View metadata, citation and similar papers at core.ac.uk
brought to you by CORE
provided by Elsevier - Publisher Connector

ELSEVIER
Available online at www.sciencedirect.com
SCIENCE DIRECT
Journal of Algebra 293 (2005) 407–426
JOURNAL OF Algebra
www.elsevier.com/locate/jalgebra

Extended plus closure and colon-capturing ☆

Raymond C. Heitmann

The University of Texas at Austin, Department of Mathematics, 1 University Station C1200, Austin, TX 78712-0257, USA

Received 4 October 2004

Available online 9 April 2005

Communicated by Craig Huneke

Abstract

Three results concerning the colon-capturing property for extended plus closures in excellent mixed characteristic rings are demonstrated. It is shown that the extended plus closure has the colon-capturing property for arbitrary sets of three parameters. It is also seen that establishing the colon-capturing property for the extended plus closure is sufficient to guarantee the existence of balanced big Cohen–Macaulay algebras. Also, for three-dimensional complete domains, the second local cohomology of R^+ is annihilated by arbitrarily small powers of every non-unit.
© 2005 Elsevier Inc. All rights reserved.

In several recent articles, [3–5], there has been progress made toward developing a mixed characteristic analog of tight closure. Here we shall present three loosely related results that advance that cause.

In [5], the following theorem was proved.

Theorem 0.1. *Let p, x, y be parameters in R, an excellent local domain. Suppose $p^N z \in (x, y)R$. Then for any rational $\varepsilon > 0$, there is a module-finite extension S of R with $p^\varepsilon z \in (x, y)S$. Thus $z \in (x, y)R^{\mathrm{epf}}$, the (full) extended plus closure of $(x, y)R$.*

☆ This research was partially supported by a grant from the National Science Foundation, DMS-0355486.
E-mail address: heitmann@math.utexas.edu.

0021-8693/$ – see front matter © 2005 Elsevier Inc. All rights reserved.
doi:10.1016/j.jalgebra.2005.03.002

In dimension three, this result asserts that the extended plus closure has the colon-capturing property. In dimension four (or higher), it yields a limited colon-capturing property—colon-capturing for sets of three parameters when one of the parameters is p. In Section 1, we show that this can be extended to arbitrary sets of three parameters.

In Section 2, we prove another variant of the original result in dimension three. The conclusion $p^\varepsilon z \in (x, y)S$ and so $p^\varepsilon z \in (x, y)R^+$ raises an obvious question. Is there something special about p, is p a natural test element, or can a similar result be proved for other ring elements? We shall show the following.

Theorem 0.2. *Let p, x, y be parameters in R, an excellent local domain of dimension three. Suppose $p^N z \in (x, y)R$. Then for any element c in the maximal ideal and any rational $\varepsilon > 0$, $c^\varepsilon z \in (x, y)R^+$.*

This result suggests that if there is to be a theory of test elements for the extended plus closure, it may resemble the theory for tight closure. Small powers of all elements which kill the colon ideal suffice; p is not special. It also gives another interesting fact. If (R, P) is a three-dimensional complete local domain of mixed characteristic, then the local cohomology module $H^2_P(R^+)$ is actually a vector space over the residue field of R^+.

Unfortunately the proof of Theorem 0.2 is quite intricate. It is patterned after the proof of Theorem 0.1. However, unlike the proof of the earlier result, which relied solely on the conducting power of p, this proof must necessarily utilize the conducting power of p in some steps and the conducting power of c in others. It seems unavoidable that any direct proof of this result which relies on constructing a polynomial will necessarily be messy.

Section 3 is devoted to generalizing the following theorem.

Theorem 0.3 [6]. *Let $R \to S$ be a local homomorphism of complete local domains of mixed characteristic and dimension at most* 3. *Then there is a commutative diagram*:

$$\begin{array}{ccc} B & \longrightarrow & C \\ \uparrow & & \uparrow \\ R & \longrightarrow & S \end{array}$$

where B is a balanced big Cohen–Macaulay algebra over R and C is a balanced big Cohen–Macaulay algebra over S.

A key ingredient in Hochster's proof was the fact that the extended plus closure has the colon-capturing property in dimension three. Since the existence of balanced big Cohen–Macaulay algebras implies the direct summand conjecture while the converse is not known, this result provides a more powerful use of colon-capturing. We want to show that if the extended plus closure has the colon-capturing property more generally, balanced big Cohen–Macaulay algebras exist more generally. Actually, we want even more than this. As it is not clear what the optimal notion for a tight closure analog in mixed characteristic is, we want to prove the result for the closure which is eventually chosen. Hence we introduce a new closure operation which contains all of the closures considered in my previous articles and demonstrate that colon-capturing for this potentially larger closure yields balanced big Cohen–Macaulay algebras. Moreover, with some additional hypothesis on the rings involved, we can even get weakly functorial algebras.

Notation and conventions

Throughout, R will be an excellent integral domain of mixed characteristic, that is, the Jacobson radical of R will contain a prime integer p which is nonzero as an element of the ring. When R is local, we will sometimes write (R, P) to indicate that the unique maximal ideal of R is P. R^+ will denote the integral closure of R in an algebraic closure of its quotient field. We will refer to $x_1, \ldots, x_n$ as a set of parameters in R provided $\mathrm{ht}(x_1, \ldots x_n)R = n$. By $H_i(x_1, x_2, x_3, R)$ and $H^i(x_1, x_2, x_3, R)$, we mean the usual Koszul homology and cohomology, respectively. By $H^i_P(R)$, we indicate local cohomology. We will make use of the usual description of local cohomology as a direct limit of Koszul cohomology. In particular, when R is integrally closed of dimension three, we know that R_c is Cohen–Macaulay for every non-unit c and so every element of the maximal ideal has a power which kills the second local cohomology module and so all of the first Koszul homology modules.

Next we recall some definitions from [4]. The last two were introduced in that article; the first has a longer history. We are shortening the closure name to extended plus closure (respectively rank one closure), dropping the word full. Also, since we are only concerned with integral domains which do not contain the rational numbers, we may state the definitions more simply.

Definition. If $x \in R$, then x is in the plus closure of I if $x \in IR^+ \cap R$. We write $x \in I^+$.

Definition. If $x \in R$, then x is in the extended plus closure of I if there exists $c \neq 0 \in R$ such that for every positive integer n, $c^{1/n}x \in (I, p^n)R^+$. We write $x \in I^{\mathrm{epf}}$.

Definition. If $x \in R$, then x is in the rank one closure of I if for every rank one valuation on R^+, every positive integer n, and every $\varepsilon > 0$, there exists $d \in R^+$ with $v(d) < \varepsilon$ such that $dx \in (I, p^n)R^+$. We write $x \in I^{\mathrm{r1f}}$.

The three sections of this article can be read independently.

1. Arbitrary parameters

We begin with a formal proof of a standard fact that we will need.

Proposition 1.1. *Let R be an excellent normal domain and suppose x, y, z are parameters. Then for some $d \in \mathbb{Z}^+$, $H_1(x^m, y^m, z^m, R)$ is isomorphic to $H_1(x^d, y^d, z^d, R)$ for all $m > d$. In particular, x^d, y^d, z^d annihilate $H_1(x^m, y^m, z^m, R)$.*

Proof. The first conclusion follows from the second. To prove the second, let $I = (x, y, z)R$ and consider the local cohomology module $H^2_I(R)$. Recall $H_1(x^m, y^m, z^m, R) \cong$

$H^2(x^m, y^m, z^m, R)$ and $H_I^2(R) \cong \lim_{m\to\infty} H^2(x^m, y^m, z^m, R)$ [1, p. 130]. We note that the limit is an ascending union and so the proposition will follow if we show x^d, y^d, z^d annihilate $H_I^2(R)$.

The associated primes of $H_I^2(R)$ are those depth two primes of R containing I, a subset of $\mathrm{Ass}(R/(x, y))$ and hence a finite set. Thus it suffices to show some power of x, y, z kill $H_I^2(R)$ locally. So we may assume R is local. Next we complete R. As R is excellent, $\widehat{R}$ is again a normal domain; in particular it is (S2). Thus by [2], $H^2_{I\widehat{R}}(\widehat{R})$ is annihilated by a power of I. It follows that for some $d \in \mathbb{Z}^+$, x^d, y^d, z^d annihilate $H_1(x^m, y^m, z^m, \widehat{R})$ for all m. But $H_1(x^m, y^m, z^m, \widehat{R}) \cong ((x^m, y^m) :_{\widehat{R}} z^m)/(x^m, y^m)\widehat{R} \cong ((x^m, y^m) :_R z^m)/(x^m, y^m)R \otimes \widehat{R}$ and so x^d, y^d, z^d annihilate $((x^m, y^m) :_R z^m)/(x^m, y^m)R \cong H_1(x^m, y^m, z^m, R)$ as desired. $\square$

The next lemma is a special case of the main theorem of this section.

Lemma 1.2. *Let R be an excellent normal local domain of mixed characteristic. Assume p, x, y, z are parameters in R and suppose $w \in R$ such that $zw \in (x, y)R$. Then $w \in (x, y)^{\mathrm{epf}}$.*

Proof. Fix $N > 0$. It suffices to show $(pz)^{1/N} w \in (x, y, p^N)R^+$. We let $S_0 = R[u, zt, p^{N^2}t]$ with $t = u^{-1}$, an augmented Rees ring. Let S be the integral closure of S_0. S is again a graded ring—the summand of degree n is $\overline{(z, p^{N^2})^n} Rt^n$.

Claim 1. *$(p, x, y)S$ is a height three ideal.*

To see this, as R and hence S_0 are excellent, it suffices to show that $(p, x, y)S_0$ is a height three ideal. Let Q be a prime ideal of S_0 which contains $(p, x, y)S_0$. If $u \notin Q$, then $(S_0)_Q$ is a localization of $S_0[u^{-1}] = R[u, u^{-1}]$ and so $\mathrm{ht}\, Q \geqslant \mathrm{ht}(Q \cap R) \geqslant 3$. If $u \in Q$, then $z \in Q$ and so $\mathrm{ht}(Q \cap R[u]) \geqslant 5$. It follows that $\mathrm{ht}\, Q \geqslant 3$ since S_0 is a polynomial ring in two variables over $R[u]$ modulo a height two prime ideal.

Claim 2. *There exist elements $p^{1/N}, \alpha, \beta \in S^+$ such that $p^{1/N}(ztw) = x\alpha + y\beta$.*

It suffices to prove the claim locally. For those maximal ideals which do not contain $(p, x, y)S$, the claim follows immediately. (If p is a unit, t is in the ring and $zwt = t(zw) \in (x, y)$.) So we may assume S is an excellent normal local domain of mixed characteristic. We may also adjoin σ with $\sigma^{p-1} = p$ and take the integral closure without destroying our hypothesis. Thus we may assume S contains σ. Now note that $zw = xa + yb$ for some $a, b \in R \subset S$. Thus $p^{N^2}ztw = (p^{N^2}t)(xa + yb) \in (x, y)S$. Next, by Proposition 1.1, there exists $d \in \mathbb{Z}^+$ such that x^d, y^d, p^d annihilate $H_1(x^m, y^m, p^m, S)$ for every m. Hence we may apply [5, Theorem 2.7] with x^d, y^d in place of x, y and $(xy)^{d-1}ztw$ in place of z to find $\gamma, \delta \in S^+$ such that $p^{1/N}(xy)^{d-1}ztw = x^d\gamma + y^d\delta$. But then $\gamma = y^{d-1}\alpha$, $\delta = x^{d-1}\beta$ for some $\alpha, \beta \in S^+$ and so the claim follows.

Claim 3. *In satisfying Claim* 2*, we may choose α, β to be homogeneous of degree* 1*.*

Let $f(T)$ be the irreducible polynomial with coefficients in $\overline{S}$, the integral closure of $S[p^{1/N}]$ which is satisfied by α. We may write $f(T) = T^M + a_1T^{M-1} + \cdots + a_M$. By [3], the integrality of β is equivalent to the conditions

$$\sum_{j=0}^{i} \binom{M-j}{i-j} a_j (ztw)^{i-j} x^j \in y^i \overline{S}$$

holding for every $i = 1, 2, \ldots, M$. As $\overline{S}$ is graded, y is homogeneous, and each term on the left-hand side except a_j is homogeneous, we may let $\overline{a_j}$ be the degree j term of a_j and we then have

$$\sum_{j=0}^{i} \binom{M-j}{i-j} \overline{a_j} (ztw)^{i-j} x^j \in y^i \overline{S}$$

holding for every $i = 1, 2, \ldots, M$. Replacing α by a root of $f(T) = T^M + \overline{a_1}T^{M-1} + \cdots + \overline{a_M}$, we get α, β to be homogeneous of degree 1 as desired.

Now $u\alpha$ and $u\beta$ are homogeneous of degree 0 and so are elements of R^+. In fact, they are elements of $\overline{(z, p^{N^2})R^+} = \overline{(z^{1/N}, p^N)^N R^+}$. By the mixed characteristic version of the Briançon–Skoda theorem [3, p. 702], this gives $u\alpha, u\beta \in (z^{1/N}, p^N)^{N-1}R^+ \subseteq (z^{(N-1)/N}, p^N)R^+$. Thus $z^{1/N}u\alpha, z^{1/N}u\beta \in (z, p^N)R^+$. Now

$$\begin{aligned}(pz)^{1/N} zw &= (pz)^{1/N} u(ztw) = xz^{1/N}u\alpha + yz^{1/N}u\beta \\ &= x\bigl(zs_1 + p^N t_1\bigr) + y\bigl(zs_2 + p^N t_2\bigr) = z(xs_1 + ys_2) + p^N(xt_1 + yt_2)\end{aligned}$$

for some $s_1, s_2, t_1, t_2 \in R^+$. Since z divides $p^N(xt_1 + yt_2)$ and no height one prime ideal contains $(z, p)R$, it follows that $(pz)^{1/N}w = xs_1 + ys_2 + p^N r$ for some $r \in R^+$. $\square$

Theorem 1.3. *Let R be an excellent normal local domain of mixed characteristic. Assume x, y, z are parameters in R and suppose $w \in R$ such that $zw \in (x, y)R$. Then $w \in (x, y)^{\mathrm{epf}}$.*

Proof. $(x, y)R$ is unchanged if x is replaced by an element of the form $x + ry$ and so we may use prime avoidance to assume $\mathrm{ht}(p, x)R = 2$. Likewise, the condition $zw \in (x, y)R$ is unaffected by replacing z by an element of the form $z + ry$ and so we may assume $\mathrm{ht}(x, z, p) = 3$. Now we write $zw = ax + by$. Next we fix a primary decomposition for $(x, z)R$ and write $(x, z)R = Q_1 \cap Q_2 \cap Q_3$ where Q_1 is the intersection of the isolated primary components, Q_2 is the intersection of the embedded primary components which contain a power of p, and Q_3 is the intersection of the embedded primary components which do not contain a power of p. (Either of the latter two may be R.) Since Q_3 is not contained in any height 3 prime which contains $(x, z, p)R$, we may choose $v \in Q_3$ such that $\mathrm{ht}(x, z, p, v) = 4$ (or $v = 1$ if $Q_3 = R$). As $b \in Q_1$, clearly $vp^n b \in (x, z)R$ for some n.

Using Proposition 1.1 and [5, Theorem 2.7], there is a module-finite extension S of R such that $p^{1/N}vb \in (x, z)S$. Next, we may use (1.2) to get $p^{1/N}b \in (x, z)^{\mathrm{epf}}$ and so it follows that $b \in (x, z)^{\mathrm{epf}}$. Thus, for some $c \in R$, $\alpha, \beta, \gamma \in R^+$, we may write $c^{1/N}b =$

$x\alpha + z\beta + p^N\gamma$. It follows that $zc^{1/N}w = c^{1/N}ax + c^{1/N}by = c^{1/N}ax + y(x\alpha + z\beta + p^N\gamma)$ and so $z(c^{1/N}w - y\beta) = x(c^{1/N}a + y\alpha) + p^N y\gamma$. Again we may apply [5, Theorem 2.7] to get $p^{1/N}y\gamma \in (x, z)R^+$. So we write $p^{1/N}y\gamma = x\sigma + z\delta$ with $\delta, \sigma \in R^+$. Substituting this into the previous equation and letting $d = N - 1/N$, we have $z(c^{1/N}w - y\beta) = x(c^{1/N}a + y\alpha) + p^d(x\sigma + z\delta)$. Rearranging, we see that $z(c^{1/N}w - y\beta - p^d\delta) \in xR^+$, so $c^{1/N}w - y\beta - p^d\delta \in xR^+$, and finally $c^{1/N}w \in (x, y, p^d)R^+$. Thus $w \in (x, y)^{\mathrm{epf}}$. □

2. Arbitrary annihilators of small order

The proof of Theorem 2.8 is patterned after the proof of [5, Theorem 2.7]. Accordingly we will begin with a collection of lemmas, mostly without proof, which either appeared in the earlier article or resemble lemmas from [5].

Definition. For a positive integer n, express $n - 1$ in base p and let $\bar{\tau}(n)$ be the sum of the digits. We take $\bar{\tau}(1) = 0$. Then define

$$\tau(n) = \frac{\bar{\tau}(n)}{p - 1}.$$

Lemma 2.1 [5, Lemma 1.6]. *Let $0 < j < i < p^L$ be integers.*

(a) *The highest power of p which divides $\binom{i-1}{j-1}$ is $\tau(j) + \tau(i - j + 1) - \tau(i)$.*
(b) *The highest power of p which divides $\binom{p^L - j}{i-j}$ is also $\tau(j) + \tau(i - j + 1) - \tau(i)$.*
(c) *The highest power of p which divides $\binom{p^L}{i}$ is $L + \tau(i + 1) - \tau(i) - \tau(2)$.*

The next is a variant of a lemma in [5].

Lemma 2.2. *Let R be an excellent integrally closed domain and let $\varepsilon > 0$ be a rational number. Let $c, x, y, z \in R$ where no height one prime ideal contains both c and y. Suppose $z = xw + yv$ where w, v are integral over $R[c^{-1}]$ and w satisfies the monic polynomial $T^n + a_1T^{n-1} + \cdots + a_n$. Further suppose $c^{\varepsilon j}a_j$ is integral over R for every j. Then $c^\varepsilon w$ and $c^\varepsilon v$ are integral over R.*

Proof. Since c^ε is integral over R, we may adjoin it to R without altering our hypothesis. After taking the integral closure, we may assume $c^{\varepsilon j}a_j \in R$. Letting $b_j = c^{\varepsilon j}a_j$, we see that $c^\varepsilon w$ satisfies $T^n + b_1T^{n-1} + \cdots + b_n$ and so is integral over R. Now this implies that $c^\varepsilon v$ is integral over $R[y^{-1}]$. By hypothesis, $c^\varepsilon v$ is also integral over $R[c^{-1}]$. Since R is integrally closed and no height one prime ideal contains both c and y, we see that $c^\varepsilon v$ is integral over R. □

Lemma 2.3 [3, Lemma 2.1]. *Suppose $x, y, z \in R$ with $y \neq 0$. Let*

$$f(T) = \sum_{i=0}^{n} a_i T^{n-i}$$

be a monic polynomial over R and suppose w is an element in an extension domain of R such that $f(w)=0$. For $0 \leqslant i \leqslant n$, set

$$b_i = (-1)^i \sum_{j=0}^{i} \binom{n-j}{i-j} a_j z^{i-j} x^j$$

and let

$$g(T) = \sum_{i=0}^{n} b_i T^{n-i}.$$

Then $g(z-xw)=0$. In particular, if each $b_i \in y^i R$, $(z-xw)/y$ is integral over R.

In the original statement of the next lemma and those that follow, it was assumed that S contained the rational numbers although the full strength of that assumption was seldom needed. Unlike the earlier article, we will not be applying the lemmas to $S = R[p^{-1}]$ and so this hypothesis is unacceptable. Accordingly, we will state the results without that hypothesis when possible. The proofs will not be affected. That assumption was used nowhere in the proof of the next lemma. In the succeeding lemma, there is division by combinatorial symbols and it is necessary to assume S contains $\mathbb{Z}_{(p)}$.

Lemma 2.4 [5, Lemma 2.2]. *Let S be an integral domain. Suppose $z = ax + by$ with $a, b, x, y \in S$ and let n be a positive integer. Further suppose $a_0 = 1, a_1, \dots, a_{k-1} \in S$ have been chosen with $k \leqslant n$ so that*

$$\sum_{j=0}^{i} \binom{n-j}{i-j} a_j z^{i-j} x^j \in y^i S \quad \textit{for } i = 1, \dots, k-1.$$

Then we may find $a_k \in S$ such that

$$\sum_{j=0}^{k} \binom{n-j}{k-j} a_j z^{k-j} x^j \in y^k S.$$

Lemma 2.5 [5, Lemma 2.5]. *Let S be an integral domain which contains $\mathbb{Z}_{(p)}$. Suppose $M > L$ are positive integers and $x, y, z, a_1, \dots, a_k$ are elements of S such that*

$$\sum_{j=0}^{i} \binom{p^L - j}{i-j} a_j z^{i-j} x^j \in y^i S \quad \textit{for all } i \leqslant k$$

where k is an integer less than p^L. If

$$\tilde{a}_j = \left(\prod_{m=0}^{j-1} \frac{p^M - m}{p^L - m} \right) a_j \quad \textit{for each } j > 0,$$

then $\tilde{a}_j = p^{M-L} q_j a_j$ *where* q_j *is a unit in* $\mathbb{Z}_{(p)}$ *and*

$$\sum_{j=0}^{i} \binom{p^M - j}{i-j} \tilde{a}_j z^{i-j} x^j = p^{M-L} q_i \sum_{j=0}^{i} \binom{p^L - j}{i-j} a_j z^{i-j} x^j \in y^i S \quad \textit{for all } i \leqslant k.$$

The last lemma in the sequence does require that S contain the rationals. Unfortunately, the hypothesis of [5, Lemma 2.6] was incorrectly worded. What appears here is the lemma which was actually proved and used in the earlier article.

Lemma 2.6 [5, Lemma 2.6]. *Let* S *be an integral domain which contains the rational numbers. Let* d, i, M, L *be integers with* $L \leqslant M$, $0 < d < i \leqslant p^M$, *and* $i - d \leqslant p^L$. *Suppose* $x, y, z, a_0, a_1, \ldots, a_{i-1}$ *are elements of* S *such that*

$$\sum_{m=0}^{h} \binom{p^L - m}{h-m} a_m z^{h-m} x^m \in y^h S$$

for all integers $h < i - d$. *Let* $\tilde{a}_j = 0$ *for* $j < d$ *and*

$$\tilde{a}_j = y^d \binom{p^L - j + d}{i-j} a_{j-d} \Big/ \binom{p^M - j}{i-j}$$

for $d \leqslant j \leqslant i - 1$. *Then*

$$\sum_{j=0}^{k} \binom{p^M - j}{k-j} \tilde{a}_j z^{k-j} x^j \in y^k S$$

for all $k < i$.

We need one additional new lemma for our proof.

Lemma 2.7. *Let* D, n *be fixed positive integers and suppose* $\varepsilon > 0$. *For an integer* $K > D$, *let* $J_i = \{j \leqslant i \mid j \leqslant p^{nK+D}$ *and* $\tau(j) > nK\}$. *If* K *is sufficiently large, then* $|J_i| < \varepsilon i$ *for all* i.

Proof. First we bound the cardinality of J_i. If $j \leqslant p^{nK+D}$ and $\tau(j) > nK$, then $\tau(p^{nK+D} - j + 1) = nK + D - \tau(j) < D$. So it is equivalent to count the set $\{j \leqslant p^{nK+D} \mid \tau(j) < D\}$. Thus we are counting ordered sequences of $nK + D$ digits, none of which exceeds $p - 1$, which sum to at most $D(p-1) - 1$. Since we merely need to bound the cardinality, we can replace the set by a larger one and drop the restriction that no digit exceeds $p - 1$. Also, adding an additional digit, we may assume that the sequence sums to exactly $D(p-1)-1$. Now the cardinality of the set is an easy computation and equals

$$\binom{nK+Dp-1}{nK+D} = \binom{nK+Dp-1}{D(p-1)-1} \leqslant (nK+Dp)^{Dp}.$$

For $i \leqslant p^{nK}$, the condition $\tau(j) > nK$ cannot be satisfied and so J_i is empty. Thus $|J_i| = 0$ for $i \leqslant p^{nK}$ and $|J_i| \leqslant (nK+Dp)^{Dp}$ for $i > p^{nK}$. It follows that $|J_i|/i$ is bounded by $(nK+Dp)^{Dp}/p^{nK}$. However, this fraction approaches zero as K goes to infinity and so the lemma is proved. □

We are now ready to prove a special case of the main result of this section. The full theorem will then easily follow.

Theorem 2.8. *Let R be a three-dimensional integrally closed excellent domain of mixed characteristic and suppose pc, x, y is a system of parameters with* $\mathrm{ht}(p,c)R = 2$. *Assume there is an element $\sigma \in R$ with $\sigma^{p-1} = p$ and that p^N, x, y, c kill $H_1(p^m, x^m, y^m, R)$ for every positive integer m. Further suppose that $z \in ((x,y) : p^N) = ((x,y) : c)$. Then for any rational $\varepsilon > 0$, there is a module-finite extension S of R with $c^{\varepsilon} z \in (x,y)S$.*

Proof. We prove the result by constructing a polynomial $f(T) = T^{p^L} + a_1 T^{p^L - 1} + \cdots + a_{p^L}$ with coefficients in $R[c^{-1}]$ such that if w is any root of $f(T)$, $v = (z - xw)/y$ is also integral over $R[c^{-1}]$. If we can accomplish this for fixed ε with $c^{\varepsilon j} a_j$ integral over R for every j, then, by Lemma 2.2, the conclusion holds with $S = R[c^{\varepsilon}, c^{\varepsilon} w, c^{\varepsilon} v]$. Thus the entire proof rests on our ability to satisfactorily choose the a_j's. Unfortunately L is not determined at the start of the process; it will be chosen in the recursive procedure.

Let $D = \lambda(H_1(p^m, x^m, y^m, R))$, the length of the homology module. Fix an integer $K > D$. After describing the recursion, we shall show that by making K arbitrarily large, we can make ε arbitrarily small. We first describe the goals of the recursive process by which we construct the polynomial. For each integer i, we choose a set Γ_i with $\Gamma_1 = \{1\}$ and, in general, $\Gamma_i = \Gamma_{i-1}$ or $\Gamma_i = \Gamma_{i-1} \cup \{i\}$. We let $G_i = |\Gamma_i|$. We also shall choose integers $F_i \geqslant F_{i-1}$ and $L_i \geqslant L_{i-1}$ with $KG_i < L_i \leqslant KG_i + D$, as well as elements $a_{1i}, \ldots, a_{ii}$ such that the following conditions are satisfied:

(1) $a_{ji} \in p^{KG_i - \tau(j)} R[c^{-1}]$ for every j;
(2) $c^{F_i} a_{ji} \in R$ for every j and $c^{F_{2j}} a_{ji} \in R$ whenever $2j < i$; and
(3) with $a_{0i} = 1$, the condition

$$\sum_{j=0}^{k} \binom{p^{L_i} - j}{k - j} a_{ji} z^{k-j} x^j \in y^k R\bigl[c^{-1}\bigr]$$

will be satisfied for each $k \leqslant i$.

The procedure ends when $i = p^{L_i}$ (something we still must demonstrate happens), at which time we get the desired polynomial $f(T)$ with $L = L_i$ and $a_j = a_{ji}$ for every j. We also let $z_1 = z$ and for $i > 1$, we let

$$z_i = p^{-E_i} c^{F_{i-1}} \sum_{j=0}^{i-1} \binom{p^{L_{i-1}} - j}{i - j} a_{j,i-1} z^{i-j} x^j$$

where $E_i = \sup\{KG_{i-1} - \tau(i) + \tau(2), 0\}$. We shall also show that

(4) $z_i \in ((x^i, y^i) : c)$.

In general, we let $Q_i = (x^i, y^i, \{(xy)^{i-n} z_n \mid n \in \Gamma_{i-1}\})R$. $Q_i/(x^i, y^i)R$ naturally embeds in $H_1(p^m, x^m, y^m, R)$ via a map which factors through $\phi_i : Q_i/(x^i, y^i)R \to Q_{i+1}/(x^{i+1}, y^{i+1})R$. This is the standard mapping used in showing the second local cohomology module is a limit of Koszul cohomology. Let D_i be the least nonnegative integer such that $p^{D_i} z_i \in Q_i$. Trivially, $D_i \leqslant D - \lambda(Q_i/(x^i, y^i)R)$ and, if $i \in \Gamma_i$, $D_i \leqslant \lambda(Q_{i+1}/(x^{i+1}, y^{i+1})R) - \lambda(Q_i/(x^i, y^i)R)$. We shall choose $L_i = L_{i-1}$ when $i \notin \Gamma_i$ and $L_i = L_{i-1} + K + D_i$ when $i \in \Gamma_i$. By the above, taking $L_1 = K + D_1$, we clearly have $L_i \geqslant L_{i-1}$ and $KG_i < L_i \leqslant KG_i + D$ by an inductive proof (provided $D_1 > 0$, a harmless assumption). To conclude this preliminary note, we point out that we shall never choose $i \in \Gamma_i$ unless $D_i > 0$. Consequently, for all i, $G_i \leqslant D$ and $L_i \leqslant (K+1)D$ and so the process must terminate.

Now we are ready to describe the recursive procedure. For the initial step ($i = 1$), we may find $a \in R$ such that $p^{D_1} z + ax \in yR$. Choose $F_1 = 0$ and $a_{11} = p^K a$. Trivially we see that the first three conditions are satisfied with $a_{11} \in p^K R = p^{K-\tau(1)} R$. By hypothesis, $z_1 \in ((x, y) : c)$.

For $i > 1$, we first demonstrate (4). As we know that

$$\sum_{j=0}^{k} \binom{p^{L_{i-1}} - j}{k - j} a_{j,i-1} z^{k-j} x^j \in y^k R\big[c^{-1}\big] \quad \text{for each } k < i,$$

Lemma 2.4 yields

$$\sum_{j=0}^{i-1} \binom{p^{L_{i-1}} - j}{i - j} a_{j,i-1} z^{i-j} x^j \in \big(x^i, y^i\big) R\big[c^{-1}\big]$$

and so $z_i \in (x^i, y^i)R[(pc)^{-1}]$. To see that $z_i \in R$, it suffices to prove it one term at a time. Also, as p and c are relatively prime, it suffices to prove each term is in $R[p^{-1}] \cap R[c^{-1}]$. That each term is in $R[p^{-1}]$ follows from $c^{F_{i-1}} a_{j,i-1} \in R$. To see that each term is in $R[c^{-1}]$, it is enough to show

$$p^{-E_i} \binom{p^{L_{i-1}} - j}{i - j} a_{j,i-1} \in R\big[c^{-1}\big].$$

By Lemma 2.1,

$$\binom{p^{L_{i-1}} - j}{i - j} \in p^{\tau(j)+\tau(i-j+1)-\tau(i)} R,$$

and so it suffices to show

$$\bigl(-KG_{i-1} + \tau(i) - \tau(2)\bigr) + \bigl(\tau(j) + \tau(i-j+1) - \tau(i)\bigr) + \bigl(KG_{i-1} - \tau(j)\bigr) \geqslant 0$$

when $j > 0$. But this is clear since the left-hand side is just $\tau(i-j+1) - \tau(2)$. For $j = 0$, we need $(-KG_{i-1} + \tau(i) - \tau(2)) + (L_{i-1} + \tau(i+1) - \tau(i) - \tau(2)) \geqslant 0$. Since $L_{i-1} \geqslant KG_{i-1} + 1$, it suffices to show that $1 + \tau(i+1) \geqslant 2\tau(2)$ and this too is clear. So $z_i \in R$ and as $z_i \in (x^i, y^i)R[c^{-1}] \cap R$, it follows that $z_i \in ((x^i, y^i) : c)$ and so (4) holds as desired.

Now we consider three cases.

Case 1. Suppose $KG_{i-1} - \tau(i) \geqslant 0$ and $z_i \notin Q_i$. Here we set $\Gamma_i = \Gamma_{i-1} \cup \{i\}$ and $F_i = F_{i-1}$. This will be the only case where $i \in \Gamma_i$, a fact we shall utilize. We have $G_i = G_{i-1} + 1$ and $L_i = L_{i-1} + K + D_i$. Let

$$a_{ji0} = \left(\prod_{m=0}^{j-1} \frac{p^{L_i} - m}{p^{L_{i-1}} - m}\right) a_{j,i-1}.$$

By Lemma 2.5, we have

$$\sum_{j=0}^{k} \binom{p^{L_i} - j}{k - j} a_{ji0} z^{k-j} x^j \in y^k R\bigl[c^{-1}\bigr]$$

for $k < i$. We also see that

$$\sum_{j=0}^{i-1} \binom{p^{L_i} - j}{i - j} a_{ji0} z^{i-j} x^j = \bigl(p^{D_i} u\bigr)\bigl(p^{E_i+K} c^{-F_{i-1}} z_i\bigr),$$

where u is the unit $(\prod_{m=1}^{i-1}(p^{L_i} - m)/(p^{L_{i-1}} - m))$. Since $p^{D_i} z_i \in Q_i$, we have a relation

$$p^{D_i} z_i + r x^i + \sum_{n \in \Gamma_i} c_n (xy)^{i-n} z_n = b y^i$$

and so

$$p^{E_i+K} c^{-F_{i-1}} u \left(p^{D_i} z_i + r x^i + \sum_{n \in \Gamma_i} c_n (xy)^{i-n} z_n\right) \in y^i R\bigl[c^{-1}\bigr].$$

Letting $a_{ii0} = p^{E_i+K} c^{-F_{i-1}} ur$, we have

$$\sum_{j=0}^{i} \binom{p^{L_i} - j}{i - j} a_{ji0} z^{i-j} x^j = p^{E_i+K} c^{-F_{i-1}} u\bigl(p^{D_i} z_i + r x^i\bigr).$$

For each $n \in \Gamma_i$, we intend to find $a_{1in}, \dots, a_{i-1,in}$ such that

$$\sum_{j=1}^{i-1} \binom{p^{L_i} - j}{i - j} a_{jin} z^{i-j} x^j = p^{E_i+K} c^{-F_{i-1}} u c_n (xy)^{i-n} z_n.$$

Then we set $a_{ji} = a_{ji0} + \sum_{n\in\Gamma_i} a_{jin}$ for $0 < j < i$ and $a_{ii} = a_{ii0}$. Clearly (1) will be proved if we show

(1′) a_{jin} is in $p^{KG_i-\tau(j)} R[c^{-1}]$ for every j, n.

Likewise (2) will follow from

(2′) $c^{F_i} a_{jin} \in R$ for every j, n and $c^{F_{2j}} a_{jin} \in R$ for every n whenever $2j < i$.

To prove (3), we note that the ith condition follows from the definition:

$$\begin{aligned}
&\sum_{j=0}^{i} \binom{p^{L_i} - j}{i - j} a_{ji} z^{i-j} x^j \\
&\quad = \sum_{j=0}^{i} \binom{p^{L_i} - j}{i - j} a_{ji0} z^{i-j} x^j + \sum_{n\in\Gamma_i} \sum_{j=1}^{i-1} \binom{p^{L_i} - j}{i - j} a_{jin} z^{i-j} x^j \\
&\quad = p^{E_i+K} c^{-F_{i-1}} u\bigl(p^{D_i} z_i + r x^i\bigr) + \sum_{n\in\Gamma_i} p^{E_i+K} c^{-F_{i-1}} u c_n (xy)^{i-n} z_n \in y^i R\bigl[c^{-1}\bigr].
\end{aligned}$$

Thus (3) will follow if we show

(3′) $\sum_{j=0}^{k} \binom{p^{L_i}-j}{k-j} a_{jin} z^{k-j} x^j \in y^k R[c^{-1}]$ for every n and $0 < k < i$.

We shall define the set $\{a_{jin}\}$ and prove (1′), (2′) and (3′) using three subcases: $n = 0$, $n = 1$, $n > 1$. For $n = 0$, $\{a_{jin}\}$ is already defined and (3′) was previously noted. Again by Lemma 2.5, $a_{ji0} \in p^{K+D_i} a_{ji} R$ and so (1′) and (2′) are trivial for $j < i$. Finally $a_{ii0} \in p^{E_i+K} c^{-F_{i-1}} R$ gives the final case since $E_i + K > KG_i - \tau(i)$ and $F_i = F_{i-1}$. For $n = 1$, we get

$$\sum_{j=1}^{i-1} \binom{p^{L_i} - j}{i - j} a_{jin} z^{i-j} x^j = p^{E_i+K} c^{-F_{i-1}} u c_n (xy)^{i-1} z$$

by choosing

$$a_{jin} = \left(p^{E_i+K} c^{-F_{i-1}} uc_n y^{i-1}\right) \Big/ \binom{p^{L_i} - j}{i-j}$$

when $j = i-1$ and $a_{jin} = 0$ otherwise. That

$$\sum_{j=1}^{k} \binom{p^{L_i} - j}{k-j} a_{jin} z^{k-j} x^j \in y^k R\left[c^{-1}\right]$$

for every $k < i$ is trivial since each $a_{jin} \in y^{i-1} R[c^{-1}]$. To see that $a_{jin} \in p^{KG_i - \tau(j)} R[c^{-1}]$, we note that $a_{jin} = 0$ unless $j = i-1$. In the latter case, we need only show

$$p^{E_i+K} \Big/ \binom{p^{L_i} - (i-1)}{1} \in p^{KG_i - \tau(i-1)} R.$$

This requires only $(KG_{i-1} - \tau(i) + \tau(2)) + K - (\tau(i-1) + \tau(2) - \tau(i)) \geqslant KG_i - \tau(i-1)$ and, as $G_i = G_{i-1} + 1$, this is an equality. $(2')$ is trivial since $F_i = F_{i-1}$.

Now we fix $n > 1$. To get

$$\sum_{j=1}^{i-1} \binom{p^{L_i} - j}{i-j} a_{jin} z^{i-j} x^j = p^{E_i+K} c^{-F_{i-1}} uc_n (xy)^{i-n} z_n,$$

we recall that

$$z_n = p^{-E_n} c^{F_{n-1}} \sum_{j=0}^{n-1} \binom{p^{L_{n-1}} - j}{n-j} a_{j,n-1} z^{n-j} x^j.$$

We can obtain the desired equality provided

$$\binom{p^{L_i} - j}{i-j} a_{jin} = p^{E_i+K} c^{-F_{i-1}} uc_n y^{i-n} p^{-E_n} c^{F_{n-1}} \binom{p^{L_{n-1}} - j - n + i}{i-j} a_{j+n-i,n-1}$$

for $j = i-n, \ldots, i-1$ and $a_{jin} = 0$ otherwise. So for $j = i-n, \ldots, i-1$,

$$\begin{aligned} a_{jin} &= p^{E_i+K} c^{F_{n-1}-F_{i-1}} uc_n y^{i-n} p^{-E_n} \binom{p^{L_{n-1}} - j - n + i}{i-j} a_{j+n-i,n-1} \Big/ \binom{p^{L_i} - j}{i-j} \\ &= \left(p^{E_i+K-E_n} c^{F_{n-1}-F_{i-1}} uc_n\right) y^{i-n} \binom{p^{L_{n-1}} - j + (i-n)}{i-j} a_{j-(i-n),n-1} \Big/ \binom{p^{L_i} - j}{i-j}. \end{aligned}$$

To prove (1′), we first note that since $n \in \Gamma_i$, the nth step used Case 1. Thus $KG_{n-1} - \tau(n) \geqslant 0$ and $E_n = KG_{n-1} - \tau(n) + \tau(2)$. Now, to see that $a_{jin} \in p^{KG_i - \tau(j)} R[c^{-1}]$, it suffices to show that

$$p^{E_i + K - E_n} \binom{p^{L_{n-1}} - j - n + i}{i - j} a_{j+n-i,n-1} \Big/ \binom{p^{L_i} - j}{i - j} \in p^{KG_i - \tau(j)} R\big[c^{-1}\big].$$

For $j + n - i > 0$, we apply Lemma 2.1 to see that this is equivalent to

$$\begin{aligned} &\big(KG_{i-1} - \tau(i) + \tau(2)\big) + K - \big(KG_{n-1} - \tau(n) + \tau(2)\big) + \tau(j + n - i) \\ &\quad + \tau(i - j + 1) - \tau(n) + \big(KG_{n-1} - \tau(j + n - i)\big) - \big(\tau(j) + \tau(i - j + 1) - \tau(i)\big) \\ &\geqslant KG_i - \tau(j). \end{aligned}$$

However, the two sides of this expression are clearly equal.

For $j + n - i = 0$, we must show

$$p^{E_i + K - E_n} \binom{p^{L_{n-1}}}{i - j} \Big/ \binom{p^{L_i} - j}{i - j} \in p^{KG_i - \tau(j)} R.$$

Using Lemma 2.1, it suffices to show

$$\begin{aligned} &\big(KG_{i-1} - \tau(i) + \tau(2)\big) + K - \big(KG_{n-1} - \tau(n) + \tau(2)\big) \\ &\quad + \big(L_{n-1} - \tau(i - j) + \tau(i - j + 1) - \tau(2)\big) - \big(\tau(j) + \tau(i - j + 1) - \tau(i)\big) \\ &\geqslant KG_i - \tau(j). \end{aligned}$$

This is equivalent to $-KG_{n-1} + \tau(n) + L_{n-1} - \tau(2) - \tau(i - j) \geqslant 0$. Since $i - j = n$, this is equivalent to $L_{n-1} \geqslant KG_{n-1} + \tau(2)$. As $\tau(2) \leqslant 1$, (1′) holds.

To prove (2′), it is enough to show $c^{F_i} c^{F_{n-1} - F_{i-1}} a_{j+n-i,n-1} \in R$ for every j and $c^{F_{2j}} c^{F_{n-1} - F_{i-1}} a_{j+n-i,n-1} \in R$ whenever $2j < i$. The first half is trivial since $c^{F_{n-1}} a_{j+n-i,n-1} \in R$. For the second half, note that $2j < i$ and $j + n - i \geqslant 0$ imply $2n > i$. Because $n \in \Gamma_i$, $n \in \Gamma_n$. Again, this is possible only if the $i = n$ step utilized Case 1. Hence $F_n = F_{n-1}$ and $G_n = G_{n-1} + 1$. Since $n \leqslant p^{L_{n-1}}$, it has at most $L_{n-1} \leqslant KG_{n-1} + D$ digits in its base p representation. As $D < K$, if $n < k \leqslant pn$, k has at most KG_n digits and so $\tau(k) < KG_{k-1}$. This means that the $i = k$ step utilizes either Case 1 or Case 2 (this is actually how the cases are defined). We have seen in Case 1 and will see in Case 2 that $F_i = F_{i-1}$. Thus, as $i < pn$, $F_{i-1} = F_n = F_{n-1}$ and we are reduced to showing $c^{F_{2j}} a_{j+n-i,n-1} \in R$ whenever $2j < i$. If $2j \geqslant n - 1$, this is clear since $F_{2j} \geqslant F_{n-1}$ while if $2j < n - 1$, it follows from the $i = n - 1$ step since $j + n - i < j$.

Finally, to prove (3′), we apply Lemma 2.6 with $R[(pc)^{-1}]$ for S, $i - n$ for d, L_i for M, L_{n-1} for L, and $a_{j,n-1}$ for a_j. This gives

$$\sum_{j=0}^{k} \binom{p^{L_i} - j}{k - j} a_{jin} z^{k-j} x^j \in y^k R\big[(pc)^{-1}\big] \quad \text{for } n \in \Gamma_i,\ 0 < k < i.$$

However, by (2′), a power of c will conduct each a_{jin} into R and since p and y are relatively prime, the sum is actually in $y^k R[c^{-1}]$ as desired.

Case 2. Suppose $KG_{i-1} - \tau(i) \geqslant 0$ and $z_i \in Q_i$. As promised, we let $\Gamma_i = \Gamma_{i-1}$ and $F_i = F_{i-1}$. Here $G_i = G_{i-1}$. The demonstration of this case is a simplified version of the previous one. Since $D_i = 0$ and $L_i = L_{i-1}$ and so $a_{ji0} = a_{j,i-1}$, there is no need to invoke Lemma 2.5. Otherwise the proof is identical to the previous case.

Case 3. Suppose $KG_{i-1} - \tau(i) < 0$. Again we let $\Gamma_i = \Gamma_{i-1}$, but this time we set $F_i = F_{i-1} + 1$. For $j < i$, set $a_{ji} = a_{j,i-1}$. Since (4) holds, we may write $cz_i = rx^i + sy^i$ with $r, s \in R$. Set $a_{ii} = -p^{E_i} c^{-F_i} r$. (1) and (2) hold trivially and the choice of a_{ii} was precisely that needed to give (3).

It only remains to show that we can pick K so that $c^{\varepsilon j} a_j$ is integral over R. We will actually prove integrality by showing $c^{\lfloor \varepsilon j \rfloor} a_j \in R$. As $c^{F_{2j}} a_j \in R$ for $j < p^L/2$ and $c^{F_{p^L}} a_j \in R$ for large j, it suffices to show $\varepsilon j > F_{2j}$. This is equivalent to showing $F_i < (\varepsilon/2)i$ and, of course, the 2 only affects the choice of K, not the existence. So we consider the sequence F_i. As i increases, F_i either remains the same (in Cases 1 and 2) or increases by one (in Case 3). Thus F_i is a counter which measures how many times the condition $KG_{i-1} - \tau(i) < 0$ holds. In this setup, $i \leqslant p^{L_{i-1}} \leqslant p^{KG_{i-1}+D}$. Since there are only finitely many choices for G_{i-1}, Lemma 2.7 asserts that we may choose K sufficiently large so that $F_i < \varepsilon i$ for all i. $\square$

Theorem 2.9. *Let (R, P) be a three-dimensional local excellent domain of mixed characteristic and suppose p, x, y is a system of parameters for R. Suppose that $z \in ((x, y) : p^N)$ and $c \in P$. Then for any rational $\varepsilon > 0$, $c^\varepsilon z \in (x, y)R^+$.*

Proof. We may adjoin, if necessary a $(p-1)$st root of p and then take the integral closure without endangering our hypothesis. Using prime avoidance, we can easily choose $c_1, c_2 \in P$ such that both x, y, pc_1c_2 and p, c_1, c_2 are systems of parameters. Replacing x, y, c_1, c_2 by powers if necessary, we may assume that each of these four elements kills $H_P^2(R)$. Now we may apply the previous theorem with $c = c_1$ and again with $c = c_2$ to get $p^\varepsilon z, c_1^\varepsilon z, c_2^\varepsilon z \in (x, y)R^+$.

Let $c \in P$ be arbitrary. For some m, $c^m \in (p, c_1, c_2)R$. It suffices to prove the conclusion with c replaced by c^{3m}. By the above, we are done if $c^{3m\varepsilon} \in (p^\varepsilon, c_1^\varepsilon, c_2^\varepsilon)R^+$. However, since $c^m \in (p, c_1, c_2)R$, we see that $c^{3m\varepsilon}$ is in the integral closure of $(p^\varepsilon, c_1^\varepsilon, c_2^\varepsilon)^3 R^+$. By [3, Theorem 2.13], $c^{3m\varepsilon} \in (p^\varepsilon, c_1^\varepsilon, c_2^\varepsilon)R^+$ as desired. $\square$

Remark. A more sweeping generalization of [5, Theorem 2.7] would draw the same conclusion from the hypothesis that p, x, y are parameters in a local excellent domain and c is contained in every embedded associated prime ideal of $(x, y)R$. The need to assume dimension three here makes this result somewhat weaker. I believe that this is an artifact of the proof, rather than a suggestion that the stronger result is not true. In the proof at hand, the finite length of $H_P^2(R)$ gives us our D. If the local cohomology is merely finitely generated but not of finite length, the procedure will still terminate and some number will necessarily play the role of D. However, it is not yet clear that this D-equivalent will not depend on K and so thwart our efforts to get smaller values of ε. In any event, a successful resolution of this problem would seem likely to add yet another level of complexity to this already burdened proof.

Corollary 2.10. *Let (R, P) be a three-dimensional complete domain of mixed characteristic. Then $H^2_{PR^+}(R^+)$ is a vector space over R^+/Q, where Q is the maximal ideal of R^+.*

Proof. If $c \in Q$, then we apply the $\varepsilon = 1$ case of Lemma 2.9 to the integral closure of $R[c]$. □

3. Big Cohen–Macaulay algebras

In the quest for a mixed characteristic analog of tight closure, the optimal definition for the closure is not yet clear. We would like theorems which assert that if our closure has the colon-capturing property, other good results follow. To circumvent the uncertainty, it seems useful to define a comparatively large closure operation—one that will have the colon-capturing property if other reasonable choices do. If we can show that demonstrating the colon-capturing property for this larger closure implies the desired results, we can also obtain the results for smaller closures. To pursue this line, we will need the following definitions.

Definition. An extended valuation v on the local domain(R, P) is a rank one valuation on the quotient field of R^+/Q for some prime ideal Q of R^+ satisfying $v(x) > 0$ for all $x \in P$.

Definition. Let $(R, P) \to (S, Q)$ be a local homomorphism of complete local domains. We may extend this map to an R-algebra homomorphism θ from R^+ to S^+ by mapping the roots of a monic polynomial over R to the roots of the image polynomial over S. The choice of θ is not unique but we fix a choice once and for all. Now let v be any extended valuation on (S, Q). By restriction, v induces an extended valuation on (R, P). We will call both extended valuations v and say v is a compatible valuation on R and S.

Definition. Let I be an ideal in R, $x \in R$ and let v be an extended valuation on R. Then x is in the v-augmented closure of I (denoted I^v) provided that, for every $\varepsilon > 0, t \in \mathbb{Z}^+$, there exists $d \in R^+$ with $v(d) < \varepsilon$ such that $dx \in (I, P^t)R^+$.

Definition. We say that the v-augmented closure satisfies the colon-capturing property for R provided that if S is a finite integral extension of R, $x_1, \dots, x_{k+1}$ is a set of parameters in S, and $u \in ((x_1, \dots, x_k) :_S x_{k+1})$, then $u \in ((x_1, \dots, x_k)S)^v$.

The basic goal of this section is to show that the colon-capturing property implies the existence of balanced big Cohen–Macaulay algebras which are weakly functorial in some settings. In [6], Hochster demonstrated the existence of weakly functorial big Cohen–Macaulay algebras for mixed characteristic domains of dimension at most three. Since the colon-capturing property is not known at this time for any of the potential closures for any dimension greater than three, we cannot improve upon his result at this time. However, should colon-capturing be demonstrated, our results here will allow us to get weakly functorial big Cohen–Macaulay algebras more generally. The methods are heavily based on Hochster's original proof.

We must first discuss the notion of partial algebra modifications developed by Hochster and used in [6]. We must revamp the notation in order to get our proofs to work, but the underlying concept remains the same. Let $X_1, \ldots, X_k$ be indeterminates and let $R[X] = R[X_1, \ldots, X_k]$. By $R[X]_{\leqslant N}$, we mean the R-submodule of $R[X]$ spanned by all monomials of total degree at most N. We will refer to $R[X]_{\leqslant N}$ as a partial algebra over R. Likewise, any finite tensor product of such objects will be called a partial algebra. So if T is a partial algebra over R, so is $T[X]_{\leqslant N} = T \otimes_R R[X]_{\leqslant N}$. Thus a partial algebra is a submodule of a polynomial ring over R defined by some perhaps complicated bound on the degrees of the monomials which appear. Of course, to any partial algebra over R, there is naturally associated a polynomial ring over R.

Definition. Let T be a partial algebra over R, A the associated polynomial ring, and $F_1, \ldots, F_n \in T$. Then $\sum_{i=1}^{n} F_i T$ is called a pseudo-ideal of (A, T).

It should be noted that a pseudo-ideal is just an R-submodule of A. While the definition depends upon T and the multiplicative structure of A, a pseudo-ideal will typically not be a subset of T and will not have a multiplicative structure.

Definition. If T is a partial algebra over R, A the associated polynomial ring, and J a pseudo-ideal of (A, T), then (A, T, J) is called an algebra triple over R.

Next we recall the definition of an algebra modification. Let A be an R-algebra. Assume $x_1, \ldots, x_{k+1}$ is a set of parameters in R with $k \geqslant 0$ and suppose $u \in ((x_1, \ldots, x_k)A :_A x_{k+1})$. Letting $F = u - \sum_{i=1}^{k} x_i X_i$, $A' = A[X_1, \ldots, X_k]/(FA[X_1, \ldots, X_k])$ is called an algebra modification of A.

Definition. Let (A, T, J) be an algebra triple over R and let $M = T/(J \cap T)$. Assume $x_1, \ldots, x_{k+1}$ is a set of parameters in R with $k \geqslant 0$ and suppose $u \in T$ with its image $\bar{u} \in ((x_1, \ldots, x_k)M :_M x_{k+1})$. Let $A' = A[X_1, \ldots, X_k]$, $F = u - \sum_{i=1}^{k} x_i X_i$, N be a fixed positive integer, $T' = T[X_1, \ldots, X_k]_{\leqslant N}$, and $J' = J[X_1, \ldots, X_k]_{\leqslant N} + FT'$. Then (A', T', J') is called an algebra triple modification of (A, T, J).

Of course, (A', T', J') is an algebra triple. We note that in this setting, $A'/J'A'$ is an algebra modification of A/JA. With our notation, we keep track of more information and this enables us to take advantage of both the algebra modification and the finiteness of $T/(J \cap T)$.

Definition. Let (A, T, J) be an algebra triple over R. Let v be an extended valuation of R. We say (A, T, J) is v-good if for every $\varepsilon > 0, t \in \mathbb{Z}^+$, we can find $d \in R^+$ with $v(d) < \varepsilon$ and an R-algebra homomorphism $\phi : A \to R^+[d^{-1}]$ such that $\phi(T) \subset d^{-1}R^+$ and $\phi(J) \subset d^{-1}P^t R^+$.

Lemma 3.1. *If the v-augmented closure satisfies the colon-capturing property for integral extensions of R, (A, T, J) is v-good, and (A', T', J') is an algebra triple modification of (A, T, J), then (A', T', J') is v-good.*

Proof. We maintain the same notation; u, F, N are as above. Thus we have a relation $x_{k+1}u = \sum_{i=1}^{k} x_i u_i + w$ with each $u_i \in T$ and $w \in J \cap T$. Now choose, if necessary, $x_{k+2}, \ldots, x_n \in P$ so that $x_1, \ldots, x_n$ is a complete system of parameters.

Fix $\varepsilon > 0$, $t \in \mathbb{Z}^+$. Choose s sufficiently large so that $P^s \subseteq (x_1^{t+1}, \ldots, x_n^{t+1})R^+$. Let $\varepsilon_1 = \varepsilon/2(N + 2)$. Since (A, T, J) is v-good, we can find $d_1 \in R^+$ with $v(d_1) < \varepsilon_1$ and a suitable $\phi_1 : A \to R^+[d_1^{-1}]$ such that $\phi_1(T) \subset d_1^{-1}R^+$ and $\phi_1(J) \subset d_1^{-1}P^s R^+$. Since $\phi_1(w) \in d_1^{-1}P^s R^+$, we get $x_{k+1}\phi_1(u) = x_1\phi_1(u_1) + \cdots + x_k\phi_1(u_k) + x_1^{t+1}r_1 + \cdots + x_n^{t+1}r_n$ in $d_1^{-1}R^+$. Multiplying through by d_1, we get $x_{k+1}d_1\phi_1(u) \in (x_1, \ldots, x_k, x_{k+1}^{t+1}, \ldots, x_n^{t+1})R^+$. Hence, for some $b \in R^+$, $x_{k+1}(d_1\phi_1(u) - bx_{k+1}^t) \in (x_1, \ldots, x_k, x_{k+2}^{t+1}, \ldots, x_n^{t+1})R^+$. By the colon-capturing property, there exists $d_2 \in R^+$ with $v(d_2) < \varepsilon_1$ such that $d_2(d_1\phi_1(u) - bx_{k+1}^t) \in (x_1, \ldots, x_k, x_{k+2}^{t+1}, \ldots, x_n^{t+1}, x_{k+1}^t)R^+$. Hence there exists $b_1, \ldots, b_k \in R^+$ such that $d_2d_1\phi_1(u) - \sum_{i=1}^{k} x_i b_i \in (x_1^t, \ldots, x_n^t)R^+ \subseteq P^t R^+$. We set $d_3 = d_1d_2$ and $d = d_3^{N+2}$; clearly $v(d) < \varepsilon$. Now we complete the diagram

$$\begin{tikzcd}
R^+[d_1^{-1}] \arrow[r] & R^+[d^{-1}] \\
A \arrow[u, "\phi_1"] \arrow[r, hook] & A' \arrow[u, "\phi"']
\end{tikzcd}$$

commutatively by taking $\phi(yX_1^{f_1} \ldots X_k^{f_k}) = \phi_1(y)(d_3^{-1}b_1)^{f_1} \ldots (d_3^{-1}b_k)^{f_k}$ for any $y \in A$. It is easy to check that ϕ has all the desired properties. Certainly $\phi(T') \subset d_1^{-1}d_3^{-N}R^+ \subset d^{-1}R^+$. Also $\phi(J[X_1, \ldots, X_k]_{\leqslant N}) \subset d_3^{-N}\phi_1(JR^+) \subset d_3^{-N-1}P^t R^+$, while $\phi(F) \in d_3^{-1}P^t R^+$ and so $\phi(FT') \subset d^{-1}P^t R^+$; hence $\phi(J') \subset d^{-1}P^t R^+$ as desired. □

Lemma 3.2. *Let $\theta : (R, P) \to (S, Q)$ be a local map of local rings and let v be a compatible valuation on R and S. Suppose (A, T, J) is an algebra triple over R which is v-good. Then $(A \otimes S, T \otimes S, J \otimes S)$ is v-good as an algebra triple over S.*

Proof. It is clear that $(A \otimes S, T \otimes S, J \otimes S)$ is an algebra triple over S. Let $\theta : R^+ \to S^+$ be the extension of θ implicit in the definition of v. For any $\varepsilon > 0$, $t \in \mathbb{Z}^+$, we find the appropriate map $\phi_1 : A \to R^+[d^{-1}]$. Composing with the map which θ induces on $R^+[d^{-1}]$, we get a homomorphism $\phi : A \to S^+[(\theta(d))^{-1}]$. Clearly $\phi(T) \subset (\theta(d))^{-1}S^+$ and $\phi(J) \subset (\theta(d))^{-1}P^t S^+$. Since $S^+[(\theta(d))^{-1}]$, $(\theta(d))^{-1}S^+$, and $(\theta(d))^{-1}P^t S^+$ are S-modules and $v(\theta(d)) = v(d)$, ϕ induces an S-module homomorphism on $A \otimes S$ which has all the desired properties. □

Theorem 3.3. *Let $R \to S$ be a local homomorphism of complete local domains. Let v be a compatible valuation on R and S. Further suppose the v-augmented closure satisfies the* *colon-capturing property for integral extensions of R and S. Then there is a commutative diagram*:

$$\begin{array}{ccc} B & \longrightarrow & C \\ \uparrow & & \uparrow \\ R & \longrightarrow & S \end{array}$$

where B is a balanced big Cohen–Macaulay algebra over R and C is a balanced big Cohen–Macaulay algebra over S.

Proof. The basic idea of the proof is the same as that used in [6,7] and the basic pattern dates back to the original proof of big Cohen–Macaulay modules in the equicharacteristic case. Suppose A is an R-algebra, $x_1, \ldots, x_n$ is a system of parameters in R, and $I = (x, \ldots, x_k)A$. If $x_{k+1}u \in I$ but $u \notin I$, we have a very specific obstruction to A being Cohen–Macaulay. This obstruction can be removed by forming an algebra modification of A. Take $A' = A[X_1, \ldots, X_k]/(u - \sum_{i=1}^{k} x_i X_i)$. Intuitively, one may simply construct a long chain of algebra modifications starting from R to obtain an R-algebra in which all of the obstructions are gone and so every system of parameters forms a regular sequence. The limit B will be a balanced big Cohen–Macaulay algebra over R unless $PB = B$ where P is the maximal ideal of R. Thus, proving the existence of B comes down to showing $1 \notin PB$. Now if the identity is in PB, the offending equation involves only finitely many elements from B and so occurs as the result of one specific modification and so the limit process does not really play a role. More formally, in [7], B is constructed as the direct limit of finitely generated algebras constructed from finite sequences of modifications and it is seen that if $1 \in PB$, we actually have $1 \in PA$ where A is formed from R via a finite sequence of algebra modifications. Likewise, C is constructed as a direct limit using algebra modifications of $B \otimes_R S$. Again following [7], the theorem is valid unless there exists a sequence of modifications $R = T_0, T_1, \ldots, T_r, U_0 = T_r \otimes_R S, U_1, \ldots, U_s$ with $1 \in QU_s$ for Q the maximal ideal of S where each T_{i+1} (respectively U_{i+1}) is an algebra modification of T_i (respectively U_i). So we simply must show such a sequence is impossible.

Assume we have such a bad double sequence of algebra modifications. Ultimately, U_s is constructed as a homomorphic image of a polynomial ring over S. The condition $1 \in QU_s$ corresponds to an equation in the polynomial ring:

$$1 = \sum_{i=1}^{n} x_i H_i + \sum_{i=1}^{m} G_i F_i,$$

where each F_i maps to the zero element in U_s because it played the role of F in a specific algebra modification. Now each modification was performed because of a relation which can be lifted to a relation in the polynomial ring of the form

$$y_{k+1}u = \sum_{i=1}^{k} y_i u_i + \sum_{i=1}^{j} G_{li} F_i,$$

where the y's and u's vary from modification to modification. There is clearly some bound for the degree of the polynomials $H_i, G_i F_i, G_{li} F_i, u, u_i$. and so polynomials of sufficiently large degree add nothing to the process. Accordingly, Hochster introduced partial algebra modifications in [6] and noted that it was sufficient to prove that there are no bad partial algebra modifications.

Thus far, this is just Hochster's proof worded differently. At this point the proofs diverge. Let $R = T_0, T_1, \ldots, T_r$, $U_0 = T_r \otimes_R S, U_1, \ldots, U_s$ be a bad sequence of algebra modifications. Then we have a corresponding bad sequence of algebra triple modifications $(R, R, (0)), (A_{11}, T_{11}, J_{11}), \ldots, (A_{1r}, T_{1r}, J_{1r}), (A_{1r} \otimes S, T_{1r} \otimes J_{1r}, J \otimes S)$, $(A_{21}, T_{21}, J_{21}), \ldots, (A_{2s}, T_{2s}, J_{2s})$. The equation

$$1 = \sum_{i=1}^{n} x_i H_i + \sum_{i=1}^{k} G_i F_i$$

immediately gives, as a relation in T_{2s}, that $1 \in QT_{2s} + J_{2s}$ since each $G_i F_i$ is in J_{2s}. Next the algebra triple $(R, R, (0))$ is trivially v-good and repeated application of the lemmas implies (A_{2s}, T_{2s}, J_{2s}) is v-good. Choose $\varepsilon = v(Q)$ and $t = 1$. We then find $d \in S^+$ with $v(d) < \varepsilon$ and a homomorphism $\phi : A_{2s} \to S^+[d^{-1}]$ such that $\phi(T_{2s}) \subset d^{-1}S^+$ and $\phi(J_{2s}) \subset d^{-1}QS^+$. Applying ϕ to our bad relation gives $1 \in Qd^{-1}S^+ + d^{-1}QS^+$. Hence $d \in QS^+$. But $v(d) < v(Q)$, a contradiction which proves the theorem. □

References

[1] W. Bruns, J. Herzog, Cohen–Macaulay Rings, Cambridge University Press, Cambridge, 1993.

[2] G. Faltings, Über die Annulatoren lokaler Kohomologiegruppen, Arch. Math. 30 (1978) 473–476.

[3] R. Heitmann, The plus closure in mixed characteristic, J. Algebra 193 (1997) 688–708.

[4] R. Heitmann, Extensions of plus closure, J. Algebra 238 (2001) 801–826.

[5] R. Heitmann, The direct summand conjecture in dimension three, Ann. of Math. 156 (2002) 695–712.

[6] M. Hochster, Big Cohen–Macaulay algebras in dimension three via Heitmann's theorem, J. Algebra. 254 (2002) 395–408.

[7] M. Hochster, C. Huneke, Infinite integral extensions and big Cohen–Macaulay algebras, Ann. of Math. 135 (1992) 53–89.