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A singular value inequality for Heinz means

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Abstract

We prove a matrix inequality for matrix monotone functions, and apply it to prove a singular value inequality for Heinz means recently conjectured by Zhan.

Keywords: Matrix inequality; Singular value; Matrix monotone function; Matrix mean

1. Introduction

Heinz means, introduced in [2], are means that interpolate in a certain way between the arithmetic and geometric mean. They are defined over $\mathbb{R}^+$ as

$$H_\nu(a, b) = (a^\nu b^{1-\nu} + a^{1-\nu} b^\nu)/2 \tag{1}$$

for $0 \leqslant \nu \leqslant 1$. One can easily show that the Heinz means are "inbetween" the geometric mean and the arithmetic mean:

$$\sqrt{ab} \leqslant H_\nu(a, b) \leqslant (a + b)/2. \tag{2}$$

Bhatia and Davis [3] extended this to the matrix case, by showing that the inequalities remain true for positive semidefinite (PSD) matrices, in the following sense:

$$|||A^{1/2} B^{1/2}||| \leqslant |||H_\nu(A, B)||| \leqslant |||(A + B)/2|||, \tag{3}$$

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where $|||\cdot|||$ is any unitarily invariant norm and the Heinz mean for matrices is defined identically as in (1). In fact, Bhatia and Davis proved the stronger inequalities, involving a third, general matrix X,

$$|||A^{1/2}XB^{1/2}||| \leqslant |||(A^{\nu}XB^{1-\nu} + A^{1-\nu}XB^{\nu})/2||| \leqslant |||(AX + XB)/2|||. \tag{4}$$

Zhan [6,7] conjectured that the second inequality in (3) also holds for singular values. Namely: for $A, B \geqslant 0$,

$$\sigma_j(H_\nu(A, B)) \leqslant \sigma_j((A + B)/2) \tag{5}$$

is conjectured to hold for all j. These inequalities have been proven in a few special cases. The case $\nu = 1/2$ is known as the arithmetic–geometric mean inequality for singular values, and has been proven by Bhatia and Kittaneh [4]. The case $\nu = 1/4$ (and $\nu = 3/4$) is due to Tao [5]. In the present paper, we prove (5) for all $0 \leqslant \nu \leqslant 1$. To do so, we first prove a general matrix inequality for matrix monotone functions (Section 3). The proof of the conjecture is then a relatively straightforward application of this inequality (Section 4).

Remark. One might be tempted to generalise the first inequality in (3) to singular values as well:

$$\sigma_j(A^{1/2}B^{1/2}) \leqslant \sigma_j(H_\nu(A, B)). \tag{6}$$

These inequalities are false, however. Consider the following PSD matrices (both are rank 2):

$$A = \begin{pmatrix} 2 & 4 & 2 \\ 4 & 8 & 4 \\ 2 & 4 & 4 \end{pmatrix}, \qquad B = \begin{pmatrix} 5 & 0 & 4 \\ 0 & 0 & 0 \\ 4 & 0 & 4 \end{pmatrix}.$$

Then $\sigma_2(A^{1/2}B^{1/2}) > \sigma_2(H_\nu(A, B))$ for $0 < \nu < 0.13$.

2. Preliminaries

We denote the eigenvalues and singular values of a matrix A by $\lambda_j(A)$ and $\sigma_j(A)$, respectively. We adhere to the convention that singular values and eigenvalues (in case they are real) are sorted in non-increasing order.

We will use the positive semidefinite (PSD) ordering on Hermitian matrices throughout, denoted $A \geqslant B$, which means that $A - B \geqslant 0$. This ordering is preserved under arbitrary conjugations: $A \geqslant B$ implies $XAX^* \geqslant XBX^*$ for arbitrary X.

A matrix function f is *matrix monotone* iff it preserves the PSD ordering, i.e., $A \geqslant B$ implies $f(A) \geqslant f(B)$. If $A \geqslant B$ implies $f(A) \leqslant f(B)$, we say f is *inversely matrix monotone*. A matrix function f is *matrix convex* iff for all $0 \leqslant \lambda \leqslant 1$ and for all $A, B \geqslant 0$,

$$f(\lambda A + (1 - \lambda)B) \leqslant \lambda f(A) + (1 - \lambda)f(B).$$

Matrix monotone functions are characterised by the integral representation [1,7]

$$f(t) = \alpha + \beta t + \int_0^\infty \frac{\lambda t}{t + \lambda}\, \mathrm{d}\mu(\lambda), \tag{7}$$

where $\mathrm{d}\mu(\lambda)$ is any positive measure on the interval $\lambda \in [0, \infty)$, α is a real scalar and β is a non-negative scalar. When applied to matrices, this gives, for $A \geqslant 0$,

$$f(A) = \alpha\mathbb{1} + \beta A + \int_0^\infty \lambda A(A + \lambda\mathbb{1})^{-1}\, \mathrm{d}\mu(\lambda). \tag{8}$$

The primary matrix function $x \mapsto x^p$ is matrix convex for $1 \leqslant p \leqslant 2$, matrix monotone and matrix concave for $0 \leqslant p \leqslant 1$, and inversely matrix monotone and matrix convex for $-1 \leqslant p \leqslant 0$ [1].

3. A matrix inequality for matrix monotone functions

In this section, we present the matrix inequality that we will use in the next section to prove Zhan's conjecture.

Theorem 1. *For $A, B \geqslant 0$, and any matrix monotone function f:*

$$Af(A) + Bf(B) \geqslant \left(\frac{A+B}{2}\right)^{1/2} (f(A) + f(B)) \left(\frac{A+B}{2}\right)^{1/2}. \tag{9}$$

Proof. Let A and B be PSD. We start by noting the matrix convexity of the function $t \mapsto t^{-1}$. Thus

$$\frac{A^{-1} + B^{-1}}{2} \geqslant \left(\frac{A+B}{2}\right)^{-1}. \tag{10}$$

Replacing A by $A + \mathbb{1}$ and B by $B + \mathbb{1}$,

$$(A + \mathbb{1})^{-1} + (B + \mathbb{1})^{-1} \geqslant 2(\mathbb{1} + (A + B)/2)^{-1}. \tag{11}$$

Let us now define

$$C_k := \frac{A^k}{A + \mathbb{1}} + \frac{B^k}{B + \mathbb{1}},$$

and

$$M := (A + B)/2.$$

With these notations, (11) becomes

$$C_0 \geqslant 2(\mathbb{1} + M)^{-1}. \tag{12}$$

This implies

$$C_0 + \sqrt{M} C_0 \sqrt{M} \geqslant 2(\mathbb{1} + M)^{-1} + 2\sqrt{M}(\mathbb{1} + M)^{-1}\sqrt{M} = 2\mathbb{1}, \tag{13}$$

where the last equality follows easily because all factors commute.

Now note: $C_k + C_{k+1} = A^k + B^k$. In particular, $C_0 + C_1 = 2\mathbb{1}$, and thus (13) becomes

$$\sqrt{M}(2\mathbb{1} - C_1)\sqrt{M} \geqslant C_1. \tag{14}$$

Furthermore, as $C_1 + C_2 = 2M$, this is equivalent with

$$C_2 \geqslant \sqrt{M} C_1 \sqrt{M}, \tag{15}$$

or, written out in full:

$$\frac{A^2}{A + \mathbb{1}} + \frac{B^2}{B + \mathbb{1}} \geqslant \left(\frac{A+B}{2}\right)^{1/2} \left(\frac{A}{A + \mathbb{1}} + \frac{B}{B + \mathbb{1}}\right) \left(\frac{A+B}{2}\right)^{1/2}. \tag{16}$$

We now replace A by $\lambda^{-1}A$ and B by $\lambda^{-1}B$, for λ a positive scalar. Then, after multiplying both sides with λ^2, we obtain that

$$\frac{\lambda A^2}{A+\lambda\mathbb{1}} + \frac{\lambda B^2}{B+\lambda\mathbb{1}} \geqslant \left(\frac{A+B}{2}\right)^{1/2} \left(\frac{\lambda A}{A+\lambda\mathbb{1}} + \frac{\lambda B}{B+\lambda\mathbb{1}}\right) \left(\frac{A+B}{2}\right)^{1/2} \tag{17}$$

holds for all $\lambda \geqslant 0$. We can therefore integrate this inequality over $\lambda \in [0, \infty)$ using any positive measure $d\mu(\lambda)$.

Finally, by matrix convexity of the square function, $((A+B)/2)^2 \leqslant (A^2+B^2)/2$ [1,7], we have, for $\beta \geqslant 0$,

$$A(\alpha\mathbb{1} + \beta A) + B(\alpha\mathbb{1} + \beta B) \geqslant \left(\frac{A+B}{2}\right)^{1/2} (2\alpha\mathbb{1} + \beta(A+B)) \left(\frac{A+B}{2}\right)^{1/2}. \tag{18}$$

Summing this up with the integral expression just obtained, and recognising representation (8) in both sides finally gives us (9). □

Weyl monotonicity, together with the equality $\lambda_j(AB) = \lambda_j(BA)$, immediately yields

Corollary 1. *For $A, B \geqslant 0$, and any matrix monotone function f:*

$$\lambda_j(Af(A) + Bf(B)) \geqslant \lambda_j\left(\frac{A+B}{2}(f(A)+f(B))\right). \tag{19}$$

4. Application: Proof of (5)

As an application of Theorem 1 we now obtain the promised singular value inequality (5) for Heinz means, as conjectured by Zhan:

Theorem 2. *For $A, B \in M_n(\mathbb{C})$, $A, B \geqslant 0$, $j = 1, \ldots, n$, and $0 \leqslant s \leqslant 1$,*

$$\sigma_j(A^s B^{1-s} + A^{1-s} B^s) \leqslant \sigma_j(A+B). \tag{20}$$

Proof. Corollary 1 applied to $f(A) = A^r$, for $0 \leqslant r \leqslant 1$, yields

$$\begin{aligned}\lambda_j(A^{r+1} + B^{r+1}) &\geqslant \frac{1}{2}\lambda_j((A+B)(A^r+B^r)) \\ &= \frac{1}{2}\lambda_j\left(\begin{pmatrix} A^{r/2} \\ B^{r/2}\end{pmatrix} (A+B) \begin{pmatrix} A^{r/2} & B^{r/2}\end{pmatrix}\right), \qquad (21) \\ &= \frac{1}{2}\lambda_j\left(\begin{pmatrix} A^{1/2} \\ B^{1/2}\end{pmatrix} (A^r+B^r) \begin{pmatrix} A^{1/2} & B^{1/2}\end{pmatrix}\right). \qquad (22)\end{aligned}$$

Tao's Theorem [5] now says that for any 2×2 PSD block matrix $Z := \begin{pmatrix} M & K \\ K^* & N\end{pmatrix} \geqslant 0$ (with $M \in M_m$ and $N \in M_n$) the following relation holds between the singular values of the off-diagonal block K and the eigenvalues of Z, for $j \leqslant m, n$:

$$\sigma_j(K) \leqslant \frac{1}{2}\lambda_j(Z). \tag{23}$$

The inequality (21) therefore yields

$$\begin{aligned}\lambda_j(A^{r+1} + B^{r+1}) &\geqslant \sigma_j\left(A^{r/2}(A+B)B^{r/2}\right) \\ &= \sigma_j(A^{1+r/2}B^{r/2} + A^{r/2}B^{1+r/2}). \qquad (24)\end{aligned}$$

Replacing A by $A^{1/(r+1)}$ and B by $B^{1/(r+1)}$ then yields (20) for $s = (1 + r/2)/(1 + r)$, hence for $0 \leqslant s \leqslant 1/4$ and $3/4 \leqslant s \leqslant 1$.

If, instead, we start from (22) and proceed in an identical way as above, then we obtain (20) for $s = (r + 1/2)/(1 + r)$, which covers the remaining case $1/4 \leqslant s \leqslant 3/4$. □

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