Factorization of complete ideals in normal birational extensions in dimension two

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Abstract

Let \((R, \mathfrak{m}_R)\) be a two-dimensional regular local ring and let \(O\) be a normal birational extension of \(R\). We relate the factorizations and semifactorization of complete \(\mathfrak{m}_O\)-primary ideals in \(O\) to the factorizations of some complete \(\mathfrak{m}_R\)-primary ideals in \(R\). To this aim and given a complete \(\mathfrak{m}_R\)-primary ideal \(I \subset R\), we show that we can associate to each complete ideal sheaf with finite cosupport on \(X = Bl_I(R)\) a complete ideal in \(R\) from which it can be recovered.

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1. Introduction

The theory of \(\mathfrak{m}_R\)-primary complete (integrally closed) ideals in a two-dimensional regular local ring \((R, \mathfrak{m}_R)\) was introduced by O. Zariski around 1938 as an algebraic translation of the classical theory of germs of plane curves developed by F. Enriques ([9], Book IV, vol. 2). One of the central points of the theory states that any complete ideal in \(R\) has a unique factorization into irreducible complete ideals (simple ideals) (see [23,24]). Since then, factorization of complete ideals has been studied by several authors, including Lipman [17–19] and Cutkosky [4–6]. In particular, if \((S, P)\) is a normal rational surface singularity, Lipman proves that there is unique...
factorization for complete ideals if and only if the completion of $\mathcal{O}_{S,P}$ is factorial (Theorem 20.1 of [17]). Despite this fact, the semigroup of complete ideals in a rational surface singularity is always semifactorial as shown in Corollary 1.6 of [3] (definition in Section 2).

This paper deals with the factorization properties of the complete ideals in two-dimensional normal local domains which birationally dominate a two-dimensional regular local ring $(R, m_R)$. By Proposition 1.2 of [17], such rings have rational singularities and by Corollary 1.2 of [15], they are not factorial unless they are regular. Normal surface singularities of this kind are known as sandwiched singularities and they are characterized as the singular points that can be obtained by blowing up a complete $m_R$-primary ideal in $R$ (see Remark 1.3 of [22]). They have been deeply studied by Spivakovsky [22] and van Straten [7], and also as a nice testing ground for the Nash Problem of arcs by Lejeune-Jalabert and Reguera in [16,21]. Relevant to our purposes will be the fact that the group of Weil divisors on the ring of a rational surface singularity $(X, Q)$ is finite (see §14 of [17]). In particular, it follows that for any Weil divisor $C$ on $(X, Q)$, there is an integer $n$ such that $nC$ is Cartier.

The framework of this paper is set up as follows. Let $I$ be a complete $m_R$-primary ideal in a regular local two-dimensional $\mathbb{C}$-algebra $R$ and let $X = Bl_I(S)$ be the surface obtained by blowing-up $I$. In Section 2, we review some definitions and facts concerning the theory of clusters of infinitely near points and the factoriality properties of complete ideals in rational surface singularities. In Section 3, we use the theory of infinitely near points and the geometry of the base points of complete ideals to associate to each complete ideal sheaf $J$ with finite cosupport in the exceptional locus of $X$ a complete $m_R$-primary ideal $H_J \subset R$, from which $J$ can be recovered. This fact induces an isomorphism between the semigroup of such ideal sheaves on $X$ and the semigroup of Cartier ideals (for $X$) in $R$ (see Definition 2.4). In Section 4 we use this result to relate the factorizations (and the semifactorization) of ideals in a birational normal extension $\mathcal{O}$ of $R$ to the factorizations of complete ideals in $R$. In particular, we see that factorizations of complete ideals of $\mathcal{O}$ into simple ideals induce and are induced in turn by factorizations of $H_J$ into irreducible Cartier ideals. In Section 5 we describe an algorithm to compute minimal systems of local generators of complete ideal sheaves on $X$ at the points of their cosupport.

2. Preliminaries

Throughout this work the base field is the field $\mathbb{C}$ of complex numbers. By a curve we mean an effective Weil divisor on a surface. $\delta_{i,j}$ stands for the Kronecker delta, i.e., $\delta_{i,j}$ equals 1 if $i = j$ and 0, otherwise.

2.1. Clusters of base points of complete ideals

The reader is referred to [2] for most of the material treated here. Let $S$ be a non-singular surface over $\mathbb{C}$ and $O \in S$. Write $(R, m_R)$ for the local two-dimensional ring $\mathcal{O}_{S,O}$. We say that a finite set $K$ of points infinitely near or equal to $O$ is a cluster if for any $p \in K$, $K$ contains all points preceding $p$. A weighted cluster is a pair $(K, v)$ obtained by assigning integral multiplicities $v = \{v_p\}$ to the points of $K$, $v_p$ being called the virtual multiplicity of $K$ at $p$. We write $p \rightarrow q$ if $p$ is proximate to $q$. If the excess of $K$ at $p$ is

$$\rho^K_p = v_p - \sum_{q \rightarrow p} v_q,$$
consistent clusters are clusters without negative excesses. They are characterized as those (weighted) clusters whose virtual multiplicities may be realized effectively by some curve on \( S \).

If \( \mathcal{K} \) is not consistent, \( \tilde{\mathcal{K}} \) is the cluster obtained from \( \mathcal{K} \) by unloading, i.e., \( \tilde{\mathcal{K}} \) is the unique consistent cluster which is equivalent to \( \mathcal{K} \) and has the same points than \( \mathcal{K} \) (see [2], §4.2 and §4.6). Strictly consistent clusters are consistent clusters having no points of virtual multiplicity zero. We write \( \mathcal{K}^+ = \{ p \in K \mid \rho p^{(K)} > 0 \} \) for the set of dicritical points of \( \mathcal{K} \).

The equations of all curves going through a weighted cluster \( \mathcal{K} \) define a complete \( mR \)-primary ideal \( H_{\mathcal{K}} \) in \( R \) (see [2], §8.3). Two weighted clusters \( \mathcal{K} \) and \( \mathcal{K}' \) are equivalent if \( H_{\mathcal{K}} = H_{\mathcal{K}'} \), and we write \( \mathcal{K} \prec \mathcal{K}' \) if \( H_{\mathcal{K}} \supset H_{\mathcal{K}'} \). Any complete \( mR \)-primary ideal \( J \) in \( R \) has a cluster of base points, denoted by \( BP(J) \), which consists of the points shared by the curves defined by generic elements of \( J \) taken with their multiplicities. Moreover, the maps \( J \mapsto BP(J) \) and \( \mathcal{K} \mapsto H_{\mathcal{K}} \) are reciprocal isomorphisms between the semigroup of complete ideals in \( R \) and the semigroup of strictly consistent clusters (see [2], 8.4.11, for details). If \( p \) is infinitely near or equal to \( O \), \( I_p \) is the simple ideal generated by the equations of the branches going through \( p \), and \( \mathcal{K}(p) \) is the weighted cluster corresponding to it by the above isomorphism. This way, if \( \mathcal{K} = BP(I) \),

\[
I = \prod_{p\in\mathcal{K}^+} I_p^{\rho_p} \tag{1}
\]

is the (Zariski) factorization of \( I \) into simple ideals (Theorem 8.4.13 of [2]). If \( \mathcal{K} = (K, \nu) \) is a consistent cluster, the colength of \( H_{\mathcal{K}} \) is

\[
l_R \left( \frac{R}{H_{\mathcal{K}}} \right) = \sum_{p \in \mathcal{K}} v_p (v_p + 1) \frac{2}{3}. \tag{2}
\]

If \( \pi_K : S_K \to S \) is the composition of the blowing-ups of all points in \( K \), write \( E_K \) for the exceptional divisor of \( \pi_K \) and \( \{ E_p \}_{p \in \mathcal{K}} \) for its irreducible components. Recall from Lemma 3.1 of [11], that if \( q \in E_K \) and \( \mathcal{K}_q \) denotes the cluster obtained by adding \( q \) with virtual multiplicity one to \( \mathcal{K} \), then \( H_{\mathcal{K}_q} \subset H_{\mathcal{K}} = I \) and \( l(I/H_{\mathcal{K}_q}) = 1 \). Moreover, every complete ideal \( J \subset I \) with \( l(I/J) = 1 \) has this form. A flag is a sequence of clusters

\[
T_0 < T_1 < \cdots < T_N, \quad \text{where } l \left( \frac{H_{T_i}}{H_{T_{i+1}}} \right) = 1 \quad \text{for } i = 0, \ldots, N - 1.
\]

If \( C \) is a curve on \( S \) and \( p \) is infinitely near or equal to \( O \), \( e_p(C) \) is the effective multiplicity of \( C \) at \( p \) and \( v_p(C) \) is the value of \( C \) relative to the divisorial valuation associated to \( E_p \). If \( \mathcal{K} \) is consistent, defined by generic elements of \( H_{\mathcal{K}} \) have effective multiplicities equal to the virtual ones and have no singularities outside \( K \): these curves are said to go sharply thought \( \mathcal{K} \). For each \( p \in \mathcal{K} \), \( v_p(C) = e_p(C) + \sum_{q \to p} v_q(C) \). If \( I \) is a complete \( mR \)-primary ideal, we write \( E_I^K = \sum_{p \in \mathcal{K}} v_p(I)E_p \) (or simply, \( E_I \) if no confusion may arise) which is the exceptional part of the total transform on \( S_K \) of curves defined by generic elements of \( I \).

Use \( | \cdot | \) as meaning intersection number and \( [\cdot \cdot \cdot]_p \) as intersection multiplicity at \( p \). We have the equality (projection formula) for \( \pi_K' : \left[ \pi_K' \cdot C \cdot D \right]_{S_K} = [C, (\pi_K)_* D]_O \), \( D \) being a curve on \( S_K \) without exceptional components. We write \( \mathcal{K}^2 = \sum_{p \in \mathcal{K}} v_p^2 \) for the self-intersection of \( \mathcal{K} \). If \( \mathcal{K} = BP(I) \), the projection formula applied to \( \pi_K' \) implies that

\[
\mathcal{K}^2 = \left| E_I \cdot E_I \right|_{S_K}. \tag{3}
\]
We will make use of the following result, which follows from the adjunction formula by direct computation.

**Lemma 2.1.** (See 6.2 of [12].) Let $K = (K, ν)$ be a consistent cluster, and write $\mathbb{K}_K$ for the canonical divisor on $S_K$. If $C$ is a curve on $S$ and $E_C$ is the exceptional part of its total transform on $S_K$, then

$$|E_C \cdot \mathbb{K}_K|_{S_K} = - \sum_{p \in K} e_p(C).$$

**2.2. Factoriality and semifactoriality of complete ideals**

Let $(X, Q)$ be a rational surface singularity. Fixed a desingularization $f : S' \to (X, Q)$, denote by $\mathbb{E}^+_S$ the (additive) semigroup of divisors $D$ on $S'$ with exceptional support for $f$ and such that $|D \cdot E| \leq 0$ for every irreducible exceptional component $E$ of $f$. Given a divisor $D \in \mathbb{E}^+_S$, the ideal $I_D \subset O_{X, Q}$ is the stalk at $Q$ of $f_*O_{S'}(-D)$. If $D = \sum_\alpha n_\alpha E_\alpha$, an element $g$ of $O_{X, Q}$ is in $I_D$ if and only if $v_\alpha(g) \geq n_\alpha$ for all $\alpha$, where $v_\alpha$ is the discrete valuation defined by $E_\alpha$. Thus, $I_D$ is defined by valuation inequalities and is complete. Moreover, Lipman showed that $O(-D)$ is generated by global sections and so, $I_DO_{S'} = O_{S'}(-D)$ (see Theorem 12.1 of [17]). Conversely, to each complete $m_Q$-primary ideal $I \subset O_{X, Q}$ such that $IO_{S'}$ is invertible we can associate the unique divisor $D_I \in \mathbb{E}_S^+$ such that $IO_{S'} = O_{S'}(-D_I)$. Since $I$ is complete, $I$ is the stalk at $Q$ of the sheaf of ideals $f_*O_{S'}(-D_I)$.

Thus, the maps $D \to I_D$ and $I \to D_I$ are reciprocal isomorphisms between $\mathbb{E}_S^+$ and the (multiplicative) semigroup $J^S_Q$ of complete $m_Q$-primary ideals $I$ such that $IO_{S'}$ is invertible ($J^S_Q$ is a semigroup because of Theorem 7.1 of [17]). We will write $J^S_Q$ for the semigroup of all complete $m_Q$-primary ideals of $O_{X, Q}$.

A well-known result due to Lipman establishes that the unique rational surface singularities for which unique factorization for complete ideals holds are the singularities of type $E_8$ and the nonsingular germs of surface (see Theorems 20.1 and 25.1 of [17]). Related to factorization, there is a weaker notion due to Göhner [13]: a commutative semigroup $(G, +)$ is **semifactorial** if every $g \in G$ can be formally expressed in a unique way as $g = \sum_{i=1}^r q_ig_i$ where the $q_i$ are positive rational numbers and the $g_i$ are extremal elements of $G$; an element $g \neq 0$ of $G$ is **extremal** if $g$ has no opposite in $G$ and if $ng = g_1 + g_2$ with $n \in \mathbb{N}$, $g_1, g_2 \in G$ then $\alpha_1g_1 = \beta_1g$ and $\alpha_2g_2 = \beta_2g$ for suitable positive integers $\alpha_i, \beta_i$, $i = 1, 2$. For rational surface singularities, the extremal elements of $\mathbb{E}_S^+$ are all multiple integers of the divisors $D_i = m_iD'_i$, where $D'_i \in \bigoplus_{i=1}^s \mathbb{Q}E_i$ is the unique $\mathbb{Q}$-Cartier divisor on $S'$ such that $|D'_i \cdot E_j|_{S'} = -\delta_{i,j}$ (Kronecker $\delta$) and $m_i$ is the smallest integer such that $m_iD'_i$ is a divisor (this is well defined because the intersection matrix $A^+_S = (|E_i \cdot E_j|_{S'})$ is negative-definite, see [20], §1, or [8]). From this, it follows that the semigroup $\mathbb{E}_S^+$ is semifactorial and also

**Theorem 2.2.** (See Corollary 1.6 of [3].) The semigroup $J^S_Q$ of $m_Q$-primary complete ideals of $O_{X, Q}$ is semifactorial: any ideal $I \in J^S_Q$ can be formally expressed in a unique way as $I = \prod_{i=1}^s I_{D_i}^{q_i}$ with $q_i \in \mathbb{Q}_+$. 


We will make use of the following formula for the colength of a complete \( m_Q \)-primary ideal \( I \subset \mathcal{O}_{X,Q} \), which is a direct consequence of the adjunction formula (see Proposition V.1.5 of [14]):

\[
I\left( \frac{\mathcal{O}_{X,Q}}{I} \right) = -\frac{1}{2} |D_I \cdot (D_I + \mathbb{K}_{S'})|_{S'}.
\]

(4)

where \( \mathbb{K}_{S'} \) is a canonical divisor on \( S' \).

2.3. Sandwiched surface singularities and birational extensions in dimension two

Sandwiched surface singularities are normal surface singularities that can be projected birationally into a non-singular surface \( S \). From a more algebraic point of view, they correspond to normal local domains which birationally dominate a two-dimensional regular local ring \( R \). Given such a domain \( O \), there is a complete \( m_R \)-primary ideal \( I \subset R \) such that \( O = R/I/a \), where \( a \) is a generic element of \( I \) and \( N \) is a height two maximal ideal of \( R/I/a \) (see [15]). Sandwiched singularities are rational singularities (Proposition 1.2 of [17]) and so, the semigroup of complete ideals in \( O \) is not factorial unless \( O \) is already regular (Corollary 1.2 of [15]). The reader is referred to [10,12,22] for proofs and some known facts about sandwiched singularities and complete ideals.

Let \( I \subset R = \mathcal{O}_{S,O} \) be a complete \( m_R \)-primary ideal, \( \mathcal{K} = BP(I) \) the cluster of base points of \( I \) and \( \pi : X = Bl_I(S) \to S \) the blowing-up of \( I \), so there is a commutative diagram

\[
\begin{array}{ccc}
S_K & \xrightarrow{f} & X \\
\downarrow{\pi_K} & & \downarrow{\pi} \\
S & & 
\end{array}
\]

where the morphism \( f \), given by the universal property of the blowing-up, is the minimal resolution of the singularities of \( X \) (see Remark I.1.4 of [22]). It is well known that there is a bijection between the set of simple ideals \( \{I_p\}_{p \in \mathcal{K}_+} \) in the (Zariski) factorization of \( I \) and the set of irreducible components of \( \pi^{-1}(O) \) (see Corollary I.1.5 of [22]). We write \( \{L_p\}_{p \in \mathcal{K}_+} \) for the set of these components. If \( J \subset R \) is a complete \( m_R \)-primary ideal, we write \( L_J = \sum_{q \in \mathcal{K}_+} v_q(J)L_q \) and if \( p \in \mathcal{K}_+ \), \( L_p = L_{I_p} \). By the projection formula applied to \( \pi \), we have

\[
|L_p \cdot L_q|_X = -\delta_{p,q}.
\]

(5)

Given a curve \( C \) on \( S \), we will write \( \tilde{C} \) and \( C^* = \tilde{C} + L_C \) for the strict and the total transform of \( C \) on \( X \), respectively. The following result will be crucial for us:

**Theorem 2.3.** (See Theorems 3.1 and 4.1 of [12].) If \( L_C = \sum_{p \in \mathcal{K}_+} a_p L_p \), then \( \tilde{C} \) is a Cartier divisor on \( X \) if and only if \( a_p \in \mathbb{Z} \) for each \( p \in \mathcal{K}_+ \). If this is the case, every dicritical base point \( p \) of the ideal \( \{g \in R \mid v_p(g) \geq v_p(H) \forall p \in \mathcal{K}_+ \} \) is already dicritical for \( I \), i.e., \( p \in \mathcal{K}_+ \). Moreover, we have that \( a_p = |\tilde{C} \cdot L_p|_X \geq 0 \).

This result inspires the following definition.
Definition 2.4. A complete \( m_R \)-primary ideal \( H \subset R \) is Cartier for \( X \) if \( L_H \in \bigoplus_{p \in K_+} \mathbb{Z} L_p \).

The following result is immediate from 2.3 and needs no proof.

Lemma 2.5. If \( H \subset R \) is a complete \( m_O \)-primary ideal, then \( L_H \in \bigoplus_{p \in K_+} \mathbb{Z} L_p \) and the base points of \( H^0 \) are contained in the cluster \( K \). Moreover, the following statements are equivalent:

(i) \( H \) is Cartier;
(ii) the strict transform on \( X \) of the curves defined by generic elements of \( H \) are Cartier divisors on \( X \);
(iii) the dicritical base points of \( H^0 \) are contained in \( K_+ \);
(iv) the sheaf \( H^0 \mathcal{O}_X \) is invertible.

In this case, \( H^0 \) is also Cartier and \( E_H = f^* (L_H) \).

The product of two Cartier ideals is again a Cartier ideal, so the set \( \text{CI}_R = \{ \text{Cartier ideals for } X \} \) with the natural product of ideals is a semigroup. The reader may notice that if \( H_1, H_2 \in \text{CI}_R \), then \( L_{H_1 H_2} = L_{H_1^0 H_2^0} \otimes \mathcal{O}_X \), and so,

\[
(H_1 H_2)^0_X = (H_1)^0_X (H_2)^0_X.
\] (6)

3. Complete ideal sheaves on a sandwiched surface

In this section, we assume that a complete \( m_R \)-primary ideal \( I \subset R \) is fixed. Keeping the notations as in the preceding section, we write \( K = (K, \nu) \) for the cluster of base points of \( I \) and \( \pi : X = Bl_I (S) \to S \) for the blowing up of \( I \). Since no confusion may arise, if \( H \subset R \) is a complete and \( m_R \)-primary ideal, we denote by \( H^0 \) the ideal \( H^0 \mathcal{O}_X \). We deal here with ideal sheaves \( J \) on \( X \) satisfying the following condition

\[
(\dagger) \quad J \text{ has finite cosupport, say } C(J) = \{ Q_1, \ldots, Q_n \}, \text{ contained in the exceptional locus of } X \text{ and for each } Q_i, \text{ the stalk } J_i = J_{Q_i} \text{ is a complete } m_{Q_i} \text{-primary ideal of } O_{X,Q_i} \text{ (by the cosupport of } J \text{ we mean the support of the sheaf } O_X J).\]

Notice that given \( J_1, J_2 \) as above, the ideal sheaf \( J_1 J_2 \) defined as the image of the natural map \( J_1 \otimes J_2 \to \mathcal{O}_X \) satisfies the condition (\( \dagger \)): its cosupport equals the union of the cosupports of \( J_1 \) and \( J_2 \), and the stalk of \( J_1 J_2 \) at any \( Q \in X \) is the product of ideals \( (J_1)_Q (J_2)_Q \subset O_{X,Q} \). We will denote by \( S_X \) the (multiplicative) semigroup of the ideal sheaves on \( X \) satisfying (\( \dagger \)).

Given an ideal sheaf \( J \) in \( S_X \), we will write

\[
\lambda(J) = \sum_{Q \in C(J)} l \left( \frac{O_{X,Q}}{J_Q} \right)
\]

for the colength of \( J \) on \( X \). Let \( f_J : S_J \to X \) be the minimal resolution of \( X \) such that the sheaf \( J \mathcal{O}_{S_J} \) is invertible (equivalently, \( S_J \) is the minimal resolution of the surface obtained by
blowing up $\mathcal{J}$ on $X$). Since $\pi_I \circ f_{\mathcal{J}} : S_{\mathcal{J}} \to S$ is a birational morphism between regular surfaces, there exists some cluster $K_{\mathcal{J}}$ (containing $K$) such that $S_{\mathcal{J}}$ is the surface obtained by blowing up all the points in $K_{\mathcal{J}}$:

\begin{equation}
\begin{array}{c}
S_{\mathcal{J}} \\
\downarrow f_{\mathcal{J}} \\
S_K \\
\downarrow \pi_K \\
X \end{array}
\end{equation}

Write $\{E_p\}_{p \in K_{\mathcal{J}}}$ for the irreducible components of the exceptional divisor $\pi_{K_{\mathcal{J}}}^{-1}(O)$ on $S_{\mathcal{J}}$. For any $Q \in X$, write $T_Q^{\mathcal{J}} = \{p \in K_{\mathcal{J}} \mid (f_{\mathcal{J}})_*(E_p) = Q\}$. Note that $T_Q^{\mathcal{J}} \neq \emptyset$ if and only if either $Q$ is a singularity of $X$ or $Q \in C(\mathcal{J})$. Note also that

\[ K_{\mathcal{J}} \setminus K_+ = \bigcup_{Q \in X} T_Q^{\mathcal{J}}. \]

We denote by $D_{\mathcal{J}}$ the (effective) exceptional divisor on $S_{\mathcal{J}}$ defined by $\mathcal{J}\mathcal{O}_{S_{\mathcal{J}}} = \mathcal{O}_{S_{\mathcal{J}}}(-D_{\mathcal{J}})$, and

\[ \mathcal{L}_{\mathcal{J}} = \sum_{p \in K_+} a_{\mathcal{J}}(p) \mathcal{L}_p \]

where $a_{\mathcal{J}}(p) = |D_{\mathcal{J}} \cdot E_p|_{S_{\mathcal{J}}} \in \mathbb{Z}_{\geq 0}$ for every $p \in K_+$. The following result will be fundamental for us.

**Theorem 3.1.** Let $\mathcal{J}$ be in $S_X$. The complete $m_R$-primary ideal $H_{\mathcal{J}} = \Gamma(X, \mathcal{J}\mathcal{O}_X(-\mathcal{L}_{\mathcal{J}}))$ in $R$ satisfies the following conditions:

(a) $H_{\mathcal{J}}$ is a Cartier ideal for $X$;
(b) $H_{\mathcal{J}}\mathcal{O}_X = \mathcal{J}\mathcal{O}_X(-\mathcal{L}_{\mathcal{J}})$, i.e., the sheaf $\mathcal{J}\mathcal{O}_X(-\mathcal{L}_{\mathcal{J}})$ is generated by global sections, and also $\mathcal{L}_{H_{\mathcal{J}}}^{\mathcal{J}} = \mathcal{L}_{H_{\mathcal{J}}} = \mathcal{L}_{\mathcal{J}}$;
(c) $l_R(H_{\mathcal{J}}^{\mathcal{J}}) = \lambda(\mathcal{J})$.

Moreover, $H_{\mathcal{J}}$ is the only complete ideal satisfying these conditions.

In order to prove 3.1, we establish a first partial result:

**Lemma 3.2.** The sheaf $\mathcal{J}\mathcal{O}_X(-\mathcal{L}_{\mathcal{J}})$ is generated by global sections. Moreover, if $H \subset R$ is a Cartier ideal for $X$ and $\mathcal{L}_H = \sum_{p \in K_+} b_p \mathcal{L}_p$, then $\mathcal{J}\mathcal{O}_X(-\mathcal{L}_H)$ is generated by global sections if and only if $b_p \geq a_{\mathcal{J}}(p)$ for all $p \in K_+$. In particular, $H_{\mathcal{J}}$ is a Cartier ideal for $X$.

**Proof.** First of all, from (5) and the definition of $\mathcal{L}_{\mathcal{J}}$, we have that $|E_p \cdot (D_{\mathcal{J}} + f_{\mathcal{J}}^*(\mathcal{L}_{\mathcal{J}}))|_{S_{\mathcal{J}}} = 0$ if $p \in K_+$, while $|E_p \cdot (D_{\mathcal{J}} + f_{\mathcal{J}}^*(\mathcal{L}_{\mathcal{J}}))|_{S_{\mathcal{J}}} = |E_p \cdot D_{\mathcal{J}} |_{S_{\mathcal{J}}} < 0$ if $p \in K_+ \setminus K_{\mathcal{J}}$. Thus,
\[ \mathcal{J} \mathcal{O}_{S_{\mathcal{J}}}(-f_{\mathcal{J}}^{*}(\mathcal{L}_{\mathcal{J}})) \] is generated by global sections (Theorem 12.1 of [17]) and so is \( \mathcal{J} \mathcal{O}_X(-\mathcal{L}_{\mathcal{J}}) \). This proves the first assertion.

Similarly, if \( b_p \geq a_{\mathcal{J}}(p) \) for all \( p \), then
\[
|E_p \cdot (D_{\mathcal{J}} + f_{\mathcal{J}}^{*}(\mathcal{L}_{H}))|_{S_{\mathcal{J}}} \leq |E_p \cdot (D_{\mathcal{J}} + f_{\mathcal{J}}^{*}(\mathcal{L}_{\mathcal{J}}))|_{S_{\mathcal{J}}} \leq 0
\]
and we conclude in the same way. Conversely, write \( H' = \Gamma(X, \mathcal{J} \mathcal{O}_X(-\mathcal{L}_{H})) \) and assume that \( \mathcal{J} \mathcal{O}_X(-\mathcal{L}_{H}) = H' \mathcal{O}_X \). Then, \( \mathcal{L}_{H'} = \mathcal{L}_H \) and if \( g \in H' \), the strict transform on \( X \) of the curve \( C : g = 0 \) is defined near each \( Q \) by an element of the stalk of \( \mathcal{J} \) at \( Q \). We infer that \( |\overline{C} \cdot L_p|_X \geq a_{\mathcal{J}}(p) \), for each \( p \in \mathcal{K}_+ \), and so \( \mathcal{L}_C \geq \mathcal{L}_{\mathcal{J}} \). By taking \( g \) to be a generic element of \( H' \), we have \( \mathcal{L}_C = \mathcal{L}_H \) and we are done. □

From now on, we will write \( \mathcal{K}^*_\mathcal{J} = (K, \sigma^0) \) for the cluster of base points of \( H^0_{\mathcal{J}} \), and \( \mathcal{K}_{\mathcal{J}} = (K_{\mathcal{J}}, \sigma) \) for the cluster of base points of \( H_{\mathcal{J}} \). Note that \( \mathcal{K}_{\mathcal{J}} \) has excess 0 at every point of \( \mathcal{K}_+ \).

**Lemma 3.3.**

(a) \( \mathcal{E}_{H_{\mathcal{J}}} = \mathcal{E}_{H^0_{\mathcal{J}}} + D_{\mathcal{J}} \).

(b) \( |D_{\mathcal{J}} \cdot D_{\mathcal{J}}|_{S_{\mathcal{J}}} = \mathcal{K}^2_{\mathcal{J}} - [\mathcal{K}_{\mathcal{J}}, \mathcal{K}^*_\mathcal{J}]_O \).

**Proof.** From 2.5 we infer that \( \mathcal{E}_{H^0_{\mathcal{J}}} = f_{\mathcal{J}}^{*}(\mathcal{L}_{\mathcal{J}}) \) and so, \( \mathcal{E}_{H_{\mathcal{J}}} = D_{\mathcal{J}} + \mathcal{E}_{H^0_{\mathcal{J}}} \) as claimed in (a). By this and (3), we have
\[
|D_{\mathcal{J}} \cdot D_{\mathcal{J}}|_{S_{\mathcal{J}}} = |\mathcal{E}_{H_{\mathcal{J}}} \cdot \mathcal{E}_{H_{\mathcal{J}}}|_{S_{\mathcal{J}}} - 2|\mathcal{E}_{H_{\mathcal{J}}} \cdot \mathcal{E}_{H^0_{\mathcal{J}}}|_{S_{\mathcal{J}}} + |\mathcal{E}_{H^0_{\mathcal{J}}} \cdot \mathcal{E}_{H^0_{\mathcal{J}}}|_{S_{\mathcal{J}}}
\]
\[
= \mathcal{K}_{\mathcal{J}}^2 - 2[\mathcal{K}_{\mathcal{J}}, \mathcal{K}_{\mathcal{J}}^0]_O + (\mathcal{K}^0_{\mathcal{J}})^2.
\]
Now, if \( C \) and \( C_o \) be curves going sharply through \( \mathcal{K}_{\mathcal{J}} \) and \( \mathcal{K}^*_\mathcal{J} \), respectively, then \( L_{C_o} = L_C \), and the strict transforms \( \overline{C} \) and \( \overline{C}_0 \) share no points on \( X \). By the projection formula applied to \( \pi \), we have \( [\mathcal{K}_{\mathcal{J}}, \mathcal{K}^0_{\mathcal{J}}]_O = [C, C_o]_O = |(\overline{C} + L_C) \cdot \overline{C}_0|_X = |L_C \cdot \overline{C}_0|_X \). Similarly, we obtain that \( (\mathcal{K}^0_{\mathcal{J}})^2 = |L_C \cdot \overline{C}_0|_X \). Therefore, \( [\mathcal{K}_{\mathcal{J}}, \mathcal{K}^0_{\mathcal{J}}]_O = (\mathcal{K}^0_{\mathcal{J}})^2 \). This proves the claim. □

**Proof of 3.1.** The assertions (a) and (b) follow directly from 3.2. The uniqueness of \( H_{\mathcal{J}} \) follows from the first equality of (b) by taking global sections. It only remains to prove (c). Write \( D_{\mathcal{J}} = \sum_{Q \in C(\mathcal{J})} D_{J_Q} \), where each \( D_{J_Q} \) is a connected divisor whose irreducible components are \( \{E_p\}_{p \in T^*_\mathcal{J}} \). According to 4 and using that the divisors \( D_{J_Q} \) do not intersect each other, we have
\[
\lambda(\mathcal{J}) = -\frac{1}{2} |D_{\mathcal{J}} \cdot (D_{\mathcal{J}} + \mathcal{K} S_{\mathcal{J}})|_{S_{\mathcal{J}}}.
\]
From (a) of 3.3 and 2.1, we have that
\[
|D_{\mathcal{J}} \cdot \mathcal{K} |_{S_{\mathcal{J}}} = |(\mathcal{E}_{H_{\mathcal{J}}} - \mathcal{E}_{H^0_{\mathcal{J}}}) \cdot \mathcal{K} |_{S_{\mathcal{J}}} = -\sum_{p \in \mathcal{K}_{\mathcal{J}}} \sigma_p + \sum_{p \in \mathcal{K}_{\mathcal{J}}} \sigma^0_p.
\]
Thus, from (b) of 3.3 and the Noether formula (Theorem 3.3.1 of [2]), we infer that
\[
\lambda(\mathcal{J}) = -\frac{1}{2} \left( \sum_p \sigma_p^2 - \sum_p \sigma_p \sigma_p^o - \sum_p \sigma_p + \sum_p \sigma_p^0 \right)
\]
\[
= -\frac{1}{2} \sum_p \sigma_p (\sigma_p - 1) + \frac{1}{2} \sum_p \sigma_p^0 (\sigma_p - 1)
\]
\[
= \ell_R \left( \frac{H_0^o}{H_{\mathcal{J}}} \right)
\]
the last equality by (2). This proves (c). ∎

**Remark 3.4.** Since \( H_{\mathcal{J}} \) is Cartier and \( \mathcal{L}_{H_0^o} = \mathcal{L}_{\mathcal{J}} \) (see 3.1), it follows that \( \rho_p^{K_0^o} = a_{\mathcal{J}}(p) \) if \( p \in K_+ \) and 0, otherwise. Then, (1) applies to give that
\[
H_0^o = \prod_{p \in K_+} I_{p}^{a_{\mathcal{J}}(p)} \subset R
\]
is the (Zariski) factorization of \( H_0^o \).

Let us quote a direct consequence of 3.1 for future reference.

**Corollary 3.5.** Let \( J' \) be a complete \( m_R \)-primary ideal such that \( H_{\mathcal{J}} \subset J' \subset H_0^o \). Then, the ideal sheaf \( J' = J' \mathcal{O}_X(\mathcal{L}_{\mathcal{J}}) \) satisfies (†) and its cosupport is contained in \( \mathcal{C}(\mathcal{J}) \). Moreover,
\[
\lambda(J') = \ell_R \left( \frac{H_0^o}{J'} \right).
\]

**Proof.** Since \( J' \) is complete, the ideal sheaf \( J' \) is so (see [17]). Since \( H_{\mathcal{J}} \subset J' \subset H_0^o \), we have that \( C(J') \subset C(\mathcal{J}) \) because \( J' \mathcal{O}_X(\mathcal{L}_{\mathcal{J}}) \rightarrow J' \mathcal{O}_X(\mathcal{L}_{\mathcal{J}}) \). Also, \( \mathcal{L}_{J'} = \mathcal{L}_{\mathcal{J}} \) and thus, \( H_0^o = H_{\mathcal{J}} \). The last equality follows from 3.1 applied to \( J' \).

To close this section, we show that there exists an isomorphism of semigroups between \( S_X \) and \( \text{CI}_R \). To this aim, we need to introduce a definition and some notation. If \( H \subset R \) is a Cartier ideal for \( X \) and \( C \) is a curve on \( S \) defined by some element of \( H \), the virtual transform of \( C \) relative to \( H \) on \( X \) is the effective divisor \( \hat{C}^H = \pi^*(C) - \mathcal{L}_H \). Observe that \( \hat{C}^H \) is always a Cartier divisor, since \( \mathcal{L}_H \) is so. Notice that the equations of such curves generate locally the sheaf \( H \mathcal{O}_X(\mathcal{L}_H) \).

For each \( m \geq 0 \), write
\[
\text{CI}_R = \left\{ \text{Cartier ideals } H \text{ for } X \mid \ell_R \left( \frac{H_0^o}{H} \right) = m \right\}
\]
and
\[
S_X^m = \left\{ \mathcal{J} \in S_X \mid \lambda(\mathcal{J}) = m \right\}.
\]
Theorem 3.6. Fixed $m \geq 0$, by associating to each ideal sheaf $\mathcal{J}$ in $S_X^m$ the complete $m_R$-primary ideal $H_{\mathcal{J}}$, we get a bijection between $S_X^m$ and $\text{CI}_R^m$. The inverse map associates to each $H \subset R$, the ideal sheaf on $X$ obtained by removing the exceptional part of $H \mathcal{O}_X$. Moreover, the maps

$$\varphi : S_X \rightarrow \text{CI}_R$$
$$\mathcal{J} \mapsto H_{\mathcal{J}}$$

and

$$\psi : \text{CI}_R \rightarrow S_X$$
$$H \mapsto H \mathcal{O}_X(\mathcal{L}_H)$$

are reciprocal isomorphisms of semigroups between $S_X$ and $\text{CI}_R$.

Proof. The maps $\mathcal{J} \rightarrow H_{\mathcal{J}}$ and $H \rightarrow H \mathcal{O}_X(\mathcal{L}_H)$ are well defined by 3.1. By (6) above, we have that $(H_{\mathcal{J}_1 \mathcal{J}_2})^o = H_{\mathcal{J}_1}^o H_{\mathcal{J}_2}^o$. From this, it follows that

$$H_1 H_2 \mathcal{O}_X(\mathcal{L}_{H_1 H_2}) = (H_1 \mathcal{O}_X(\mathcal{L}_{H_1}))(H_2 \mathcal{O}_X(\mathcal{L}_{H_2})).$$

Thus, $\psi(H_1 H_2) = \psi(H_1) \psi(H_2)$. On the other hand, since $\mathcal{L}_{\mathcal{J}_1 \mathcal{J}_2} = \mathcal{L}_{\mathcal{J}_1} + \mathcal{L}_{\mathcal{J}_2}$, we have that $H_{\mathcal{J}_1 \mathcal{J}_2} = H_{\mathcal{J}_1} H_{\mathcal{J}_2}$. Hence, $\varphi(\mathcal{J}_1 \mathcal{J}_2) = \varphi(\mathcal{J}_1) \varphi(\mathcal{J}_2)$.

We show that both maps are reciprocal bijections between $S_X$ and $\text{CI}_R$. If $\mathcal{J} \in S_X$, we know that $\mathcal{L}_{H_{\mathcal{J}}} = \mathcal{L}_{\mathcal{J}}$ and $H_{\mathcal{J}} \mathcal{O}_X = \mathcal{J} \mathcal{O}_X(\mathcal{L}_{\mathcal{J}})$. Thus, $\mathcal{J} = H_{\mathcal{J}} \mathcal{O}_X(\mathcal{L}_{\mathcal{J}}) = \mathcal{J} = H_{\mathcal{J}} \mathcal{O}_X(\mathcal{L}_{H_{\mathcal{J}}}) = \mathcal{J}$. If $H \subset R$ is a Cartier ideal for $X$, the cosupport of the sheaf $\mathcal{J} = H \mathcal{O}_X(\mathcal{L}_H)$ is composed of the points on $X$ shared by the virtual transforms relative to $H$ of the curves defined by elements of $H$, and hence it is finite. Moreover, we have that $\mathcal{L}_{\mathcal{J}}$ equals the exceptional divisor $\mathcal{L}_H$ and hence, $\mathcal{J} \mathcal{O}_X(\mathcal{L}_H) = H \mathcal{O}_X$. We deduce that $H_{\mathcal{J}} = H$. □

Remark 3.7. By taking $m = 1$ and identifying each point $Q \in X$ with the ideal sheaf $\mathcal{M}_Q$ generated by the maximal ideal $m_Q$ of $\mathcal{O}_{X,Q}$, the preceding theorem generalizes Theorem 3.5 of [10]: any sheaf $\mathcal{J}$ of $S_X^1$ is $\mathcal{M}_Q$ for some $Q$, and the ideal $H_{\mathcal{M}_Q} (= \varphi(\mathcal{M}_Q))$ is just the ideal $I_Q$ of codimension one in $I$. Recall by the way that any such ideal has the form $H_{K_q}$, for some $q \in E_K$, and that $Q$ is regular if and only if $K_q$ is consistent.

4. Factorization of complete ideals in normal birational extensions of $R$

Fixed a birational normal extensions $\mathcal{O}$ of $R$, we pay attention here to the factorizations and semifactorization of complete $m_{\mathcal{O}}$-primary ideals in $\mathcal{O}$ and relate them to the factorization of some complete $m_R$-primary ideals in $R$. As claimed above, such extensions can be understood as the local rings of points on a surface obtained by blowing-up a complete $m_R$-primary ideal in $R$ and hence, they have rational surface singularities. Then, it is well known that the group $\text{CI}(\mathcal{O})$ of Weil divisors on $\mathcal{O}$ is finite (see §14 of [17]).

Fix a complete $m_{\mathcal{O}}$-primary ideal $I \subset R$ such that $\mathcal{O}$ is the local ring of some $Q \in X$, i.e., $\mathcal{O} = R[I/a]_m_{\mathcal{O}}$, for some generic element $a \in I$ (it is enough to take a generic $a \in I$ such that the strict transform on $X$ of the curve $\{a = 0\}$ does not go through $Q$). Take a complete $m_{\mathcal{O}}$-primary ideal $J \subset \mathcal{O}$ and write $\mathcal{J}$ for the ideal sheaf on $X$ generated by $J$, $\mathcal{J} = J \mathcal{O}_X$. Clearly,
\( \mathcal{J} \) satisfies the condition \((\dagger)\) and \( C(\mathcal{J}) = \{Q\} \). Keeping the notations of the preceding section, for each \( p \in T^\mathcal{J}_Q \), let \( \gamma_p \) be a generic branch going through \( p \). From the finiteness of \( Cl(\mathcal{O}) \), there is some integer \( n \) such that \( n \gamma_p \) is a Cartier divisor on \( X \). Write \( m_p \) for the least of these integers. In virtue of 2.5, \( m_p \) is also the least integer \( n \) such that \( I^n_p \) is a Cartier ideal for \( X \). Moreover, the ideal sheaf

\[
\mathcal{J}(p) = I^{m_p}_p \mathcal{O}_X(m_p \mathcal{L}_p)
\]

satisfies \((\dagger)\) and its cosupport is also \( \{Q\} \). Write \( J_p \subset \mathcal{O}_{X,Q} \) for the stalk of \( \mathcal{J}(p) \) at \( Q \). Because of the minimality of \( m_p \), \( J_p \) is simple and, as explained in Section 2.2, the extremal elements of \( E^+_S \) are the (integer) multiples of the divisors \( D_{J_p} \) defined by \( J_p \mathcal{O}_S = \mathcal{O}_S(-D_{J_p}) \). We infer from this that the ideals \( \{J_p\} \subset T^\mathcal{J}_Q \) are the extremal elements of \( J^S_Q \) (see Proposition 1.4 of [3]).

The following theorem relates the factorization of complete \( m_R \)-primary ideals in \( R \) to the semifactorization of complete \( m_Q \)-primary ideals in \( \mathcal{O}_{X,Q} \).

**Theorem 4.1.** Let \( J \subset \mathcal{O}_{X,Q} \) be a complete \( m_Q \)-primary ideal and \( \mathcal{J} = J \mathcal{O}_X \). If

\[
H_{\mathcal{J}} = \prod_p I^{\alpha_p}_p \tag{7}
\]

is the (Zariski) factorization of \( H_{\mathcal{J}} \subset R \) into simple ideals, then \( J = \prod_p J^{\alpha_p}_p m_p \) is the semifactorization of \( J \) (in the sense of 2.2).

**Proof.** Write \( m = \text{lcm}_{\alpha_p > 0} \{m_p\} \). From the equality (7) above, we have that

\[
m \mathcal{L}_{H_{\mathcal{J}}} = \sum_{p \in T^\mathcal{J}_Q} m \alpha_p \mathcal{L}_p.
\]

and by 3.1, \( \mathcal{J} = H_{\mathcal{J}} \mathcal{O}_X(\mathcal{L}_{H_{\mathcal{J}}}) \). Therefore, \( \mathcal{J}^m \mathcal{O}_X(-m \mathcal{L}_{H_{\mathcal{J}}}) = H^m_{\mathcal{J}} \mathcal{O}_X = (\prod I^{m \alpha_p}_p) \mathcal{O}_X \). Since the ideals \( J_p \subset \mathcal{O}_{X,Q} \) are the stalks at \( Q \) of the ideals sheaves given by \( I^{m \alpha_p}_p \mathcal{O}_X(m_p \mathcal{L}_p) \), it follows from the equality above that the stalk of \( \mathcal{J}^m \) at \( Q \) is

\[
J^m = \prod_p J^{m \alpha_p}_p m_p
\]

and by allowing rational exponents, \( J = \prod_p J^{\alpha_p}_p m_p \). This completes the proof. \( \square \)

It is clear that the above semifactorization of \( J \) is a factorization into simple complete ideals of \( \mathcal{O}_{X,Q} \) if and only if \( \alpha_p \in (m_p) \), for each \( p \in T^\mathcal{J}_Q \). In this case, there is only one factorization of \( J \) into simple ideals. The next result shows that, in general, the factorizations of complete ideals in \( \mathcal{O} \) can be read of in terms of the Cartier ideals for \( X \).

First, we need a definition.

**Definition 4.2.** A Cartier ideal for \( X \) is an irreducible Cartier ideal for \( X \) if it cannot be obtained as the product of two proper Cartier ideals for \( X \).
Theorem 4.3. Given a complete $m_Q$-primary ideal $J \subset O_{X,Q}$, each factorization of $J$ into complete $m_Q$-primary ideals

$$J = \prod_{i=1}^{r} J_{i}^{{a}_{i}}$$

induces a factorization of $H_J$ into Cartier ideals for $X$

$$H_J = \prod_{i=1}^{r} H_{J_i}^{{a}_{i}},$$

where $J_i = J_i O_X$. Moreover, each factorization of $H_J$ into Cartier ideals for $X$ has this form. In particular, $J$ is a simple complete $m_Q$-primary ideal if and only if $H_J$ is an irreducible Cartier ideal for $X$.

Proof. By 3.6, each factorization of $J$ as above gives rise to a factorization $H_J = \prod_{i=1}^{r} H_{J_i}^{{a}_{i}}$. Conversely, assume that $H_J = \prod_{i=1}^{r} H_{J_i}^{{a}_{i}}$ where each $H_i$ is a Cartier ideal for $X$. Then, $L_{H_i} \in \bigoplus_{p \in \mathbb{K}_+} \mathbb{Z}L_p$ and by 3.6, the ideal sheaf $H_i O_X(L_{H_i})$ is in $S_X$ and satisfies the condition $(\dagger)$. Therefore, its stalk $J_i$ at $Q$ is a complete $m_Q$-primary ideals in $O_{X,Q}$. Since $H_J O_X = \prod_{i=1}^{r} H_i^{{b}_{i}} O_X$ and $L_J = L_{H_J} = \sum_{i=1}^{s} {b}_{i} L_{H_i}$, we deduce $H_J O_X(L_{H_J}) = \sum_{i=1}^{s} H_i^{{b}_{i}} O_X(L_{H_i})$. By taking the stalks at $Q$, we deduce that $J = \prod_{i=1}^{r} J_i^{{b}_{i}}$ and the claim follows.

Remark 4.4. In virtue of the above result, irreducible Cartier ideals for $X$ correspond to simple ideals in the local ring of some point in the exceptional locus of $X$. Recall from [10] that if $q \in E_K$ does not lie on a component $E_p$ with $\rho_p = 0$, the cluster $K_q$ is consistent and $H_{K_q}$ is $I_Q$, for some regular point $Q \in X$. Then, there is a bijection between the points infinitely near or equal to $Q$, $q \geq Q$, and the simple ideals in $O_{X,Q}$

$$u \mapsto I_u^{(Q)} = \text{ideal generated by } \{ h \in O_{X,Q} \mid C: h = 0 \text{ goes through } u \}.$$ 

In this case, if $J = I_u^{(Q)}$, the irreducible Cartier ideal for $X$ $H_J$ is just the simple ideal $I_u \subset R$. From this, it follows that any simple ideal $I_u$ where $u$ is infinitely near or equal to some regular point of $X$ is an irreducible Cartier ideal for $X$.

In virtue of 4.3, $J \subset O_{X,Q}$ has unique factorization into simple $m_Q$-primary ideals if and only if $H_J$ has unique factorization into irreducible Cartier ideals for $X$. On the other hand, the equality $L_J = L_{J_1} + L_{J_2}$ implies that any factorization of $J$ induces a decomposition of $L_J$ into elements of $\bigoplus_{p \in \mathbb{K}_+} \mathbb{Z}L_p$. Therefore,

$$L_J \text{ is irreducible } \Rightarrow J \subset O_{X,Q} \text{ is simple } \Rightarrow H_J \text{ is irreducible Cartier ideal for } X$$

In the following example, we show that the converse to the first implication does not hold in general.
Example 4.5. Take \( \mathcal{O} = R[y^3/x] \) and the complete \( m_R \)-primary ideal \( I = (x, y^3) \subset R \). The cluster of base points of \( I \) is composed of three points, the origin \( O = p_1 \), \( p_2 \) in the first neighborhood of \( O \) and \( p_3 \) in the first neighborhood of \( p_2 \). The three of them are free, have virtual multiplicity one and lie over the curve \( x = 0 \) (see Fig. 1). The surface \( X = Bl_1(S) \) has only one exceptional component \( L_{p_3} \) and one singularity, say \( Q \). Indeed, \( \mathcal{O} \simeq \mathcal{O}_{X, Q} \). Since \( \mathcal{K}^+ = \{ p_3 \} \), for any ideal \( J \subset R \), we have that \( L^2 J_{2} = a L^2_{p_3} \), where \( a = v_p(J) \). Since \( L^3_{p_3} = 3 L^3_{p_3} \), a complete \( m_O \)-primary ideal \( H \) in \( R \) is Cartier relative to \( X \) if and only if \( a \in (3) \).

Write \( \mathcal{I}_Q = \varphi(\mathcal{M}_Q) = \Gamma(X, \mathcal{M}_Q \mathcal{O}_X) \). The Zariski factorization of \( \mathcal{I}_Q \) is \( \mathcal{I}_Q = I_{p_1} I_{p_2} \). Thus, \( v_{p_3}(\mathcal{I}_Q) = v_{p_3}(I_{p_1}) + v_{p_3}(I_{p_2}) = 3 \). Hence, \( \mathcal{I}_Q \) is an irreducible Cartier ideal. Analogously, it can be seen that \( I_{p_3} \) is also irreducible as a Cartier ideal for \( X \). On the other hand, if \( q \) is the point in the first neighborhood of \( p_2 \) proximate to \( p_1 \), we have that \( \mathcal{J} = m_2 Q \) is an irreducible Cartier ideal for \( X \), while \( L_{\mathcal{J}} = 6 L_{p_3} \) is not irreducible in \( \mathbb{Z} L_{p_3} \). Now, if \( \mathcal{J} = m^3 Q \subset \mathcal{O}_{X, Q} \), we have that \( H_{\mathcal{J}} = I^3_{p_1} I^3_{p_2} \). By 4.3, the factorizations \( H_{\mathcal{J}} = I^3_{Q} \) and \( H_{\mathcal{J}} = I^3_{p_1} I^3_{p_2} \) induce two different factorizations of \( \mathcal{J} \) into simple ideals of \( \mathcal{O}_{X, Q} \): \( \mathcal{J} = m^3 Q \) and \( \mathcal{J} = J_1 J_2 \), where \( J_1, J_2 \) are the stalks at \( Q \) of the ideal sheaves on \( X \) given by \( I^3_{p_1} \mathcal{O}_X(3 L_{p_3}) \) and \( I^3_{p_2} \mathcal{O}_X(6 L_{p_3}) \).

5. Systems of generators of a complete ideal in normal birational extensions of \( R \)

Back to considering the situation of Section 3, fix an ideal sheaf \( \mathcal{J} \in \mathbb{S}_X \). This section is devoted to describe an algorithm to compute minimal system of generators for the stalks of \( \mathcal{J} \) in the points of its cosupport \( C(\mathcal{J}) \). To this aim, we introduce a couple of objects associated to \( \mathcal{J} \). The first one is the complete ideal

\[ H'_{\mathcal{J}} = \Gamma(X, \mathcal{J} \mathcal{O}_X(-L_{\mathcal{M}\mathcal{J}})) \subset R \]

where \( \mathcal{M} \) denotes the ideal sheaf defined by \( \mathcal{M} = \prod_{Q \in C(\mathcal{J})} \mathcal{M}_Q \). The second object is just the cluster of base points of this ideal,

\[ \mathcal{K}'_{\mathcal{J}} = BP(H'_{\mathcal{J}}). \]

The reader may notice that both clusters \( \mathcal{K}'_{\mathcal{J}} \) and \( \mathcal{K}_{\mathcal{M}\mathcal{J}} = BP(H_{\mathcal{M}\mathcal{J}}) \) have the same set of points, \( K_{\mathcal{J}} \). By 3.2, the sheaf \( \mathcal{J} \mathcal{O}_X(-L_{\mathcal{M}\mathcal{J}}) \) is generated by global sections and so, \( H'_{\mathcal{J}} \mathcal{O}_X = \mathcal{J} \mathcal{O}_X(-L_{\mathcal{M}\mathcal{J}}) \). The monomorphism of sheaves

\[ \mathcal{M} \mathcal{J} \mathcal{O}_X(-L_{\mathcal{M}\mathcal{J}}) \hookrightarrow \mathcal{J} \mathcal{O}_X(-L_{\mathcal{M}\mathcal{J}}) \]
induces inclusions of ideals in \( R, H_{\mathcal{MJ}} \subset H'_{\mathcal{J}} \subset H_{\mathcal{MJ}}\) and so,

\[ K'_{\mathcal{J}} < K_{\mathcal{MJ}}. \]

We have also that

\[ v_p(H'_{\mathcal{J}}) \begin{cases} < v_p(H_{\mathcal{MJ}}) & \text{if } p \in T^\mathcal{J}_Q \text{ for some } Q \in C(\mathcal{J}); \\ = v_p(H_{\mathcal{MJ}}) & \text{otherwise}. \end{cases} \] (8)

The algorithm suggested here follows the idea of [1] and constructs a flag of clusters

\[ T_0 = K'_{\mathcal{J}} < T_1 < \cdots < T_N = K_{\mathcal{MJ}}. \]

where \( N = l(H'_{\mathcal{MJ}}). \) Before describing it, notice that in virtue of (8), for all \( i \in \{1, \ldots, N\} \) there

\[ L_{T_i} = L_{\mathcal{MJ}} = L_{\mathcal{M}} + L_{\mathcal{J}}. \] (9)

It follows that every ideal \( H_{T_i} \) is a Cartier ideal for \( X \).

Now, we are able to describe the algorithm. Write \( T_0 = K'_{\mathcal{J}}. \) Since \( K'_{\mathcal{J}} < K_{\mathcal{MJ}} \), there is

some point \( p \) such that \( v_p(H_{\mathcal{MJ}}) > v_p(H'_{\mathcal{J}}). \) Take any such \( p \). By (8), necessarily \( p \in T^\mathcal{J}_Q \) for some \( Q \in C(\mathcal{J}) \). Write \( Q_1 \) for this point, and choose any point \( q \) in the first neighborhood of \( p \), not already in \( K_{\mathcal{J}} \). Then, define \( T_1 \) as the cluster obtained by adding \( q \) to \( T_0 \) counted once and unloading, if necessary. For \( i \geq 1 \) and once defined the cluster \( T_i \), proceed similarly: choose any \( q \) in the first neighborhood of a point \( p \) such that \( v_p(H_{\mathcal{MJ}}) > v_p(H_{T_i}) \), not already in \( K_{\mathcal{J}} \), and define \( T_{i+1} \) as the cluster obtained by adding \( q \) to \( T_i \) counted once and unloading, if necessary. Then, since \( l(H'_{T_i}) = 1 \) for each \( i \), this procedures comes to an end. Thus, if \( J_1 \) is the stalk of \( \mathcal{J} \) at \( Q_1 \) and \( N_1 = l(J_1/m_{Q_1} J_1) \), we obtain a flag \( T_0 = K'_{\mathcal{J}} < T_1 < \cdots < T_{N_1} \). Moreover,

\[ v_p(H_{T_{N_1}}) = v_p(H_{\mathcal{MJ}}), \quad \text{if } p \in T^\mathcal{J}_{Q_1}. \]

Repeat this procedure for the remaining points of \( C(\mathcal{J}) \), say \( Q_2, \ldots, Q_n. \)

In this way, if \( N_i = l(J_i/m_{Q_i} J_i) \) for each \( i \in \{1, \ldots, n\} \) and \( M_i = \sum_{k=1}^{i-1} N_k \) (take \( M_1 \) as 0) we obtain a flag

\[ T_{M_0} < \cdots < T_{M_2} < \cdots < T_{M_{n-1}} < \cdots < T_{M_n} = K_{\mathcal{MJ}}, \]

where for \( i = 0, \ldots, n - 1 \),

\[ v_p(H_{M_{i+1}}) = v_p(H_{\mathcal{MJ}}), \quad \text{if } p \in T^\mathcal{J}_{Q_j} \text{ for some } j \leq i. \]

The next lemma says that the number of steps to be performed in the algorithm equals the colength of \( \mathcal{J} \).

**Lemma 5.1.** \( N = \sum_{i=1}^{n} l(J_i/m_{Q_i} J_i). \)
Proof. We have that

\[ N = \mathfrak{l}\left( \frac{H'_J}{H_{M,J}} \right) = \mathfrak{l}\left( \frac{H'^o_{M,J}}{H_{M,J}} \right) - \mathfrak{l}\left( \frac{H'^o_{M,J}}{H'_J} \right). \]

Applying 3.5 to $H'_J$ and $H_{M,J}$, we have that

\[ N = \mathfrak{l}\left( \frac{H'_J}{H_{M,J}} \right) = \sum_{i=1}^{n} \mathfrak{l}\left( \frac{\mathcal{O}_{X,Q_i}}{J_i} \right) - \sum_{i=1}^{n} \mathfrak{l}\left( \frac{\mathcal{O}_{X,Q_i}}{m_{Q_i} J_i} \right) = \sum_{i=1}^{n} \mathfrak{l}\left( J_i \right) \]

as claimed. \(\square\)

The following proposition shows how to pick curves on \(S\) in order the equations of their virtual transforms give rise to minimal systems of generators of the stalks \(J_i\) of \(\mathcal{J}\).

**Proposition 5.2.** Let \(h = 0\) be a generic element of \(H^o_{M,J}\), and for each \(j = 0, \ldots, N - 1\), let \(C_j\): \(h_j = 0\) be a curve going through \(T_j\) but going not through \(T_{j+1}\). Then \(\{\frac{h_{M_j}}{h}, \ldots, \frac{h_{M_j+N_j-1}}{h}\}\) is a minimal system of generators of \(J_i \subset \mathcal{O}_{X,Q_i}\).

**Proof.** Fixed \(i \in \{1, \ldots, n\}\) and \(j \in \{1, \ldots, N_i\}\), write \(H^i_j = H_{T_{M_j+j}}\). Then, the subflag \(T_{M_i} < T_{M_i+1} < \cdots < T_{M_i+N_i}\) gives a filtration of ideals in \(R\),

\[ H^i_0 \supset H^i_1 \supset \cdots \supset H^i_{N_i} \quad (10) \]

with \(\mathfrak{l}(H^i_j / H^i_{j+1}) = 1\), for \(j = 0, \ldots, N_i - 1\). Since we know that \(\mathcal{L}_{H^i_j} = \mathcal{L}_{H_J}\) (see (9) above), 3.5 applies to \(H_J\) to give that the sheaf \(\mathcal{H}^i_j = H^i_j \mathcal{O}_X(\mathcal{L}_{M,J})\) has finite co-support contained in \(\{Q_1, \ldots, Q_n\}\) and also

\[ \lambda(\mathcal{H}^i_j) = \mathfrak{l}_R\left( \frac{H'^o_{M,J}}{H^i_j} \right). \]

Therefore, if \(j \in \{0, \ldots, N_i - 1\}\), \(\lambda(\mathcal{H}^i_{j+1}) = \lambda(\mathcal{H}^i_j) + 1\). Note that if \(k \neq i\) and \(j \in \{0, \ldots, N_i - 1\}\), then

\[ (\mathcal{H}^i_j)_{Q_k} = (\mathcal{H}^i_{j+1})_{Q_k}. \]

Hence, if we write \(H^i_Q\) for the stalk of \(\mathcal{H}^i_j\) at \(Q_i\), we have that \(\mathfrak{l}(\frac{H^i_Q}{H^i_{j+1}}) = 1\). It follows that the filtration (10) induces a filtration of ideals in \(\mathcal{O}_{X,Q_i}\):

\[ J_i \supset H^i_{Q_i} \supset \cdots \supset H^i_{N_i} = m_{Q_i} J_i. \]
In order to obtain a minimal system of generators of $J_i$, it is enough to pick for $j = 0, \ldots, N_i - 1$, curves $C_j$: $h_j = 0$ on $S$ going through $T_i$ but not through $T_{i+1}$. Then, the strict transform $\tilde{C}_j$ on $X$ is defined locally by an element in $H^Q_{i}$ not in $H^Q_{i+1}$. This completes the proof. 

**Remark 5.3.** Using the algorithm suggested in [1], a minimal system of generators $\{g_0, \ldots, g_m\}$ for $HJ$ can be obtained, i.e., a minimal system of global sections generating the sheaf $J_{OX}(L_J)$. By removing the exceptional part of the total transform of the curves defined by these elements (by dividing them by a generic element $HJ$) we obtain local generators for the complete ideals $J_i$, $i = 1, \ldots, n$. However, in general, this is far from being a minimal system of generators for the ideals $J_i$. 

To close this section, we illustrate the above procedure with an example.

**Example 5.4.** Take the complete $m_R$-primary ideal $I = (x, y^2)$ in $R = \mathbb{C}[x, y]$. The cluster $K = BP(I)$ is composed of two points, the origin $O$ and $p$ in the first neighborhood of $O$, both of them with virtual multiplicity one and over the curve $x = 0$. The surface $X = Bl_I(S)$ has only one singularity $Q_1$ and one exceptional component $L_p$. Now, the ideal $I' = (x, y^3) \subset R$ is complete and has codimension one in $I$. By the bijection of 3.6, the ideal $I'$ maps to the ideal sheaf of one (regular) point on $L_p$, say $Q_2$ (see 3.7). Let $\mathcal{J}$ be the ideal sheaf on $X$ with cosupport $C(\mathcal{J}) = \{Q_1, Q_2\}$, whose stalks are $J_1 = m_{Q_1}$, $J_2 = m_{Q_2}^2$. As above, write $\mathcal{M} = \mathcal{M}_{Q_1, Q_2}$. It is clear that $\mathcal{J}$ and $\mathcal{M}\mathcal{J}$ satisfy the condition (†). We have $\mathcal{L}_{\mathcal{M}\mathcal{J}} = 5L_p = 10L_p$. Let $q_2$ be the point in the first neighborhood of $p$ identified with $Q_2$. The Enriques diagram of $K_{\mathcal{J}}$ is shown
Fig. 3. The Enriques diagrams of the clusters \( \{T_i\}_{i=1,...,6} \) obtained by applying the algorithm of Section 5.

in (a) of Fig. 2. The cluster \( K'_{\mathcal{T}} \) consists of the points \( O, p \) and \( q_2 \), with virtual multiplicities 6, 4 and 2, respectively. The cluster \( K_{\mathcal{M},\mathcal{T}} \) has points \( O \) and \( p \) with virtual multiplicities 5, 5.

Fig. 3 shows the Enriques diagrams of the clusters obtained by the algorithm described in this section. Let \( h \) be a generic element of \( H^0_{\mathcal{M},\mathcal{T}} = I^5 \), and for \( i = 0, \ldots, 5 \), let \( C_i \): \( h_i = 0 \) where

\[
\begin{align*}
  h_0 &= (x^2 - y^2)(x + y^2)(x^4 - y^6), & h_3 &= y(x^2 - y^2)(x - 2y)(x + y^2)(x^4 - y^6), \\
  h_1 &= y(x + y)(x^2 - y^4)(x^4 - y^6), & h_4 &= y(x^2 - y^2)(x - 2y)(x + y^2)(x - y^3)(x + 2y^3), \\
  h_2 &= y(x - y)(x^2 - y^4)(x^4 - y^6), & h_5 &= y(x^2 - y^2)(x - 2y)(x + y^2)(x^2 - y^6).
\end{align*}
\]

Then, each \( C_i \) goes through \( T_i \) but not through \( T_{i+1} \). 5.2 applies to give that \( \{h_0/h, h_1/h, h_2/h\} \) and \( \{h_3/h, h_4/h, h_5/h\} \) are local generators of \( J_1 \) and \( J_2 \), respectively.

Acknowledgments

I want to thank Eduard Casas-Alvero for all the conversations we held on this topic during the elaboration of my Ph.D. thesis. I also want to thank Ana Reguera and Olivier Piltant for their kind suggestions.

References