Exposed Faces of Dual Cones and Peak-Set Criteria for Function Spaces

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This paper gives conditions for a closed subcone of a weak* closed cone in a Banach dual space to be exposed by a weak* continuous linear functional. This set-up is applied to the study of complex-valued and vector-valued continuous function spaces on a compact Hausdorff space to deduce peak-set criteria. In the case of complex-valued function spaces peak-set conditions are given in terms of gages on certain cones of complex measures. These conditions are shown to imply various known peak-set criteria involving annihilating measures.

In this paper we represent various types of function spaces (real, complex, and Banach space valued) as partially ordered Banach spaces and use abstract ordered Banach space theorems to deduce peak-set criteria related to those in [2, 5, 7, 8].

In Section 1 we consider an ordered Banach space $E$ with cone $P$ and a second cone $Q \supset P$. Then if $P^*$ and $Q^*$ are the dual cones in $E^*$ we have $Q^*$ a weak* closed subcone of $P^*$. We give necessary and sufficient conditions for $Q^*$ to be a face of $P^*$ and for $Q^*$ to be a semiexposed face of $P^*$. These extend related results of Jameson [12] and Ellis [10]. From this we derive further conditions for $Q^*$ to be semiexposed in terms of certain gages on $P^*$. These conditions are satisfied if, for example, $P^*$ is decomposable at $Q^*$ where the notion of "decomposability" is an adaptation to cones of the concept for compact convex sets introduced in [4].

In Section 2 we consider spaces $M$ of functions from a compact Hausdorff space $X$ into a Banach space $E$. We show that a variety of hull conditions on a closed subset $F \subseteq X$ imply that an appropriate cone in $M^*$ is decomposable at a subcone related to $F$. From these we derive that $F$ is a generalized peak set with respect to $M$. For
example, let $E$ be a Hilbert space with an element $u$ of norm one and suppose $M$ contains the function identically $u$ and $M |_F$ is closed in $C(F, E)$. Then if we can choose from a bounded set in $M$ functions arbitrarily close to 0 on $F$ and satisfying $\Re\langle f(y), u \rangle \geq 1$ outside neighborhoods of $F$ then $F$ is a generalized peak-set with "peak value" $u$. This extends a more restrictive condition considered in [5, 9] where $E = C$ and is equivalent to the condition of [8, Lemma 2.1] if $E = C$ and $F$ is a point. This is also related to the $\frac{1}{4} - \frac{3}{4}$ theorem of Bishop for function algebras [11, 1.1].

In Section 3 we apply the results of Section 1 to certain cones of measures on $X$ to derive results when $E = C$. Specifically we consider cones of complex measures taking values in a nontrivial cone of $C$ symmetric about the positive reals.

We express our peak-set conditions in terms of gages on these measure cones and in this fashion obtain generalizations of results of Alfsen and Hirschberg [2] and Briem [7] which are expressed in terms of annihilating measures of $M$.

1.

Let $E$ be a Banach space with dual $E^*$. If $A \subset E(E^*)$ then the polar set, $A^0$, is the set of elements in $E^*(E)$ which are less than or equal to one on $A$. We denote $A_r = \{ a \in A : \| a \| \leq r \}$. If $P$ is a closed convex cone in $E$ then the set $P^*$ is the weak* closed convex dual cone $-P^0$ in $E^*$. If $P^*$ is a weak* closed cone in $E^*$ then the corresponding predual cone is $P = -(P^*)^0$ and by the bipolar theorem the two notions of $P^*$ are consistent. We shall use repeatedly the bipolar theorem and the various formulas of the "polar-calculus" [6].

In the following we shall assume that $(E, P)$ is an ordered Banach space with closed convex positive cone $P$ and that $Q$ is a closed convex cone containing $P$. The relation $\leq$ always refers to the $P$-ordering; i.e., $x \leq y$ if and only if $y - x \in P$. In this set-up $Q^*$ is a weak* closed subcone of $P^*$. Let $N$ be the weak* closed linear span of $Q^*(N = (Q^* - Q^*)^\perp)$ and let $M = Q \cap -Q$. Then $N$ and $M$ are mutually polar and $Q^*$ is said to be self-determined if $N \cap P^* = Q^*$. Thus, from the definition and the "polar-calculus" we have the following proposition.

1.1. Proposition. The subcone $Q^*$ is self-determined in $P^*$ if and only if $Q = (P - M)^\perp$ in $E$.

We say $Q^*$ is a face of $P^*$ if $0 \leq x \leq y$ and $y \in Q^*$ implies $x \in Q^*$. 
If \( Q^* \) is self-determined then \( Q^* \) is a face of \( P^* \) if and only if \( N \) is an order ideal. It follows from a result of Jameson [12, 3.11] that if \( Q^* \) is self-determined then \( Q^* \) is a face if and only if \( M \) is nearly directed.

A slight modification of this yields a necessary and sufficient condition for \( Q^* \) to be a (not necessarily self-determined) face of \( P^* \).

**Definition.** The cone \( Q \) is *approximately \( P \)-directed* if given \( x \in Q \) and \( \varepsilon > 0 \) there exist \( y \in Q \) and \( z, w \in E \), such that

\[
y + z \leq x \quad \text{and} \quad y + w \leq 0.
\]

1.2. **Theorem.** The subcone \( Q^* \) is a face of \( P^* \) if and only if \( Q \) is approximately \( P \)-directed.

**Proof.** \( Q^* \) is a face of \( P^* \) if and only if \( (Q^* - P^*) \cap P^* = Q^* \). Since \( Q^* - P_r^* \) is weak* closed, by taking polars we have that this is equivalent to

\[
Q \subseteq \{p + [Q \cap (E - P)^{-}]^{-} \}^{-} \quad \text{for all} \quad s > 0,
\]

which in turn is equivalent to \( Q \) being approximately directed.

**Definition.** The cone \( Q^* \) is *exposed* in \( P^* \) if there exists an \( x \in M \cap P \) for which \( p(x) > 0 \) for all \( p \in P^* \setminus Q^* \). We say \( Q^* \) is *semiexposed* if for each \( p \in P^* \setminus Q^* \) there exists \( x_p \in M \cap P \) with \( p(x_p) > 0 \).

If \( Q^* \) is a weak* \( G_\delta \) then \( Q^* \) is exposed if and only if it is semiexposed. This is always the case if, for example, \( E \) is separable.

**Definition.** The cone \( Q \) is *approximately \( MP \)-directed* if given \( x \in Q \) and \( \varepsilon > 0 \) there exist \( y \in M \) and \( z \in E \), such that

\[
y + z \leq x \quad \text{and} \quad y \leq 0.
\]

1.3. **Theorem.** The following are equivalent: (1) \( Q^* \) is semi-exposed in \( P^* \);

(2) \( Q^* = P^* \cap (N - P^*)^\circ \) (weak* closure);

(3) \( Q \) is approximately \( MP \)-directed.

**Proof.** \( Q^* \) is semiexposed in \( P^* \) if and only if

\[
Q^* = \{p \in P^* : p \equiv 0 \text{ on } P \cap M\} = \{p \in P^* : p \leq 0 \text{ on } P \cap M\} = \{p \in P^* : p \leq 1 \text{ on } P \cap M\} = P^* \cap (P \cap M)^\circ = P^* \cap (N - P^*)^\circ,
\]
so (1) and (2) are equivalent. By taking polars, (2) is equivalent to

(4) \( Q = (P - P \cap M)^- \),

which is equivalent to \( Q \) being approximately \( MP \)-directed.

1.4. **Corollary.** If \( Q^* \) is semiexposed then \( Q^* \) is a self-determined face.

1.5. **Corollary.** If \( Q^* \) is a self-determined face of \( P^* \) and \( N - P^* \) is weak* closed then \( Q^* \) is semiexposed.

**Proof.** Since \( Q^* = P^* \cap (Q^* - P^*) \) and \( Q^* = N \cap P^* \) we have

\[
Q^* = P^* \cap (Q^* - P^*) = P^* \cap (N \cap P^* - P^*) = P^* \cap (N - P^*)^-.
\]

1.6. **Corollary.** If \( Q^* \) is self-determined and \( M \) is directed \((M = M \cap P - M \cap P)\) then \( Q^* \) is a semiexposed face.

**Proof.**

\[
Q = (P - M)^- = (P + M \cap P - M \cap P)^- = (P - M \cap P)^-,
\]

and, hence, \( Q \) is approximately \( MP \)-directed.

We define for \( y \in E^* \)

\[
\| y \|_N = \inf \{ \| y - n \| : n \in N \}.
\]

For \( y \in P^* \) we define

\[
\| y \|_N^+ = \inf \{ \| y - n \| : n \in N \text{ and } n \leq y \}
\]

and

\[
\rho(y) = \inf \{ \| y - q \| : q \in Q^* \text{ and } q \leq y \}.
\]

Then for \( y \in P^* \) we have \( 0 \leq \| y \|_N \leq \| y \|_N^+ \leq \rho(y) \leq \| y \| \) and each of these functionals is subadditive, positive homogeneous and weak* lower-semicontinuous. Thus, the infimum in each of the formulas is always attained. We say two nonnegative functions, \( \varphi \) and \( \psi \), are equivalent if \( a \varphi \leq \psi \leq b \varphi \) for some positive numbers \( a \) and \( b \).

1.7. **Proposition.** If \( \| \cdot \|_N \) and \( \| \cdot \|_N^+ \) are equivalent on \( P^* \) then \( N - P^* \) is weak* closed.
Proof. By the Krein-Smulyan theorem it suffices to show 
\((N - P^*) \cap E_1^*\) is weak* closed. If \(e = n - p, \| e \| \leq 1, n \in N\) and 
\(p \in P^*\) then \(\| p \|_N \leq 1\), and, hence, \(\| p \|_N^+ \leq \alpha\) for some \(\alpha\). Thus,
\[
p' = p - n'; \quad p' \in P_\alpha^* \quad \text{and} \quad n' \in N.
\]
Thus,
\[
e - n - n' - p' \in N - P_\alpha^* \quad \text{and so} \quad (N - P^*) \cap E_1^* = (N - P_\alpha^*) \cap E_1^*,
\]
and the latter set is weak* closed.

1.8. Corollary. If \(Q^*\) is a self-determined face of \(P^*\) for which 
\(\| \cdot \|_N\) and \(\| \cdot \|_N^+\) are equivalent then \(Q^*\) is semiexposed.

If \(P\) is a normal cone then from the Grosberg-Krein theorem [13]
there is a \(K > 0\) such that each \(x \in E^*\) can be written as 
\(x = p_1 - p_2; \quad p_i \in P^*\) and \(\| p_1 \| + \| p_2 \| \leq K \| x \|\). Thus, \(co(P_1^* \cup -P_1^*)\) absorbs 
\(E_1^*\) so that \(E\) is order isomorphic to \(A_0(P_1^*)\) (all weak* continuous 
affine functions on \(P_1^*\) vanishing at 0) with the sup-norm. Also \(M\) 
is order isomorphic to \(A_0(qP_1^*)\) where \(q : E^* \to E^*/N\) is the quotient 
map.

1.9. Proposition. Let \(p\) be the Minkowski functional of \(qP_1^*\) in 
\(E^*/N\). Then \(\| \cdot \|_N^+ = p \circ q\) on \(P^*\).

Proof. If \(y \in P^*\) then 
\[
p(qy) = \inf\{r : q(y) \in rp(P_1^*)\} = \inf\{r : y \in P_r^* + N\} = \| y \|_N^+.
\]

We will say a functional \(\varphi\) is \(\alpha\)-additive on \(P^*\) if 
\(\varphi(a) + \varphi(b) \leq \alpha \varphi(a + b)\) and \(\varphi\) is totally \(\alpha\)-additive if 
\(\sum_{i=1}^{n} \varphi(a_i) \leq \alpha \varphi(\sum_{i=1}^{n} a_i)\) for 
any positive integer \(n\).

We can now use Proposition 1.9 and the preceding remarks to 
deduce the following result from [3, Proposition 2.4].

1.10. Theorem. If \(P\) is a normal cone, \(Q^*\) a self-determined 
subcone of \(P^*\) and \(\| \cdot \|_N^+\) is \(\alpha\)-additive on \(P^*\) then \(Q^*\) is semiexposed.

Proof. Since \(\| \cdot \|_N^+\) is \(\alpha\)-additive it follows that, in the terminology 
of [3], \(qP_1^*\) is \((\alpha, 2)\)-additive at 0. Thus, \(A_0(qP_1^*)\) is directed, and, 
then, \((M, M \cap P)\) is directed so that \(Q^*\) is semiexposed by Corollary 
1.6.

We consider now the functional \(p_\alpha\) on \(P^*\). Clearly \(p_\alpha^{-1}(0) = Q^*\).
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If \( p_\alpha(y) = \| y - x \|, \ y \in Q^* \) and \( x \leq y \) then \( p_\alpha(y - x) = \| y - x \| \) so that

\[ y = (y - x) + x \in C + Q^* \quad \text{where} \quad C = \{ z \in P^* : p_\alpha(z) = \| z \| \}.

1.11. Proposition. If \( \{ p \in P^* : \| p \|_N^+ = 0 \} \subset p_\alpha^{-1}(0) \) then \( Q^* \) is self-determining.

Proof.

\[ \{ p \in P^* : \| p \|_N^+ = 0 \} = N \cap P^*. \]

The next result is proved in [6, Theorem 3.2].

1.12. Theorem. If \( p_\alpha \) and \( \| \cdot \|_N \) are equivalent on \( P^* \) then each relatively weak* continuous linear functional on \( N \), nonnegative on \( Q^* \), has a weak* continuous extension contained in \( P \).

Definition. We shall say \( P^* \) is decomposable at \( Q^* \) if there is an \( h \in E^{**} \) such that \( N \subset h^{-1}(0) \) and \( h \) is equivalent to \( p_\alpha \) on \( P^* \). Since \( P^* = Q^* + C \), this is equivalent to having an \( h \in E^{**} \) with \( N \subset h^{-1}(0) \) and \( P^* = Q^* + K \) where \( K = \{ z \in P^* : h(z) \geq \| z \| \} \).

1.13. Theorem. Let \( P \) be a normal cone. The following are equivalent:

(1) \( P^* \) is decomposable at \( Q^* \);

(2) \( p_\alpha \), \( \| \cdot \|_N^+ \) and \( \| \cdot \|_N \) are equivalent and each is totally \( \alpha \)-additive on \( P^* \) for some \( \alpha > 0 \),

(3) \( \| \cdot \|_N^+ \) is totally \( \alpha \)-additive and equivalent to \( p_\alpha \) on \( P^* \).

Proof. If (1) holds then \( N \subset h^{-1}(0) \) and \( p_\alpha \leq ah \) on \( P^* \) so that for \( y \in P^* \), \( p_\alpha(y) \leq ah(y) = ah(y - n) \leq a \| h \| \| y - n \| \) for any \( n \in N \). Hence, \( p_\alpha \leq a \| h \| \cdot \| \cdot \|_N \), and all three are equivalent. Since \( h \) is linear each is totally \( \alpha \)-additive for some \( \alpha \). If (3) holds it follows from Proposition 1.9 and [3, Proposition 2.3] that there is an \( h_0 \) linear on \( E^*/N \) and equivalent to the quotient norm on \( P^* \). Since \( P \) is normal this means \( h_0 \) is bounded on \( E^*/N \) and hence \( h = h_0 \circ q \) is a bounded linear functional on \( E^* \), vanishing on \( N \), and equivalent to \( \| \cdot \|_N^+ \) and, hence, \( p_\alpha \) on \( P^* \).

1.14. Corollary. If \( P \) is normal and \( P^* \) is decomposable at \( Q^* \) then \( Q^* \) is a semiexposed face and has the monotone extension property of Theorem 1.12.

We note that if \( Q^* - Q^* \) is already weak* closed then \( P^* \) is
decomposable at $Q^*$ if and only if there is an $h \in E^{**}$ equivalent to $p_\mathcal{O}$ on $P^*$. Necessary and sufficient conditions for this to happen are well known in the case where $P^*$ has a weak* compact base (cf. Alfsen [1, II.5.5 and II.5.9]). In general if $X$ is weak* compact and convex in $E^*$ and $N = \text{lin}(X)$ then $N = \bigcup_{n=1}^{\infty} nB_X$ where $B_X = \text{co}(X \cup -X)$. Define $\| \cdot \|_1$ on $N$ with unit ball $B_X$ so that

$$\| n \|_1 = \inf \{ a_1 + a_2; n = a_1 x_1 - a_2 x_2 ; x_i \in X, a_i \geq 0 \}.$$ 

Let $\theta : E \rightarrow C(X)$ be the restriction map and let $M = \theta E$ with uniform closure $\tilde{M}$ in $C(X)$. Define $\Phi : (N, \| \cdot \|_1) \rightarrow M^* - \tilde{M}^*$ by $\Phi(n)(\theta f) = n(f)$. Then $\Phi$ is an isometry and 

$$\theta^* \circ \Phi = I : (N, \| \cdot \|_1) \rightarrow N \subset E^*.$$ 

Theorem 1.15 now follows in the same manner as [3, Theorem 3.1].

1.15. Theorem. The following are equivalent: (1) $N = \text{lin}(X)$ is weak* closed in $E^*$; 

(2) $N$ is norm closed; 

(3) $\| \cdot \|$ and $\| \cdot \|_1$ are equivalent on $N$; 

(4) $E \mid_X$ is uniformly closed in $C(X)$.

If $X = Q_1^*$ then $\text{lin}(Q_1^*) = Q^* - Q^*$ is weak* closed if and only if $E \mid_{Q_1^*}$ is uniformly closed in and, hence, equal to $A_0(Q_1^*)$.

If $X$ is a compact convex subset of a locally convex space let $E = A(X)$, the space of continuous affine functions on $X$ with $P$ the cone of nonnegative functions in $E$. Let $F$ be a closed convex subset of $X$ and let $Q$ be the cone of functions in $E$ nonnegative on $F$. Then $P^*$ has the canonical embedding of $X$ in $A(X)^*$ as a weak* compact base and similarly $Q^* = \bigcup_{r \geq 0} rF$. Clearly $F$ is semiexposed in $X$ if and only if $Q^*$ is semiexposed in $P^*$. Thus, from Theorem 1.3 we have the following.

1.16. Theorem. The set $F$ is semiexposed in $X$ if and only if for each $f \in A(X)$, $f \mid_F \geq 0$ and $\epsilon > 0$ there exists $g \in A(X)$ with $g \equiv 0$ on $F$ and $g \leq 0 \land (f + \epsilon)$.

This extends a result of Ellis [10] which gives a similar necessary and sufficient condition for a self-determined face to be semiexposed.

In [4] we define $X$ to be decomposable at $F$ if there is an $h \in A(X)^{**}$ with $h \equiv 0$ on the weak* closed linear span of $F$ and $X = \text{co}(K \cup F)$, $K = \{ x \in X : h(x) \geq 1 \}$. 


1.17. **Theorem.** The set $X$ is decomposable at $F$ if and only if $P^*$ is decomposable at $Q^*$ in $A(X)^*$.

**Proof.** Let $P^*$ be decomposable at $Q^*$ under $h$ where $p_Q \ll h \ll \alpha p_Q$. We have $P^* = C + Q^*$ where $C = \{z \in P^*: p_0(z) = \|z\|\}$ so that

$$X = \text{co}((X \cap C) \cup F) \quad \text{and} \quad h \geq 1 \quad \text{on} \quad X \cap C.$$  

Conversely, if $X$ decomposable at $F$ under $h$ then $y \in X$ implies

$$y = \lambda z + (1 - \lambda)x; \quad z \in \{z' \in X: h(z') \geq 1\} \quad \text{and} \quad x \in F.$$  

Let $K = \{z: h(z) \geq \|z\|\}$. Then $K = \bigcup_{r \geq 0} r\{z' \in X: h(z') \geq 1\}$ so that $P^* = Q^* + K$.

2.

Let $X$ be a compact Hausdorff space and $F$ a closed subset. Let $E$ be a Banach space (real or complex) and let $M$ be a uniformly closed subspace of continuous functions from $X$ to $E$. We say $F$ is an **EM-peak-set** if there is an $f \in M$ such that

$$f|_F = \ast, \quad \|u\| = 1, \quad \text{and} \quad \|f(x)\| < 1 \quad \text{for all} \quad x \in X \setminus F.$$  

We will say $F$ is a **generalized EM-peak-set** if it is an intersection of peak-sets.

We consider first the case where $E$ is real and ordered by a closed convex normal cone $P$ with $\text{int} P \neq \emptyset$, say $e + E_t \subseteq P (t > 0)$. Thus, if

$$S_e = \{s \in E^* : s \geq 0 \text{ on } P \text{ and } s(e) = 1\}$$

then $S_e$ is a weak* compact convex base for $P^*$ and $E$ is order isomorphic to $A(S_e)$ under the map $\theta x(s) = s(x)$. Let

$$P = \{f \in M : f(X) \subseteq P\}.$$  

Then $P$ is a closed convex normal cone in $M$. Define $\Phi: X \times S_e \rightarrow M^*$ by $\Phi(x, s)(f) = s(f(x))$. Then $\Phi$ is a continuous map of $X \times S_e$ into $M^*$ with the weak* topology. Let

$$Z = \text{co}(\Phi(X \times S_e)) (\text{weak* closure})$$

so that $Z$ is a weak* compact convex set in $M^*$.

2.1. **Proposition.** The dual cone $P^*$ in $M^*$ equals $(\bigcup_{r \geq 0} rZ)^\ast$ (weak* closure).
Proof. Let $Q^* = (\bigcup_{r \geq 0} rZ)^-$ with predual cone $Q$. Since $\Phi(x, s) \in \mathcal{P}^*$ for all $(x, s) \in X \times S_c$, $Q^* \subset \mathcal{P}^*$. If $f \in Q$ then $\Phi(x, s)(f) \geq 0$ for all $s \in S_c$ so that $f(x) \in P$ for all $x \in X$, and, hence, $Q \subset \mathcal{P}$. Thus, $\mathcal{P}^* \subset Q^*$.

If $F$ is closed in $X$ let $M_F = \{f \in M : f \equiv 0 \text{ on } F\}$. We say $F$ is MP-exposed if there is an $f \in M_F \cap \mathcal{P}$ with $f(X\backslash F) \subset \text{int } P$. Say $F$ is semiexposed if it is the intersection of exposed sets.

Define $F$ to be a strong MP-hull if there is a $K > 0$ such that for each open $V \supset F$ there is an $f_V \in M_F$ with $\|f_V\| \leq K$ and for each $y \in X\backslash V$

$$f_V(y) \geq \epsilon \quad (\epsilon \text{ a fixed point in int } P).$$

We say $F$ is a weak MP-hull if there is a $K > 0$ such that for each open $V \supset F$ and $\epsilon > 0$ there is an $f_{V, \epsilon} \in M$ with $\|f_{V, \epsilon}\| \leq K$, $\|f_{V, \epsilon}(x)\| \leq \epsilon$ for all $x \in F$ and $f_{V, \epsilon}(y) \geq \epsilon$ for all $y \in X\backslash V$.

We say $F$ is an interpolation set if $M|_F$ is uniformly closed in $C(F, E)$. Note that this differs from some authors' usage in that we do not require that $M|_F$ to be all of $C(F, E)$.

2.2. Proposition. If $\hat{F} = \overline{\text{co }} \Phi(F \times S_c)$ then $F$ is an interpolation set if and only if $\text{lin}(\hat{F})$ is weak* closed in $M^*$.

Proof. Since $E$ is isomorphic to $A(S_c)$ the space $M|_F \subset C(F, E)$ is isomorphic to $M|_{\Phi(F \times S)} \subset C(F \times S_c)$ which is isomorphic to $M|_F$ since $\text{ext } \hat{F} \subset \Phi(F \times S_c)$. Thus, the proposition follows from Theorem 1.15.

2.3. Theorem. Let $\hat{F} = \text{co } \Phi(F \times S_c)$. If $F$ is a strong MP-hull then (1) $F \times S_c = \Phi^{-1}(\hat{F})$,

(2) $\hat{F}$ is a decomposable face of $Z$.

Proof. Consider $\{h_V\}_{F \subset V}$ as a net in the ball of radius $h$ in $M^{**}$ and let $h$ be a weak* limit point. Since $h_V \in M_F$ we have $h \equiv 0$ on $M_F^0$ in $M^*$ which is a weak* closed subspace containing $\hat{F}$. We show next that

$$Z = \text{co } (K \cup \hat{F}) \quad \text{where } K = \{z \in Z : h(z) \geq 1\}.$$

Let $z \in Z$ and let $\mu$ be a probability measure on $Z$ which represents $z$ with respect to $M$ [14; 5, Theorem 3] and supp $\mu \subset (\text{ext } Z)^{-\subset} \Phi(X \times S_c)$. Assume first $\mu(\Phi(F \times S_c)) = 0$ and given $\epsilon > 0$ let $W$ be a neighbourhood of $\Phi(F \times S_c)$ in $Z$ with $\mu(W) < \epsilon$. By a standard
compactness argument there is an open $V_0 \supset F$ in $X$ such that $V_0 \times S_e \subset \Phi^{-1}(W)$. Let $F \subset V \subset V_0$, $V$ open.

(*) If $\Phi(x, s) \in \Phi(X \times S_e)\setminus W$ then $x \in X \setminus V_0$ and so $h_\nu(x) \geq \varepsilon$, and, hence, $\Phi(x, s)(h_\nu) \geq s(e) = 1$ for all $s \in S_e$. Thus

$$h_\nu(z) = \int_{\Phi(X \times S_e)} h_\nu \, d\mu \geq \int_W h_\nu \, d\mu + (1 - \mu(W)) \geq 1 - (k + 1) \varepsilon,$$

Since $\varepsilon$ is arbitrary this shows $h(z) \geq 1$. Now given $y \in Z$ and $\mu$ representing $y$ with supp $\mu \subset \Phi(X \times S_e)$, by considering the restriction measures of $\mu$ to $\Phi(F \times S_e)$ and its complement, $\mu$ can be written as a convex combination of probability measures $\mu_1$ and $\mu_2$ with supp $\mu_1 \subset \Phi(F \times S_e)$ and $\mu_2(\Phi(F \times S_e)) = 0$. Thus $y$ is a convex combination of some $z$ and $x$ with $x \in \Phi(F \times S_e) = F$ and $h(z) \geq 1$. Clearly $h$ can be carried to a bounded linear functional on $A(Z)^*$ which decomposes $Z$ at $\hat{F}$. This proves (2). Statement (1) follows since (*) implies that for $x \in X \setminus F$ and $s \in S_e$, $h(\Phi(x, s)) \geq 1$.

2.4. Corollary. If $F$ is a strong MP-hull then $F$ is semi-MP-exposed.

Proof. If we take $M = M_F$ then $\Phi(F \times S_e) = \{0\}$ and Theorem 2.3 shows $Z$ is decomposable at $\{0\}$ by a function $h \in M^{**}$. Since $M$ is isomorphic to $A_0(Z)$ we have from [3, Proposition 2.4] that $P' = \bigcup_{n=1}^\infty nZ$ is weak* closed and, hence, equal to $P^*$. Furthermore $h$ is equivalent to $\| \cdot \|$ on $P^*$ so that the cone $P^*$ is decomposable at $\{0\}$. Thus, $\{0\}$ is semiexposed and by Theorem 2.3(1) $\Phi((X \setminus F) \times S_e)$ is disjoint from $\{0\}$. Thus for each $y \in X \setminus F$, $\Phi(\{y\} \times S_e)$ is a compact subset of $P^* \setminus \{0\}$, and, hence, there is an $h_y \in M_F \cap P$ with $h_y(\Phi(y, s)) = s(h_y(y)) > 0$ for all $s \in S_e$. Thus, $h_y(y) \in \text{int } P$.

Let $\tilde{e}$ denote the function identically equal to $e$ on $X$.

2.5. Corollary. If $\tilde{e} \in M$ and $F$ is a strong MP-hull then $\mathcal{Q}^*$ is a decomposable subcone of $P^*$ where $\mathcal{Q}^* = \bigcup_{r \geq 0} r\hat{F}$.

Proof. If $\tilde{e} \in M$ then $Z$ is a compact base for $P^*$ and $M$ is isomorphic to $A(Z)$. Thus, $\hat{F}$ decomposable in $Z$ implies by Theorem 1.17 that $\mathcal{Q}^*$ is a decomposable subcone of $P^*$.

2.6. Corollary. If $\tilde{e} \in M$ and $F$ is a weak MP-hull and interpolation set then $P^*$ is decomposable at $\mathcal{Q}^*$ and $F$ is semi-MP-exposed.
Proof. Consider \( \{ h_{v,\epsilon} : F \subseteq V \text{ and } \epsilon > 0 \} \) as a net in \( M^{**} \). Then the same proof as Theorem 2.3 shows that a weak* limit point \( h \) in \( M^{**} \) is identically 0 on \( \bar{F} \) and \( Z = \text{co}(F \cup \{ z \in Z : h(z) \geq 1 \}) \). By Proposition 2.2 this shows that \( Z \) is decomposable at \( \bar{F} \), and, therefore, \( P^{**} \) is decomposable at \( \bar{Q}^{*} \). Thus, as in Corollary 2.4 for each \( y \in X \setminus F \) there is an \( h_{y} \in \bar{P} \), \( h_{y} \equiv 0 \) on \( \bar{F} \) and \( h_{y} > 0 \) on \( \Phi(\{y\} \times S_{v}) \). Thus,

\[
h_{y} \in M_{F} \cap \bar{P} \quad \text{and} \quad h_{y}(y) \in \text{int} \, P.
\]

In certain cases semiexposed sets are generalized peak-sets. Let \( \| e \| = 1 \) in \( E \). We will say \( P \) is internal at \( e \) if there is a constant \( C > 0 \) such that \( 0 \neq x \in P \) and \( \| x \| \leq C \) implies \( \| e - x \| < 1 \).

2.7. Proposition. If \( \bar{e} \in M \), \( F \) semi-MP-exposed and \( P \) internal at \( e \) then \( F \) is a generalized EM-peak-set.

Proof. If \( y \in X \setminus F \) and \( h_{y} \in M_{F} \cap \bar{P} \) with \( h_{y}(y) \in \text{int} \, P \) then let

\[
f_{y} = \bar{e} - (C/\| h_{y} \|) h_{y}.
\]

As an example let \( E \) be any Banach space (real or complex) and let \( 0 < r < 1 \). Fix some \( u \) with \( \| u \| = 1 \) and define

\[
P_{r} = \bigcup_{s \geq 0} s(u + E_{r}).
\]

2.8. Proposition. (1) \( P_{r} \) is a closed convex cone,

(2) \( u \in \text{int} \, P_{r} \),

(3) \( P_{r} \) is normal,

(4) \( P_{r} \) is internal at \( u \).

Proof. Clearly \( P_{r} \) is a convex cone. If \( z_{n} \in P_{r} \) and \( z_{n} \to z \) we have \( z_{n} = s_{n}(u + y_{n}) \); \( s_{n} \geq 0 \), \( \| y_{n} \| \leq r \). Then \( s_{n} = \| z_{n} \|/\| u + y_{n} \| \) so that \( \| z_{n} \|/(1 + r) \leq s_{n} \leq \| z_{n} \|/(1 - r) \). If \( z \neq 0 \) then by going to a subsequence we can assume \( s_{n} \to s, 0 < s < \infty \). Thus, \( y_{n} \to y = (z - su)/s \), and, hence, \( z = s(u + y) \in P_{r} \). Obviously \( u \in \text{int} \, P_{r} \). Since \( \text{int} \, E_{(1-r)} \cap (u + E_{r}) = \emptyset \), there is an \( f \in E^{*} \) with \( f \leq 1 \) on \( E_{(1-r)} \) and \( f \geq 1 \) on \( u + E_{r} \). Hence,

\[
\| z \|/(1 + r) \leq f(z) \leq \| z \|/(1 - r) \quad \text{for all } z \in P_{r}.
\]

Thus, if \( 0 \leq x \leq y \) then \( \| x \| \leq (1 + r)f(x) \leq (1 + r/1 - r) \| y \| \) so that \( P_{r} \) is normal. If \( 0 \neq z \in P_{r} \) and \( \| z \| \leq 1 - r \) then
\[ z = s(u + y), \quad s > 0 \quad \text{and} \quad \| y \| \leq r \quad \text{so that} \quad s \leq \| z \|(1 - r) \leq 1. \]

Since \[ \| (x/s) - u \| = \| y \| < r < 1, \] we have \[ z - u = s(x/s - u) + (1 - s)u, \] and, hence, \[ \| z - u \| \leq sr + (1 - s) < 1. \] Hence, (4) holds.

For a fixed \( u \in E \) with \( \| u \| = 1 \) we define \( F \) to be a strong \( ME_r \)-hull if \( \bar{u} \in M \) and there is a \( K > 0 \) such that for each open \( V \supset F \) there is \( h_v \in M_F \) with \( \| h_v \| \leq K \) and \( \| h_v(x) - u \| \leq r \) for all \( x \in X \setminus V \). Define \( F \) to be a weak \( ME_r \)-hull if for each open \( V \supset F \) and \( \epsilon > 0 \) there is \( h_v, \epsilon \in M \) with \( \| h_v, \epsilon \| \leq K, \quad \| h_v, \epsilon(x) \| \leq \epsilon \) on \( F \) and \( \| h_v, \epsilon(y) - u \| \leq \epsilon \) on \( X \setminus V \).

If \( E \) is a complex space we consider \( E \) and \( E^* \) in duality as real spaces so that if \( S_u = \{ s \in E^* : \Re s \geq 0 \} \) on \( P_r \) and \( \Re s(u) = 1 \) then \( E \) is real isomorphic to \( A(S_u) \) under the map \( \theta x(s) = \Re s(x) \). If \( \bar{u} \in M \) then \( M \) is isomorphic to \( A(Z) \) again under the map \( f \mapsto \Re f \). Since \( f_u(x) \to f(x) \) if and only if \( \Re s(f_u(x)) \to \Re s(f(x)) \) uniformly on \( S_u \) we still have \( F \) is an interpolation set if and only if the real linear span of \( \tilde{F} \) is weak* closed.

2.9. Proposition. If \( F \) is a strong (weak) \( ME_r \)-hull then \( F \) is a strong (weak) \( MP_{r'} \)-hull for any \( r'/r \).

Proof. Let \( s_0 = r' - r \). If \( \| x - u \| \leq r \) then \( x = u + y, \quad \| y \| \leq r \) so that \( x - s_0 u \in u + E_{r'} \), and, hence, \( (1/s_0)x \geq u \) with respect to the \( P_{r'} \) ordering. The conclusion now follows easily.

2.10. Theorem. If \( F \) is a strong \( ME_r \)-hull \((0 < r < 1)\) then \( F \) is a generalized peak set. If \( F \) is a weak \( ME_r \)-hull and an interpolation set then \( F \) is a generalized peak set.

Proof. By Proposition 2.9 and Corollaries 2.4 and 2.6 \( F \) is semi-\( MP_{r'} \)-exposed \((1 > r' > r)\) and, hence, a generalized peak-set since \( P_{r'} \) is internal at \( u \).

Now let \( E \) be a Hilbert space with \( \| u \| = 1 \). In this case we will say \( F \) is a strong \( ME \)-hull if \( \bar{u} \in M \) and there is a \( K > 0 \) such that for each open \( V \supset F \) there is an \( h_v \in M_F \) with \( \| h_v \| \leq K \) and \( \Re \langle h_v(y), u \rangle \geq 1 \) for all \( y \in X \setminus V \). We define weak \( ME \)-hull analogously.

2.11. Proposition. If \( F \) is a strong (weak) \( ME_r \)-hull then \( F \) is a strong (weak) \( ME_r \)-hull for some \( r < 1 \).

Proof. If \( x \in E \) and \( \| x \| \leq K, \Re \langle x, u \rangle \geq 1 \), then
\[
\| x/K^2 - u \|^2 = \langle x/K^2 - u, x/K^2 - u \rangle
= \langle x, x \rangle/K^4 - (2/K^2) \Re \langle x, u \rangle + 1 \leq 1 - 1/K^2.
\]

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Thus, if \( r = 1 - 1/K^2 \), then \( r < 1 \) and \( F \) is a strong (weak) \( ME_r \)-hull.

2.12. **Theorem.** If \( E \) is a Hilbert space and \( F \) is a strong \( ME \)-hull then \( F \) is a generalized peak set. If \( F \) is an interpolation set and a weak \( ME \)-hull then \( F \) is a generalized peak-set.

3.

We now consider the case where \( E \) is the complex numbers \( C \) and \( u = 1 \). We consider the collection of cones \( \{P_r\}_{r<1} \) but it is convenient to represent them in a different fashion. Thus, let \( 0 < \alpha_0 < 1 \) and let \( C_{\alpha} = \{ z \in C : \text{Re } z \geq \alpha_0 | z | \} \). Then \( C_{\alpha} \) is a closed proper subcone of \( C \) symmetric about \( R^+ \) with \( 1 \in \text{int } C_{\alpha} \). If \( z \in C_{\alpha} \) we will say \( z \geq 0(\alpha) \). Let \( \alpha \) and \( \tilde{\alpha} \) denote the end-points of the intersection of \( C_{\alpha} \) with \( \text{Re } z = \alpha_0 \) (say, \( \text{Im } \alpha > 0 \)) and let \( \beta_0 = (1 - \alpha_0^2)^{1/2} \). Then

\[
C_{\beta} = C_{\alpha}^* = \{ z \in C : \text{Re } \alpha z, \text{Re } \tilde{\alpha} z \geq 0 \}.
\]

If \( \beta, \tilde{\beta} \) are the end-points of \( \text{Re } z = \beta_0 \) intersected with \( C_{\beta} \) then (with \( \text{Im } \beta > 0 \)) \( \text{Re } \beta \alpha = \text{Re } \beta \tilde{\alpha} = 0 \) and \( \text{Re } \beta \alpha = \text{Re } \beta \tilde{\alpha} = \gamma > 0 \). If \( a \in C_{\alpha} \) then \( a = (\text{Re } \beta \alpha) x + (\text{Re } \beta \alpha) \tilde{\alpha} \) uniquely as \( r \alpha + s \tilde{\alpha} \) (\( r, s \geq 0 \)). Conversely, \( a = r \alpha + s \tilde{\alpha} \) with \( r, s \geq 0 \) implies \( a \in C_{\alpha} \).

If \( X \) is compact Hausdorff and \( f \) a continuous complex function on \( X \) we say \( f \geq 0(\alpha) \) if \( f(X) \subset C_{\alpha} \). If \( \mu \) is a (regular Borel) complex measure we say \( \mu \geq 0(\alpha) \) if \( \mu \) takes values in \( C_{\alpha} \).

3.1. **Proposition.** The measure \( \mu \geq 0(\alpha) \) if and only if \( \mu = \alpha \mu_1 + \tilde{\alpha} \mu_2 \) (uniquely), \( \mu_1, \mu_2 \) positive measures.

**Proof.** Clearly \( \mu_1, \mu_2 \) positive implies \( \alpha \mu_1 + \tilde{\alpha} \mu_2 \geq 0(\alpha) \). Conversely, let \( \mu_1 = \text{Re } (\beta \mu)/\tau, \mu_2 = \text{Re } \beta \mu/\tau \) where \( \beta_0 = (1 - \alpha_0^2)^{1/2} \). Then \( \mu_1, \mu_2 \) are positive and \( \mu = \alpha \mu_1 + \tilde{\alpha} \mu_2 \).

Let \( M \) be a closed subspace of \( C(X, C) \) containing the constants and separating points. Let \( P_0 \) be the cone of functions \( f \) in \( M \) such that \( f \geq 0(\beta) \) (\( 0 < \beta_0 < 1 \)). If \( I \) is the intersection of \( \text{Re } z = \alpha_0 \) with \( C_{\alpha}(\alpha_0 = (1 - \beta_0^2)^{1/2}) \) then \( I \) is the "state space" of \( (C, C_{\beta}) \), i.e., \( I = \{ z \in C : \text{Re } \beta z, \text{Re } \tilde{\beta} z \geq 0 \) and \( \text{Re } z = 1 \} \). Thus, the map \( \Phi \) of Section 2 takes \( X \times I \) onto \( I \cdot \varphi(X) \) where \( \varphi \) is the usual evaluation map into \( M^* \).

Thus,

\[
Z = co(\alpha S \cup \tilde{\alpha} S) \quad \text{where} \quad S = \{ s \in M^* : \| s \| = 1 = s(1) \}.
\]
If $F$ is closed in $X$ let $\tilde{F} = \overline{co} \varphi(F) \subset S$ and thus $\hat{F} = co(\alpha \tilde{F} \cup \tilde{\alpha} \tilde{F})$. We call the dual cone $P_\alpha^*$ in $M^*$ so that

$$P_\alpha^* = \{ \varphi \in M^*: \text{Re} \varphi(f) \geq 0 \text{ for all } f \in P_\alpha \}.$$ 

Then $P_\alpha^*$ has $Z$ as its base and $M$ is real isomorphic to $A(Z)$ under the map $f(z) \to \text{Re} z(f)$.

Considering $C$ as a Hilbert space we have that $F$ is a strong hull if the functions $\{h_\nu\}$ in $M_F$ can be chosen with $\text{Re} h_\nu \geq 1$ on $X \setminus V$ and the definition of weak hull is taken analogously. Thus, applying the results of Section 2 we have the following.

3.2. THEOREM. If $F$ is a strong hull or an interpolation set and weak hull then $\tilde{F}$ is a decomposable face of $Z$ and $F$ is a generalized peak-set.

We now give conditions for $F$ to be a generalized peak-set in terms of the measures on $X$. If $\mu$ is a regular Borel measure on $X$ we write $\|\mu\|$ for the total variation of $\mu$ and say $r(\mu) = w \in M^*$ where $w$ is the unique element in $M^*$ such that $w(f) = \int_X f \, d\mu$. We say $\mu$ is a boundary measure if $|\mu|$ is maximal in the usual sense [1, 14] and denote by $M^\perp$ the space of annihilating measures of $M$. We denote by $\partial X$ the Choquet boundary ($= \varphi^{-1}(\text{ext } S)$).

Theorem 3.2 is an extension of the results of [5] and generalizes the hull conditions of Curtis and Figa-Talamanca [9] and Curtis [8, Lemma 2.1].

3.3. PROPOSITION. (1) If $\mu$ is a measure $\geq 0(\alpha)$ then $\|\mu\| \geq \text{Re} \mu(X) \geq \alpha_0 \|\mu\|$.

(2) If $\mu \geq 0(\alpha)$ then $r(\mu) = w \in P_\alpha^*$ and $\|\mu\| \geq \|w\| \geq \alpha_0 \|\mu\|$.

(3) If $w \in P_\alpha^*$ then there is a boundary measure $\mu \geq 0(\alpha)$ such that $r(\mu) = w$.

(4) If $\varphi \in P_\alpha^*$ then $\|\varphi\| \geq \text{Re} \varphi(1) \geq \alpha_0 \|\varphi\|$.

(5) There is a constant $\alpha'$ such that for each $w \in M^*$ there is a $\mu = \mu_1 - \mu_2$ with $\mu_i$ boundary measures $\geq 0(\alpha)$, $\|\mu\| < \alpha' \|w\|$ and $r(\mu) = w$.

Proof. If $X$ is a disjoint union of measurable sets $F_i (i \leq n)$ then for $\mu \geq 0(\alpha)$

$$\alpha_0 \sum_{i=1}^{n} |\mu(F_i)| \leq \text{Re} \sum_{i=1}^{n} \mu(F_i) = \text{Re} \mu(X)$$

so (1) holds.
If \( \mu = \alpha \mu_1 + \tilde{\alpha} \mu_2 \) with \( \mu_1, \mu_2 \) positive then \( r(\mu_i) = r_i s_i \) \( (r_i \geq 0, s_i \in S) \) so that \( r(\mu) = w = r_1 \alpha s_1 + r_2 \tilde{\alpha} s_2 \in P_a^* \). Then

\[ \| \mu \| \geq \| w \| \geq \Re w(1) = \Re \mu(X) \geq \alpha_0 \| \mu \|. \]

If \( w \in P_a^* \) then \( w = r_1 \alpha s_1 + r_2 \tilde{\alpha} s_2 \) so that by the Choquet–Bishop–de Leeuw theorem \([1, 14]\) there are boundary probability measures \( r(\mu_i) = s_i \). Hence, \( \mu = r_1 \alpha \mu_1 + r_2 \tilde{\alpha} \mu_2 \) satisfies (3). (4) now follows from (3), (2), and (1).

Since \( M \) is real isomorphic to \( A(Z) \) there is a number \( K \) such that \( w = z_1 - z_2 ; z_i \in P_a^* \) and \( \| z_1 \| + \| z_2 \| \leq K \| w \| \). Thus, (5) holds with \( \alpha' = K/\alpha_0 \).

We define \( \| \mu \|_M = \inf\{\| \mu' \| : \mu' \in \mu + M^\perp \} \). If \( \text{supp } \mu \subset C \) then \( \| \mu \|_{M|_F} = \inf\{\| \mu' \| : \text{supp } \mu' \subset C \text{ and } \mu' \in \mu + M^\perp \} \). Since \( r(\mu) = w \) is the quotient map \( C(X)^*/M^\perp \to M^*, \| \mu \|_M = \| w \| \) where \( r(\mu) = w \). Similarly if \( \text{supp } \mu \subset C \) then \( \| \mu \|_{M|_F} = \| w \|, r(\mu) = w \in (M|_F)^* \). Let \( N = \text{real lin}(\tilde{F}) = \text{complex lin}(\tilde{F}) \). In the terminology of Theorem 1.15 \((N, \| \cdot \|_1) \) is isometric to \( A(Z)^* \) which is real isomorphic to \((M|_F)^* \). Thus, if \( n \in N \) and \( r(\mu) = n \) with \( \text{supp } \mu \subset C \) the norm \( \| n \|_2 = \| \mu \|_{M|_F} \) on \( N \) as the dual of \( M|_F \) is equivalent to \( \| \cdot \|_1 \). The next proposition now follows immediately from Theorem 1.15.

### 3.4. Proposition

The set \( F \) is an interpolation set if and only if \( \| \cdot \|_M \) and \( \| \cdot \|_{M|_F} \) are equivalent on the measures with support in \( F \).

We shall assume now that \( F \) is an interpolation set so that for each \( n \in N = \text{lin}(\tilde{F}) = (\text{lin}(\tilde{F}))^\perp \) there is a \( \mu \) on \( F \) with \( r(\mu) = n \). Also we have \( Q^* = \bigcup_{r_* \geq 0} r\tilde{F} \) as a weak* closed subcone of \( P_a^* \).

For \( \mu_0 \geq O(\alpha) \) on \( X \) define

\[
p_F(\mu_0) = \inf\{\| \mu \|_{X|F} : \mu \in \mu_0 + M^\perp \text{ and } \mu \geq O(\alpha)\}.
\]

\[
q_F(\mu_0) = \inf\{\| \mu \|_{X|F} : \mu \in \mu_0 + M^\perp \text{ and } \mu \mid_{X|F} \geq O(\alpha)\}.
\]

For any \( \mu_0 \)

\[
\tau_F(\mu_0) = \inf\{\| \mu \|_{X|F} : \mu \in \mu_0 + M^\perp\}
\]

### 3.5. Proposition

If \( \mu \geq O(\alpha) \) then

\[
\| r(\mu) \|_N^\perp \leq q_F(\mu) \leq (1/\alpha_0) \| r(\mu) \|_N^\perp,
\]

\[
p_O(r(\mu)) \leq p_F(\mu) \leq (1/\alpha_0) p_O(r(\mu)),
\]
and for any $\mu$

$$
\| r(\mu) \|_N \leq \tau_F(\mu) \leq \alpha \| r(\mu) \|_N.
$$

**Proof.** Given $\mu_0 \geq O(\alpha)$ and $\mu \in \mu_0 + M^\perp$ with $\mu |_{X \cdot F} \geq O(\alpha)$ write $\mu = \mu |_F + \mu |_{X \cdot F}$. Then $r(\mu_0) = r(\mu) = r(\mu |_F) + r(\mu |_{X \cdot F}) = n + z \in N + P_a^*$ so that $\| r(\mu_0) \|_N^\perp \leq \| z \| \leq \| \mu |_{X \cdot F} \|$, and, hence, $\| r(\mu_0) \|_N^\perp \leq q_F(\mu_0)$. If $r(\mu_0) = y = n + z \in N + P_a^*$, let $r(\mu_1) = n$ (supp $\mu_1 \subset F$) and $r(\mu_2) = z$ ($\mu_2 \geq O(\alpha)$). If $\mu = \mu_1 + \mu_2$ then $\mu \in \mu_0 + M^\perp$ and $\mu |_{X \cdot F} = \mu_2 |_{X \cdot F} \geq O(\alpha)$. Thus, $q_F(\mu_0) \leq \| \mu |_{X \cdot F} \| = \| \mu_2 |_{X \cdot F} \| \leq (1/\alpha_0) \| z \|$ and, hence, $q_F(\mu_0) \leq (1/\alpha_0) \| r(\mu_0) \|_N^\perp$. The second equivalence is proved similarly. The equivalence for $\tau_F$ follows as well using Proposition 3.3 (5).

We denote the $M$-convex hull of $F$ by $k(F)$ so that

$$
k(F) = \{ y \in X : | f(y) | \leq 1 \text{ whenever } \| f \|_F \leq 1 \text{ and } f \in M \}.
$$

Then [2, Proposition 5.1; 7, Lemma 1] $k(F) = \varphi^{-1}(\tilde{F})$ ($\varphi$ the evaluation map and $\tilde{F} = \partial \varphi(F)$). Let $h(F) = \{ y \in X : f(y) = 0 \text{ whenever } f \in M \text{ and } f \equiv 0 \text{ on } F \}$. Thus, $k(F) \subset h(F)$ and $h(F) = \varphi^{-1}(N)$.  

3.6. **Theorem.** Let $F$ be an interpolation set for $M$ and let $0 < \alpha_0 < 1$. (1) If $p_F(\mu) = 0$ whenever $q_F(\mu) = 0$ then $k(F) = h(F)$.

(2) If $p_F$ is equivalent to $\tau_F$ on the cone of measures $\geq O(\alpha)$ then each $f \in M |_F$ with $f \geq O(\beta)$ extends to a $g \in M$, $g \geq O(\beta)$.

(3) If $q_F$ is $\alpha'$-additive for some $\alpha'$ on the measures $\geq O(\alpha)$ then $h(F)$ is a generalized peak set.

(4) If $p_F$ and $q_F$ are equivalent and totally $\alpha'$-additive for some $\alpha'$ then $\tilde{F}$ is a decomposable face of $Z$ and $k(F)$ is a generalized peak-set with the $\beta$-positive extension property of (2).

**Proof.** (1) follows from Propositions 3.5 and 1.11 since $N \cap P_a^* = Q_\alpha^*$ implies $N \cap Z = \tilde{F}$, and, hence, $N \cap (\alpha S) = \tilde{F} \cap (\alpha S) = \tilde{F}$. Note that $h(x) \in C_\beta$ if and only if $\Re \alpha \varphi(x)(h) \geq 0$ and $\Re \tilde{\alpha} \varphi(x)(h) \geq 0$. Similarly $h(x) = 0$ if and only if $\Re \alpha h(x) = \Re \tilde{\alpha} h(x) = 0$ and $h(x) \in \text{int } C_\beta$ if and only if $\Re \alpha h(x) > 0$ and $\Re \tilde{\alpha} h(x) > 0$. Thus, (2) follows from Theorem 1.12. Statement (3) follows from Theorem 1.10 applied to the subcone $N \cap P_a^* \subset F$ since $y \in X | h(F)$ implies $\alpha \varphi(y)$, $\tilde{\alpha} \varphi(y) \in P_a^* \cap N$. Statement (4) follows from Theorem 1.13.

We show next how these results are connected with peak-set conditions due to Alfsen and Hirsberg [2] and Briem [7]. Let $F \subset \partial X$. 

Then Briem's condition says that there is a \( c, 0 \leq c < 1 \) such that for each boundary measure \( \mu \in M^\perp \) there is a \( \eta \in M^\perp \) with support in \( F \) with
\[
(P_0) \quad \| \mu \| F + \eta \| \leq c \| \mu \| _{X \setminus F} \|.
\]

If \( F = \{x\} \) the left side may be replaced with \( |\mu|_X \).

For any closed \( F \) in \( X \) we will say \( F \) satisfies \((P)\) if
\[
(P) \quad \text{for each } \mu \in M^\perp | \mu(F) | \leq c \| \mu \| _{X \setminus F} \| (c < 1).
\]

If \( F \subset \partial X \) then \( F \) satisfies \((P')\) if
\[
(P') \quad \text{for each boundary measure } \mu \in M^\perp \mu(F) \leq c \| \mu \| _{X \setminus F} \|.
\]

Since \( |\mu(F)| = |\mu|_E(1) = |(\mu \| F + \eta)(1)| \leq \| \mu \| _F + \eta \| \) for any \( \eta \in M^\perp \) \((P')\) is weaker than \((P_0)\).

3.7. THEOREM. If \( F \) is an interpolation set satisfying \((P)\) then for \( c < \alpha_0 < 1 \) \( p \) and \( q \) are equivalent and totally \( \alpha' \)-additive for some \( \alpha' \).

Proof. Given \( \mu_0 \geq O(\alpha) \) let \( \mu_1 \geq O(\alpha) \) and \( \mu_2 \in \mu_0 + M^\perp \). Take any \( \mu_2 \in \mu_1 \| X \setminus F + M^\perp \) and \( \mu_2 \| X \setminus F \geq O(\alpha) \). Let \( \mu = \mu_2 - \mu_1 \| X \setminus F \in M^\perp \).

Then
\[
\alpha_0 \| \mu \| _X \| \leq \| \mu \| _{X \setminus F} \| - \| \mu \| _{X \setminus F} \| - \| \mu_0(X \setminus F) \|
\leq \| \mu_2(X) - \mu_2(X \setminus F) \| = \| \mu_2(F) \| = \| \mu(F) \| \leq c \| \mu \| _{X \setminus F} \|
\leq c(\| \mu \| _X \| + \| \mu_2 \| _{X \setminus F} \|).
\]

Thus, \( \| \mu \| _X \| \leq (1 + c/\alpha_0 - c) \| \mu \| _{X \setminus F} \|. \) This shows \( p(F)(\mu_0) \leq (1 + c/\alpha_0 - c) q(F)(\mu_0) \).

From the definition of \( q \) we have \( q(F)(\nu) = q(F)(\nu \| X \setminus F) \) whenever \( \nu \in X \setminus F \). Therefore, \( q(F)(\mu(X \setminus F)) = q(F)(\mu_0) \) and so \( p \) and \( q \) are equivalent.

Let \( \mu_0 = \sum_{i=1}^n \mu_i \geq O(\alpha) \) and choose any \( \mu_i \in \mu_i + M^\perp \) with \( \mu_i \| X \setminus F \geq O(\alpha) \). Choose any \( \mu_i \in \sum \mu_i \| X \setminus F + M^\perp \) with \( \mu_i \| X \setminus F \geq O(\alpha) \) and let \( \mu = \mu_0 - \sum \mu_i \| X \setminus F \). Then
\[
\alpha_0 \sum \| \mu_i \| _X \| - \| \mu_0 \| _X \| \leq \sum \text{Re} \| \mu_i \| _X \| - \| \mu_0 \| _X \|
= \text{Re} \| \mu_0 \| _X \| - \| \mu_0 \| _X \| \leq | \mu_0(F) | 
= | \mu(F) | \leq c \left( | \mu_0 \| _X \| + \sum \| \mu_i \| _X \| \right).
\]
Thus, $\sum ||\mu'_i|_{X\setminus F}|| \leq (1 + c/\alpha_0 - c) ||\mu'_0|_{X\setminus F}||$ so that

$$\sum q_F(\mu_i) \leq \frac{1 + c}{\alpha_0 - c} q_F\left(\sum \mu'_i|_{X\setminus F}\right) = \frac{1 + c}{\alpha_0 - c} q_F(\mu_0).$$

If $F$ is an interpolation set and $F \subseteq \partial X$ then we can define $p'_F$, $q'_F$ and $\tau'_F$ at a measure $\mu_0 \geq O(\alpha)$ exactly as before except considering only boundary measures $\mu \in \mu_0 + M^\perp$. Then by Proposition 3.3 the new functionals will be equivalent with the originals. In this case Theorems 3.6 and 3.7 hold with the new functionals as well as the old. Also the condition $(P')$ implies the conclusion of Theorem 3.7 as well (same proof). Alfsen and Hirsberg [2] show that if $(P)$ holds with $c = 0$ and $F$ an interpolation set then $F$ is a generalized peak set. Finally, if $(P_0)$ holds then $F$ is already an interpolation set (Briem [7]). But $(P_0)$ is a considerably stronger condition than $(P)$ or $(P')$. For example if $X$ is a square in $\mathbb{R}^2$, $M = A(X)$ and $F$ two adjacent vertices then $(P_0)$ fails but $(P)$ or $(P')$ holds.

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**REFERENCES**