Excellent non-orientable spanning surfaces with distinct boundary slopes

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Abstract

We give a construction of a knot that bounds non-orientable spanning surfaces with distinct boundary slopes, each of which has a hyperbolic exterior. As an application, we show that integral accidental slopes for a knot can take distinct values.

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1. Introduction

A compact orientable 3-manifold $X$ is said to be excellent [12] if it is irreducible, $\partial$-irreducible, atoroidal, anannular, and not homeomorphic to a 3-ball. A subset $\Sigma \subset X$ is totally-knotted (excellent, respectively) in $X$ if the exterior $E(\Sigma) = X - \bar{N}(\Sigma)$ is irreducible and $\partial$-irreducible. ($E(\Sigma)$ is excellent, respectively.)

Let $K$ be a knot in a 3-manifold $X$. A spanning surface for $K$ is a compact connected surface $S$ embedded in $X$ with $S \cap K = \partial S = K$. By a slight abuse of language, a spanning surface $S$ for a knot $K$ in $X$ is totally-knotted (excellent, respectively) if it is totally-knotted (excellent, respectively) as a subset of $X$. 
In this paper, we describe some properties of knots with excellent spanning surfaces, and give a method to construct a hyperbolic knot which bounds excellent non-orientable spanning surfaces with distinct boundary slopes.

Let $X$ be a 3-manifold with $\partial X$ a union of some tori. An isotopy class of a simple closed curve $\gamma$ in $\partial X$ is called a boundary slope if there exists an incompressible and $\partial$-incompressible surface $S$ properly embedded in $X$ such that $\gamma$ is isotopic to a component of $\partial S$ in $\partial X$. We denote by $s(S)$ the boundary slope of a properly embedded surface $S$. If $X$ is a knot exterior in some 3-manifold and the knot bounds an orientable spanning surface, then isotopy classes of simple closed curves in $\partial X$ are represented by rational numbers and $\infty$ as usual, where 0 represents the boundary slope of orientable spanning surfaces and $\infty$ means the meridional slope. For homological reasons, the boundary slope of any spanning surface is an even integer.

In [2], Hatcher showed that for each component $T$ of $\partial X$, the number of slopes of incompressible and $\partial$-incompressible surfaces such that all boundary components are contained in $T$ is finite, by using branched surface theory developed by Floyd and Oertel [1]. As a consequence, it can be shown that all but finitely many Dehn surgery along a small knot, that is, a knot without closed incompressible non-$\partial$-parallel surfaces in the exterior, produce non-Haken 3-manifolds.

In [3], Hatcher and Oertel showed that each rational number is realized as a boundary slope for some Montesinos knot, and gave an algorithm to calculate boundary slopes of Montesinos knots. In [5], Ichihara and Ozawa studied strongly essential surfaces in knot exteriors in $S^3$. Here a properly embedded surface $S$ in the knot exterior $E(K)$ is said to be strongly essential if it is incompressible, $\partial$-incompressible, and some component of $E(K) - \hat{N}(S)$ is $\partial$-irreducible. In [5], it was shown that the number of components of strongly essential surfaces is at most two, the boundary slope of a strongly essential surface is integral or $\infty$, and some applications to Dehn surgery were given.

Here we remark that some (fixed) knot can bound infinitely many totally-knotted spanning surfaces [13, Theorem 1.5], up to isotopy. But it can be shown by [14, Theorem 1.1] that the number of isotopy classes of excellent spanning surfaces for a fixed knot is finite. Furthermore, we show here that only hyperbolic knots bound excellent spanning surfaces of negative Euler characteristics (Proposition 3.1) and if $K$ bounds an excellent spanning surface of negative Euler characteristic, then $K$ has the excellent covering property, that is, every finite fold regular branched covering space along $K$ is excellent (cf. Proposition 3.2).

The main theorem is the following.

**Theorem 1.1.** For any finite set of even integers $\{a_1, \ldots, a_n\}$ and for any closed connected 3-manifold $M$, there exists an excellent knot in $M$ which bounds excellent non-orientable spanning surfaces $F_1, \ldots, F_n$ such that $s(F_i) = a_i$.

It is known that [4, Theorem 4] there is no upper bound on the number of boundary slopes of minimal genus non-orientable spanning surfaces. Theorem 1.1 implies that there is no upper bound on the number of boundary slopes of excellent spanning surfaces.

In fact, strongly essential surfaces produce essential closed surfaces in the knot complement with accidental peripherals (see [5] and Section 5). As an application
of Theorem 1.1, a counterexample is given (Corollary 5.2) to a conjecture on the uniqueness of integral accidental slopes of closed essential surfaces in knot complements [5, Conjecture 3.2], which was inspired by [5, Theorem 3.1].

This paper is organized as follows: In Section 2, we prepare some lemmas needed in the proof of Theorem 1.1. In Section 3, we describe some properties and examples of knots with excellent spanning surfaces. In Section 4, we prove Theorem 1.1, and in Section 5, we make some remarks on accidental closed surfaces in knot complements.

2. Preliminaries

Through this paper, unless stated otherwise, all manifolds are assumed to be compact, and 3-manifolds are orientable. See [7] for basic terminology in 3-dimensional topology which is not stated here.

We show the following “gluing lemma” needed later.

Lemma 2.1. Let $M$ be an irreducible 3-manifold. Let $F_1$ and $F_2$ be disjoint homeomorphic surfaces in $\partial M$. Suppose that $\partial M - (\partial F_1 \cup \partial F_2)$ is incompressible in $M$ and for each $\partial$-reducing disk $D$ of $M$, $|\partial D \cap (\partial F_1 \cup \partial F_2)| \geq 4$. Then the manifold $M'$ obtained by gluing $F_1$ to $F_2$ is irreducible and $\partial$-irreducible.

Proof. Let $F$ be the surface properly embedded in $M'$ obtained by gluing $F_1$ and $F_2$. We consider $M = M' - N(F)$ as the result of cutting $M'$ along $F$. It is easy to see that $F$ is incompressible and $\partial$-incompressible in $M'$.

Let $E$ be a reducing sphere in $M'$. If $E \cap F = \emptyset$, then $E$ is contained in $M$. Since $M$ is irreducible, $E$ bounds a 3-ball in $M$. Thus, $E$ also bounds a 3-ball in $M'$ and in this case $E$ is not a reducing sphere of $M'$. Hence we assume that $E \cap F \neq \emptyset$ and $|E \cap F|$ is minimal among all reducing spheres of $M'$. Let $E'$ be an innermost disk in $E$ with respect to $E \cap F$. Since $F$ is incompressible, $\partial E'$ bounds a disk $E''$ in $F$. By the irreducibility of $M$, the sphere $E' \cup E''$ bounds a 3-ball on the side not containing $F$ and $\partial M'$. Thus $E$ is isotopic to a sphere $E^*$ with $|E \cap F| > |E^* \cap F|$. This contradicts the minimality of $|E \cap F|$. Therefore $M'$ is irreducible.

Let $D$ be a $\partial$-reducing disk of $M'$. If $D \cap F = \emptyset$, then $D$ is a compressing disk of $\partial M - (\partial F_1 \cup \partial F_2)$ and this is a contradiction. Thus we suppose $D \cap F \neq \emptyset$ and assume $|D \cap F|$ is minimal among all $\partial$-reducing disks. By an innermost disk argument, we may assume $D \cap F$ consists of arcs. Let $\alpha$ be an outermost arc and $D'$ be the corresponding outermost disk of $D$ with respect to $D \cap F$. Since $F$ is $\partial$-incompressible, there is a disk $D''$ in $F$ such that $D'' \cap D' = \alpha$. Since $\partial M - (\partial F_1 \cup \partial F_2)$ is incompressible in $M$, for the disk $D_1 = D'' \cup D'$, $\partial D_1$ bounds a disk $D_2$ in $\partial M'$. By the irreducibility of $M$, the sphere $D_1 \cup D_2$ bounds a 3-ball and $D$ is isotopic to a disk $D^*$ with $|D \cap F| > |D^* \cap F|$. This is a contradiction to the minimality of $|D \cap F|$. This completes the proof. □

Lemma 2.2. Let $M$ be an irreducible, $\partial$-irreducible, and atoroidal 3-manifold. Let $F_1$ and $F_2$ be disjoint homeomorphic surfaces in $\partial M$ each component of which has negative Euler characteristic. Suppose that:
\[ \partial M = (\partial F_1 \cup \partial F_2) \text{ is incompressible in } M, \]
\[ \text{there is no essential annulus } A \text{ in } M \text{ such that a component of } \partial A \text{ is contained in } F_1 \]
\[ \text{and} \]
\[ \text{there is no essential annulus whose boundary is contained in } \partial M - (F_1 \cup F_2). \]

Then the manifold \( M' \) obtained by gluing \( F_1 \) to \( F_2 \) is excellent and not Seifert fibered.

**Proof.** Let \( F \) be the surface properly embedded in \( M' \) obtained by gluing \( F_1 \) and \( F_2 \). We consider \( M = M' - \hat{N}(F) \) as the result of cutting \( M' \) along \( F \), and it is easy to see that \( F \) is incompressible and \( \partial \)-incompressible. By Lemma 2.1, \( M' \) is irreducible and \( \partial \)-irreducible.

Let \( T \) be an essential torus in \( M' \). Since \( M \) is atoroidal and \( F \) has no annular or toral component, \( T \) may be isotoped to intersect \( F \) so that each component \( T' \) of \( T - \hat{N}(F) \) forms an essential annulus in \( M \) or an annulus parallel to an annulus \( A' \in \partial M \). In the latter case, \( A' \) is a union of three annuli, two of them are some collar neighborhoods \( C_1 \) and \( C_2 \) of \( \partial F_i \) and the other is the closure \( C_3 \) of an annular component of \( \partial M - (F_1 \cup F_2) \). By pushing \( T' \) to \( C_3 \), we obtain an essential annulus \( A \) properly embedded in \( M' \). It is easy to see that \( A \) is incompressible since \( T \) is incompressible. If \( A \) is \( \partial \)-parallel, then \( T \) is \( \partial \)-parallel, or \( T \) bounds a solid torus. Thus, \( A \) is essential in \( M' \) and we will deal with essential annuli later. Hence we may assume \( T' \) is an essential annulus in \( M \). However this contradicts the condition that there is no essential annulus with some boundary component contained in \( F_1 \).

Let \( A \) be an essential annulus in \( M' \). By the same argument as above, we may assume that \( A \cap F \) consists of essential arcs of \( A \) and \( |A \cap F| \) is minimal among such essential annuli. Let \( D \) be a component of \( A - \hat{N}(F) \). Since \( M \) is \( \partial \)-irreducible, \( \partial D \) bounds a disk \( D' \in \partial M \). By the incompressibility of \( F \) in \( M' \), \( D' \cap F \) is a rectangular disk or two bi-gonal disks. If \( E = D' \cap F \) is a rectangular disk, then \( |A \cap F| > 1 \) and \( A \) is isotopic to the annulus obtained by replacing \( D \) by \( E \). Using the 3-ball \( B \) bounded by the sphere \( D \cup D' \) derived from the irreducibility of \( M' \). By a slight isotopy, we can reduce \( |A \cap F| \) and this contradicts the minimality of \( |A \cap F| \). If \( E = D' \cap F \) consists of bi-gonal two disks, then each component is a \( \partial \)-compressing disk of \( A \) and thus \( A \) is inessential. If \( A \cap F = \emptyset \), then by the hypothesis \( A \) is \( \partial \)-parallel in \( M \) and since \( F \) has no annular component, the parallel annulus in \( \partial M \) does not contain any component of \( F_i \), and thus \( A \) is also \( \partial \)-parallel in \( M' \).

Now \( M' \) is irreducible, \( \partial \)-irreducible, atoroidal, and anannular. Since each component of \( F \) is incompressible and has negative Euler characteristic, \( M' \) is not a 3-ball. Therefore \( M' \) is excellent. Suppose that \( M' \) is Seifert fibered. Then \( F \) is a horizontal surface, that is, each component of \( M' - \hat{N}(F) \) is an I-bundle. If \( F \) is closed, then each component of \( M' - \hat{N}(F) \) is an I-bundle over a closed surface with negative Euler characteristic and has an essential annulus. If \( F \) has non-empty boundary, then each component of \( M' - \hat{N}(F) \) is a handlebody, contradicting the \( \partial \)-irreducibility of \( M \). This completes the proof. \( \square \)

The following is a consequence of arguments of Myers [12, Theorem 1.1] or Kawauchi [9, Theorem 1.1]. The construction of \( \tau \) is explicit.

**Lemma 2.3.** Let \( M \) be a connected 3-manifold with non-empty boundary, without spherical boundary component. For any two point \( p_1 \) and \( p_2 \) in \( \partial M \), there is an excellent arc \( \tau \)
such an excellent arc can be constructed from a given Heegaard decomposition of \( M \).

3. The excellent covering property

**Proposition 3.1.** If a knot \( K \) bounds an excellent spanning surface \( S \) with negative Euler characteristic \( \chi(S) < 0 \), then \( K \) is excellent.

**Proof.** Splitting along \( \partial N(S) \), the knot exterior \( E(K) \) is decomposed into two 3-manifolds \( E(S) \) and \( N(S) - \hat{N}(K) \). The characteristic Seifert pair of \( (N(S) - \hat{N}(K), \partial(N(S) - \hat{N}(K))) \) consists of two components \( M_1 \) and \( M_2 \), such that \( M_1 \) is an \( S^1 \times S^1 \times I \) and \( M_2 \) is an \( I \)-bundle over a non-orientable surface with connected boundary which is homeomorphic to \( S \). In fact, \( N(S) - \hat{N}(K) \) is obtained from \( M_1 \) and \( M_2 \) by identifying an incompressible annulus on \( \partial M_1 \) with the vertical annulus of \( M_2 \). Using Lemma 2.1, we can show that \( N(S) \) is irreducible and \( \partial \)-irreducible. Each essential torus and annulus \( T \) in \( N(S) - \hat{N}(K) \) may be isotoped into \( M_1 \) or \( M_2 \). Since \( M_2 \) is an \( I \)-bundle over \( S \), all incompressible torus is isotoped off \( M_2 \). Thus \( T \) is a torus parallel to \( \partial N(K) \), or an annulus connecting \( \partial N(K) \) to \( \partial N(S) \) or such that \( \partial T \subset \partial N(S) \). Thus, Lemma 2.2 implies that \( E(K) \) is excellent.

**Proposition 3.2.** Let \( K \) be a knot in a 3-manifold \( M \). If \( K \) bounds an excellent spanning surface \( S \) with \( \chi(S) < 0 \), then every finite fold branched covering space of \( M \) along \( K \) such that each degree of the upstairs branching sets is greater than one is excellent and not Seifert fibered.

**Proof.** Let \( p : M' \to M \) be such a finite fold branched covering along \( K \). By the Torus-Annulus Theorem [7], it can be seen that each component of \( p^{-1}(E(S)) \) is excellent. By the condition on the branched covering, each component \( H \) of \( p^{-1}(N(S) - N(K)) \) forms a book of \( I \)-bundles each sheet of which has negative Euler characteristic. By the same argument as that of [14, Lemma 4.1], we can show that \( H \) is irreducible, \( \partial \)-irreducible and atoroidal. Thus by Lemma 2.2, \( M' \) is excellent and not Seifert fibered.

We say a spanning surface \( S \) for a knot \( K \) is free if \( E(S) \) is a handlebody.

**Proposition 3.3.** Each non-trivial knot with a free spanning surface \( F \) with \( \chi(F) = -1 \) does not bound excellent spanning surfaces.

**Proof.** Suppose there exists a non-trivial knot \( K \) in a 3-manifold \( M \) which bounds a genus one free spanning surface \( F \) and an excellent spanning surface \( S \). Let \( p : M' \to M \) be a 2-fold covering space along \( K \). Clearly \( M' \) is obtained from two copies of \( E(F) \), which is a genus two handlebody, by gluing their boundaries together and \( M' \) is a closed 3-manifold of Heegaard genus at most two. On the other hand, \( M' \) is obtained from two copies of \( E(S) \), thus \( M' \) contains a closed separating acylindrical surface, that is, an incompressible surface without essential annuli in the exterior. However it is known that a closed 3-manifold...
of Heegaard genus at most two does not contain separating acylindrical surfaces (cf. [8, Theorem 6]). This completes the proof. □

It is known that some knot does not bound free incompressible spanning surfaces [10], and some knot does not bound totally-knotted spanning surfaces, fibered knots for example. However, there does exist a simple knot which bounds a genus one, free spanning surface $S$ and a genus one, totally-knotted spanning surface $S'$ [11]. By Lemma 3.3, such an $S'$ cannot be excellent. According to the author’s unpublished work, there exists a higher genus knot bounding a free spanning surface and an excellent spanning surfaces each of which is of minimal genus.

4. Proof of Theorem 1.1

Proof of Theorem 1.1. Put $m = \max\{(\max[a_1, \ldots, a_n] - \min[a_1, \ldots, a_n])/2, 1\}$. Let $(B, \tau = t_1 \cup t_2 \cup t_3)$ be an excellent 3-string tangle. We can construct such a tangle by Lemma 2.3. Let $D_1 \cup D_2 \cup D_3$ be disjoint union of disks in $\partial B$ such that $\partial t_i \subset D_i$, and let $p_1 \cup p_2$ be two points in $\partial B - (D_1 \cup D_2 \cup D_3)$. We call the 4-tuple $(B, \tau, D_1 \cup D_2 \cup D_3, p_1 \cup p_2)$ a node.

Take mutually disjoint $m$ nodes $B^{(1)}, B^{(2)}, \ldots, B^{(m)}$ $(B^{(i)} = (B^{(i)}, \tau^{(i)}, D_1^{(i)} \cup D_2^{(i)} \cup D_3^{(i)}, p_1^{(i)} \cup p_2^{(i)}))$ in $M$. Then connect the $8m$ points $\bigcup \partial \tau^{(i)} \cup p_1^{(i)} \cup p_2^{(i)}$ with $7m$ arcs $\sigma = s_1 \cup \cdots \cup s_{7m}$ outside the nodes $B^{(i)}$’s as indicated in Fig. 1. Then we obtain an $m$-component graph $\bigcup \tau^{(i)} \cup \sigma$ such that $p_j^{(i)}$ is a vertex of degree one for each $i, j$ and each point of $\partial \tau^{(i)}$ is a vertex of degree three.

Put $\Sigma = \bigcup \tau^{(i)} \cup \bigcup \partial B^{(j)} \cup \sigma$.

Then $\Sigma$ is a connected 2-polyhedron such that $E(\Sigma)$ consists of $m + 1$ components; $m$ exteriors of excellent 3-string tangles in 3-balls and one 3-manifold $X$ whose boundary is a connected surface of genus $6m + 1$. By Lemma 2.3, we can choose $\sigma$ so that $X$ is excellent, and thus $\Sigma$ is excellent.

![Fig. 1. $\Sigma = \bigcup \tau^{(i)} \cup \bigcup \partial B^{(i)} \cup \sigma$.](image)
Let $\Sigma'$ be a polyhedron obtained from $\Sigma$ by tubing along some components $\tau_T$ of $\tau^{(i)}$'s inside the node and splitting the strings as shown in Fig. 2. In fact this operation is not unique and may produce a simple closed curve, but we always perform the splittings so that the resultant polyhedron $\Sigma'$ is connected by adding a half-twist to the new strings if necessary. Let $D_T$ be the union of the disks of $\bigcup D^{(i)}_j$ each of which contains the boundaries of $\tau_T$. Put $D_R = \bigcup D^{(i)}_j - D_T$ and $P = D_R - \partial \sigma$. We denote by $T$ the once-punctured tori obtained from $D_T$ by this operation.

**Lemma 4.1.** $\Sigma'$ is excellent.

**Proof.** Notice that $E(\Sigma')$ is obtained from $E(\Sigma)$ by gluing certain planar surfaces $P'$ in $\partial E(\Sigma)$ together such that each component of $P'$ has three boundary components. To adapt Lemma 2.2, we show that $\partial E(\Sigma) - \partial P'$ satisfies the condition on Lemma 2.2. By the construction of $\Sigma$, each component of $\partial P'$ is non-separating in $\partial E(\Sigma)$ thus it is not contractible in $\partial E(\Sigma)$. Suppose $\partial E(\Sigma) - \partial P'$ is compressible and let $R$ be a compressing disk. Since $\Sigma$ is excellent, $\partial R$ bounds a disk $R'$ in $\partial E(\Sigma)$ containing some component of $\partial P'$. However, in this case the innermost one is contractible in $\partial E(\Sigma)$. This is a contradiction. Now since $\Sigma$ is excellent, we can apply Lemma 2.2 to show $\Sigma'$ is excellent. $\square$

Let $\Sigma''$ be a polyhedron obtain from $\Sigma'$ by removing $P$ and splitting strings in the node as shown in Fig. 3. We can also perform this operation so that the resultant $\Sigma''$ is connected.
Note that $N(\Sigma'')$ is homeomorphic to a handlebody, because $\Sigma''$ is the union of a surface with boundary and several arcs. Later, $N(\Sigma'')$ will match a regular neighborhood of a desired excellent spanning surface for some fixed knot.

**Lemma 4.2.** $\Sigma''$ is excellent.

**Proof.** Let $\Sigma^T$ be the polyhedron obtained from $\Sigma$ by performing the tubing-splitting operation as shown in Fig. 2 along all components of $\bigcup D^{(i)}$. By Lemma 4.1, $\Sigma^T$ is excellent. Notice that $E(\Sigma'')$ is obtained from $E(\Sigma^T)$ by removing the once-punctured tori $T$ pictured in Fig. 2; this can be viewed instead, however, as gluing together the two corresponding punctured tori $T^\pm$ in $\partial E(\Sigma^T)$. To adapt Lemma 2.2, it is sufficient to show that $\partial E(\Sigma^T) - \partial T^\pm$ is incompressible since $\Sigma^T$ is excellent. Suppose there exists a compressing disk $R$ of $\partial E(\Sigma^T) - \partial T^\pm$. Since $\Sigma^T$ is excellent, $\partial R$ bounds a disk $R'$ in $\partial E(\Sigma^T)$. By the construction of $\Sigma^T$, it is easy to see that any component of $\partial T^\pm$ is not contractible in $\partial E(\Sigma^T)$. Thus, $R'$ does not contain any component of $\partial T^\pm$. This implies that $\partial E(\Sigma^T) - \partial T^\pm$ is incompressible in $E(\Sigma^T)$. Now by Lemma 2.2, $\Sigma''$ is excellent. $\square$

By embedding an oriented simple closed curve in $N(\Sigma)$, we can construct a knot $K$ as follows. Double every one of the arcs $s_j$ outside of the $B^{(i)}$ and make four parallel copies of the arcs $t^{(i)}_j$ inside of $B^{(i)}$, to end up with the number of arcs in Fig. 4. Then we have, in

![Fig. 4. A schematic picture of $B^{(i)}$.](image)
each $D^{(i)}_j$, two sets of four arcs in $B^{(i)}$ and two sets of four arcs outside of $B^{(i)}$ meeting, and for each $p^{(i)}_k$ we have two arcs inside and two arcs outside meeting. The splitting operation amounts to pairing up these arcs, inside and outside. Adding half-twists suitably to them, we get a knot $K$ in $N(\Sigma)$. Then $K$ has an orientation so that each of $K \cap N(t^{(i)}_j)$ and $K \cap N(s_j)$ forms coherently oriented two arcs and $K$ is viewed around each node $B^{(i)}$ as illustrated in Fig. 4.

Now let us consider how to bound non-orientable spanning surfaces to $K$. Outside $B^{(i)}$'s, $K$ forms $7m$ pairs of two strings each of which cobounds a band. In $B^{(i)}$, $K$ forms six pairs of two arcs in $N(\tau^{(i)})$ and two trivial arcs $\alpha_1^{(i)}$ and $\alpha_2^{(i)}$ as in Fig. 4.

There are two forms for surfaces $F^{(i)}_\alpha$ cobounded by $\alpha_1^{(i)}$ and $\alpha_2^{(i)}$ with boundary slopes differing by two as shown in Fig. 5, six bands are not drawn in the pictures and the corresponding $D_R$ are indicated, one of them is the union of a Möbius band with a single “linked” handle and six bands, the other is the union of a disk with two “linked” handles and six bands.

Sewing up the $7m$ bands outside $B^{(i)}$’s, the $6m$ bands inside $B^{(i)}$ and $F^{(i)}_\alpha$’s, we obtain a non-orientable spanning surface $F$ for $K$. Hence $K$ bounds $2m$ non-orientable spanning surfaces, and the difference of boundary slopes is contributed by the crossing indicated by the dotted circle.

In each node, we choose the crossing indicated by the dotted circle in Fig. 4 so that the number of all positive crossings coincides with $\max\{a_i, 0\}$. Thus, we can construct a non-orientable spanning surface with boundary slope an arbitrary even number between $\min\{a_i\}$ and $\max\{a_i\}$ and with the Euler characteristic between $-3m$ and $-4m$ for the fixed knot $K$.

For each spanning surface $F$ bounded by $K$ as above, we can construct a $\Sigma''$ from $\Sigma$ so that $E(F)$ is homeomorphic to $E(\Sigma'')$, because the complement of each piece above, after being pushed into $B^{(i)}$, is one or two of the disks $D^{(i)}_1$, $D^{(i)}_2$ and $D^{(i)}_3$. Thus by Lemma 4.2, $F$ is excellent. Hence by Proposition 3.1, $K$ is excellent. This completes the proof of Theorem 1.1.

In Fig. 6, we exhibit an example of $\Sigma$ in $S^3$ when $m = 1$. Then $S^3 - \tilde{N}(\Sigma)$ consists of two components each of which is homeomorphic to the exterior of Suzuki’s Brunnian $\theta_n$-curve which is known to be hyperbolic [15]. Then, we obtain such a knot as in Fig. 7.
5. Closed accidental surfaces

Let $K$ be a knot in $S^3$. An essential closed surface $S$ in $E(K)$ is said to be accidental if there is an embedded annulus $A$ with $\partial A = l' \cup l$ such that $A \cap S = l'$ and $A \cap \partial E(K) = l$. It is known that the slope determined by $l$ is independent of the choice of $A$ [5, Theorem 1.2]. Hence such a slope is called an accidental slope for $S$. Furthermore it is known that any accidental slope is integral or $\infty$ [5, Lemma 2.5.3] and an example of a knot admitting accidental surfaces of accidental slopes $0$ and $\infty$ is given in [5, Fig. 1].

On the other hand, mutually disjoint accidental surfaces have the same accidental slopes [5, Theorem 3.1]. In [6], Ichihara and Ozawa estimated an upper bound on the difference of integral accidental slopes as follows:

**Theorem 5.1** (cf. [6, Theorem 3.2]). Let $S_1$ and $S_2$ be accidental surfaces with integral accidental slopes $s_1$ and $s_2$ in $E(K)$. Then $|s_1 - s_2| \leq \min\{-\chi(S_1), -\chi(S_2)\}$. 
The knot illustrated in Fig. 8 is constructed by a method different from the one given in Section 4—it produces lower genus spanning surfaces than the original construction in Section 4 but does not yield excellent spanning surfaces—which bounds non-orientable spanning surfaces $F_1$ and $F_2$, both of them are totally knotted, such that $|s(F_1) - s(F_2)| = 2$, $\chi(F_1) = -3$, $\chi(F_2) = -2$. Therefore its complement contains two accidental closed incompressible surfaces $S_1$ and $S_2$ with integral accidental slopes differing by two such that $\chi(S_1) = -6$ and $\chi(S_2) = -4$. At this writing, as far as I know, the best-possibility of Theorem 5.1 is unknown.

As a consequence of Theorem 1.1, we obtain the following corollary.

**Corollary 5.2.** For any finite set of even integers $\{a_1, \ldots, a_n\}$ and for any closed connected 3-manifold $M$, there exists an excellent knot in $M$ such that each $a_i$ is an accidental slope of some closed essential accidental surface in the complement.

**Proof.** Let $K$ be an excellent knot in a 3-manifold $M$ obtained in Theorem 1.1 for given $\{a_i\}$. Since each spanning surface $F_i$ for $K$ is excellent, the closed surface $S_i = \partial N(F_i)$ is incompressible in $M - K$, and has an accidental annulus disjoint from $F_i$. Thus the accidental slope of $S_i$ coincides with $s(F_i) = a_i$. \[\square\]

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**References**