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Topology and descriptive set theory

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Abstract

This paper consists essentially of the text of a series of four lectures given by the author in the Summer Conference on General Topology and Applications, Amsterdam, August 1994.

Instead of attempting to give a general survey of the interrelationships between the two subjects mentioned in the title, which would be an enormous and hopeless task, we chose to illustrate them in a specific context, that of the study of Borel actions of Polish groups and Borel equivalence relations. This is a rapidly growing area of research of much current interest, which has interesting connections not only with topology and set theory (which are emphasized here), but also to ergodic theory, group representations, operator algebras and logic (particularly model theory and recursion theory).

There are four parts, corresponding roughly to each one of the lectures. The first contains a brief review of some fundamental facts from descriptive set theory. In the second we discuss Polish groups, and in the third the basic theory of their Borel actions. The last part concentrates on Borel equivalence relations.

The exposition is essentially self-contained, but proofs, when included at all, are often given in the barest outline.

Keywords: Polish spaces; Borel sets; Analytic sets; Polish groups; Borel actions; Borel equivalence relations

1. Some basic descriptive set theory

1.1. Polish spaces

Descriptive set theory is the study of "definable sets" (such as Borel, analytic, projective, etc.) in Polish spaces. Recall that a *Polish space* is a separable completely

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metrizable topological space. Many standard spaces studied in mathematics are Polish. For example, $\mathbb{R}^n$, $\mathbb{C}^n$, $\mathbb{R}^{\mathbb{N}}$, $\mathbb{C}^{\mathbb{N}}$, the *Hilbert cube* $I^{\mathbb{N}}$ ($I = [0,1]$), the *Cantor space* $\mathcal{C} = 2^{\mathbb{N}}$, the *Baire space* $\mathcal{N} = \mathbb{N}^{\mathbb{N}}$, any separable Banach space, etc.

Compact, metrizable spaces are Polish and so are all locally compact, metrizable and K_σ (i.e., countable unions of compact) spaces. Also the completion of any separable metric space is Polish.

The following are some basic closure properties: The direct product or sum of a countable family of Polish spaces is Polish. A subspace X of a Polish space Y is Polish iff it is G_δ in Y.

Certain Polish spaces have important universality properties. First, the Hilbert cube $I^{\mathbb{N}}$ is *universal* for all Polish spaces in the sense that every Polish space is homeomorphic to a (necessarily G_δ) subspace of $I^{\mathbb{N}}$. Also every Polish space is homeomorphic to a closed subspace of $\mathbb{R}^{\mathbb{N}}$.

The Baire space $\mathcal{N}$ is *surjectively universal* in the sense that for each (nonempty) Polish space X there is a continuous surjection $f : \mathcal{N} \to X$. (One can make a more precise assertion here, that we will use in the proof of Theorem 2.11. Let d_X be a complete compatible metric for X and let d be the standard metric on $\mathcal{N}$, i.e.,

$$d(x,y) = 2^{-n-1},$$

where for $x \neq y$ in $\mathcal{N}$, n is least with $x(n) \neq y(n)$. Then there is a surjection $f : \mathcal{N} \to X$ which is Lipschitz, i.e., $d_X(f(x), f(y)) \leqslant d(x,y)$.) Moreover, there is a closed subset $F \subseteq \mathcal{N}$ and a continuous bijection $g : F \to X$.

1.2. The Choquet criterion

There is a useful characterization of Polish spaces, due to Choquet, in terms of games (which are descendants of the so-called *Banach–Mazur games*).

Let X be a nonempty topological space. The *strong Choquet game* G_X^s is defined as follows:

$$\begin{array}{lllll} \mathrm{I} & x_0, U_0 & & x_1, U_1 & \\ & & & & \cdots \\ \mathrm{II} & & V_0 & & V_1 \end{array}$$

Player I starts by playing an open nonempty set $U_0 \subseteq X$ and an element $x_0 \in U_0$. Player II then plays a nonempty open set $V_0 \subseteq U_0$ with $x_0 \in V_0$. Next I plays a nonempty open set $U_1 \subseteq V_0$ and an element $x_1 \in U_1$; II follows by playing a nonempty open set $V_1 \subseteq U_1$, with $x_1 \in V_1$, etc. We say that II wins this run of the game if $\bigcap_n V_n$ $(= \bigcap_n U_n) \neq \emptyset$. Otherwise, I wins.

A *strategy* for II is a "rule" that tells him how to play, for each n, his nth move V_n, given I's previous moves $(x_0, U_0), (x_1, U_1), \ldots, (x_n, U_n)$. Technically, this can be defined as a sequence of functions $f_0, f_1, \ldots, f_n, \ldots$, where f_n maps the set of all sequences $((x_0, U_0), (x_1, U_1), \ldots, (x_n, U_n))$, with U_i open and $x_i \in U_i$, into the set of nonempty open sets, and which have the following properties:

(i) $f_0((x_0, U_0)) = V_0 \subseteq U_0$ and $x_0 \in V_0$;

(ii) If $((x_0, U_0), (x_1, U_1))$ is such that for $f_0((x_0, U_0)) = V_0$, we have $U_1 \subseteq V_0$, then $f_1((x_0, U_0), (x_1, U_1)) = V_1 \subseteq U_1$ and $x_1 \in V_1$;

etc.

We say that a strategy for II is *winning* if II wins every run of the game in which he follows this strategy. (Similarly, we define strategies and winning strategies for player I.)

A nonempty topological space X is called *strong Choquet* if II has a winning strategy in G^s_X.

It is easy to see that every Polish space is strong Choquet. In fact, we have

Theorem 1.1 (Choquet). *Let X be a nonempty topological space. Then the following are equivalent*:

(i) *X is Polish*;

(ii) *X is second countable, T_1, regular and strong Choquet.*

We will not give the proof here, but the main part is to show that if X is nonempty separable metrizable and strong Choquet, then X is G_δ in its completion.

One immediate corollary of Theorem 1.1 is, for example, Hausdorff's Theorem that if X is Polish, Y separable metrizable and $f : X \to Y$ is a continuous open surjection, then Y is Polish.

1.3. Baire category

Baire category arguments are a basic tool in descriptive set theory. Recall that a set A in a topological space X is *nowhere dense* in X if its closure has empty interior and *meager* in X if $A = \bigcup_n A_n$, with A_n nowhere dense. The complement of a meager set is called *comeager*. A topological space X is *Baire* if every nonempty open set is non-meager or equivalently the intersection of countably many dense open sets is dense. By the *Baire Category Theorem* every Polish space is Baire. In fact, every strong Choquet space is also Baire.

A set $A \subseteq X$ has the *Baire property* (BP) if there is an open set $U \subseteq X$ with $A \triangle U$ $(= (A \setminus U) \cup (U \setminus A))$ meager. The sets with the BP form a σ-algebra containing all the open sets. A function $f : X \to Y$ is *Baire measurable* if $f^{-1}(U)$ has the BP in X for each open $U \subseteq Y$. It is easy to show that if $f : X \to Y$ is Baire measurable, where X is Baire and Y second countable, then there is a dense G_δ set $G \subseteq X$ such that $f|G$ is continuous.

1.4. Borel and projective sets

We will define now the basic concepts of descriptive set theory. Let X be a Polish space. As usual the class of *Borel sets* of X is defined as the smallest σ-algebra of subsets of X closed under complements and countable unions (so also under countable intersections). We denote by $\mathbf{B}(X)$ this class. It can be ramified in a transfinite hierarchy of length ω_1, the first uncountable ordinal, the so-called *Borel hierarchy*, as follows: Let for $1 \leqslant \xi < \omega_1$,

$$\Sigma^0_1(X) = \text{open}, \qquad \Pi^0_1(X) = \text{closed};$$

$$\Sigma^0_\xi(X) = \{\bigcup_{n\in\mathbb{N}} A_n : A_n \text{ is } \Pi^0_{\xi_n}(X) \text{ for } \xi_n < \xi\};$$

$$\Pi^0_\xi(X) = \text{the complements of } \Sigma^0_\xi(X) \text{ sets.}$$

So (for any fixed X) $\Sigma^0_2 = F_\sigma$, $\Pi^0_2 = G_\delta$, $\Sigma^0_3 = G_{\delta\sigma}$, $\Pi^0_3 = F_{\sigma\delta}$, etc.

Then we have the following picture

$$\begin{array}{ccccccccc} \Sigma^0_1 & \Sigma^0_2 & & \Sigma^0_\xi & & \Sigma^0_\eta & & \\ & & \cdots & & \cdots & & \cdots \\ \Pi^0_1 & \Pi^0_2 & & \Pi^0_\xi & & \Sigma^0_\eta & & \end{array}$$

where $1 \leqslant \xi \leqslant \eta < \omega_1$ and any class is contained in every class to the right of it (properly if X is uncountable) and

$$\mathbf{B}(X) = \bigcup_{1\leqslant\xi<\omega_1} \Sigma^0_\xi(X) = \bigcup_{1\leqslant\xi<\omega_1} \Pi^0_\xi(X).$$

We next define the *projective sets* in Polish spaces. These are obtained from the Borel sets by the operations of projection (or equivalently continuous image) and complementation. More precisely, given a Polish space X the *analytic sets* in X are those of the form $f(A)$, for $f : Y \to X$ continuous from a Polish space Y into X, and A Borel in Y. Equivalently, they are the sets of the form $\mathrm{proj}_X(B)$, for $B \subseteq X \times Z$ Borel, where Z is a Polish space. We denote their class by $\Sigma^1_1(X)$. The *co-analytic sets*, whose class is denoted by $\Pi^1_1(X)$, are the complements of the analytic sets. We define then inductively for $n \geqslant 1$, the classes $\Sigma^1_{n+1}(X)$, $\Pi^1_{n+1}(X)$, simultaneously for all Polish spaces X, as follows: For $A \subseteq X$,

$$\begin{aligned} A \in \Sigma^1_{n+1}(X) &\Leftrightarrow \exists \text{ Polish } Y, \text{ continuous } f : Y \to X, \text{ and } B \in \Pi^1_n(Y) \\ &\qquad \text{with } f(B) = A; \\ A \in \Pi^1_{n+1}(X) &\Leftrightarrow X \setminus A \in \Sigma^1_{n+1}(X). \end{aligned}$$

We have the following picture:

$$\begin{array}{ccccccc} & \Sigma^1_1 & \Sigma^1_2 & & \Sigma^1_n & \Sigma^1_{n+1} & \\ \mathbf{B} & & & \cdots & & & \cdots \\ & \Pi^1_1 & \Pi^1_2 & & \Pi^1_n & \Pi^1_{n+1} & \end{array}$$

where again every class is contained to any class to the right of it (properly if X is uncountable). Let

$$\mathbf{P}(X) = \bigcup_n \Sigma^1_n(X) = \bigcup_n \Pi^1_n(X).$$

The sets in $\mathbf{P}(X)$ are called the *projective sets* of X, and the hierarchy defined above the *projective hierarchy*.

One can proceed further to define and study transfinite extensions of the projective hierarchy and even more complex "definable sets". In this paper however, we will restrict ourselves just to the first level of the projective hierarchy, consisting of the Borel, analytic and co-analytic sets.

1.5. Basic facts about Borel sets and functions

A fundamental result concerning Borel sets is that they can be viewed as clopen sets in an appropriate extension of the underlying topology. More precisely we have:

Theorem 1.2. *Let X be a Polish space with topology $\mathcal{T}$ and $A \subseteq X$ a Borel set. Then there is a Polish topology $\mathcal{T}_A$ on X extending $\mathcal{T}$ (i.e., $\mathcal{T}_A \supseteq \mathcal{T}$) such that A is clopen in $\mathcal{T}_A$ and $\mathcal{T}, \mathcal{T}_A$ have the same Borel sets.*

Proof. This is based on the following two lemmas.

Lemma. *If $F \subseteq X$ is closed and $\mathcal{T}_F$ is the topology generated by $\mathcal{T} \cup \{F\}$, then $\mathcal{T}_F$ is Polish (and clearly F is clopen in it).*

This follows easily from the fact that $\mathcal{T}_F$ is the direct sum of the relative topologies on F and $X \setminus F$, which are Polish.

Lemma. *If $(\mathcal{T}_n)$ is a sequence of Polish topologies on X extending $\mathcal{T}$ (i.e., $\mathcal{T}_n \supseteq \mathcal{T}$) so that $\mathcal{T}_n, \mathcal{T}$ have the same Borel sets, then the topology $\mathcal{T}_\infty$ generated by $\bigcup_n \mathcal{T}_n$ is Polish and $\mathcal{T}_\infty, \mathcal{T}$ have the same Borel sets.*

To see this, consider the diagonal map $\varphi : X \to \prod_n X_n$, where $X_n \equiv X$, given by $\varphi(x) = (x, x, \ldots)$. Then $\varphi(X)$ is closed in $\prod_n (X_n, \mathcal{T}_n)$, so is Polish. But φ is a homeomorphism of $(X, \mathcal{T}_\infty)$ with $\varphi(X)$.

Using these lemmas it is easy to check that the class of $A \subseteq X$ satisfying the theorem is a σ-algebra containing all open sets. □

It is routine to extend Theorem 1.2 to show that for a countable sequence (A_n) of Borel sets, there is a Polish topology extending the underlying one, but with the same Borel sets, in which every A_n is clopen.

One can therefore essentially view Borel sets as clopen sets. This has several consequences. For example, it solves the cardinality problem for Borel sets: Every Borel set in a Polish space is either countable or contains a Cantor set (Alexandrov, Hausdorff). This is clear since, by standard arguments, every uncountable Polish space contains a Cantor set, and clopen sets in Polish spaces are Polish.

Recall now that a function $f : X \to Y$ between topological spaces is *Borel* if the inverse image of any open set is Borel. We have now an analogous result for Borel functions.

Theorem 1.3. *Let X be a Polish space with topology $\mathcal{T}$ and $f : X \to Y$ a Borel function, where Y is second countable. Then there is a Polish topology $\mathcal{T}_f \supseteq \mathcal{T}$ having the same Borel sets, so that $f : (X, \mathcal{T}_f) \to Y$ is continuous.*

Proof. Let $\{U_n\}$ be an open basis for Y. Apply the remark following Theorem 1.2 to $\{f^{-1}(U_n)\}$. $\square$

By similar arguments one can show that if $f : X \to Y$ is a bijection between Polish spaces with f, f^{-1} Borel (actually the first condition implies the second as we will see soon), then there are Polish extensions of the given topologies on X, Y with the same Borel sets, for which f becomes a homeomorphism.

Actually both Theorem 1.2 and Theorem 1.3 characterize exactly the Borel sets and functions as we will see soon, after we state another fundamental result.

In general a continuous image of a Borel set is (analytic but) not Borel. However we have the following basic result.

Theorem 1.4 (Lusin–Souslin). *Let X, Y be Polish and $f : X \to Y$ a Borel function. Then if $A \subseteq X$ is Borel and $f|A$ is injective, $f(A)$ is Borel.*

The proof of this result is based on the Lusin Separation Theorem for analytic sets, see Theorem 1.7, and we will omit it here.

There are many consequences of this theorem. First it implies that an injective Borel map $f : X \to Y$ between Polish spaces sends Borel sets to Borel sets and so a Borel bijection $f : X \to Y$ is necessarily a *Borel isomorphism*, i.e., has a Borel inverse. Furthermore, it shows that all uncountable Polish spaces are Borel isomorphic (therefore when studying questions at the Borel level we can work, without loss of generality, with any particularly chosen one). To see this, notice that if X is uncountable Polish, then there is a Borel injection of $\mathcal{N}$ into X (since $\mathcal{N} \subseteq \mathcal{C}$) and also by Section 1.1 there is a continuous bijection $f : F \to X$, where $F \subseteq \mathcal{N}$ is closed, so by taking its inverse there is a Borel injection of X into $\mathcal{N}$. Using a Schröder–Bernstein argument we can obtain a Borel bijection of X with $\mathcal{N}$.

It follows also that if a set X has two Polish topologies $\mathcal{T}, \mathcal{T}'$ with $\mathcal{T}$ consisting of Borel sets in $\mathcal{T}'$ (in particular if $\mathcal{T} \subseteq \mathcal{T}'$), then $\mathcal{T}, \mathcal{T}'$ have the same Borel sets. (Just look at the identity map $\mathrm{id}_X : (X, \mathcal{T}') \to (X, \mathcal{T})$.) In particular, we see that Theorems 1.2 and 1.3 can be strengthened to:

Theorem 1.2′. *If X is Polish and $A \subseteq X$, then A is Borel iff there is a Polish extension of the topology of X in which A is clopen.*

Theorem 1.3′. *If X is Polish, Y second countable and $f : X \to Y$, then f is Borel iff there is a Polish extension of the topology of X in which f is continuous.*

We already mentioned that the cardinality problem can be solved for Borel sets. In fact, Borel sets and functions have many more regularity properties. For example, it is clear that every Borel set has the BP and all Borel functions are Baire measurable. A much deeper regularity property of Borel sets is contained in the Borel Determinacy Theorem of Martin. Given $A \subseteq \mathcal{N}$ consider the game G_A defined as follows:

$$\begin{array}{llllll} \mathrm{I} & n_0 & & n_2 & & \\ & & & & & \cdots \\ \mathrm{II} & & n_1 & & n_3 & \end{array}$$

Player I starts by playing $n_0 \in \mathbb{N}$; II plays then $n_1 \in \mathbb{N}$; I follows by playing $n_2 \in \mathbb{N}$, etc. Then I wins this run of this game iff $x = (n_0, n_1, \ldots) \in A$. Otherwise II wins.

The notions of strategy and winning strategy are defined as before. We say that A is *determined* iff one of the players has a winning strategy in G_A. We now have the following theorem.

Theorem 1.5 (Martin). *Every Borel set $A \subseteq \mathcal{N}$ is determined.*

1.6. Analytic and co-analytic sets

Recall that, given a Polish space X, the analytic sets in X are the continuous images of Borel sets (from some Polish space into X). It is easy to see that they can be also characterized as the sets of the form $\mathrm{proj}_X(B)$, with $B \subseteq X \times Y$ Borel, Y a Polish space, or of the form $\mathrm{proj}_X(F)$, with $F \subseteq X \times \mathcal{N}$ closed. They turn out also to be the sets of the form $f(\mathcal{N})$, with $f : \mathcal{N} \to X$ continuous.

Clearly every Borel set is analytic, i.e.,

$$\mathbf{B}(X) \subseteq \mathbf{\Sigma}_1^1(X).$$

This inclusion is proper for X uncountable, in fact, $\mathbf{\Sigma}_1^1(X)$ is not closed under complements in this case. This is because for each uncountable X, there is an analytic set $\mathcal{U} \subseteq X \times X$ which is *universal*, in the sense that

$$\mathbf{\Sigma}_1^1(X) = \{\mathcal{U}_x : x \in X\}$$

(here $\mathcal{U}_x = \{y : (x, y) \in \mathcal{U}\}$). Then by a standard diagonalization argument the set

$$\mathcal{A} = \{x : (x, x) \in \mathcal{U}\}$$

is analytic, but its complement is not. On the other hand analytic sets are closed under countable intersections and unions, and continuous images.

There are many interesting examples of analytic (or co-analytic) sets which occur in nature. For instance, the set of uncountable compact sets is analytic but not Borel in

$\mathcal{K}(X)$ (the Polish space of compact subsets of X with the Vietoris topology), when X is uncountable (Hurewicz). The set of differentiable functions in $C([0,1])$ is co-analytic but not Borel (Mazurkiewicz); the set of simply connected compact sets in $\mathcal{K}(\mathbb{R}^2)$ is co-analytic but not Borel (Becker), etc.

One can also describe analytic sets in a form related to that of Theorem 1.2′ for the Borel sets.

Theorem 1.6. *Let X be a nonempty Polish space and $A \subseteq X$. Then A is analytic iff there is a topology $\mathcal{T}_A$ extending the topology of X, such that $\mathcal{T}_A$ is second countable and strong Choquet, in which A is open.*

The part of this result concerning the existence of such a $\mathcal{T}_A$ for any analytic A seems to be folklore, while the converse is due to Becker.

One of the fundamental properties of analytic sets is the following:

Theorem 1.7 (The Lusin Separation Theorem). *If X is Polish and $A, B \subseteq X$ are disjoint analytic, then there is a Borel set C separating A from B.*

As an immediate corollary we have

Theorem 1.8 (Souslin). *Let X be Polish. Then A is Borel iff A is both analytic and co-analytic.*

It is an immediate corollary, for example, that a function $f : X \to Y$ between Polish spaces is Borel iff its graph is Borel iff its graph is analytic. Because if graph(f) is analytic and $U \subseteq Y$ is open, then

$$x \in f^{-1}(U) \quad \Leftrightarrow \quad \exists y\,[(x,y) \in \operatorname{graph}(f) \;\&\; y \in U],$$

and

$$x \notin f^{-1}[U] \quad \Leftrightarrow \quad \exists y\,[(x,y) \in \operatorname{graph}(f) \;\&\; y \notin U],$$

so $f^{-1}(U)$ is both analytic and co-analytic, thus Borel.

Finally, analytic and co-analytic sets share many of the regularity properties of the Borel sets. For example, they are universally measurable (i.e., μ-measurable for any σ-finite Borel measure μ) and have the Baire property. One can also solve the cardinality problem for analytic sets: Every uncountable analytic set contains a Cantor set (Souslin). The same holds for co-analytic sets but the proof requires adding to ZFC a (very mild) large cardinal axiom (Solovay). Also from a large cardinal axiom it follows that all analytic (and thus all co-analytic) sets in $\mathcal{N}$ are determined (Martin).

General references for the results in this section are [16], [21] and [13].

2. Polish groups

2.1. Metrizable groups

A *topological group* is a group $(G, \cdot, 1)$ together with a topology such that $(x, y) \mapsto xy^{-1}$ is continuous (from G^2 into G). A well-known result of Birkhoff and Kakutani asserts that a topological group is metrizable iff it is Hausdorff and admits a countable neighborhood basis at the identity 1. Moreover, every such group admits a left-invariant compatible metric d (but not necessarily a (two-sided) invariant one).

There is a canonical procedure for completing a metrizable group. Let G be such a group and d be a left-invariant compatible metric. Then if $D(x,y) = d(x,y) + d(x^{-1}, y^{-1})$, D is also a compatible metric and if $(\overline{G}, \overline{D})$ is the completion of (G, D), then multiplication extends uniquely to $\overline{G}$, so that it becomes a topological group (with compatible metric $\overline{D}$).

2.2. Polish groups

Definition 2.1. A *Polish group* is a topological group whose topology is Polish.

Thus a Polish group admits a compatible complete metric and a compatible left-invariant metric, but it may not admit a compatible left-invariant complete metric (an example is the group S_∞ discussed below.) It follows also from Section 2.1 that every separable metrizable group embeds densely in a Polish group.

Examples. (i) Every Hausdorff second countable locally compact group is Polish.

(ii) The additive group of a separable Banach space is Polish.

(iii) The measure algebra of Lebesgue measure m on $[0,1]$ under Boolean addition and with the topology induced by the usual metric $d(A, B) = m(A \triangle B)$ is Polish.

Many examples of Polish groups arise as groups of "symmetries" of mathematical structures. Here are a few particular cases:

(iv) The *symmetric group* S_∞ on $\mathbb{N}$, with the topology it inherits as a subspace of the Baire space $\mathcal{N}$.

(v) The *unitary group* $U(H)$ of a separable Hilbert space, with the strong (or equivalently the weak) topology.

(vi) The *group of homeomorphisms* $H(X)$ of a compact metrizable space, with the topology it inherits as a subspace of $C(X, X)$.

(vii) The *group of isometries* $\mathrm{Iso}(X, d)$ of a complete separable metric space (X, d), with the topology generated by the maps $f \mapsto f(x)$, $x \in X$.

2.3. Closure properties

We collect here a few general closure properties of Polish groups.

(i) A closed subgroup of a Polish group is Polish. Note also that a G_δ subgroup of a Polish group is actually closed, a fact which follows easily from the Baire Category Theorem.

(ii) The product of countably many Polish groups is Polish.

(iii) The quotient of a Polish group by a closed subgroup is a Polish space. In particular, the quotient of a Polish group by a normal closed subgroup is a Polish group.

2.4. *Automatic continuity and openness*

The following basic results are proved by Baire category arguments and are actually valid in more general contexts.

Theorem 2.2 (Banach–Kuratowski–Pettis). *Let G be a Polish group and $A \subseteq G$ a nonmeager set having the Baire property. Then $A^{-1}A$ contains an open neighborhood of* 1. (*In particular, if A is also a subgroup, A is open, thus clopen.*)

Proof. Let U be open such that $A \triangle U$ is meager. Let V be an open neighborhood of 1 and $g \in G$ such that $gVV^{-1} \subseteq U$. Then $V \subseteq A^{-1}A$. $\square$

Some basic applications of this result are the following results about homomorphisms.

Theorem 2.3. *Let $\varphi : G \to H$ be a homomorphism between Polish groups. If φ is Baire measurable* (*e.g., Borel*), *then φ is continuous.*

Proof. To show that φ is continuous at 1, let U be an open neighborhood of $1 \in H$ and choose another open neighborhood V of $1 \in H$ with $V^{-1}V \subseteq U$. Let $\{h_n\}$ be dense in H. Then $\bigcup_n (h_n V) = H$, so for some n $\varphi^{-1}(h_n V)$ is nonmeager and has the Baire property. Apply the preceding theorem to conclude that $\varphi^{-1}(U)$ contains an open neighborhood of $1 \in G$. $\square$

Theorem 2.4. *Let $\varphi : G \to H$ be a continuous homomorphism between Polish groups. If $\varphi(G)$ is not meager, then φ is open. In particular, if φ is onto, then $G/\ker(\varphi)$ is isomorphic to H.* (*Two topological groups are* isomorphic *if there is an algebraic isomorphism between them which is also a homeomorphism.*)

Proof. Similar to the proof of the preceding result, using the fact that for each open set $U \subseteq G$, $\varphi(U)$ is analytic, so has the Baire property. $\square$

2.5. *Borel transversals*

Let us finally note the following two basic descriptive set theoretic facts.

Theorem 2.5. *Let $\varphi : G \to H$ be a continuous homomorphism between Polish groups. The canonical map $\varphi^* : G/\ker(\varphi) \to H$ is a continuous injective homomorphism of the*

Polish group $G/\ker(\varphi)$ onto $\varphi(G)$, thus $\varphi(G)$ is a Borel subgroup of H.

The proof is evident.

Theorem 2.6. *Let G be a Polish group and $H \subseteq G$ a closed subgroup. Then there is a Borel function $s : G/H \to G$ such that $s(xH) \in xH$, i.e., s is a* Borel selector *for the (left) cosets of H. In particular, there is a Borel subset of G (namely $s(G/H)$) meeting each coset in exactly one point (i.e., a* Borel transversal *for the cosets of H).*

Proof. For each Polish space X, let $\mathcal{F}(X)$ be the set of closed subsets of X. On $\mathcal{F}(X)$ we consider the σ-algebra $\mathcal{E}$ generated by the sets $\{F \in \mathcal{F}(X) : F \cap U \neq \emptyset\}$, for U open in X. Then $(\mathcal{F}(X), \mathcal{E})$ is called the *Effros Borel space* of $\mathcal{F}(X)$. There is a Polish topology on $\mathcal{F}(X)$ whose class of Borel sets is precisely $\mathcal{E}$. To see this, let $\overline{X}$ be a metrizable compactification of X and note that the map $F \in \mathcal{F}(X) \mapsto \overline{F}$ (= the closure of F in $\overline{X}$) is injective and has range a G_δ, therefore Polish, subspace of $\mathcal{K}(\overline{X})$, the space of compact subsets of $\overline{X}$ with the Vietoris topology, which is a compact metrizable space. Transfer back this Polish topology to $\mathcal{F}(X)$ via the inverse of $F \mapsto \overline{F}$ and note that its Borel sets are exactly those in $\mathcal{E}$.

So we can view $\mathcal{F}(X)$ as a Polish space. The following is a basic selector theorem about $\mathcal{F}(X)$.

Theorem 2.7 (Kuratowski–Ryll-Nardzewski). *There is a Borel function $f : \mathcal{F}(X) \to X$ such that $f(F) \in F$ for all $F \neq \emptyset, F \in \mathcal{F}(X)$.*

Proof. Fix a basis of nonempty open sets $\{U_n\}$ and a compatible complete metric d for X. For each $F \neq \emptyset$, inductively define a subsequence $n_0 < n_1 < n_2 < \cdots$ such that $\overline{U_{n_{i+1}}} \subseteq U_{n_i}$, $\mathrm{diam}(U_{n_{i+1}}) < 2^{-i-1}$, $U_{n_{i+1}} \cap F \neq \emptyset$ and n_{i+1} is least with these properties and let $\{f(F)\} = \bigcap_i U_{n_i}$. $\square$

To complete the proof of Theorem 2.6, note that the identity map $\varphi : G/H \to \mathcal{F}(G)$ is Borel, since for $U \subseteq G$ open, $xH \cap U \neq \emptyset \Leftrightarrow x \in UH$, and UH is open. Put then $s = f \circ \varphi$, where f is as in the preceding result. Then s is a Borel selector and its range $s(G/H) = T$ is a Borel transversal, since s is injective Borel. $\square$

2.6. Universal Polish groups

The Hilbert cube $I^{\mathbb{N}}$ is universal among Polish spaces, in the sense that every Polish space is homeomorphic to a subspace of $I^{\mathbb{N}}$. The following result establishes an analogous fact for Polish groups.

Theorem 2.8 (Uspenskii [26]). *The Polish group $H(I^{\mathbb{N}})$ of homeomorphisms of the Hilbert cube $I^{\mathbb{N}}$ is a universal Polish group, i.e., every Polish group is isomorphic to a (necessarily closed) subgroup of $H(I^{\mathbb{N}})$.*

Proof. Let X be a separable Banach space. Denote by $\mathrm{LIso}(X)$ the group of linear isometries of X. It is a closed subgroup of $\mathrm{Iso}(X,d)$, where d is the metric associated with the norm of X, so it is Polish.

Given a Polish group G we will find first a separable Banach space X, so that G is isomorphic to a closed subgroup of $\mathrm{LIso}(X)$. To do this let $d \leqslant 1$ be a left-invariant metric compatible with the topology of G. For $g \in G$, let $f_g : G \to \mathbb{R}$ be defined by $f_g(h) = d(g,h)$. Then $f_g \in C_b(G)$, the (not necessarily separable) Banach space of all bounded continuous real functions on G with the supremum norm. Let X be the closed linear subspace of $C_b(G)$ generated by $\{f_g : g \in G\}$, so that X is separable. For $g \in G$, let $U_g : C_b(G) \to C_b(G)$ be defined by $U_g(f)(h) = f(g^{-1}h)$. Then $U_g(X) \subseteq X$ and if $T_g = U_g|X$, $T_g \in \mathrm{LIso}(X)$ and $g \mapsto T_g$ is an isomorphism of G with a (necessarily closed) subgroup of $\mathrm{LIso}(X)$.

Let now $K = B_1(X^*)$ be the unit ball of the dual X^* of X with the weak*-topology, so that K is compact metrizable. For $S \in \mathrm{LIso}(X)$, let $S^* \in \mathrm{LIso}(X^*)$ be its adjoint. Then $S^*|K \in H(K)$. Put for $T \in \mathrm{LIso}(X)$, $h(T) = (T^{-1})^*|K \in H(K)$. Then h is an isomorphism of $\mathrm{LIso}(X)$ with a (necessarily closed) subgroup of $H(K) = H(B_1(X^*))$.

We use now the following basic result in infinite-dimensional topology, see Bessaga and Pelczynski [2].

Theorem 2.9 (Keller's Theorem). *Let X be a separable infinite-dimensional Banach space. Then $B_1(X^*)$ with the weak*-topology is homeomorphic to the Hilbert cube $I^{\mathbb{N}}$.*

So if X is infinite-dimensional, we have shown that G is isomorphic to a subgroup of $H(I^{\mathbb{N}})$. In the finite-dimensional case, $B_1(X^*)$ is homeomorphic to I^n, for some n and since $H(I^n)$ is isomorphic to a subgroup of $H(I^{\mathbb{N}})$ we are done as well. □

We conclude this section with an interesting open problem and some related results. We call a Polish group G *surjectively universal* if for every Polish group H there is a continuous surjective homomorphism $\varphi : G \to H$ or equivalently $H = G/N$, where N is a closed normal subgroup of G.

Problem 2.10. Is there a surjectively universal Polish group?

A positive answer is known (and seems to be a folklore result) if instead of all Polish groups one restricts attention to the following subclasses:

(i) All Polish groups that admit an invariant compatible metric;

(ii) All abelian Polish groups.

The proof in both cases is based on the existence of an interesting invariant metric on the free group $F(X)$ with set of generators X, where (X,d) is a metric space, called the *Graev metric*.

Given a nonempty set X, let $X^{+1} = \{x^{+1} : x \in X\}, X^{-1} = \{x^{-1} : x \in X\}$ be two disjoint copies of X and let $e \notin X^{+1} \cup X^{-1}$. Put $\overline{X} = X^{+1} \cup X^{-1} \cup \{e\}$. Let $W(X)$ be the set of finite sequences (words) $(\overline{x_0}, \ldots, \overline{x_n})$, where $\overline{x_i} \in \overline{X}$. We write $\overline{x_0} \cdots \overline{x_n}$ instead of $(\overline{x_0},\dots,\overline{x_n})$. A word w is *irreducible* if $w=e$ or else $w=x_0^{\varepsilon_0}x_1^{\varepsilon_1}\cdots x_n^{\varepsilon_n}$, with $\varepsilon_i=\pm 1$, $x_i\in X$ and $(x_i=x_{i+1}\Rightarrow \varepsilon_i=\varepsilon_{i+1})$. Denote by $F(X)$ the set of irreducible words. To each $w\in W(X)$ we associate a unique irreducible word obtained by successively replacing any $x^{\varepsilon}x^{-\varepsilon}$ in w by e and eliminating e from any occurrence of the form w_1ew_2, where at least one of w_1,w_2 is nonempty. Denote by w' this reduced word. As usual we turn $F(X)$ into a group, the *free group* with set of generators X, by defining

$$wv=(w^\frown v)',$$

$$1=e,$$

$$(x_0^{\varepsilon_0}x_1^{\varepsilon_1}\cdots x_n^{\varepsilon_n})^{-1}=x_n^{-\varepsilon_n}x_{n-1}^{-\varepsilon_{n-1}}\cdots x_1^{-\varepsilon_1}x_0^{-\varepsilon_0},$$

$$e^{-1}=e$$

(here $^\frown$ denotes concatenation of words). We view X as a subset of $F(X)$, identifying x with x^{+1}. Any map $f:X\to G$, where G is a group, extends uniquely to a homomorphism $\hat{f}:F(X)\to G$, given by $f(e)=1_G$,

$$f(x_0^{\varepsilon_0}\cdots x_n^{\varepsilon_n})=f(x_0)^{\varepsilon_0}\cdots f(x_n)^{\varepsilon_n}.$$

Assume now (X,d) is a metric space with $d<1$. We define a metric $\overline{d}$ on $\overline{X}$ by copying the metric d on X^{+1},X^{-1} and defining the distance between any point of X^{+1} with any point of X^{-1} to be 1, and similarly between e and any point of $X^{+1}\cup X^{-1}$. If $v=\overline{v_0}\cdots\overline{v_n}$, $w=\overline{w_0}\cdots\overline{w_n}$ are in $W(X)$, and have the same length, put

$$\rho(v,w)=\sum_{i=0}^{n}\overline{d}(\overline{v_i},\overline{w_i}).$$

Call a word $w\in W(X)$ *trivial* if $w'=e$. A *trivial extension* of a reduced word $w\in F(X)$ is any trivial word if $w=e$, while if $w=\overline{x_0}\cdots\overline{x_n}$, with $\overline{x_i}\in X^{+1}\cup X^{-1}$, it is any word of the form $y=A_0\overline{x_0}A_1\overline{x_1}\cdots A_n\overline{x_n}A_{n+1}$, with each A_i trivial or empty. In particular, $y'=w$.

Finally, the *Graev metric* on $F(X)$ is defined by

$$\delta(u,v)=\inf\{\rho(u^*,v^*):u^*,v^*\text{ are trivial extensions of }u,v\text{ resp., of the same length}\}.$$

The basic result, due to Graev [10] is that δ is an invariant metric on $F(X)$ extending d. The main argument in the proof is to show that $\delta(u,v)=0\Rightarrow u=v$. The following elegant proof is due to Swierczkowski. Given distinct $u,v\in F(X)$ it is enough to show that if $S=\{x\in X:x^{+1}\text{ or }x^{-1}\text{ occurs in }u\text{ or }v\}$, then

$$\delta(u,v)\geqslant\min\{1,\min\{d(x,y):x\neq y\text{ in }S\}\}. \qquad (*)$$

(This, by the way, also shows that δ extends d.) Assume $(*)$ fails, and let u^*,v^* be trivial extensions of u,v of the same length with $\rho(u^*,v^*)$ strictly less than this minimum. In particular, $\rho(u^*,v^*)<1$. Let $u^*=\overline{u_0}\cdots\overline{u_n}$, $\overline{v}^*=\overline{v_0}\cdots\overline{v_n}$, with $\overline{u_i},\overline{v_i}\in\overline{X}$.

Since $\rho(u^*, v^*) < 1$, each $(\overline{u_i}, \overline{v_i})$ is (e, e) or of the form (x^{+1}, y^{+1}), (x^{-1}, y^{-1}) for $x, y \in X$.

Consider the graph on X with edges the pairs (x, y) for which (x^{+1}, y^{+1}) or (x^{-1}, y^{-1}) is one of the $(\overline{u_i}, \overline{v_i})$. Since $\rho(u^*, v^*) < d(x, y)$ for $x \neq y$ in S, every component of the graph contains at most one element of S. So let $f : X \to X$ be such that $f(x)$ is the unique element of S in the component of x, if it exists, and some fixed element of X, otherwise. Extend f in the obvious way to $\overline{X}$ and then to words from $\overline{X}$ and call this extension $\overline{f}$. Then $\overline{f}(x) = x$ if $x \in S$, $\overline{f}(\overline{u_i}) = \overline{f}(\overline{v_i})$, and $\overline{f}$ maps trivial words to trivial words. So $\overline{f}(u^*), \overline{f}(v^*)$ are trivial extensions of u, v resp., but $\overline{f}(u^*) = \overline{f}(v^*)$, so $u = v$, a contradiction.

Note finally that if (X, d) is separable, so is $(F(X), \delta)$.

Denote now by $(\overline{F}(X), \overline{\delta})$ the completion of $(F(X), \delta)$. Then $\overline{F}(X)$, with the topology induced by $\overline{\delta}$, is Polish and admits the invariant compatible metric $\overline{\delta}$.

Take now $X = \mathcal{N}$ with its usual metric d given by

$$d(x, y) = 2^{-n-1}$$

where, for $x \neq y$ in $\mathcal{N}, n$ is least such that $x(n) \neq y(n)$.

Theorem 2.11. *The group $\overline{F}(\mathcal{N})$ is surjectively universal in the class of all Polish groups which admit an invariant compatible metric. The group $\overline{F}(\mathcal{N})/\overline{N}$, when N is the commutator subgroup of $\overline{F}(\mathcal{N})$, is surjectively universal in the class of all abelian Polish groups.*

Proof. Let H be Polish admitting an invariant compatible metric d_H. Then, using Sections 2.1 and 2.3(i), it is not hard to see that d_H is also complete. Let, by Section 1.1, $\varphi : \mathcal{N} \to H$ be a *Lipschitz surjection*, i.e., $d_H(\varphi(x), \varphi(y)) \leqslant d(x, y)$. Let $\hat{\varphi} : F(\mathcal{N}) \to H$ be the associated (surjective) homomorphism. It is not hard to check that $\hat{\varphi}$ is also Lipschitz (in the metrics δ, d_H), so $\hat{\varphi}$ has a unique extension $f : \overline{F}(\mathcal{N}) \to H$ to a surjective homomorphism, which is also Lipschitz (in the metrics $\overline{\delta}, d_H$), thus continuous, so the proof is complete.

The result about abelian groups is an immediate corollary. □

3. Borel actions of Polish groups

3.1. Basic concepts

Let G be a group and X a set. An action of G on X is a map $\alpha : G \times X \to X$, usually written as $\alpha(g, x) = g \cdot_\alpha x$, or just $g \cdot x$ if there is no danger of confusion, such that $1 \cdot x = x, g \cdot (h \cdot x) = gh \cdot x$. The *orbit* of a point $x \in X$ is the set $G \cdot x = \{g \cdot x : g \in G\}$. We denote by $X/G = \{G \cdot x : x \in X\}$ the set of orbits and by E_G the associated equivalence relation, whose equivalence classes are the orbits, i.e., $x \, E_G \, y \iff \exists g \in G(g \cdot x = y)$. A subset $Y \subseteq X$ is called *invariant* if $y \in Y \Rightarrow G \cdot y \subseteq Y$.

The *stabilizer* of a point $x \in X$ is the subgroup $G_x \subseteq G$ defined by $G_x = \{g \in G : g \cdot x = x\}$. There is a canonical bijection between G/G_x and $G \cdot x$ given by $gG_x \mapsto g \cdot x$.

3.2. Continuous actions

Suppose now G is a topological group and X a topological space. Then an action $(g,x) \mapsto g \cdot x$ is *continuous* if it is continuous as a function from $G \times X$ into X.

Examples. (i) The following are continuous actions of a topological group G on itself: The *left-action* $g \cdot x = gx$, the *right-action* $g \cdot x = xg^{-1}$ and the conjugation action $g \cdot x = gxg^{-1}$.

(ii) Let X be a Polish locally compact space. Denote by $\mathcal{F}(X)$ the set of closed subsets of X. The *Fell topology* on $\mathcal{F}(X)$ has as basis the sets of the form $\{F \in \mathcal{F}(X) : F \cap K = \emptyset \;\&\; F \cap U_1 \neq \emptyset \;\&\; \cdots \;\&\; F \cap U_n \neq \emptyset\}$ for K compact in X and U_i open in X. If $\hat{X} = X \cup \{\infty\}$ is the one-point compactification of X, the map $F \in \mathcal{F}(X) \mapsto F \cup \{\infty\} \in \mathcal{F}(\hat{X})$ is a homeomorphism of this space with the closed subspace of $\mathcal{F}(\hat{X})$ (with the Vietoris topology) consisting of all $\hat{F} \in \mathcal{F}(\hat{X})$ containing ∞. So $\mathcal{F}(X)$ is compact metrizable.

Let now G be a Polish locally compact group. Then $\mathcal{S}(G)$, the set of all closed subgroups of G is a closed subspace of $\mathcal{F}(G)$, so also compact metrizable. The left, right and conjugation actions of G on $\mathcal{F}(G)$ and the conjugation action of G on $\mathcal{S}(G)$ are all continuous.

(iii) If X is a compact metrizable space, $H(X)$ acts continuously on X by $h \cdot x = h(x)$. Similarly for (X,d) a complete separable metric space and the group $\mathrm{Iso}(X,d)$.

(iv) Let G be a countable (discrete) group and X a topological space. Then G acts continuously on X^G (with the product topology) by the *left-shift action* $g \cdot p(h) = p(g^{-1}h)$, the right-shift action $g \cdot p(h) = p(hg)$ and the *conjugation-shift action* $g \cdot p(h) = p(g^{-1}hg)$.

The verification that an action is continuous can be sometimes simplified by using the following fact (see, e.g., [13, 9.14]: If G is a Polish group and X is metrizable, then an action $(g,x) \mapsto g \cdot x$ of G on X is continuous iff it is separately continuous (i.e., $x \mapsto g \cdot x$ is continuous for each g and $g \mapsto g \cdot x$ is continuous for each x).

Assume now G is a Polish group acting continuously on a Polish space X. Given $x \in X$, each stabilizer G_x is clearly closed and the canonical bijection $gG_x \mapsto g \cdot x$ of the Polish space G/G_x onto the orbit $G \cdot x$ is continuous. It follows that each orbit $G \cdot x$ is Borel. (However it can be shown by examples that the associated equivalence relation E_G, which is clearly analytic, as a subset of X^2, might not always be Borel.) It is natural to ask further under what conditions this canonical bijection is a homeomorphism. The answer is given by the following basic result of Effros.

Theorem 3.1 (Effros [6]). *Let G be a Polish group, X a Polish space and $(g,x) \mapsto g \cdot x$ a continuous action of G on X. The following are equivalent for each $x \in X$:*

(i) *The canonical continuous bijection* $gG_x \mapsto g \cdot x$ *of* G/G_x *with* $G \cdot x$ *is a homeomorphism*;

(ii) $G \cdot x$ *is not meager in its relative topology*;

(iii) $G \cdot x$ *is* G_δ *in* X.

Proof. It is enough to prove (ii) $\Rightarrow$ (i). The following elegant argument is due to Becker.

Let $\psi : G \cdot x \to G/G_x$ be the inverse of the canonical map, i.e., $\psi(g \cdot x) = gG_x$. Clearly ψ is Baire measurable, since the canonical map, being a continuous injection, sends Borel sets to Borel sets by 1.4, thus ψ is continuous on a dense G_δ set $C \subseteq G \cdot x$. If ψ is not continuous, let $g_i, g_\infty \in G$ be such that $g_i \cdot x \to g_\infty \cdot x$, but $g_i G_x \not\to g_\infty G_x$ (in G/G_x). The same holds for hg_i, hg_∞ for all $h \in G$. So it is enough to find $h \in G$ such that $hg_i \cdot x, hg_\infty \cdot x \in C$ to obtain a contradiction. For that it is clearly sufficient to show that given $g \in G$, $hg \cdot x \in C$ for comeager many $h \in G$ or, equivalently, given $y \in G \cdot x$, there are comeager many $h \in G$ for which $h \cdot y \in C$. Now $C^* = \{y :$ for comeager many $h \in G$, $h \cdot y \in C\}$ is an invariant under the action set, so it is enough again to show that $G \cdot x \cap C^* \neq \emptyset$. Letting $\forall^*_Z$ mean "for comeager many in Z", we have $\forall h \in G \forall^*_{G\cdot x} y (h \cdot y \in C)$, so by the Kuratowski–Ulam Theorem (see, e.g., [13, 8.41], we have $\forall^*_{G\cdot x} y \forall^*_G h (h \cdot y \in C)$, so there is some $y \in G \cdot x$ (since $G \cdot x$ is not meager) with $y \in C^*$ and we are done. $\square$

The preceding result has interesting applications to topology (see, e.g., [7,23] and references contained therein). For instance, it follows that if (X, d) is compact metric, then X is homogeneous iff for each $\varepsilon > 0$ there is $\delta > 0$ such that if $d(x, y) < \delta$, then $\exists g \in H(X)(g(x) = y \;\&\; \forall z \in X(d(g(z), z) < \varepsilon))$ (Jones, Hagopian).

There is a related result which concerns the global structure of the orbit space X/G. Suppose G is a Polish group and X a Polish space on which G acts continuously. Consider the orbit space X/G with the quotient topology. In general this space is quite bad. For example, it may not be even T_0. In fact one has the following theorem.

Theorem 3.2 (Effros [6]). *Let* G *be a Polish group and* X *a Polish space on which* G *acts continuously. The following are equivalent.*

(i) X/G *is* T_0;

(ii) *every orbit* $G \cdot x$ *is* G_δ;

(iii) E_G *is* G_δ.

Proof. (i) $\Rightarrow$ (iii) The space X/G is second countable, so fix a basis $\{U_n\}$ for it. Put $\pi(x) = G \cdot x$. Then

$$x \, E_G \, y \;\Leftrightarrow\; \forall n (x \in \pi^{-1}(U_n) \;\Leftrightarrow\; y \in \pi^{-1}(U_n)),$$

so, since $\pi^{-1}(U_n)$ is open, E_G is G_δ.

(iii) $\Rightarrow$ (ii) Obvious.

(ii) $\Rightarrow$ (i) If $p = G \cdot x$, $q = G \cdot y$ are distinct, then $\overline{G \cdot x} \neq \overline{G \cdot y}$, since $G \cdot x$, $G \cdot y$ are G_δ. Then $\overline{\{p\}} \neq \overline{\{q\}}$, since $\overline{G \cdot x} = \pi^{-1}(\overline{\{p\}})$, $\overline{G \cdot y} = \pi^{-1}(\overline{\{q\}})$. $\square$

3.3. Borel actions

It will be convenient here to introduce the concept of a *standard Borel space*, i.e., a pair $(X, \mathcal{S})$ consisting of a set X and a σ-algebra $\mathcal{S}$ such that there is a Polish topology $\mathcal{T}$ on X whose class of Borel sets is exactly $\mathcal{S}$. We refer to the elements of $\mathcal{S}$ as the Borel sets of X as well. Except for the obvious example of a Polish space with its Borel sets, other interesting examples of standard Borel spaces include:

(i) The Effros Borel space on $\mathcal{F}(X)$, X a Polish space; see the proof of Theorem 2.6.

Note here that if G is a Polish locally compact group, the Effros Borel space on $\mathcal{F}(G)$ is exactly the Borel space of the Fell topology (see Example (ii) of Section 3.2).

(ii) The space $(L(H), \mathcal{S})$, where $L(H)$ is the set of bounded linear operators on a separable Hilbert space H and $\mathcal{S}$ the class of Borel sets is the strong (equivalently: weak) topology on $L(H)$. (This topology is not Polish if H is infinite-dimensional, so the fact that this is a standard Borel space is not automatic. It uses the fact that the unit ball of $L(H)$ is compact metrizable in the weak topology.)

(iii) If X is a standard Borel space and $Y \subseteq X$ a Borel set, then Y equipped with the σ-algebra of Borel subsets of it is also standard. This follows from Theorem 1.2. Also countable products of standard Borel spaces (equipped with the product σ-algebras) are standard.

Let now G be a Polish group and X a standard Borel space. An action $(g, x) \mapsto g \cdot x$ of G on X is *Borel* if it is Borel as a function from (the standard Borel product space) $G \times X$ into X. (A function between standard Borel spaces is called Borel if the inverse image of any Borel set is Borel.)

For example, if G is a Polish group, the left, right and conjugation actions on the Effros Borel space $\mathcal{F}(G)$ of all closed subsets of G are all Borel. Similarly, the conjugation action of G on $\mathcal{S}(G)$, the space of closed subgroups of G (which is a Borel subset of $\mathcal{F}(G)$, thus standard) is Borel. Also if H is a (complex) separable Hilbert space and $U(H)$ its unitary group, then $U(H)$ acts on $L(H)$ by conjugation

$$g \cdot T = g \circ T \circ g^{-1}.$$

This is a Borel action. Its corresponding equivalence relation is *unitary equivalence* of operators.

As we will see next, among all Borel actions of a Polish group G on standard Borel spaces there is a universal one. If G acts in a Borel way on standard Borel spaces X, Y a *Borel G-embedding* is a Borel injection $\pi : X \to Y$ such that $\pi(g \cdot x) = g \cdot \pi(x)$. It is a *Borel G-isomorphism* if it is also onto. Note that if π is a Borel G-embedding, then $\pi(X)$ is a Borel invariant subset of Y, so that π is a Borel G-isomorphism between X and $\pi(X)$. Also, by a Schröder–Bernstein argument, if each of X, Y Borel G-embeds in the other, X and Y are Borel G-isomorphic. A Borel action of G on a standard Borel space Y is *universal* if for all Borel actions of G on a standard Borel space X, there is a Borel G-embedding of X into Y. Clearly Y is uniquely determined, if it exists, up to Borel G-isomorphism.

Theorem 3.3 (Becker–Kechris [1]). *For each Polish group G there is a universal Borel action of G on a standard Borel space.*

Proof. Consider the standard Borel space $\mathcal{F}(G)$ and the product space $U_G = \mathcal{F}(G)^{\mathbb{N}}$. Consider also the left-action of G on U_G, $g \cdot (F_n) = (gF_n)$. We will show that this is universal.

For each $A \subseteq G$, let

$$E(A) = \{g \in G : \forall \text{ open neighborhoods } V \text{ of } g,\ V \cap A \text{ is not meager}\}.$$

Then $E(A)$ is closed and if A has the Baire property, $E(A) \triangle A$ is meager.

Given now a Borel action of G on a standard Borel space X, let $\{S_n\}$ be a sequence of Borel sets in X separating points. Define $\pi : X \to \mathcal{U}_G$ by

$$\pi(x) = (E(\{g : g \cdot x \in S_n\})^{-1})_n.$$

This is a Borel G-embedding of X into $\mathcal{U}_G$. $\square$

In case G is locally compact the preceding result is due to Mackey [17] and Varadarajan [27].

3.4. Topological realization of Borel actions

It is always possible to view any Borel action of a Polish group as a continuous one. This is the content of the next result.

Theorem 3.4 (Becker–Kechris [1]). *Let G be a Polish group and X a standard Borel space. If $(g,x) \mapsto g \cdot x$ is a Borel action of G on X, then there is a Polish topology $\mathcal{T}$ on X which generates the Borel structure of X such that the action is continuous with respect to this topology.*

Proof. The key tool in this, as well as many other results on Polish group actions, is the notion of *Vaught Transform*. For $A \subseteq X$ and $U \subseteq G$ open define

$$A^{\triangle U} = \{x : \text{for nonmeager many } g \in U,\ g \cdot x \in A\}.$$

(One also defines the transform

$$A^{*U} = \{x : \text{for comeager many } g \in U,\ g \cdot x \in A\}$$

but we will not need it here.) It can be calculated that if A is Borel, so is $A^{\triangle U}$. Fix now a countable basis $\mathcal{B}$ for G. One can see, using the type of construction outlined in the proof of Theorem 1.2, that there is a countable Boolean algebra $\mathcal{C}$ of Borel subsets of X such that

(i) $A \in \mathcal{C} \Rightarrow A^{\Delta U} \in \mathcal{C}$, $\forall U \in \mathcal{B}$;

(ii) the topology generated by $\mathcal{C}$ is Polish.

Let now $\mathcal{T}$ be the topology generated by $\{A^{\Delta U} : A \in \mathcal{C},\ U \in \mathcal{B}\}$ (which is contained in the topology generated by $\mathcal{C}$).

It can be first checked by direct calculations that the action is continuous in $\mathcal{T}$. To show that $\mathcal{T}$ is Polish we apply the Choquet criterion, see Theorem 1.1. Finally, $\mathcal{T}$ consists of Borel sets, thus it generates exactly the Borel structure of X. □

Here is an immediate corollary (which can be also proved directly).

Theorem 3.5 (Miller [20]). *Let G be a Polish group, X a standard Borel space and $(g,x) \mapsto g \cdot x$ a Borel action of G on X. Then each stabilizer G_x is closed and every orbit $G \cdot x$ is Borel.*

It also follows that the universal action of Theorem 3.3 can be always assumed to be continuous, with the underlying space Polish. When G is locally compact the underlying space can be also taken to be compact (see [17,27]), but this is unknown in the general case. (It holds though for the (nonlocally compact) symmetric group S_∞—see [1].)

Let us mention one more application, this time in the context of so-called *paradoxical decompositions*. Our basic reference will be the book of Wagon [28].

Let G be a group acting on a set X. Given $A, B \subseteq X$ we put

$$A \sim B,$$

if there are partitions $A = \bigcup_{i=1}^n A_i$, $B = \bigcup_{i=1}^n B_i$ for some n, and $g_i \in G$, $1 \leqslant i \leqslant n$, with $g_i \cdot A_i = B_i$. We call X *G-paradoxical* if $X \sim A \sim B$ with $A \cap B = \emptyset$. For example, S^2 is G-paradoxical, where G is the group of rotations of S^2 (*The Banach–Tarski Paradox*). A *finitely additive probability measure* on X is a map $\varphi : \mathcal{P}(X) \to [0,1]$, where $\mathcal{P}(X)$ is the power set of X, such that $\varphi(X) = 1$ and $\varphi(A \cup B) = \varphi(A) + \varphi(B)$, if $A \cap B = \emptyset$. It is *G-invariant* if $\varphi(g \cdot A) = \varphi(A)$, $\forall g \in G$. One has now the following basic result of Tarski:

Theorem 3.6 (Tarski). *Let a group G act on a set X. Then X is not G-paradoxical iff there is a G-invariant finitely additive probability measure on X.*

The problem has been raised whether there is an analog of Tarski's Theorem for countably additive probability measures.

For that consider now a measurable space $(X, \mathcal{A})$, i.e., a set with a σ-algebra. Let a group G act on X so that $A \in \mathcal{A} \Rightarrow g \cdot A \in \mathcal{A}$. For $A, B \in \mathcal{A}$ we put now

$$A \sim_\infty B,$$

if there are partitions $A = \bigcup_{i\in I} A_i, B = \bigcup_{i\in I} B_i$, with I countable, $A_i, B_i \in \mathcal{A}$, and $g_i \in G$, $i \in I$, such that $g_i \cdot A_i = B_i$. We say again that X is *countably G-paradoxical* if $X \sim_\infty A \sim_\infty B$, with $A, B \in \mathcal{A}$, $A \cap B = \emptyset$. A (countably additive) probability measure $\mu : \mathcal{A} \to [0,1]$ on $(X, \mathcal{A})$ is *G-invariant* if $\mu(g\cdot A) = \mu(A), \forall g \in G$. If such a measure exists clearly X is not countably G-paradoxical. The converse turns out to be false in this generality, as various pathologies can arise. However, it turns out to be true in most regular situations. Below by a *Borel probability measure* on a standard Borel space we mean a (countably additive) probability measure on the σ-algebra of its Borel sets.

Theorem 3.7 (Becker–Kechris [1]). *Let G be a Polish group, X a standard Borel space and $(g \cdot x) \mapsto g \cdot x$ a Borel action of G on X. Then X is not countably G-paradoxical iff there is a G-invariant Borel probability measure on X.*

Proof. It was shown by Nadkarni [22] that this result holds for $G = \mathbb{Z}$ and his proof can be readily adapted to any countable group G. For an arbitrary Polish G, let G' be a countable dense subgroup of G. By 3.4 we can assume that the action is continuous. If X is not G-paradoxical, it is obviously not G'-paradoxical, so by Nadkarni's Theorem it has a G'-invariant Borel probability measure, which is then G-invariant by the continuity of the action. □

3.5. *The topological Vaught conjecture*

The following basic dichotomy concerning the quotient space of a co-analytic equivalence was proved by Silver. For an equivalence relation E on a set X, we denote by X/E the set of E-equivalence classes.

Theorem 3.8 (Silver [25]). *Let X be a Polish space and E a co-analytic (as a subset of X^2) equivalence relation on X. Then either X/E is countable or else there is a Cantor set $C \subseteq X$ consisting of pairwise E-inequivalent elements.*

Using Silver's result, Burgess established the following version for analytic equivalence relations.

Theorem 3.9 (Burgess [3]). *Let X be a Polish space and E an analytic equivalence relation on X. Then either $|X/E| \leqslant \aleph_1$ or else there is a Cantor set $C \subseteq X$ consisting of pairwise E-inequivalent elements.*

The first alternative "$|X/E| \leqslant \aleph_1$" cannot be improved here to "X/E is countable", as various examples show.

The equivalence relation E_G induced by a continuous action of a Polish group G on a Polish space X is analytic (and is generally not Borel). However, the following is still open.

The topological Vaught conjecture (Miller). *Let G be a Polish group acting continuously on a Polish space X. Then either X/G is countable or else there is a Cantor set $C \subseteq X$ consisting of pairwise E_G-inequivalent elements.*

This conjecture is motivated by and generalizes a famous long-standing conjecture in model theory known as *Vaught's Conjecture*, which is also open at the time of writing. (See, e.g., [1] for more on this.)

4. Borel equivalence relations

4.1. The Glimm–Effros dichotomy

Consider a Polish space X and a Borel equivalence relation E on X (i.e., E is Borel as a subset of X^2). We call E *smooth* if there is a Polish space Y and a Borel map $f : X \to Y$ such that

$$x E y \iff f(x) = f(y).$$

This means that one can classify elements of X up to E-equivalence by explicitly computable invariants of a fairly concrete nature. Note also that E is smooth iff it has a countable Borel *separating family*, i.e., a family (A_n) of Borel sets such that $x E y \iff \forall n(x \in A_n \Leftrightarrow y \in A_n)$. One of the canonical examples of such a situation is the case $X = \mathbb{C}^{n^2}$ = the space of $n \times n$ matrices from $\mathbb{C}$, E = the equivalence relation of similarity and $f(A)$ = the Jordan canonical form of A. In this particular case we actually have that f is a *Borel selector*, i.e., $f : X \to X$, $f(x)\, E\, x$, $x E y \iff f(x) = f(y)$, but in general the existence of a Borel selector is a stronger condition. Notice also that the existence of a Borel selector is equivalent to the existence of a *Borel transversal*, i.e., a Borel set $T \subseteq X$ which meets every equivalence class in exactly one point.

A standard example of a nonsmooth equivalence relation is the classical *Vitali equivalence relation*: $X = [0, 1]$, $E_V = \{(x, y) : x - y \in \mathbb{Q}\}$. To see that it is not smooth let us note the following general fact, after we introduce some terminology.

Given a Polish space X and a Borel equivalence relation E on X we say that a Borel probability measure μ on X is *E-ergodic* if $\mu(A) = 0$ or $\mu(A) = 1$ for any E-invariant Borel set $A \subseteq X$ (where A is E-invariant iff $x \in A \;\&\; y E x \Rightarrow y \in A$) and that μ is *E-nonatomic* if $\mu([x]_E) = 0$, for any E-equivalence class $[x]_E$.

It is easy to see now that a Borel equivalence relation which admits an ergodic nonatomic Borel probability measure μ is not smooth. Otherwise, if $f : X \to Y$ demonstrates the smoothness of E, then the Borel probability measure $f\mu(A) = \mu(f^{-1}(A))$ on Y takes only values in $\{0, 1\}$ and $f\mu(\{y\}) = 0$ for all $y \in Y$, which gives easily a contradiction.

Since the usual Lebesgue measure on $[0, 1]$ is E_V-ergodic, the Vitali equivalence relation is not smooth.

Many times it is easier to work with a combinatorial manifestation of E_V. Consider the Cantor space $X = 2^{\mathbb{N}}$ and the equivalence relation $E_0 = \{(x, y) : \exists n \forall m \geqslant n(x(m) = y(m))\}$. It can be actually shown that E_0, E_V are Borel isomorphic, but it is easy to see directly that the usual coin-tossing measure on X is E_0-ergodic (this is the Zero–One Law) and E_0-nonatomic, so E_0 is nonsmooth.

In the case where the Borel equivalence relation E is F_σ and induced by a continuous action of a Polish group G on a Polish space X, Effros [6,7], following up on work of Glimm [9] (who dealt with the case where G, X are locally compact) proved a basic dichotomy concerning the equivalence relation E, which asserts that exactly one of the following two alternatives occurs: Either E is smooth or else E contains a "copy" of E_0. More precisely, given a Borel equivalence relation E on a Polish space X we write

$$E_0 \sqsubseteq E,$$

if there is a continuous embedding $f : 2^{\mathbb{N}} \to X$ such that

$$x \, E_0 \, y \Leftrightarrow f(x) \, E \, f(y).$$

We now have the following theorem.

Theorem 4.1 (The Glimm–Effros Dichotomy; Effros [6,7]). *Let G be a Polish group and X a Polish space on which G acts continuously. If the associated equivalence relation E_G is F_σ, then exactly one of the following holds*:

(I) *E_G is smooth*;
(II) $E_0 \sqsubseteq E_G$.

Moreover, each of these alternatives has the following equivalent versions:

(a) (I) *is equivalent to*:
(I^{i}) *X/G is T_0*;
(I^{ii}) *Every orbit is G_δ*;
($\mathrm{I}^{\mathrm{iii}}$) *Every orbit is locally closed* (*i.e., the difference of two closed sets*);
(I^{iv}) *E_G is G_δ*;
(I^{v}) *Every orbit is of the second category in its relative topology*;
(I^{vi}) *For each x, the canonical map $gG_x \mapsto g \cdot x$ is a homeomorphism*;
($\mathrm{I}^{\mathrm{vii}}$) *There is a Borel selector for E_G*.
(*For* (I^{i})–(I^{vi}) *compare with Theorems* 3.1 *and* 3.2.)

(b) (II) *is equivalent to*
(II^{i}) *E_G admits an ergodic nonatomic Borel probability measure.*

Note that when G is Polish locally compact, E_G is automatically F_σ, so all the preceding hold for continuous actions of such groups.

The Glimm–Effros dichotomy was first discovered in the representation theory of locally compact groups and C^*-algebras, more particularly in the context of Glimm's proof [8], and the subsequent simplification by Effros [6], of the so-called Mackey "smooth dual iff type I" conjecture. It has since found many other applications, for example in ergodic theory (see, e.g., [12,14,15,24,29]).

4.2. An application to topology

An appropriate form of a Glimm–Effros type dichotomy has been recently proved and applied by Solecki to settle an old problem in continua theory. An earlier application of the original Glimm–Effros dichotomy in this context can be found in [23].

Let X be an *indecomposable continuum*, i.e., one that is not the union of two proper subcontinua. Let E be the equivalence relation whose classes are the composants of X, where a *composant* is the union of all proper subcontinua containing a given point of X. The following is an old problem (see, e.g., [19]): Is there a Borel selector for E? A negative answer in special cases has been obtained in [4,23,5] (see also references therein). Now it is known that each composant is dense and there are continuum many composants, and Rogers [23] shows that E is a K_σ equivalence relation. So the following general result (which is actually a particular case of Solecki's theorem) solves the problem in full generality.

Theorem 4.2 (Solecki). *Let X be a Polish space and E a K_σ equivalence relation on X, with at least two equivalence classes, such that all of its equivalence classes are dense. Then $E_0 \sqsubseteq E$, so E is not smooth (and thus has no Borel selector).*

4.3. The Glimm–Effros dichotomy for Borel equivalence relations

The Glimm–Effros dichotomy has been extended the last few years to the very general context of an arbitrary Borel equivalence relation. More precisely one has the following:

Theorem 4.3 (Harrington–Kechris–Louveau [11]). *Let X be a Polish space and E a Borel equivalence relation. Then exactly one of the following holds:*

(I) *E is smooth;*

(II) *$E_0 \sqsubseteq E$.*

Moreover, each of the alternatives has the following equivalent formulations.

(a) (I) *is equivalent to*

(I$^{\mathrm{i}}$) *There is a Polish topology $\mathcal{T}$ on X, extending its given one, in which E is G_δ in $(X^2, \mathcal{T}^2)$;*

(I$^{\mathrm{ii}}$) *Same as* (I$^{\mathrm{i}}$) *but with E closed in $(X^2, \mathcal{T}^2)$.*

(b) (II) *is equivalent to*

(II$^{\mathrm{i}}$) *E admits an ergodic, nonatomic Borel probability measure.*

Although this result has a totally classical descriptive set theoretic formulation, the proof requires the use of methods of the so-called *effective descriptive set theory* and so is fundamentally dependent on the *theory of computability* on the integers. (The proof of the original Glimm-Effros Dichotomy 4.1 uses only classical topological and combinatorial constructions.) No other proof of this result is known.

4.4. *Effective descriptive set theory*

We will give here a brief sketch of the main ideas of effective descriptive set theory leading up to the description of the so-called *Gandy-Harrington topology* which is the crucial tool in the proof of 4.3. For a more detailed introduction the reader can consult [21,18,11].

For simplicity, and without any essential loss of generality (since all uncountable Polish spaces are Borel isomorphic), we will work below in a concrete setting, namely the Baire space $\mathcal{N}$. The canonical basis for this space consists of the sets

$$N_s = \{x \in \mathcal{N} : s \subseteq x\},$$

where $s \in \mathbb{N}^{<\mathbb{N}} = \bigcup_n \mathbb{N}^n$ is a finite sequence and $s \subseteq x \Leftrightarrow x$ extends s (i.e., if $s \in \mathbb{N}^n$, $x(i) = s(i)$, for $i < n$). Any open set in $\mathcal{N}$ is of the form $G = \bigcup_{n \in \mathbb{N}} N_{s_n}$, where (s_n) is a sequence of elements of $\mathbb{N}^{<\mathbb{N}}$.

A set $G \subseteq \mathcal{N}$ is called *effectively open* if $G = \bigcup_{n \in \mathbb{N}} N_{s_n}$, where $n \mapsto s_n$ is *computable.* Intuitively this means that there is an algorithm which for each n computes, in finitely many steps, s_n. This can be formalized by defining precisely what is meant by algorithm here (e.g., in terms of Turing machines).

Notice that there are only countably many effectively open sets. Some examples are: N_s $(s \in \mathbb{N}^{<\mathbb{N}})$, $\{x \in \mathbb{N} : x(n) \text{ is odd for some } n\}$, etc.

It will be convenient also to consider the discrete space $\mathbb{N}$ and call a subset $G \subseteq \mathbb{N}$ *effectively open* if $G = \{k_n : n \in \mathbb{N}\}$ $(= \bigcup_n \{k_n\})$, where $n \mapsto k_n$ is computable. (These sets are also called, for obvious reasons, *effectively enumerable* in computability theory.) Finally, we extend this to define effectively open sets in product spaces $X = X_1 \times \cdots \times X_k$, where each X_i is $\mathcal{N}$ on $\mathbb{N}$. For example, if $X = \mathcal{N} \times \mathcal{N} \times \mathbb{N}$, the canonical basis for X consists of sets of the form $N_s \times N_t \times \{k\}$, with $s, t \in \mathbb{N}^{<\mathbb{N}}$, $k \in \mathbb{N}$. A set $G \subseteq X$ is effectively open if $G = \bigcup_n N_{s_n} \times N_{t_n} \times \{k_n\}$, where $n \mapsto (s_n, t_n, k_n)$ is computable.

We call a set G (in one of these product spaces) *effectively closed* if its complement is effectively open.

A function $f : X \to \mathcal{N}$ is *effectively continuous* if the set

$$G_f = \{(x, s) : f(x) \in N_s\}$$

is effectively open in $X \times \mathbb{N}^{<\mathbb{N}}$, where we identify $\mathbb{N}^{<\mathbb{N}}$ here with $\mathbb{N}$ via some canonical computable bijection. This just means that $f^{-1}(N_s)$ is effectively open for each $s \in \mathbb{N}^{<\mathbb{N}}$, *uniformly* on s. Similarly, $f : X \to \mathbb{N}$ is effectively continuous if

$$G_f = \{(x, k) : f(x) \in \{k\}\} = \{(x, k) : f(x) = k\}$$

is effectively open in $X \times \mathbb{N}$. Finally, we call $f : X \to Y = Y_1 \times \cdots \times Y_n$ effectively continuous if $f = (f_1, \ldots, f_n)$ where each $f_i : X \to Y_i$ is effectively continuous.

For example, the map $f(x,n) = x(n)$ from $\mathcal{N} \times \mathbb{N}$ into $\mathbb{N}$ is effectively continuous and the bijection $(x,y) \mapsto (x(0), y(0), x(1), y(1), \ldots)$ is an effective homeomorphism (i.e., effectively continuous with effectively continuous inverse) of $\mathcal{N}^2$ with $\mathcal{N}$, etc.

We define next effectively analytic sets. Recall that the analytic sets in X can be characterized as the projections of closed sets in $X \times \mathcal{N}$. We say that $A \subseteq X$ is *effectively analytic* if $A = \mathrm{proj}_X(B)$, where $B \subseteq X \times \mathcal{N}$ is effectively closed. It is *effectively co-analytic* if its complement is effectively analytic.

One can also define the concept of an effectively Borel set but the definition is somewhat more complex and we will omit here. It turns out again though that a set is effectively Borel iff it is effectively analytic and co-analytic (Kleene).

How are these notions of effective descriptive set theory related with the classical ones we introduced in Section 1? The key is the concept of *relativization*.

Given an *arbitrary* $p \in \mathcal{N}$ we say that a function $n \mapsto k_n$ ($k_n \in \mathbb{N}$) is *computable in* (*or relative to*) p if there is an algorithm for computing, in finitely many steps, for each n the value k_n, where the algorithm is allowed to contain instructions asking for the value of p at certain arguments (which, as p itself is not computable, is thought to be produced by an "oracle" for p). Obviously p (i.e., $n \mapsto p(n)$) is computable in p. Also, for example,

$$n \mapsto n \cdot 2^{p(n)}$$

is computable in p. Similarly, we define what it means for $n \mapsto s_n$ ($s_n \in \mathbb{N}^{<\mathbb{N}}$) to be computable in p by identifying $\mathbb{N}^{<\mathbb{N}}$, via a computable bijection, with $\mathbb{N}$. Also instead of the parameter p being in $\mathcal{N} = \mathbb{N}^{\mathbb{N}}$ we can consider the case where p is, e.g., in $(\mathbb{N}^{<\mathbb{N}})^{\mathbb{N}}$ via the same identification.

Replacing throughout in the earlier definitions "computable" by "computable in p" we can define the relativized concepts: effectively open in p, effectively closed in p, effectively continuous in p, effectively analytic in p, effectively co-analytic in p and effectively Borel in p (which again turns out to be the same as effectively analytic and co-analytic in p). It is clear that every effectively open in p set is (classically) open and it is easy to check that if G is open, then it is effectively open in *some* p. For example, if $G \subseteq \mathcal{N}$ is open, so that $G = \bigcup_n N_{s_n}$, for some (s_n), then G is effectively open in p, where $p = (n \mapsto s_n)$. So we have the following basic relation between the classical and the (relativized) effective notions:

$$* = \bigcup_{p \in \mathcal{N}} (\text{effectively } * \text{ in } p),$$

where $*$ = open, closed, continuous, Borel, analytic or co-analytic.

In view of this fact, *effective descriptive set theory*, i.e., the study of the effective concepts introduced above, appears as a refinement of the classical theory. Results proved for the effective notions imply immediately corresponding classical ones. However in the effective theory we have additional powerful tools and ideas from computability theory, which are not available in the classical context. These can be used to provide often much simpler (once the basic effective theory is understood) proofs of many known results in descriptive set theory, but also prove new ones for which no other classical-type method is known. An example of this is the preceding Theorem 4.3.

4.5. *The Gandy–Harrington topology*

Recall from Theorem 1.6 that for every analytic set $A \subseteq X$, X a Polish space, there is a second countable strong Choquet topology, extending the topology of X, in which A is open.

Let us now take X to be one of the product spaces as in Section 4.4. Then A is effectively analytic in some $p \in \mathcal{N}$. Consider the topology with basis all effectively analytic in p subsets of X (they are closed under finite intersections, so this is indeed a basis). This is called the *Gandy–Harrington topology in* p. Denote it by $\mathcal{T}_p$. It is clearly second countable and extends the usual topology of X. It can be shown that it is also strong Choquet. (It is not however Polish (it lacks regularity), but gets very close to being so—there is a dense open set whose relative topology is Polish.) Thus $\mathcal{T}_p$ becomes a canonical (once p is chosen) choice of a second countable strong Choquet topology which makes A (basic) open. In fact it has a lot of other remarkable properties, which are a consequence of the detailed effective theory (which we are not developing here), and which account for its striking success.

The idea of the Gandy–Harrington topology was introduced in the 1960's by Gandy, but was not used too much until Harrington used it to give, in the 1970's, a much simpler proof of Silver's Theorem 3.8. (By the way, Silver's original proof used in some form or other metamathematical forcing techniques, so no classical-type proof of that theorem is known.) After that the Gandy–Harrington topology has become an important tool in descriptive set theory and in particular is used in the proof of Theorem 4.3. We are ready now to give a quick outline of the main steps in the proof of that theorem.

Let X be a Polish space and E a Borel equivalence relation on X. We can assume that X is uncountable, thus Borel isomorphic to $\mathcal{N}$, so there is no loss of generality to assume that $X = \mathcal{N}$. Let then $p \in \mathcal{N}$ be such that E is effectively analytic and co-analytic (i.e., effectively Borel) in p. Let $\mathcal{T} = \mathcal{T}_p$ be the Gandy–Harrington topology in p for the space $\mathcal{N}$. Consider the space $(\mathcal{N}^2, \mathcal{T}^2)$. There are two possibilities: Either E is closed in this space or it is not. In the first case it can be shown that E is smooth (in fact in an effective sense, i.e., there is an effectively Borel in p function $f : \mathcal{N} \to \mathcal{N}$ such that $xEy \Leftrightarrow f(x) = f(y)$). In the second case, let $\overline{E}$ be its closure in $(\mathcal{N}^2, \mathcal{T}^2)$, so that $E \neq \overline{E}$, thus $\{x : E_x \neq \overline{E_x}\} = V \neq \emptyset$ (where for $A \subseteq \mathcal{N}^2$, $A_x = \{y : (x, y) \in A\}$). Using the effective theory again it turns out that V is effectively analytic in p, so open in $\mathcal{T}$. Let us give a brief sketch of how this is proved, which can perhaps give some feeling of the effective methods.

Since $x \in V \Leftrightarrow \exists y((x, y) \in \overline{E}\ \&\ (x, y) \notin E)$, V is the projection of $\overline{E} \cap (\mathcal{N}^2 \setminus E)$, so, as effectively analytic in p sets are closed under finite intersections and projections, it will be enough to show that $\overline{E}$ is effectively analytic in p. We claim first that

$$(x, y) \in \overline{E} \Leftrightarrow \text{for all } E\text{-invariant effectively Borel in } p \text{ sets } C \subseteq \mathcal{N} \qquad (*)$$

$$x \in C \iff y \in C$$

(This by the way implies also that $\overline{E}$ is an equivalence relation and a G_δ set in $\mathcal{T}^2$, facts which turn out to be important in the rest of the proof.) To see this, note first that the direction $\Rightarrow$ of the claim is straightforward. For the other direction, it is enough to show that if $(x,y) \notin \overline{E}$, then there is an effectively Borel in p E-invariant set C such that $x \in C$, $y \notin C$. Since $(x,y) \notin \overline{E}$, let A, B be effectively analytic in p sets such that $x \in A$, $y \in B$ and $(A \times B) \cap E = \emptyset$, so $[A]_E \cap [B]_E = \emptyset$, where $[D]_E = \{x \in \mathcal{N} : \exists y(y \in D \,\&\, xEy)\}$ is the *E-saturation* of $D \subseteq \mathcal{N}$. Now $[A]_E, [B]_E$ can be easily computed to be effectively analytic in p sets, so by the effective version of the Lusin Separation Theorem 1.7, we can find an effectively Borel in p set C_0 with $[A]_E \subseteq C_0$, $C_0 \cap [B]_E = \emptyset$. Then $[A]_E \subseteq [C_0]_E$, and $[C_0]_E \cap [B]_E = \emptyset$. By the same argument we can find an effectively Borel in p set C_1, such that $[C_0]_E \subseteq C_1$ and $C_1 \cap [B]_E = \emptyset$, etc. Put $C = \bigcup_n C_n$. Then $x \in C$, $y \notin C$ and C is E-invariant. Although in general an arbitrary countable union of effectively Borel in p sets might not be effectively Borel in p, the effective version of the Lusin Separation Theorem contains a certain "uniformity", which with some additional effective arguments can be used to guarantee that C is indeed effectively Borel in p and complete the proof of the claim $(*)$.

It remains to see how $(*)$ implies that E is effectively analytic in p. This depends on a fundamental parametrization result for the effectively Borel in p sets: There is an effectively co-analytic in p set $W \subseteq \mathbb{N} \times \mathcal{N}$, an effectively co-analytic in p set $P \subseteq \mathbb{N} \times \mathcal{N}$ and an effectively analytic in p set $S \subseteq \mathbb{N} \times \mathcal{N}$ such that

$$n \in W \Rightarrow P_n = S_n \quad (=: D_n)$$

and

$$\{D_n : n \in W\} = \text{the class of effectively Borel in } p \text{ subsets of } \mathcal{N}.$$

Using this we obtain

$$(x,y) \in \overline{E} \iff \forall n(n \in W \Rightarrow (x \in D_n \iff y \in D_n)),$$

which with some easy calculations, which make use of basic closure properties of the effectively analytic in p sets, shows that $\overline{E}$ is effectively analytic in p.

We conclude now the outline of the proof as follows:

Clearly E is dense in $V^2 \cap \overline{E}$ and further applications of the effective theory show that E is also meager in $V^2 \cap \overline{E}$. This is used to prove that there is continuous embedding $f : 2^{\mathbb{N}} \to \mathcal{N}$ with $x\,E_0\,y \iff f(x)\,E\,f(y)$, i.e., $E_0 \sqsubseteq E$. The density condition is used to make sure that $x\,E_0\,y \Rightarrow f(x)\,E\,f(y)$ and the meagerness to insure that $x \not\!E_0\, y \Rightarrow f(x) \not\!E\, f(y)$. The actual definition of f involves a combinatorial construction whose origins go back to the proof of the original Glimm–Effros dichotomy.

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