel to the normal line to $\partial\Omega$ at $(0, 0)$.

By Lemma 1 above, choose a unit vector $v_j \in \mathbf{C}^2$ for each j such that

$$0 < K^{-1} \leqslant \|\partial\varphi_j(q)\, v_j\| \leqslant K \tag{2}$$

By choosing a subsequence of $\{\varphi_j\}$ if necessary, we assume without loss of generality that

$$\lim_{j\to\infty} v_j = \hat{v} \quad \text{and} \quad \nu_j := \partial\varphi_j(q)\, v_j \to \hat{\nu} \quad \text{as} \quad j \to \infty. \tag{3}$$

Then we claim

LEMMA 2. *$\hat{\nu} = (0, b)$ for some nonzero complex number b.*

Proof. For each j, there exists a constant $\delta > 0$ such that the linear analytic disk $h: \Delta \to \Omega$ defined by

$$h_j(\zeta) = q + \zeta\, \delta v_j,$$

where Δ denotes the open unit disk in $\mathbf{C}$. Then, for every j,

$$\|(\varphi_j \circ h_j)'(0)\| = \delta \, \|\partial\varphi_j(q)\, v_j\| \geqslant \delta K^{-1} > 0.$$

Since $\varphi_j \circ h_j(0) \to p \in \partial\Omega$, every subsequential limit of $\varphi_j \circ h_j$ defines a nontrivial analytic variety contained in $\partial\Omega$ passing through $p = (0, 0)$. By convexity of Ω and the Maximum Principle, the limit variety is Euclidean flat. The assumption (1) then implies that this analytic variety is a subdomian of the w-plane. Therefore, the result follows.

Another consequence of Lemma 1 is that there exist, for each j, a unit vector $u_j \in \mathbf{C}^2$ such that

$$\partial\varphi_j(q)\, u_j = \lambda_j u_j. \tag{4}$$

Let us again choose a subsequence of $\{\varphi_j\}$ if necessary, so that we may assume that $u_j \to \hat{u}$ as $j \to \infty$. Then, with $\hat{v}$ defined in (3) above, we obtain

LEMMA 3. *$\hat{v}$ and $\hat{u}$ above are lineary independent over* $\mathbf{C}$.

Proof. Suppose that $a\hat{v} - \hat{u} = 0$. *Then*

$$\begin{aligned} 0 &= \lim_{j \to \infty} (a[\partial\varphi_j(q)]^{-1} v_j - \lambda_j [\partial\varphi_j(q)]^{-1} u_j) \\ &= \lim_{j \to \infty} [\partial\varphi_j(q)]^{-1} (av_j - \lambda_j u_j). \end{aligned}$$

Since $\|\partial\varphi_j(q)\| \leqslant K$ for all j, it follows that

$$0 = \lim_{j \to \infty} (av_j - \lambda_j u_j) = a\hat{v}$$

Therefore, $a = 0$, and the assertion follows.

The lemma above in particular implies that $\hat{u}$ is not tangential to the boundary of Ω.

Now consider the effect of the convex-scaling of the slices

$$S_j := \bar{\Omega} \cap \{\zeta u_j + \varphi_j(q) \in \mathbf{C}^2 \mid \zeta \in \mathbf{C}\}$$

by $[\partial\varphi_j(q)]^{-1}$. The set

$$\begin{aligned} H &:= \lim_{j \to \infty} [\partial\varphi_j(q)]^{-1} (S_j - \varphi_j(q)) \\ &= \lim_{j \to \infty} \frac{1}{\lambda_j} (S_j - \varphi_j(q)) \end{aligned} \tag{5}$$

is either the whole complex line $\mathbf{C}\hat{u}$ or a closed half plane in $\mathbf{C}\hat{u}$. Also notice that the point $(0,0) \in H$ and that $(0,0) = \omega(q) \in \omega(\Omega)$. Since $\omega(\Omega)$ cannot contain a complex line in it, the set H above is a half plane contained in $\mathbf{C}\hat{u}$. Moreover, the boundary of H, which is a straight line extending indefinitely, is contained in the boundary of $\omega(\Omega)$.

Now, consider $c_j \in \mathbf{C}$ for each j which is the complex number with the smallest modulus such that $c_j u_j + \varphi_j(q) \in \partial S_j$. Let us denote by

$$s_j = c_j u_j + \varphi_j(q). \tag{6}$$

The preceeding discussion in particular implies that

$$[\partial\varphi_j(q)]^{-1}(s_j - \varphi_j(q))$$

is a bounded (in fact convergent) sequence. Therefore, we obtain

LEMMA 4. *With s_j defined in (11) above, the holomorphic mappings*

$$\sigma_j(z) := [\partial\varphi_j(q)]^{-1}(\varphi_j(z) - s_j)$$

form a normal family, whose every subsequential limit is a one-to-one holomorphic mapping. Moreover, every subsequence of $\{\sigma_j(\Omega)\}_j$ has a subsequence converging to a domain biholomorphic to Ω in the sense of local Hausdorff set-convergence.

Throughout the rest of the section, we work with σ_j defined in the lemma above. Without loss of generality, by choosing subsequences, we may assume that $\lim \sigma_j = \sigma$ and $\lim \sigma_j(\Omega) = \sigma(\Omega)$, where the latter limit is in terms of local Hausdorff set-convergence. Then we will show that $\sigma(\Omega)$ is biholomorphic to the bidisk.

We claim

LEMMA 5. *Let H_0 be the half plane*

$$H_0 = H - \lim_{j \to \infty} [\partial\varphi_j(q)]^{-1}(s_j - \varphi_j(q)),$$

where H and s_j are as in (5) and (6) above. Then, for each boundary point x of $\sigma(\Omega)$, the closure of the domain $\sigma(\Omega)$ contains the parallel translation $H_0 + x$ of H_0 in such a way that the boundary $\partial(H_0 + x)$ of $H_0 + x$ is contained in boundary of $\sigma(\Omega)$.

Proof. Let x be a boundary point of $\sigma(\Omega)$. Since H_0 is contained in the closure of $\sigma(\Omega)$, so is the convex hull C_x of $\{x\} \cup H_0$. It is easy to see that C_x contains $H_0 + x$. Moreover, the boundary of the half plane $H_0 + x$ is a straight line passing through x. Since x is a boundary point of $\sigma(\Omega)$, and since the domain $\sigma(\Omega)$ is convex, the boundary of $H_0 + x$ is contained in the boundary of $\sigma(\Omega)$.

The lemma above also implies that the domain $\sigma(\Omega)$ is, up to an appropriate complex linear change of complex coordinates of $\mathbf{C}^2$, a product of a straight line and a convex domain in $\mathbf{R}^3$. And hence, the automorphism group of Ω contains a noncompact 1-parameter subgroup. However, it is *not* yet enough to conclude that $\sigma(\Omega)$ is a product domain in complex sense.

To reach at the desired conclusion, we observe.

LEMMA 6. *s_j defined in (6) above approaches $p = (0, 0)$, as $j \to \infty$.*

Proof. By choice of s_j, the sequence

$$\{[\partial\varphi_j(q)]^{-1}(s_j - \varphi_j(q))\}_j$$

is bounded. Namely, the sequence

$$\begin{aligned}\{[\partial\varphi_j(q)]^{-1}(s_j - \varphi(q))\}_j &= \{[\partial\varphi_j(q)]^{-1}(c_j u_j)\\ &= \frac{c_j}{\lambda_j} u_j\end{aligned}$$

has to be bounded, where c_j and λ_j are as in (6) and Lemma 1, respectively. Since $\lambda_j \to 0$ and u_j is bounded, c_j must tend to 0 as $j \to \infty$. This means that

$$\begin{aligned}\lim_{j\to\infty} s_j &= \lim_{j\to\infty} c_j u_j + \varphi_j(q)\\ &= \lim_{j\to\infty} \varphi_j(q)\\ &= p = (0, 0).\end{aligned}$$

Now, we use the hypothesis of Theorem 1 that the boundary point $p = (0, 0)$ admits a neighborhood U, say, in which the boundary of Ω is Levi flat. It is known that $\partial\Omega$ is then foliated by analytic varieties, and the foliation is C^∞ smooth. Notice that, since Ω is convex, by the Maximum Principle every analytic subvariety of the boundary of Ω is Euclidean flat. By choosing a subsequence, let us assume that $s_j \in U \cap \partial\Omega$ for all j. Denote by V_j the maximal analytic subvariety of $\partial\Omega$ passing through s_j. By the preceding discussion, each V_j is a convex subdomain of a complex hyperplane in $\mathbf{C}^2$. In this sense, due to smoothness of the foliation, there exists $\delta > 0$ such that

$$\operatorname{dis}(s_j, \partial V_j) > \delta \tag{7}$$

for all j. Define

$$\Sigma := \lim_{j\to\infty} [\partial\varphi_j(q)]^{-1}(V_j - s_j);$$

then it has to be a bounded subdomain of a complex hyperplane of $\mathbf{C}^2$. For, otherwise, (7) will imply that Σ is a whole complex hyperplane, which is not allowed since $\sigma(\Omega)$ cannot contain a complex line, being biholomorphic to Ω. Now we prove the following claim which finally completes the proof of Theorem 1 in complex dimension two:

LEMMA 7. $\overline{\sigma(\Omega)} = \bar{\Sigma} + H_0$.

Proof. $\bar{\Sigma} + H_0 \subset \overline{\sigma(\Omega)}$ follows immediately from Lemma 5, convexity of $\bar{\Sigma}$, and the fact that both $\bar{\Sigma}$ and H_0 are contained in $\overline{\sigma(\Omega)}$.

To show the reverse inclusion, let $w_j \in \mathbf{C}^2$ be a unit vector tangent to V_j, for each j, where V_j is the aforementioned maximal analytic subvariety of $\partial\Omega$ passing through s_j. Then the discussion in the paragraph containing (7) above implies that there exists $K > 0$ such that

$$\|[\partial\varphi_j(q)]^{-1} w_j\| \leqslant K, \qquad \forall j. \tag{8}$$

Therefore, choosing a subsequence again if necessary, we may assume that $\{t_j := [\partial\varphi_j(q)]^{-1} w_j\}_j$ converges. In particular, there exists $j_0 > 0$ such that u_j and w_j are linearly independent over $\mathbf{C}$ for every $j > j_0$, where u_j is as in (4).

Now let $y \in \overline{\sigma(\Omega)}$. Then, recall that $\overline{\sigma(\Omega)}$ is the local Hausdorff set-limit of the sequence of the closure of the convex domains

$$[\partial\varphi_j(q)]^{-1}(\Omega - s_j)$$

Hence, there exists a sequence $\{x_j\}_j$ in $\bar{\Omega}$ such that

$$y = \lim_{j\to\infty} (x_j - s_j).$$

Since u_j and w_j span $\mathbf{C}^2$ for each j, there exist $a_j, b_j \in \mathbf{C}$ such that

$$x_j - s_j = a_j u_j + b_j w_j \tag{9}$$

for each $j > j_0$. Thus,

$$y = \lim_{j\to\infty} \left(\frac{a_j}{\lambda_j} u_j + b_j t_j\right)$$

Note that the $\{t_j\}_j$ converges to the vector which is complex tangent to the boundary, whereas $\{u_j\}$ to the vector which is not complex tangent to the boundary as observed above. Hence, both limits

$$\lim_{j\to\infty} \frac{a_j}{\lambda_j} u_j$$

and

$$\lim_{j\to\infty} b_j t_j$$

must exist and be finite separately. In particular, $a_j \to 0$ as $j \to \infty$ since $\lambda_j \to \infty$. This implies that

$$\lim_{j\to\infty} x_j = \lim_{j\to\infty} b_j w_j$$

from (9), and that $\operatorname{dis}(b_j w_j, V_j) \to 0$ as $j \to \infty$. Then by the estimate (8), it follows that $\lim_{j\to\infty} b_j t_j \in \Sigma$. Hence, $\sigma(\Omega)$ is contained in the set

$$\mathbf{C} H_0 + \Sigma \subset \{\zeta \hat{u} + s \mid \zeta \in \mathbf{C}, \hat{u} \in H_0, s \in \Sigma\}.$$

Since $\Sigma \subset \partial(\sigma(\Omega))$, Lemma 5 now implies that

$$\overline{\sigma(\Omega)} = H_0 + \bar{\Sigma},$$

as desired.

5. Higher Dimensions

Following the arguments of the two dimensional case in the preceding section, we now give the proof of Theorem 1 in all dimensions.

For $n \geqslant 3$, let $\Omega \subset \mathbf{C}^n$ be a bounded convex domain, and let p a boundary point of Ω such that $\lim_{j\to\infty} \varphi_j(q) = p$ for some $q \in \Omega$ and some sequence $\{\varphi_j\}_j \subset \operatorname{Aut}(\Omega)$. Transforming Ω by a complex affine rigid motion of $\mathbf{C}^n$, we may assume without loss of generality that $p = (0, ..., 0) \in \mathbf{C}^n$ and that $\partial\Omega$ is tangent to the hyperplane $\{(z_1, ..., z_n) \in \mathbf{C}^n \mid \operatorname{Im} z_1 = 0\}$. Assume also that $\partial\Omega$ is Levi flat in a neighborhood of p.

Since $\partial\Omega$ admits a C^∞ foliation by $(n-1)$ dimensional complex analytic subvarieties in the neigborhood of p where $\partial\Omega$ is Levi flat, the arguments of *Case* 1 of the preceding section generalizes naturally to every complex dimension $n \geqslant 2$. Consequently, it follows that for any subsequence $\{\varphi_{j_k}\}_k$, we must have

$$\lim_{k\to\infty} \|\partial\varphi_{j_k}(q)\| \neq 0.$$

As before, a standard normal family argument shows that the sequence $\{\|\partial\varphi_j(q)\|\}_j$ is bounded. Moreover, if φ is any subsequential limit of $\{\varphi_j\}_j$, then $\varphi(\Omega)\subset\bar{\Omega}$ and $\varphi(q)\in\partial\Omega$. This implies $\varphi(\Omega)\in\partial\Omega$ by the Maximum Principle. In particular,

$$\lim_{k\to\infty}\det(\partial\varphi_j(q))=0,$$

where $\partial\varphi_j(q)$ is understood as a $n\times n$ complex nonsingular matrix for each j. This shows that as in the case of two dimensions, we have at least one eigendirection of $\partial\varphi_j(q)$ that scales to infinity.

Denote by $X(p)$ the maximal (in the sense of inclusion) complex variety of $\partial\Omega$ passing through $p\in\partial\Omega$. By convexity and the Maximum Principle, the $(n-1)$ dimensional variety $X(p)$ is in fact a convex subdomain containing the origin p of the complex hyperplane $\{(z_1, ..., z_n)\in\mathbf{C}^n \mid z_1=0\}$. In fact, all the analytic leaves that foliates $\partial\Omega$ near p are, by the same reason, Euclidean flat. By convexity and by the smoothness of the foliation, there exists $r>0$ such that, for any j, the analytic disk $h_j\colon \Delta\to\mathbf{C}^n$ defined by

$$h_j(z)=zv+\varphi_j(q)$$

satisfies $h_j(\Delta)\subset\Omega$, whenever $v\in X(p)$ and $\|v\|<r$. Then the vectors

$$\xi_j:=(\varphi_j^{-1}\circ h_j)'(0)$$

satisfy

$$C^{-1}\leqslant\|\xi_j\|\leqslant C$$

for some constant $C>0$ independent of j. By taking a subsequence if necessary, define

$$w(v):=\lim_{j\to\infty}\frac{\xi_j}{\|\xi_j\|}.$$

Choose an orthonormal basis $v_2, ..., v_n$ of the hyperplane $\{\operatorname{Im} z_1=0\}$ that contains $X(p)$, then we have the following

LEMMA 8 (Lemma 4 of [16]). *The vectors* $w_2:=w(v_2), ..., w_n:=w(v_n)$ *are linearly independent over* $\mathbf{C}$.

Proof. For clarity, we repeat the proof of [16] here. Let $a_2, ..., a_n\in\mathbf{C}$ be such that

$$a_2w_2+\cdots+a_nw_n=0.$$

Then

$$\begin{aligned} 0 &= \sum_{l=2}^{n} a_l w_l = \sum_{l=2}^{n} a_l \lim_{j\to\infty} \frac{(\varphi_j^{-1} \circ h_j)'(0)}{\|(\varphi_j^{-1} \circ h_j)'(0)\|} \\ &= \sum_{l=2}^{n} a_l \lim_{j\to\infty} \frac{1}{\|(\varphi_j^{-1} \circ h_j)'(0)\|} \cdot \partial\varphi_j(q)^{-1} v_l \\ &= \sum_{l=2}^{n} \partial\varphi_j(q)^{-1} \left[\sum_{l=2}^{n} \frac{a_l}{\|(\partial\varphi_j(q)^{-1} v_l\|} \cdot v_l \right]. \end{aligned}$$

Since $\{\|\partial\varphi_j(q)\|\}$ is bounded, since $\{\|\partial\varphi_j(q)^{-1}\|\}$ is also bounded when restricted to the span of $X(p)$ by above, and since $v_2, ..., v_n$ are linearly independent over **C**, we can conclude that $a_l = 0$ for all $l = 2, ..., n$. This completes the proof of the lemma.

A consequence of this lemma is that there exists one and only one eigen-direction, which is transversal to $X(p)$, of the scaling by $\{\varphi_j\}$ which scales to infinity. Moreover, $X(p_j)$'s will be scaled by a bounded sequence to a convex domain in a certain codimension 1 hypersurface in $\mathbf{C}^n$. Hence, a line by line imitation of the arguments of *Case* 2 of the preceding section yields the proof of Theorem 1 in every dimension.

6. Proof of Theorems 3 and 4

Let Ω be a bounded domain in $\mathbf{C}^n$, and assume that the boundary $\partial\Omega$ is C^∞ smooth near p. Suppose that there exist a point $q \in \partial\Omega$ and a sequence $\{\varphi_j\}_j \subset \text{Aut}(\Omega)$ such that $\varphi_j(q) \to p$ as $j \to \infty$. We say the automorphism orbit $\{\varphi_j(q)\}_j$ accumulates at p *nontangentially to* $\partial\Omega$, if there exists a constant $C > 0$ independent of j such that

$$\text{dis}(\varphi_j(q), p) \leqslant C \cdot \text{dis}(\varphi_j(q), \partial\Omega), \qquad \forall j.$$

If such an orbit exists, if $\partial\Omega$ is variety-free at p and if Ω is convex near p, then a very special convex scaling is valid as follows:

We start with

Lemma 9. *Let the bounded domain Ω be pseudoconvex near $p \in \partial\Omega$ and assume that $\partial\Omega$ is variety-free at p. If there exist $q \in \Omega$ and $\{\varphi_j\}_j \subset \text{Aut}(\Omega)$ such that $\varphi_j(q) \to p$ as $j \to \infty$, then $\|\partial\varphi_j(q)\| \to 0$ as $j \to \infty$.*

Proof. Suppose there exists a sequence of unit vectors v_j such that $\|\partial\varphi_j(q) v_j\| \geqslant r > 0$ for all j, where r is independent of j. By a standard normal family argument, it follows that the sequence $\{\|\partial\varphi_j(q) v_j\|\}_j$ is bounded. And hence there exists a subsequence $\{\|\partial\varphi_j(q)\,v_j\|\}_j$, in which we use the same indices that converges to a certain nozero vector $\vec{w}$. Since $q \in \Omega$, we can choose $t > 0$ independent of j such that $\zeta t v_j + q \in \Omega$ for all $\zeta \in \mathbf{C}$ with $|\zeta| < 1$. Define by

$$h_j : \Delta \to \Omega : \zeta \mapsto \zeta t v_j + q$$

for each j. Then, a subsequence of $\{\varphi_j \circ h_j\}_j$ converges to a complex analytic mapping $g : \Delta \to \bar{\Omega}$. By pseudoconvexity of Ω near p, $g(\Delta) \subset \partial\Delta$ by choosing a smaller t if necessary, since $g(0) = p \in \partial\Omega$. Since $g'(0) = \vec{w} \neq 0$, this contradicts the fact that $\partial\Omega$ is variety-free at p.

Note that this lemma enables us to apply the convex scaling of Proposition 4 of Section 2. However, we can refine the scaling method even further as follows:

PROPOSITION 5. *Let $\Omega \subset \mathbf{C}^n$ be a bounded domain with a variety-free boundary point p admitting an open ball U centered at p such that $\Omega \cap U$ is convex. Suppose that there exists an automorphism orbit $\{\varphi_j(q)\}$, for some $\{\varphi_j\}_j \subset \mathrm{Aut}(\Omega)$ and some $q \in \Omega$, accumulating at p nontangentially to $\partial\Omega$. Then, the sequence*

$$\tau_j(z) := \partial\varphi_j(q)^{-1}\,(\varphi_j(z) - p)$$

is a normal family, whose every subsequential limit is a one-to-one holomorphic mapping into $\mathbf{C}^n$. Moreover, every subsequence of $\{\tau_j\}_j$ has a convergent subsequence whose subsequential limit of the sequence of domains $\tau_j(\Omega \cap U)$ converges to a convex domain that is biholomorphic to Ω, where the set-limits are taken in the sense of local Hausdorff set-convergence.

Proof. We will simply show that the sequence

$$\partial\varphi_j(q)^{-1}\,(p - \varphi_j(q))$$

is bounded, since all the other assertions then follow from the arguments of Section 2.

Note that, by definition of nontangential orbits, there exist a constant $C > 0$ and the truncated round cone

$$C_p := \{x \in \mathbf{C}^2 \mid \mathrm{dis}(x, p) < C \cdot \mathrm{dis}(x, \partial\Omega)\} \cap U$$

is contained in Ω and such that $\varphi_j(q)$ tends to p in such a way that $\{\varphi_j(q)\} \subset C_p$, for all j.

By smoothness of $\partial\Omega$ near p, shrinking the neighborhood U of p if necessary, we may assume that C_p is contained in a cone contained in Ω with the same vertex at p but with a slightly larger aperture. Let

$d_j = \|p - \varphi_j(q)\|$, for each j. Therefore, there exists a constant $k > 0$ independent of j such that for each p_j the open ball $B_{k\,d_j}(\varphi_j(q))$ centered at $\varphi_j(q)$ with radius $k\,d_j$ is contained in Ω. Let B be the open unit ball centered at the origin in $\mathbf{C}^n$. Then define $f_j\colon B \to \Omega$ by

$$f_j(z) = \varphi_j^{-1}(k\,d_j z + \varphi_j(q)).$$

Note that $f_j(0) = q$ for all j. Therefore, by a normal family argument, there exists a constant $K > 0$ such that

$$\|\partial f_j(0)\| \leqslant K, \qquad \forall j.$$

which in turn implies that

$$\|\partial \varphi_j(q)^{-1}\| \leqslant \frac{K}{k\,d_j}, \qquad \forall j.$$

Consequently, we get

$$\|\partial \varphi_j(q)^{-1}\,(p - \varphi_j(q))\| \leqslant \frac{K}{k\,d_j} \cdot d_j = \frac{K}{k}, \qquad \forall j.$$

Then the remaining assertions follow by the triangle inequality, Lemma 9, Propositions 1 and 4, and the arguments of Section 2.

Now, we are ready for

Proof of Theorem 3. Let $\Omega \subset \mathbf{C}^2$ and $p \in \partial\Omega$ be as in the hypotheses of Theorem 3. Assume the contrary that there exists a nontangential automorphism orbit $\{\varphi_j(q)\}_j$, for some $\{\varphi_j\}_j \subset \mathrm{Aut}(\Omega)$ and $q \in \Omega$, which approaches p nontangentially to $\partial\Omega$. To reach at a contradiction, it is enough to show, due to Proposition 5, that convex scaling by τ_j, induced from a certain subsequence of $\{\varphi_j\}_j$, in Proposition 5 does not yield a limit domain that is biholomorphic to Ω. The proof will then be complete.

Since p is a boundary point of convex exponential type (see Definition preceding the statement of Theorem 3 in Section 1), we have, by an complex affine change of coordinates, and open neighborhood U of p in $\mathbf{C}^n$ and C^∞ smooth strictly convex positive real-valued function $\Phi\colon U \to \mathbf{R}$ defining $\Omega \cap U$, up to a linear change of complex coordinates of U, in such a way that

(1) $p = (0, 0)$,

(2) $\Omega \cap U = \{(z_1, z_2) \in U \mid \mathrm{Im}\, z_1 > \Phi(z_2) + O(|z_1 z_2| + |z_1|^2\}$, and

(3) $\displaystyle\lim_{z \to 0} \frac{\Phi(\lambda z)}{\Phi(z)} = \infty$ for any $\lambda > 1$.

Denote by

$$\partial\varphi_j(q) = \begin{pmatrix} a_{11,j} & a_{12,j} \\ a_{21,j} & a_{22,j} \end{pmatrix}$$

for each j. Then for every $R > 0$, there exists $j_0 > 0$ such that for all $j > j_0$ the domain

$$\tau_j(\Omega) \cap B_R(0) = \tau_j(\Omega \cap U) \cap B_R(0)$$

is defined by, inside $B_R(0)$, the inequality

$$\begin{aligned} \operatorname{Im}(a_{11,j} z_1 + a_{12,j} z_2) > \Phi(a_{21,j} z_1 + a_{22,j} z_2) + O(|(a_{11,j} z_1 + a_{12,j} z_2) \\ \times (a_{21,j} z_1 + a_{22,j} z_2)|, |a_{11,j} z_1 + a_{12,j} z_2|^2) \qquad (10) \end{aligned}$$

Now we extract subsequences from $\{\varphi_j\}_j$ several times, which we denote by the same notation, as follows:

First, we choose a subsequence so that the induced convex-scaling sequence $\{\tau_j\}_j$ converges locally uniformly and so that the sequence of sets $\tau_j(\Omega)$ converges in local Hausdorff sense. Then we further extract subsequences from $\{\varphi_j\}_j$ so that we have a sequence of positive numbers ε_j and complex numbers α_{1l}, $(l = 1, 2)$ such that

(i) $\lim_{j\to\infty}(a_{1l,j}/\varepsilon_j) = \alpha_{1l}$, $(l = 1, 2)$;

(ii) At least one of the two numbers α_{11}, α_{12} in (i) has modules 1.

Then we consider the normalized limit of the expression (10) above as follows:

$$\operatorname{Im}(\alpha_{11} z_1 + \alpha_{12} z_2) > \lim_{j\to\infty} \frac{1}{\varepsilon_j} \Phi(a_{21,j} z_1 + a_{22,j} z_2). \qquad (11)$$

Note that the O term vanishes when we take the limit of $(z_1, z_2) \in B_R(0)$.

Observe that, if the right-hand side of (11) is identically zero, then since $R > 0$ is arbitrary, the expression converges to the expression defining $\tau(\Omega)$. But, then we arrive at the conclusion that $\tau(\Omega)$ contains a complex line in its closure. Since $\tau(\Omega)$ is convex, it also contains a complex line. Then $\tau(\Omega)$ cannot be biholomorphic to any bounded domain in $\mathbf{C}^2$, which obviously contradicts the conclusion of Proposition 5 that $\tau(\Omega)$ is biholomorphic to Ω. Therefore, the right hand side of (11) cannot be identically zero. Neither can it be identical infinity, since $\tau(\Omega)$ cannot be empty. If 0 and ∞ are the only values the right hand side of (11) takes, then it is easy to see from the arguments of the preceding section that $\tau(\Omega)$ is biholomorphic to a product of two bounded convex domains in $\mathbf{C}$, and hence its interior and the boundary are completely foliated by analytic varieties. However, a theorem of Remmert and Stein [24] (also [1, 13, 15, 17]) implies that such a domain cannot be biholomorphic to a domain with a strongly pseudoconvex boundary point. On the other hand, since $p \in \partial\Omega$ is variety-free, Ω necessarily admits a strongly pseudoconvex boundary point near p. This yields again that $\tau(\Omega)$ cannot be biholomorphic to Ω, another contradiction.

Therefore, due to convexity of $\tau(\Omega)$, we have a point $p_0 = (z_1^0, z_2^0) \in \tau(\Omega)$ and a relatively compact neighborhood V of p_0 in $\tau(\Omega)$ such that

$$\frac{1}{K} \leqslant \lim_{j \to \infty} \frac{1}{\varepsilon_j} \Phi(a_{21,j} z_j + a_{22,j} z_2) \leqslant K, \qquad \forall (z_1, z_2) \in V$$

for some constant $K > 0$ independent of $(z_1, z_2) \in V$. Since V is an open neighborhood of p_0, we can choose $\lambda > 1$ such that $(\lambda z_1^0, \lambda z_2^0)$ is still contained in V. But then

$$\begin{aligned} K &\geqslant \lim_{j \to \infty} \frac{1}{\varepsilon_j} \Phi(a_{21,j} \lambda z_1 + a_{22,j} \lambda z_2) \\ &= \lim_{j \to \infty} \frac{1}{\varepsilon_j} \Phi(a_{21,j} z_1 + a_{22,j} z_2) \cdot \frac{\Phi[\lambda(a_{21,j} z_1 + a_{22,j} z_2)]}{\Phi(a_{21,j} z_1 + a_{22,j} z_2)} \\ &\geqslant K^{-1} \lim_{j \to \infty} \frac{\Phi[\lambda(a_{21,j} z_1 + a_{22,j} z_2)]}{\Phi(a_{21,j} z_1 + a_{22,j} z_2)} \\ &= \infty, \end{aligned}$$

which is absurd.

This argument shows that there exists a subsequence of $\{\varphi_j\}_j$ which does not admit the convex scaling sequence $\{\tau_j\}_j$ as specified in Proposition 5, contradicting the conclusion of Proposition 5, if one assumes the existence of an automorphism orbit approaching $p = (0, 0)$ nontangentially to $\partial\Omega$. Hence, our final conclusion is that there cannot be an automorphism orbit approaching p nontangentially to the boundary of Ω, which completes the proof of Theorem 3.

Proof of Theorem 4. With the hypotheses of Theorem 4 stated in Section 1, the convex-scaling of Proposition 5 with $\tau_j(z) = [\partial\varphi_j(q)]^{-1} (\varphi_j(z) - p)$ for *fixed* $p \in \partial\Omega$ in the preceding section applies. Then, with this special type of scaling, a line-by-line imitation of the proof of Theorem 1 in complex dimension two (Section 4) provides the proof of Theorem 4.

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