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Cyclic matrices of weighted digraphs[☆]

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ABSTRACT

In this paper, we address several properties of the so-called augmented cyclic matrices of weighted digraphs. These matrices arise in different applications of digraph theory to electrical circuit analysis, and can be seen as an enlargement of basic cyclic matrices of the form BWB^T, where B is a cycle matrix and W is a diagonal matrix of weights. By using certain matrix factorizations and some properties of cycle bases, we characterize the determinant of augmented cyclic matrices, via Cauchy–Binet expansions, in terms of the so-called *proper cotrees*. In the simpler context defined by basic cyclic matrices, we obtain the dual result of Maxwell's determinantal expansion for weighted Laplacian (nodal) matrices. Additional relations with nodal matrices are also discussed. We apply this framework to the characterization of the differential–algebraic circuit models arising from loop analysis, and also to the analysis of branch-oriented models of circuits including charge-controlled memristors.

1. Introduction

The analysis of electrical and electronic circuits has driven a lot of research in different branches of mathematics. Many theoretical results within dynamical systems, the theory of differential and differential–algebraic equations, matrix analysis and, notably, graph theory, have been motivated by electrical circuit applications. The impact of electrical circuit theory in applied mathematics becomes more relevant with the increasing use of nonlinear devices in electronics in the last decades.

Nodal and cyclic matrices. In particular, several properties of graphs involving e.g. cycles, cutsets, trees or digraph matrices have shown up in the investigation of different features of electrical circuits; cf. [3,5,8,16]. In this context, a remarkable result of Maxwell makes it possible to express the determinant of the *nodal matrix* AWA^T of a weighted digraph in terms of a sum of weight products extended over the digraph spanning trees (see [4,8]). The matrices A and W capture the digraph branch-node incidence relations and the branch weights, respectively. This result can be seen as a consequence of the Cauchy–Binet formula [21] and the fact that the incidence matrix A is totally unimodular (i.e. that $\det A_K \in \{0, \pm 1\}$ for every square submatrix A_K). If, in particular, W is the identity matrix, one obtains Kirchhoff's matrix-tree theorem for the Laplacian matrix.

Less attention has been paid to *cyclic matrices*, having the form

$$B_c = BWB^T. \quad (1)$$

Here B is a cycle matrix (also termed a "loop matrix" in circuit theory), which needs not be totally unimodular. Cyclic matrices arise, for instance, in loop or mesh analyses of electrical circuits [12], and their non-singularity (invertibility) is usually the

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key requirement for the unique solvability of the circuit equations. Moreover, in most real applications one is faced with the so-called *augmented* cyclic matrices of the form

$$B_a = \begin{pmatrix} B_1 W_1 B_1^T & B_0 \\ -B_0^T & 0 \end{pmatrix}, \tag{2}$$

where B_0, B_1 and W_1 are certain submatrices of the cycle and weight matrices B, W. The augmented setting displays additional difficulties and drives the analysis beyond the dual case of Maxwell-type formulas. To avoid terminological misunderstandings, we will often refer to matrices of the form depicted in (1) as *basic* cyclic matrices in order to distinguish them from the augmented ones (2).

In this paper, we address several properties of the cyclic matrices introduced above. Specifically, we will show that the non-singularity of augmented cyclic matrices (2) can be tackled in terms of the cotrees associated with the so-called *proper* trees. In our analysis, we will benefit from the invariance of the absolute value of the determinant of cotree submatrices (an explicit statement of this property in connection to the minimum cycle basis problem can be found in [27,28]; cf. also [7,25]). This graph-theoretic result will be combined with the Cauchy–Binet formula and its application requires the use of smart matrix factorizations. Additionally, we show how the determinant of an augmented cyclic matrix can be expressed in terms of the twig and link weights defined by a single tree. In the simpler setting of basic cyclic matrices (1), one obtains the dual result of the Maxwell-type expansion mentioned above. Moreover, when the weight matrix W (which will be assumed to be diagonal throughout the paper) is non-singular, a close relation between the basic and augmented cyclic matrices and their nodal counterparts will be proved to hold. These results will be discussed in Section 3.

Electrical circuits and memristors. In Section 4, we will apply this framework to the characterization of different features of electrical circuit models. We will show that the properties of basic and augmented cyclic matrices here discussed are of interest not only in the solvability of the models arising in loop analyses of electrical circuits; they also apply to the index characterization of several differential–algebraic circuit models, a problem which has received much recent attention [15,24,36,42,44–46]. Although a detailed discussion can be found in Section 4, it is worth indicating here that the presence e.g. of capacitors, inductors or current sources drives loop analysis models beyond the context defined by the basic cyclic matrix (1); Eq. (17) provides an example of an augmented matrix of the form (2) arising in the analysis of nonlinear circuits with reactive elements.

Our results will apply in particular to branch-oriented models of nonlinear circuits including a recently discovered device known as a memory-resistor or *memristor*, under a charge-control assumption. The existence of the memristor, which is a nonlinear device defined by a flux–charge characteristic, was predicted by Leon Chua in 1971 [11]; its actual appearance at the nanometer scale [43] has driven a lot of attention to this circuit element. The research has been further motivated by the announcement by HP that the next generation of commercial memory chips will be based on the memristor [1]. See also [13,22,23,30–33,38,39,42] and references therein.

2. Background

In this section, we compile certain notions and properties coming from digraph theory which will be useful later; see [2,3,5,14,16] for detailed introductions to graph and digraph theory. Note that by *tree*, we implicitly mean "spanning tree" and, similarly, a *cotree* is implicitly assumed to be defined by a spanning tree. Tree and cotree branches will be called *twigs* and *links*, respectively. When defining proper trees in Section 3 we will make use of the fact that, if J and K are disjoint sets of branches, there exists a tree comprising all the J-branches and no K-branch if and only if J has no loops and K has no cutsets.

2.1. Cycle bases

Consider a connected digraph $\mathcal{G}$ with m branches and n nodes. Let us call a closed path without self-intersections a *loop*, and assume without further explicit mention that loops are oriented. Assign to every loop a vector $u = (u_1, \ldots, u_m) \in \mathbb{R}^m$ defined componentwise as

$$u_j = \begin{cases} 1 & \text{if branch } j \text{ is in the loop with the same orientation} \\ -1 & \text{if branch } j \text{ is in the loop with the opposite orientation} \\ 0 & \text{if branch } j \text{ is not in the loop.} \end{cases}$$

We will call the subspace of $\mathbb{R}^m$ spanned by all these vectors the *cycle space* (cf. [5,9]). Its dimension is given by the cyclomatic number $p = m - n + 1$.

A *cycle basis* is a set of loops whose incidence vectors define a basis of the cycle space. We will call a matrix B whose rows are defined by the incidence vectors of a cycle basis a *cycle matrix*. In circuit theory this is usually termed a *reduced loop matrix* and, sometimes, simply a *loop matrix*. A cycle basis is said to be *totally unimodular* if its cycle matrix B is totally unimodular, that is, if each non-vanishing subdeterminant of B is either $+1$ or -1 [28].

Given a (spanning) tree, it is well known that every link defines a unique loop together with some twigs; these twigs are defined by the unique path in the tree which connects the incident nodes of the link. We will assume that the loop has the same orientation as the link and call it a *fundamental loop*. A cycle basis (as well as its associated cycle matrix) is called *strictly fundamental* if it is defined by the fundamental loops of a tree. Every strictly fundamental cycle matrix is known to be totally unimodular (cf. [28]).

2.2. Digraph matrices

The $(p \times p)$-submatrices of the cycle matrices $B \in \mathbb{R}^{p\times m}$ introduced above are known to be non-singular if and only if their columns are defined by the branches of a cotree (see e.g. [3]). More is true, as stated in the following result borrowed from [27,28].

Lemma 1. *Any two non-singular $(p \times p)$-submatrices $\tilde{B}, \hat{B}$ of a given cycle matrix B verify* $\det \tilde{B} = \pm \det \hat{B}$.

This means that $|\det \tilde{B}| = k > 0$ for all the submatrices of a given B defined by cotrees. This constant k is called in [28] the *determinant* of the corresponding cycle basis. Note that, if B is the strictly fundamental matrix defined by a given tree, the submatrix $\tilde{B}$ defined by the branches of the corresponding cotree is an identity matrix and, therefore, $k = |\det \tilde{B}| = 1$. Actually, the identity $k = 1$ holds for all totally unimodular cycle matrices.

An important role will also be played by the so-called *incidence* and *cutset* matrices. The entries of the (reduced) incidence matrix $A = (a_{ij}) \in \mathbb{R}^{(n-1)\times m}$ are defined as

$$a_{ij} = \begin{cases} 1 & \text{if branch } j \text{ leaves node } i \\ -1 & \text{if branch } j \text{ enters node } i \\ 0 & \text{if branch } j \text{ is not incident with node } i. \end{cases}$$

In order to define the cutset matrix, assign each (oriented) cutset a vector $v = (v_1, \dots, v_m)$ with

$$v_j = \begin{cases} 1 & \text{if branch } j \text{ is in the cutset with the same orientation} \\ -1 & \text{if branch } j \text{ is in the cutset with the opposite orientation} \\ 0 & \text{if branch } j \text{ is not in the cutset.} \end{cases}$$

The subspace of $\mathbb{R}^m$ spanned by these vectors is called the *cut space* and has dimension $n - 1$, provided that the digraph is connected. A cutset matrix is any $(n-1) \times m$ matrix whose rows are defined by $n-1$ linearly independent vectors associated with $n - 1$ cutsets. In particular, the choice of a tree yields a system of $n - 1$ linearly independent *fundamental* cutsets, each one defined by a twig together with some links. The cutset matrix will be, in this case, said to be *strictly fundamental*.

The rows of the incidence matrix A can be checked to define a basis of the cut space. This means that, given any cutset matrix Q, a relation of the form $A = PQ$ holds for some non-singular matrix P. Additionally, an $(n - 1) \times (n - 1)$ submatrix $\tilde{A}$ of A is non-singular if and only if their columns correspond to the branches of a tree (see for instance [3,10]). In this case, it is $\det \tilde{A} = \pm 1$. In the light of the relation $A = PQ$ mentioned above, it then follows that $\tilde{A} = P\tilde{Q}$ and, therefore, among the $(n - 1) \times (n - 1)$ submatrices $\tilde{Q}$ of a cutset matrix Q, only those corresponding to trees are non-singular. Moreover, the relation $\det \tilde{Q} = \pm(\det P)^{-1}$ holds for all of them. This means that Lemma 1 is also valid for cutset matrices, that is, there exists a constant $\kappa > 0$ such that $|\det \tilde{Q}| = \kappa$ for all non-singular $(n - 1) \times (n - 1)$ submatrices of Q. In particular, for a strictly fundamental cutset matrix it is $\kappa = 1$; the matrix P arising in the relation $A = PQ$ verifies in this case $\det P = \pm 1$.

Finally, for later use we compile below a characterization of the existence of certain types of loops and cutsets in terms of the cycle matrix (cf. [37, Lemmas 5.7 and 5.8]).

Lemma 2. *A set K of branches of does not contain loops if and only if $B_{\mathcal{G}-K}$ has full row rank. It does not include cutsets if and only if B_K has full column rank.*

3. Cyclic matrices and cotrees

3.1. Basic and augmented cyclic matrices

In different problems arising in electrical circuit theory, it is important to assess the non-singularity of cyclic matrices. When all weights are positive, the basic cyclic matrix $B_c = BWB^T$ in (1) is positive definite and hence non-singular. Recall that the weight matrix W is diagonal; we will denote the entries in the diagonal by $w_i, i = 1, \dots, m$. In many practical situations some of the weights may become negative (e.g. when the weight matrix comes from active devices in electrical circuits), and the characterization of the non-singularity of B_c is more intricate. In this setting the problem can be addressed in terms of cotrees using determinantal expansions. This is also the case for augmented cyclic matrices, which can be actually understood to cover basic ones as a particular case.

Determinantal expansions for cyclic matrices will be based on the Cauchy–Binet formula, stated as Lemma 3 (see [21]). Within this statement, the index sets α and β, together with $\omega = \{1, \dots, p\}$, are used to specify certain $p \times p$ submatrices of D, E, F: the first and second superscripts specify the rows and columns defining each submatrix (e.g. $D^{\omega,\alpha}$ is the submatrix of D defined by all the rows and the columns indexed by α).

Lemma 3 (*Cauchy–Binet*)**.** *Consider a product of three matrices $D \in \mathbb{R}^{p\times m}$, $E \in \mathbb{R}^{m\times m}$, $F \in \mathbb{R}^{m\times p}$, with $p \leq m$. Then*

$$\det DEF = \sum_{\alpha,\beta} \det D^{\omega,\alpha} \det E^{\alpha,\beta} \det F^{\beta,\omega}, \tag{3}$$

where α and β range over all possible subsets of $\{1, \ldots, m\}$ with cardinality p.

The form of the augmented cyclic matrix (2) assumes a splitting of the cycle matrix B in three submatrices B_0, B_1 and B_2, each one comprising certain columns of B. The submatrices B_0 and B_1 are those which enter the upper-right and upper-left blocks of (2), respectively, whereas the submatrix B_2 comprises all the columns (if any) of B which are not present either in B_0 or in B_1. The digraph branches corresponding to columns in B_i will be termed type-i branches ($i = 0, 1, 2$). In practice, both the splitting of B and the taxonomy of branches are motivated by the different nature of these in actual applications; for instance, in electrical circuit modeling these types may correspond to resistive, capacitive or inductive branches. In turn, W_1 is a diagonal matrix whose (diagonal) entries are defined by the type-1 weights, that is, the weights of the type-1 branches.

In this context, a tree is called *proper* if it includes all the type-2 branches and (maybe) some of the type-1 branches, but no type-0 branch. Accordingly, a *proper cotree* includes all the type-0 branches, possibly some of the type-1 branches, and no type-2 branch. The existence of a proper (co)tree requires the absence of cutsets defined by type-0 branches and of loops defined by type-2 branches.

Theorem 1. *For the determinant of the augmented cyclic matrix (2) not to vanish, there must exist at least one proper cotree. If this is the case, the determinant equals the sum of type- 1 weight products extended over the set of proper cotrees, up to a positive constant which does not depend on the actual weight values. If B is totally unimodular (or, in particular, if it is strictly fundamental) then this constant is* 1.

Proof. We will proceed by means of the Cauchy–Binet formula using a factorization of the augmented cyclic matrix (2) of the form

$$B_a = DEF,$$

with

$$D = \begin{pmatrix} B_1 & B_0 & 0 \\ 0 & 0 & I_0 \end{pmatrix}, \qquad E = \begin{pmatrix} W_1 & 0 & 0 \\ 0 & 0 & I_0 \\ 0 & -I_0 & 0 \end{pmatrix}, \qquad F = D^T = \begin{pmatrix} B_1^T & 0 \\ B_0^T & 0 \\ 0 & I_0 \end{pmatrix}. \tag{4}$$

The order of the identity matrices I_0 in (4) equals the number of type-0 branches. The form of the matrix D shows that $(B_1 B_0)$ must have full row rank for B_a to be non-singular. This precludes the existence of loops formed by type-2 branches, according to Lemma 2. Additionally, the non-singularity of B_a requires B_0 to have full column rank (cf. (2)); again in the light of Lemma 2, this rules out the existence of cutsets defined by type-0 branches. The existence of a proper (co)tree then follows as a necessary condition for the non-vanishing of the determinant of B_a.

Recall from Lemma 3 the notation for the submatrices of D, E, F entering the Cauchy–Binet formula (3). When D, E and F take the form displayed in (4), the presence of the I_0 matrices imposes certain restrictions on the submatrices $D^{\omega,\alpha}$, $E^{\alpha,\beta}$ and $F^{\beta,\omega}$ yielding non-vanishing determinants in the Cauchy–Binet expansion. It is not difficult to check that the blocks B_0 and B_0^T and all the identity matrices I_0 must be present in these submatrices for their determinants not to vanish. Using the diagonal form of W, it follows that any non-null determinant must be defined by submatrices having the structure

$$D^{\omega,\alpha} = \begin{pmatrix} \tilde{B}_1 & B_0 & 0 \\ 0 & 0 & I_0 \end{pmatrix}, \qquad E^{\alpha,\beta} = \begin{pmatrix} \tilde{W} & 0 & 0 \\ 0 & 0 & I_0 \\ 0 & -I_0 & 0 \end{pmatrix}, \qquad F^{\beta,\omega} = \begin{pmatrix} \tilde{B}_1^T & 0 \\ B_0^T & 0 \\ 0 & I_0 \end{pmatrix},$$

where $\tilde{B}_1$ is a submatrix of B_1 defined by some of its columns, $\tilde{W}$ being the corresponding weight matrix. Note that $F^{\beta,\omega} = (D^{\omega,\alpha})^T$.

The set of columns of B_1 entering $\tilde{B}_1$ must make $(\tilde{B}_1 B_0)$ non-singular. According to the discussion in Section 2.2, this means that the branches defined by the columns entering $\tilde{B}_1$ together with the type-0 branches must define a cotree. Moreover, since it contains all the type-0 branches and no type-2 branch, it will be a *proper* cotree. Note that $\det D^{\omega,\alpha} = \det F^{\beta,\omega} = \pm k$, where the constant k stands for $|\det(\tilde{B}_1 B_0)|$, and recall that this constant is the same for all digraph cotrees. Additionally, by exchanging the columns of the I_0-blocks in $E^{\alpha,\beta}$ one can easily check that $\det E^{\alpha,\beta} = \det \tilde{W}$, this determinant being defined by the product of weights in the branches defined by $\tilde{B}_1$. Altogether, these remarks show that the Cauchy–Binet determinantal expansion reads in this case

$$\det DEF = k^2 \sum_{\alpha\in\Gamma_{\mathrm{p}}} \prod_{i\in\tilde{\alpha}} w_i, \tag{5}$$

where Γ_{p} is the family of index sets defined by proper cotrees, and $\tilde{\alpha} \subseteq \alpha$ specifies the indices within α which correspond to type-1 branches. Up to the positive constant k^2, the right-hand side of (5) equals the sum of type-1 weight products extended over the set of proper cotrees. Finally, when the cycle matrix B is totally unimodular (or, in particular, strictly fundamental), we have $k^2 = k = 1$. □

From Theorem 1, it follows that the augmented cyclic matrix B_a will be non-singular if and only if the sum of type-1 weight products in proper cotrees does not vanish. Note that in the statement of Theorem 1, we implicitly assume that the type-1 weight product of a proper cotree without type-1 branches (i.e. just defined by type-0 branches) is set to 1.

By noticing that in the absence of type-0 and type-2 branches all trees are proper, from Theorem 1 we may derive the corresponding result for basic cyclic matrices (1). We give an independent proof because of the intrinsic interest of this case.

Corollary 1. *The determinant of the basic cyclic matrix BWB^T equals, up to a positive constant, the sum of weight products extended over the set of digraph cotrees. If B is totally unimodular (or, in particular, if it is strictly fundamental) then this constant is* 1.

Proof. Now, applying the Cauchy–Binet formula to the cyclic matrix BWB^T, we get

$$\det BWB^T = \sum_{\alpha,\beta} \det B^{\omega,\alpha} \det W^{\alpha,\beta} \det(B^T)^{\beta,\omega}. \tag{6}$$

As explained in Section 2.2 above, $B^{\omega,\alpha}$ has a non-vanishing determinant if and only if the branches specified by α define a cotree. Additionally, the diagonal nature of W requires, for $\det W^{\alpha,\beta}$ not to vanish, that $\alpha = \beta$. Denoting by Γ the family of index sets which correspond to cotrees, we may therefore write (6) as

$$\det BWB^T = \sum_{\alpha\in\Gamma} [\det B^{\omega,\alpha}]^2 \det W^{\alpha,\alpha}.$$

The result then follows from the fact that $\det B^{\omega,\alpha} = \pm k$ for all cotrees (cf. Lemma 1), so that

$$\det BWB^T = k^2 \sum_{\alpha\in\Gamma} \det W^{\alpha,\alpha} = k^2 \sum_{\alpha\in\Gamma} \prod_{i\in\alpha} w_i.$$

In particular, for a totally unimodular cycle matrix B the constant k amounts to 1, showing that in these cases the determinant of the cyclic matrix matches exactly the sum of weight products in the digraph cotrees. □

From Corollary 1, it follows immediately that the basic cyclic matrix $B_c = BWB^T$ is non-singular if and only if the sum of weight products extended over the set of digraph cotrees does not vanish.

The sum arising in Theorem 1 can be computed in terms of a single proper tree $\mathcal{T}$, as stated in Theorem 2. However, in contrast to the sum in Theorem 1, which does not involve any coefficients coming from the digraph cotrees, the determinant arising in Theorem 2 involves information which is specific to $\mathcal{T}$. More precisely, we make use of the matrix $K = (k_{ij})$ whose entries relate the type-1 twigs and links defined by $\mathcal{T}$ as follows:

$$k_{ij} = \begin{cases} 1 & \text{if the } j\text{-th type-1 twig belongs to the fundamental loop defined} \\ & \text{by the } i\text{-th type-1 link with the same orientation} \\ -1 & \text{if the } j\text{-th type-1 twig belongs to the fundamental loop defined} \\ & \text{by the } i\text{-th type-1 link with the opposite orientation} \\ 0 & \text{otherwise.} \end{cases} \tag{7}$$

Theorem 2. *Assume that the digraph has at least one proper tree $\mathcal{T}$. The sum of type-* 1 *weight products in proper cotrees arising in Theorem* 1 *equals*

$$\det(W_{1_{co}} + KW_{1_{tr}}K^T), \tag{8}$$

where $W_{1_{co}}$ and $W_{1_{tr}}$ comprise the weights of W_1 which correspond to links and twigs of $\mathcal{T}$, and $K = (k_{ij})$ is the matrix whose entries are defined in (7).

Proof. Consider the strictly fundamental cycle matrix associated with $\mathcal{T}$, writing it as

$$\begin{pmatrix} 0 & K & I & L \\ I & M & 0 & N \end{pmatrix} \tag{9}$$

for certain submatrices K (defined componentwise in (7)), L, M and N. The first group of columns corresponds to type-0 branches (all of which are links); in turn, the second and third groups of columns correspond to type-1 twigs and type-1 links, whereas those coming from the last group correspond to type-2 ones (which are twigs).

Since $k = 1$ for a strictly fundamental cycle matrix, the sum of type-1 weight products in proper cotrees matches exactly the determinant of the corresponding augmented cyclic matrix, which reads

$$\begin{pmatrix} W_{1_{co}} + KW_{1_{tr}}K^T & KW_{1_{tr}}M^T & 0 \\ MW_{1_{tr}}K^T & MW_{1_{tr}}M^T & I \\ 0 & -I & 0 \end{pmatrix}, \tag{10}$$

the upper-left block coming from the product

$$\begin{pmatrix} K & I \\ M & 0 \end{pmatrix} \begin{pmatrix} W_{1_{tr}} & 0 \\ 0 & W_{1_{co}} \end{pmatrix} \begin{pmatrix} K^T & M^T \\ I & 0 \end{pmatrix}.$$

The result then follows from the fact that the determinant of (10) can be easily checked to equal that of $W_{1_{co}} + KW_{1_{tr}}K^T$. □

This result will be used in the proof of Theorem 3 and also in the index characterization of memristive circuits addressed in Section 4.2 (cf. (21)).

Remark. The reader can easily derive the corresponding result for the basic cyclic matrix $B_c = BWB^T$; in this case the sum of weight products in cotrees characterizing (up to a positive constant) the determinant of BWB^T in Corollary 1 can be computed from a single tree as $W_{co} + KW_{tr}K^T$, where the matrix K now relates *all* the twigs and links in the digraph. This property is obtained in a straightforward manner by working with the strictly fundamental cycle matrix $(K\ I)$.

3.2. Cyclic and nodal matrices

A natural question arises from the results discussed so far, namely, how are they related to Maxwell's determinantal expansions of nodal matrices and the augmented variants considered in [40]. When the weights entering the cyclic matrix BWB^T do not vanish, its non-singularity is closely related to that of the nodal matrix $AW^{-1}A^T$; the same will happen with augmented matrices. These results are detailed in Theorem 3.

Within the second assertion of Theorem 3 we make use of the augmented nodal matrix

$$A_n = \begin{pmatrix} A_1W_1^{-1}A_1^T & A_2 \\ -A_2^T & 0 \end{pmatrix}, \tag{11}$$

where the submatrices A_i of the incidence matrix A are defined by the columns which correspond to type-i branches. The same notational criterion will be applied to the submatrices of the cutset matrix Q.

Theorem 3. *Assume that all weights are non-null. Then the cyclic matrix $B_c = BWB^T$ in (1) is non-singular if and only if it is $AW^{-1}A^T$.*

In the augmented setting defined by (2), assume that there exists at least one proper tree, and that the type-1 weights do not vanish. Then, the augmented cyclic matrix B_a in (2) is non-singular if and only if it is the augmented nodal one (11).

Proof. The proof of the claim involving the basic matrices BWB^T and $AW^{-1}A^T$ can be derived in a straightforward manner from the one detailed below for augmented matrices, and therefore we leave the details to the reader in this regard.

Let us then consider the augmented matrices (2) and (11). Fix a proper tree $\mathcal{T}$, and use the fact that the augmented cyclic matrix (2) is non-singular if and only if it is the matrix $W_{1_{co}} + KW_{1_{tr}}K^T$ arising in Theorem 2. This matrix is the Schur complement [21] of $W_{1_{tr}}^{-1}$ in

$$\begin{pmatrix} W_{1_{co}} & K \\ -K^T & W_{1_{tr}}^{-1} \end{pmatrix}. \tag{12}$$

This means that the non-singularity of (2) amounts to that of (12).

In the setting of Theorem 2, the strictly fundamental cutset matrix associated with $\mathcal{T}$ reads

$$Q = \begin{pmatrix} -M^T & I & -K^T & 0 \\ -N^T & 0 & -L^T & I \end{pmatrix}$$

(cf. (9)). Denote

$$Q_0 = \begin{pmatrix} -M^T \\ -N^T \end{pmatrix}, \qquad Q_1 = \begin{pmatrix} I & -K^T \\ 0 & -L^T \end{pmatrix}, \qquad Q_2 = \begin{pmatrix} 0 \\ I \end{pmatrix}.$$

In turn, splitting the incidence matrix A as $(A_0A_1A_2)$, the relation $A = PQ$ (with $\det P = \pm 1$) detailed in Section 2.2 yields $A_i = PQ_i$ for $i = 0, 1, 2$. This makes it possible to rewrite the augmented nodal matrix (11) as

$$\begin{pmatrix} A_1W_1^{-1}A_1^T & A_2 \\ -A_2^T & 0 \end{pmatrix} = \begin{pmatrix} P & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} Q_1W_1^{-1}Q_1^T & Q_2 \\ -Q_2^T & 0 \end{pmatrix} \begin{pmatrix} P^T & 0 \\ 0 & I \end{pmatrix}. \tag{13}$$

Some simple computations show that the second matrix in the right-hand side of (13) reads

$$\begin{pmatrix} W_{1_{tr}}^{-1} + K^TW_{1_{co}}^{-1}K & K^TW_{1_{co}}^{-1}L & 0 \\ L^TW_{1_{co}}^{-1}K & L^TW_{1_{co}}^{-1}L & I \\ 0 & -I & 0 \end{pmatrix}.$$

The determinant of this matrix equals that of $W_{1_{tr}}^{-1} + K^T W_{1_{co}}^{-1} K$, which is the Schur complement of $W_{1_{co}}$ in (12). The non-singularity of the augmented nodal matrix (11) is then equivalent to that of (12) and therefore to that of (2), as we aimed to show. □

Remark. Using the properties of Schur complements (see [21]), the sum of type-1 weight products in proper cotrees can be written as

$$\det(W_{1_{co}} + KW_{1_{tr}}K^T) = \det \begin{pmatrix} W_{1_{co}} & K \\ -K^T & W_{1_{tr}}^{-1} \end{pmatrix} \det W_{1_{tr}},$$

but in turn

$$\det \begin{pmatrix} W_{1_{co}} & K \\ -K^T & W_{1_{tr}}^{-1} \end{pmatrix} = \det W_{1_{co}} \det(W_{1_{tr}}^{-1} + K^T W_{1_{co}}^{-1} K)$$

and therefore

$$\det(W_{1_{co}} + KW_{1_{tr}}K^T) = \det W_1 \det(W_{1_{tr}}^{-1} + K^T W_{1_{co}}^{-1} K).$$

Without going into technical details, this shows that the sum of type-1 weight products in proper cotrees equals the sum of products of inverse type-1 weights in proper trees coming from the determinantal expansion of the augmented nodal matrix (11) (cf. [38]), up to the factor $\det W_1$.

The requirement of invertibility of the weight matrix makes an important difference between the cyclic and nodal settings considered above. For instance, in Section 4.2, the type-1 weights will model electrical resistances and memristances; for nonlinear devices, these quantities are defined in an incremental sense and may vanish at certain working points, making the inverse magnitudes (conductances and memductances, respectively) undefined. Under these assumptions, the nodal matrices $AW^{-1}A^T$ and A_n in (11) are not defined; by contrast, the results based on cyclic matrices discussed in Section 3.1 can be applied even when (some of) the incremental resistances and/or memristances do vanish.

4. Applications in circuit theory

Loop analysis of nonlinear electrical circuits (cf. [12]) has received less attention in the differential–algebraic context than nodal techniques [15,17,18,35,37,45], being however preferred when the circuit devices are mostly current-controlled. We illustrate in Section 4.1, how different properties of loop analysis models can be tackled by means of the results discussed in Section 3. Note in particular that our approach makes it possible to accommodate in the index analysis nonlinear resistors (that is, devices with a non-dynamic relation between voltage and current) with negative incremental resistance, such as tunnel diodes. Analogously, active memristors, which display a negative memristance at certain working ranges, can be included in the characterization of branch-oriented models discussed in Section 4.2.

4.1. Loop analysis

Consider a connected electrical circuit with m branches and n nodes. Fix $m - n + 1$ linearly independent loops, denote by B the associated cycle matrix, and assign a *loop current* j_k, $k = 1, \ldots, m - n + 1$, to each one of these loops. When the circuit is planar, these loop currents can be taken as the ones defined by the *meshes*, that is, the loops encircling the different faces in a planar description of the circuit; note, however, that the circuit need not be planar for the loop analysis to be feasible.

Denote by j the vector of loop currents. The branch currents i can be computed from j simply as $i = B^T j$. The loop analysis begins with the description of Kirchhoff's voltage law in the form $Bv = 0$, and then proceeds by replacing as far as possible the branch voltages of current-controlled devices in terms of branch currents and, eventually, of loop currents.

We show below how the results of Section 3 can be used in the characterization of several properties of time-domain circuit models based on loop analysis. Note in particular that our framework makes it possible to include active devices in the models. For the sake of simplicity, we begin by illustrating how Corollary 1 can be used to characterize the unique solvability of linear resistive circuits. In the presence of nonlinear devices, the circuit equations are usually set up in the form of a differential–algebraic equation (DAE); cf. [15,17,18,24,35–37,41,44–46]. A major issue in the characterization of DAE circuit models is the characterization of their *index* [6,19,26,29,34,37]. Theorem 1 will make it possible to characterize index one configurations in loop analysis models of nonlinear RLC circuits. We assume throughout that all circuits are well-posed, namely, that they do not have either voltage source loops or current source cutsets.

4.1.1. Linear resistive circuits

Consider a linear resistive circuit excited by independent voltage sources. Split the cycle matrix B as $(B_r\ B_u)$, the subscripts r and u standing for resistors and voltage sources, respectively. Letting R and $v_s(t)$ describe the (diagonal) matrix of resistances and the voltage source excitations, the circuit equations can be easily seen to read

$$B_r R B_r^T j + B_u v_s(t) = 0. \quad (14)$$

These equations are uniquely solvable for the loop currents j if and only if the cyclic matrix $B_r R B_r^T$ is non-singular. If this is the case, we have $j = -(B_r R B_r^T)^{-1} B_u v_s(t)$ and the branch variables are given by $i_r = B_r^T j$, $i_u = B_u^T j$, $v_r = R B_r^T j$. It is worth emphasizing that some of the resistances may take on negative values; otherwise $B_r R B_r^T$ would be positive definite and hence non-singular, rendering the problem trivial.

In the presence of negative resistances, the non-singularity of the cyclic matrix $B_r R B_r^T$ can be characterized via Corollary 1. Indeed, since the well-posedness of the circuit precludes voltage source loops, it follows that B_r is a cycle matrix of the digraph obtained after contracting voltage source branches. Provided that the weights are defined by the individual circuit resistances, we immediately get from Corollary 1 the following unique solvability characterization.

Proposition 1. *Consider a well-posed linear resistive circuit excited by independent voltage sources. The loop analysis Eq. (14) describing this circuit are uniquely solvable if and only if the sum of resistance products extended over all resistive cotrees does not vanish.*

4.1.2. Nonlinear RLC circuits

Let us now consider a circuit composed of resistors, inductors, capacitors and independent voltage and current sources. Both resistors and inductors can be nonlinear and are assumed to be defined by C^1 maps of the form $v_r = \gamma(i_r)$, $\varphi_l = \eta(i_l)$, where φ_l is the vector of magnetic fluxes in the inductors. Capacitors may also be nonlinear and are defined by a C^1 charge-voltage characteristic $q_c = \psi(v_c)$. We denote by $R(i_r)$, $L(i_l)$ and $C(v_c)$ the incremental resistance, inductance and capacitance matrices $\gamma'(i_r)$, $\eta'(i_l)$, $\psi'(v_c)$. The resistance matrix is assumed to be diagonal, the k-th incremental resistance R_k depending only on the branch current of the k-th resistor. Coupling effects are allowed among inductors and among capacitors, and we only assume that $L(i_l)$ and $C(v_c)$ are non-singular matrices. Finally, the excitation terms coming from the voltage and current sources are denoted by $v_s(t)$ and $i_s(t)$, respectively.

Split the cycle matrix B as $(B_r B_l B_c B_i B_u)$, where the subscript r (resp. l, c, i, u) signals resistive (resp. inductive, capacitive, current source, voltage source) branches. The loop analysis equations then read

$$L(i_l)i_l' = v_l \tag{15a}$$

$$C(v_c)v_c' = B_c^T j \tag{15b}$$

$$0 = B_r \gamma(B_r^T j) + B_l v_l + B_c v_c + B_i v_i + B_u v_s(t) \tag{15c}$$

$$0 = i_l - B_l^T j \tag{15d}$$

$$0 = i_s(t) - B_i^T j. \tag{15e}$$

System (15) is a differential–algebraic equation of the form

$$M(x)x' = f(x, y) \tag{16a}$$

$$0 = g(x, y), \tag{16b}$$

with $x = (i_l, v_c)$, $y = (j, v_l, v_i)$. Background on general DAEs can be found in [6,19,26,34,37]. Provided that the matrix $M(x)$ is non-singular (a condition which holds for (15) if and only if the inductance and capacitance matrices $L(i_l)$ and $C(v_c)$ are non-singular), the DAE (16) is said to be *index one* if the matrix of partial derivatives g_y is non-singular. In this situation, a straightforward application of the implicit function theorem makes it possible to describe the local system dynamics in the form $M(x)x' = f(x, \varphi(x))$, where $y = \varphi(x)$ comes from (16b). The characterization of index one DAEs is also important with regard to the numerical simulation of the system dynamics.

Theorem 1 allows for a full characterization of the situations in which the DAE (15) is index one, as detailed in Proposition 2. It is worth emphasizing that our framework makes it possible to accommodate situations in which some of the resistances may vanish (ruling out a nodal description of the circuit) or become negative. In the statement of Proposition 2, a VC-loop is a loop defined by voltage sources and/or capacitors only, and an IL-cutset is a cutset defined just by current sources and/or inductors. In this context, a proper tree will include all capacitors and voltage sources, and neither inductors nor current sources. This will be a consequence of the splitting of branches detailed below.

Proposition 2. *Assume that $L(i_l)$, $C(v_c)$ are non-singular matrices, and that $R(i_r)$ is diagonal. Then, the DAE (15) is index one if and only if*

- *the circuit exhibits neither VC-loops nor IL-cutsets, and*
- *the sum of resistance products in proper cotrees does not vanish.*

Proof. The matrix of partial derivatives of (15) with respect to j, v_l, v_i reads

$$\begin{pmatrix} B_r R B_r^T & B_l & B_i \\ -B_l^T & 0 & 0 \\ -B_i^T & 0 & 0 \end{pmatrix} \tag{17}$$

which is an augmented cyclic matrix of the form (2). The type-1 (resp. type-0; type-2) branches in (2) correspond to resistors (resp. inductors and current sources; capacitors and voltage sources). The existence of a proper tree, which in this setting includes all capacitors and voltage sources and neither inductors nor current sources, requires the absence of VC-loops and IL-cutsets. The rest of the proof is a straightforward application of Theorem 1. □

In particular, when all resistors are strictly passive (so that all resistances are positive), the sum of resistance products in proper cotrees is positive; this means that, in a strictly passive setting, the index one nature of the circuit model relies only on the absence of VC-loops and IL-cutsets.

4.2. Memristive circuits

The results presented in Section 3 apply not only to the characterization of loop analysis equations. We show below that they also apply to the so-called branch-oriented circuit models, which do not include loop currents or node potentials but just branch voltages and currents as variables; cf. [20,24,36,37,44]. Specifically, we illustrate how to characterize index one configurations of branch-oriented models of circuits including charge-controlled memristors.

Memristors were postulated by Leon Chua in 1971 as the fourth basic circuit element, besides resistors, capacitors and inductors [11]. Memristive devices are characterized by a nonlinear flux–charge relation which, in a charge-controlled setting, reads $\varphi_m = \phi(q_m)$. By differentiating this relation one gets the voltage–current relation $v_m = M(q_m)i_m$, where $M(q_m) = \phi'(q_m)$ is the so-called *memristance*, which depends on the charge q_m.

The physical realization of memristors had to wait, however, until 2008, when certain nanoscale devices were reported to exhibit a memristive characteristic [43]. In these devices the memristance has the form $M(q_m) = k_1 - k_2 q_m$, where k_1 and k_2 are physical constants, k_2 being significant at the nanometer scale. This has motivated much recent research on memristive systems (cf. [1,13,22,23,30–33,38,39,42] and references therein).

In particular, the use of active memristors in the design of nonlinear oscillators has been proposed in [22]. Active memristors are those for which the memristance becomes negative at certain operating ranges. This fact motivates the use of the framework discussed in Section 3 for the index analysis of the corresponding circuit models.

In a branch-oriented framework, the circuit model takes the form

$$C(v_c)v_c' = i_c \tag{18a}$$

$$L(i_l)i_l' = v_l \tag{18b}$$

$$q_m' = i_m \tag{18c}$$

$$0 = v_m - M(q_m)i_m \tag{18d}$$

$$0 = v_r - \gamma(i_r) \tag{18e}$$

$$0 = B_c v_c + B_l v_l + B_m v_m + B_r v_r + B_u v_s(t) + B_i v_i \tag{18f}$$

$$0 = Q_c i_c + Q_l i_l + Q_m i_m + Q_r i_r + Q_u i_u + Q_i i_s(t). \tag{18g}$$

Notice the description of Kirchhoff laws by means of the cycle and cutset matrices as $Bv = 0$, $Qi = 0$ in (18f) and (18g).

In the setting of Proposition 3, the existence of a proper tree (comprising again all capacitors and voltage sources, and neither inductors nor current sources) will arise as a necessary condition for (18) to be index one. This makes it possible to recast the model in terms of the strictly fundamental cycle and cutset matrices defined by a proper tree $\mathcal{T}$. Write these matrices as

$$B = \begin{pmatrix} K_{11} & K_{12} & K_{13} & I_{m_{co}} & 0 & 0 \\ K_{21} & K_{22} & K_{23} & 0 & I_{r_{co}} & 0 \\ K_{31} & K_{32} & K_{33} & 0 & 0 & I_{li} \end{pmatrix} \tag{19}$$

$$Q = \begin{pmatrix} I_{m_{tr}} & 0 & 0 & -K_{11}^T & -K_{21}^T & -K_{31}^T \\ 0 & I_{r_{tr}} & 0 & -K_{12}^T & -K_{22}^T & -K_{32}^T \\ 0 & 0 & I_{cu} & -K_{13}^T & -K_{23}^T & -K_{33}^T \end{pmatrix}. \tag{20}$$

For the sake of notational simplicity we have joined together the entries corresponding to inductors and current sources, and also those coming from capacitors and voltage sources.

Now type-1 branches correspond to resistors and memristors, whereas type-0 and type-2 branches are defined by inductors and current sources, and capacitors and voltage sources, respectively. The type-1 weights are then the incremental resistances and memristances.

Proposition 3. *Let the capacitance and inductance matrices $C(v_c)$, $L(i_l)$ be non-singular. Assume that resistors and memristors are current-controlled and charge-controlled, respectively, and that neither resistors nor memristors display coupling effects. In this setting, the circuit model (18) is index one if and only if*

- *the circuit exhibits neither VC-loops nor IL-cutsets, and*
- *the sum of resistance–memristance products in proper cotrees does not vanish.*

Proof. In this case the result will be an easy consequence of Theorem 2. Indeed, using the form of B and Q depicted in (19)–(20), the index one nature of the circuit relies on the non-singularity of the matrix

$$\begin{pmatrix} I_{m_{tr}} & 0 & 0 & 0 & 0 & -M_{tr} & 0 & 0 & 0 & 0 \\ 0 & I_{m_{co}} & 0 & 0 & 0 & 0 & -M_{co} & 0 & 0 & 0 \\ 0 & 0 & I_{r_{tr}} & 0 & 0 & 0 & 0 & -R_{tr} & 0 & 0 \\ 0 & 0 & 0 & I_{r_{co}} & 0 & 0 & 0 & 0 & -R_{co} & 0 \\ K_{11} & I_{m_{co}} & K_{12} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ K_{21} & 0 & K_{22} & I_{r_{co}} & 0 & 0 & 0 & 0 & 0 & 0 \\ K_{31} & 0 & K_{32} & 0 & I_{li} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I_{m_{tr}} & -K_{11}^T & 0 & -K_{21}^T & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -K_{12}^T & I_{r_{tr}} & -K_{22}^T & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -K_{13}^T & 0 & -K_{23}^T & I_{cu} \end{pmatrix}.$$

Some easy computations show that the non-singularity of this matrix is equivalent to that of

$$\begin{pmatrix} M_{co} + K_{11}M_{tr}K_{11}^T + K_{12}R_{tr}K_{12}^T & K_{11}M_{tr}K_{21}^T + K_{12}R_{tr}K_{22}^T \\ K_{21}M_{tr}K_{11}^T + K_{22}R_{tr}K_{12}^T & R_{co} + K_{21}M_{tr}K_{21}^T + K_{22}R_{tr}K_{22}^T \end{pmatrix},$$

that is,

$$\begin{pmatrix} M_{co} & 0 \\ 0 & R_{co} \end{pmatrix} + \begin{pmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{pmatrix} \begin{pmatrix} M_{tr} & 0 \\ 0 & R_{tr} \end{pmatrix} \begin{pmatrix} K_{11}^T & K_{21}^T \\ K_{12}^T & K_{22}^T \end{pmatrix}. \tag{21}$$

This matrix has the form displayed in (8). The result then follows in a simple manner from Theorem 2. □

When the incremental resistances and memristances do not vanish, then the inverse magnitudes (that is, the conductances G and the memductances M^{-1}) are well-defined. In this case, using Theorem 3 the index one nature of (18) can be equivalently expressed in terms of the non-singularity of

$$\begin{pmatrix} M_{tr}^{-1} & 0 \\ 0 & G_{tr} \end{pmatrix} + \begin{pmatrix} K_{11}^T & K_{21}^T \\ K_{12}^T & K_{22}^T \end{pmatrix} \begin{pmatrix} M_{co}^{-1} & 0 \\ 0 & G_{co} \end{pmatrix} \begin{pmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{pmatrix}, \tag{22}$$

the determinant of which equals the sum of conductance–memductance products in proper trees, as detailed in [38]. The characterization stated in Proposition 3 holds, however, without recourse to this inversion, that is, without assuming that the inverse descriptions of current-controlled resistors or charge-controlled memristors do exist. Proposition 3 is therefore applicable to problems in which (some of) the conductances/memductances are not well-defined.

5. Summary

We have tackled in this paper several features of cyclic matrices of the forms depicted in (1) and (2). A key result in our analysis is a property of cycle matrices, namely, the invariance of the absolute value of the determinant of the submatrices defined by cotrees. This property makes it possible to characterize the non-singularity of basic cyclic matrices (1) in terms of cotrees, making use of the determinantal expansions resulting from the Cauchy–Binet formula. In turn, the use of proper cotrees allows for the extension of these results to the augmented setting defined by (2), which arises in real applications and displays additional difficulties. We have also characterized the sum of cotree weights emanating from the Cauchy–Binet formula in terms of a single cotree. Additionally, these results have been shown to be closely related to Maxwell's determinantal expansions of nodal matrices.

These results are of interest in different modeling techniques for electrical circuits, e.g. in loop and branch-oriented analysis. Indeed, unique solvability properties and index one configurations have been addressed for the models resulting from loop analysis techniques, which are preferred to nodal analysis methods when the circuit devices are mostly current-controlled. We have shown as well how to characterize index one configurations of branch-oriented circuit models when they include a recently discovered device known as a *memristor*, of a potentially great interest in electrical and electronic engineering, under a charge-control assumption.

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