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New eighth-order iterative methods for solving nonlinear equations

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ABSTRACT

In this paper, three new families of eighth-order iterative methods for solving simple roots of nonlinear equations are developed by using weight function methods. Per iteration these iterative methods require three evaluations of the function and one evaluation of the first derivative. This implies that the efficiency index of the developed methods is 1.682, which is optimal according to Kung and Traub's conjecture [7] for four function evaluations per iteration. Notice that Bi et al.'s method in [2] and [3] are special cases of the developed families of methods. In this study, several new examples of eighth-order methods with efficiency index 1.682 are provided after the development of each family of methods. Numerical comparisons are made with several other existing methods to show the performance of the presented methods.

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1. Introduction

Solving nonlinear equations is one of the most important problems in numerical analysis. In this paper, we consider iterative methods to find a simple root of a nonlinear equation $f(x) = 0$, where $f : D \subset R \to R$ for an open interval D is a scalar function.

The classical Newton's method for a single non-linear equation is written as

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}. \tag{1}$$

This is an important and basic method [1], which converges quadratically.

To improve the local order of convergence and efficiency index, many modified methods have been proposed in the open literature, see [2–6] and references therein. Chun and Ham [4] developed a family of sixth-order methods by weight function methods. Kou et al. [5] presented a family of variants of Ostrowski's method with seventh-order convergence. Kung and Traub [7] conjectured that a multipoint iteration without memory, based on n evaluations, could achieve optimal convergence order 2^{n-1}. Kung and Traub [7] also provided two families of multipoint iterations based on n evaluations. For the case $n = 4$, the methods can be written as follows:

$$\begin{aligned} y_n &= x_n + \beta f(x_n), \\ z_n &= y_n - \beta \frac{f(x_n)f(y_n)}{f(y_n) - f(x_n)}, \\ w_n &= z_n - \frac{f(x_n)f(y_n)}{f(z_n) - f(x_n)} \left[\frac{y_n - x_n}{f(y_n) - f(x_n)} - \frac{z_n - y_n}{f(z_n) - f(y_n)} \right], \end{aligned} \tag{2}$$

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$$x_{n+1} = w_n - \frac{f(x_n)f(y_n)f(z_n)}{f(w_n)-f(x_n)}\left\{\frac{1}{f(w_n)-f(y_n)}\left[\frac{w_n-z_n}{f(w_n)-f(z_n)} - \frac{z_n-y_n}{f(z_n)-f(y_n)}\right] - \frac{1}{f(z_n)-f(x_n)}\left[\frac{z_n-y_n}{f(z_n)-f(y_n)} - \frac{y_n-x_n}{f(y_n)-f(x_n)}\right]\right\},$$

where β is a constant, and

$$\begin{aligned} y_n &= x_n - \frac{f(x_n)}{f'(x_n)}, \\ z_n &= y_n - \frac{f(x_n)f(y_n)}{[f(x_n)-f(y_n)]^2}\frac{f(x_n)}{f'(x_n)}, \\ x_{n+1} &= z_n - \frac{f(x_n)f(y_n)f(z_n)\{f(x_n)^2+f(y_n)[f(y_n)-f(z_n)]\}}{[f(x_n)-f(y_n)]^2[f(x_n)-f(z_n)]^2[f(y_n)-f(z_n)]}\frac{f(x_n)}{f'(x_n)}. \end{aligned} \tag{3}$$

Bi et al. presented a family of eighth-order convergence methods [2] (see (14) therein):

$$\begin{aligned} y_n &= x_n - \frac{f(x_n)}{f'(x_n)}, \\ z_n &= y_n - \frac{2f(x_n)-f(y_n)}{2f(x_n)-5f(y_n)}\frac{f(y_n)}{f'(x_n)}, \\ x_{n+1} &= z_n - H(\mu_n)\frac{f(z_n)}{f[z_n,y_n]+f[z_n,x_n,x_n](z_n-y_n)}, \end{aligned} \tag{4}$$

where $\mu_n = \frac{f(z_n)}{f(x_n)}$ and $H(t)$ represents a real-valued function with $H(0) = 1, H'(0) = 2$ and $|H''(0)| < \infty$. Another new family of eighth-order methods [3] (see (13) therein) is given by:

$$\begin{aligned} y_n &= x_n - \frac{f(x_n)}{f'(x_n)}, \\ z_n &= y_n - h(\mu_n)\frac{f(y_n)}{f'(x_n)}, \\ x_{n+1} &= z_n - \frac{f(x_n)+(\gamma+2)f(z_n)}{f(x_n)+\gamma f(z_n)}\frac{f(z_n)}{f[z_n,y_n]+f[z_n,x_n,x_n](z_n-y_n)}, \end{aligned} \tag{5}$$

where $\gamma \in R$ is constant, $\mu_n = \frac{f(y_n)}{f(x_n)}$ and $h(t)$ represents a real-valued function with $h(0) = 1, h'(0) = 2,\ h''(0) = 10$ and $|h'''(0)| < \infty$.

Recently, using the Hermite interpolation polynomial of the third order, Petković [8] provided a general class of n-point iterative methods with optimal order of convergence 2^{n-1} and optimal computational efficiency $2^{\frac{n-1}{n}}$. Here the n includes $n-1$ evaluations of the function and one evaluation of the derivative. Thukral and Petković [9] developed a family of optimal eighth-order convergence methods by weight function methods.

In this paper, in three theorems we present three families of eighth-order iterative methods derived by using the method of weight functions. The method of weight functions can be applied to construct families of iterative methods for nonlinear equations. Some applications of the weight function methods can be found in [2–4,6], where some kind of weight functions were used. In this paper, we apply a few weight functions to construct families of iterative methods with high convergence order and high efficiency index. In terms of computational cost, they require the evaluations of three functions and one first order derivative per iteration. This gives 1.682 as the efficiency index of the presented methods. The new families of eighth-order methods agree with the conjecture of Kung and Traub [7] for the case $n = 4$.

The conditions in the theorems are general and basic. For a specific iterative method, one only needs to choose the functions such that the conditions in the theorem are satisfied. In particular, one could first write down the preferable functions with undetermined coefficients, then uses the conditions in the theorem to determine the coefficients in the functions. Examples of the specific iterative methods are presented after each theorem. The presented methods are comparable with Newton's method and other methods. The efficacy of the methods is tested on a number of numerical examples. Notice that Bi et al.'s method in [2,3] are special cases of the presented eighth-order methods. Based on the weight function methods, new eighth-order methods with high efficiency index 1.682 are derived.

2. The methods and analysis of convergence

Based on (3)–(5), we consider the following three-step iteration scheme by using the method of weight functions:

$$\begin{aligned} y_n &= x_n - \frac{f(x_n)}{f'(x_n)}, \\ z_n &= x_n - \frac{f(x_n)}{f'(x_n)} G\left(\frac{f(y_n)}{f(x_n)}\right), \\ x_{n+1} &= z_n - \frac{f(z_n)}{f'(x_n)}\left[H\left(\frac{f(y_n)}{f(x_n)}\right) + V\left(\frac{f(y_n)}{f(x_n)}\right) W\left(\frac{f(z_n)}{f(y_n)}\right)\right], \end{aligned} \tag{6}$$

where $G(t)$, $H(t)$, $V(t)$ and $W(t)$ represent the real-valued functions. The order of convergence of the preceding method is analyzed in the following Theorem 1.

Theorem 1. *Assume that functions G, H, V, W, f are sufficiently differentiable and f has a simple zero $x^* \in D$. If the initial point x_0 is sufficiently close to x^*, then the methods defined by (6) converge to x^* with eighth-order under the conditions $G(0) = 1$, $G'(0) = 1$, $G''(0) = 4$, $H(0) = 1 - \frac{W(0)}{W'(0)}$, $H'(0) = 2(1 - \frac{2W(0)}{W'(0)})$, $H''(0) = \frac{1}{3}(6 - 3W(0)V''(0) + G'''(0))$, $H'''(0) = \frac{1}{4}(-96 + 8G'''(0) - 4W(0)V'''(0) + G^{(4)}(0))$, $V(0) = \frac{1}{W'(0)}$, $V'(0) = \frac{4}{W'(0)}$ and $W'(0) \neq 0$.*

Proof. Let $e_n = x_n - x^*$, $s_n = y_n - x^*$, $a_n = \frac{f(y_n)}{f(x_n)}$, $d_n = z_n - x^*$, $b_n = \frac{f(z_n)}{f(y_n)}$, $p_n = \frac{f(z_n)}{f'(x_n)}$ and $c_k = \frac{f^{(k)}(x^*)}{k! f'(x^*)}$, $k = 2, 3, \ldots$. Using Taylor expansion and taking into account $f(x^*) = 0$, we have

$$\begin{aligned} f(x_n) &= f'(x^*)\left[e_n + c_2 e_n^2 + c_3 e_n^3 + c_4 e_n^4 + c_5 e_n^5 + c_6 e_n^6 + c_7 e_n^7 + c_8 e_n^8 + O(e_n^9)\right], \\ f'(x_n) &= f'(x^*)\left[1 + 2c_2 e_n + 3c_3 e_n^2 + 4c_4 e_n^3 + 5c_5 e_n^4 + 6c_6 e_n^5 + 7c_7 e_n^6 + 8c_8 e_n^7 + O(e_n^8)\right], \\ \frac{f(x_n)}{f'(x_n)} &= e_n - c_2 e_n^2 + 2(c_2^2 - c_3)e_n^3 + (7c_2c_3 - 4c_2^3 - 3c_4)e_n^4 + 2(4c_2^4 - 10c_2^2c_3 + 3c_3^2 + 5c_2c_4 - 2c_5)e_n^5 \\ &\quad + (-16c_2^5 + 52c_2^3c_3 - 33c_2c_3^2 - 28c_2^2c_4 + 17c_3c_4 + 13c_2c_5 - 5c_6)e_n^6 - \alpha_7 e_n^7 - \alpha_8 e_n^8 + O(e_n^9), \\ s_n &= c_2 e_n^2 - 2(c_2^2 - c_3)e_n^3 - (7c_2c_3 - 4c_2^3 - 3c_4)e_n^4 - 2(4c_2^4 - 10c_2^2c_3 + 3c_3^2 + 5c_2c_4 - 2c_5)e_n^5 \\ &\quad - (-16c_2^5 + 52c_2^3c_3 - 33c_2c_3^2 - 28c_2^2c_4 + 17c_3c_4 + 13c_2c_5 - 5c_6)e_n^6 + \alpha_7 e_n^7 + \alpha_8 e_n^8 + O(e_n^9), \\ f(y_n) &= f'(x^*)\left[s_n + c_2 s_n^2 + c_3 s_n^3 + c_4 s_n^4 + O(e_n^9)\right], \\ a_n &= c_2 e_n + (-3c_2^2 + 2c_3)e_n^2 + (8c_2^3 - 10c_2c_3 + 3c_4)e_n^3 + (-20c_2^4 + 37c_2^2c_3 - 8c_3^2 - 14c_2c_4 + 4c_5)e_n^4 \\ &\quad + (48c_2^5 - 118c_2^3c_3 + 55c_2c_3^2 + 51c_2^2c_4 - 22c_3c_4 - 18c_2c_5 + 5c_6)e_n^5 + \beta_6 e_n^6 + \beta_7 e_n^7 + O(e_n^8), \end{aligned} \tag{7}$$

where

$$\begin{aligned} \alpha_7 &= -2(16c_2^6 - 64c_2^4c_3 + 63c_2^2c_3^2 - 9c_3^3 + 36c_2^3c_4 - 46c_2c_3c_4 + 6c_4^2 - 18c_2^2c_5 + 11c_3c_5 + 8c_2c_6 - 3c_7), \\ \alpha_8 &= 64c_2^7 - 304c_2^5c_3 + 408c_2^3c_3^2 - 135c_2c_3^3 + 176c_2^4c_4 - 348c_2^2c_3c_4 + 75c_3^2c_4 + 64c_2c_4^2 - 92c_2^3c_5 \\ &\quad + 118c_2c_3c_5 - 31c_4c_5 + 44c_2^2c_6 - 27c_3c_6 - 19c_2c_7 + 7c_8, \end{aligned}$$

and β_6 and β_7 are functions of $c_2, \ldots, c_8$.

Expanding $G\left(\frac{f(y_n)}{f(x_n)}\right)$ about 0 yields

$$\begin{aligned} G(a_n) &= G(0) + G'(0)a_n + \frac{G''(0)}{2!}a_n^2 + \frac{G'''(0)}{3!}a_n^3 + \frac{G^{(4)}(0)}{4!}a_n^4 + \frac{G^{(5)}(0)}{5!}a_n^5 + \frac{G^{(6)}(0)}{6!}a_n^6 + \frac{G^{(7)}(0)}{7!}a_n^7 + O(e_n^8), \\ z_n - x^* &= e_n - \frac{f(x_n)}{f'(x_n)}G(a_n) \\ &= (1 - G(0))e_n + c_2(G(0) - G'(0))e_n^2 + (2c_3(G(0) - G'(0))) - \frac{1}{2}c_2^2(4G(0) - 8G'(0) \\ &\quad + G''(0))e_n^3 + O(e_n^4). \end{aligned} \tag{8}$$

With the conditions $G(0) = 1$, $G'(0) = 1$, $G''(0) = 4$, we have

$$d_n = \gamma_4 e_n^4 + \gamma_5 e_n^5 + \gamma_6 e_n^6 + \gamma_7 e_n^7 + \gamma_8 e_n^8 + O(e_n^9), \tag{9}$$

where

$$\begin{aligned} \gamma_4 &= -c_2c_3 + c_2^3\left(5 - \frac{1}{6}G'''(0)\right), \\ \gamma_5 &= -2c_3^2 - 2c_2c_4 - c_2^2c_3(-32 + G'''(0)) + c_2^4\left(-36 + \frac{5}{3}G'''(0) - \frac{1}{24}G^{(4)}(0)\right), \end{aligned}$$

$$\gamma_6 = -7c_3c_4 + c_2(-3c_5 - 2c_3^2(-33 + G'''(0))) - \frac{3}{2}c_2^2c_4(-32 + G'''(0)) + \frac{1}{3}c_2^3c_3(-786 + 37G'''(0) - G^{(4)}(0)) - \frac{1}{120}c_2^5(-20400 + 1240G'''(0) - 65G^{(4)}(0) + G^{(5)}(0))$$

and γ_7 and γ_8 are functions of $c_2, \ldots, c_7, G'''(0), \ldots, G^{(7)}(0)$. The complex expressions are omitted here.

$$\begin{aligned}
f(z_n) &= f'(x^*)(d_n + c_2\gamma_4^2e_n^8 + O(e_n^9)),\\
p_n &= \gamma_4e_n^4 + (\gamma_5 - 2\gamma_4c_2)e_n^5 + (\gamma_6 - 2\gamma_5c_2 + 4\gamma_4c_2^2 - 3\gamma_4c_3)e_n^6 + (\gamma_7 - 2\gamma_6c_2 + 4\gamma_5c_2^2 - 8\gamma_4c_2^3\\
&\quad - 3\gamma_5c_3 + 12\gamma_4c_2c_3 - 4\gamma_4c_4)e_n^7 + (\gamma_8 - 8\gamma_5c_2^3 + 16\gamma_4c_2^4 - 3\gamma_6c_3 + 9\gamma_4c_3^2 + 4c_2^2(\gamma_6 - 9\gamma_4c_3)\\
&\quad - 4\gamma_5c_4 + c_2(\gamma_4^2 - 2\gamma_7 + 12\gamma_5c_3 + 16\gamma_4c_4) - 5\gamma_4c_5)e_n^8 + O(e_n^9),\\
b_n &= \left(-c_3 - \frac{1}{6}c_2^2(-30 + G'''(0))\right)e_n^2 + \frac{1}{24}(-48c_4 - 16c_2c_3(-30 + G'''(0)) - c_2^3(624 - 32G'''(0)\\
&\quad + G^{(4)}(0)))e_n^3 + \frac{1}{120}(-120c_2c_4(-29 + G'''(0)) - 40(9c_5 + c_3^2(-57 + 2G'''(0)))\\
&\quad - 10c_2^2c_3(1560 - 86G'''(0) + 3G^{(4)}(0)) - c_2^4(-11160 + 820G'''(0) - 55G^{(4)}(0) + G^{(5)}(0)))e_n^4 + O(e_n^5).
\end{aligned} \tag{10}$$

Expanding $H\left(\frac{f(y_n)}{f(x_n)}\right)$, $V\left(\frac{f(y_n)}{f(x_n)}\right)$, $W\left(\frac{f(z_n)}{f(y_n)}\right)$ about 0 yields

$$\begin{aligned}
H(a_n) &= H(0) + H'(0)a_n + \frac{H''(0)}{2!}a_n^2 + \frac{H'''(0)}{3!}a_n^3 + \frac{H^{(4)}(0)}{4!}a_n^4 + O(e_n^5),\\
V(a_n) &= V(0) + V'(0)a_n + \frac{V''(0)}{2!}a_n^2 + \frac{V'''(0)}{3!}a_n^3 + \frac{V^{(4)}(0)}{4!}a_n^4 + O(e_n^5),\\
W(b_n) &= W(0) + W'(0)b_n + \frac{W''(0)}{2!}b_n^2 + O(e_n^5).
\end{aligned} \tag{11}$$

Using (7)–(11), we have

$$\begin{aligned}
x_{n+1} - x^* &= d_n - p_n[H(a_n) + V(a_n)W(b_n)]\\
&= R_4e_n^4 + R_5e_n^5 + R_6e_n^6 + R_7e_n^7 + R_8e_n^8 + O(e_n^9).
\end{aligned} \tag{12}$$

With the Conditions $H(0) = 1 - V(0)W(0)$, $H'(0) = 2 - W(0)V'(0)$, $V(0) = \frac{1}{W'(0)}$, $H''(0) = \frac{1}{3}(6 - 3W(0)V''(0) + G'''(0))$, $V'(0) = \frac{4}{W'(0)}$, $H'''(0) = \frac{1}{4}(-96 + 8G'''(0) - 4W(0)V'''(0) + G^{(4)}(0))$, there are

$$\begin{aligned}
R_4 &= \gamma_4(1 - H(0) - V(0)W(0)) = 0,\\
R_5 &= \gamma_4c_2(2 - H'(0) - W(0)V'(0)) = 0,\\
R_6 &= \frac{1}{6}\gamma_4(6c_3(-1 + V(0)W'(0)) + c_2^2(-3(-12 + H''(0) + W(0)V''(0)) + V(0)W'(0)(-30 + G'''(0)))) = 0,\\
R_7 &= \frac{1}{24}\gamma_4c_2(24c_3(-4 + V'(0)W'(0)) + c_2^2(384 + 4V'(0)W'(0)(-30 + G'''(0))\\
&\quad - 8G'''(0) - 4H'''(0) - 4W(0)V'''(0) + G^{(4)}(0))) = 0,\\
R_8 &= -\frac{1}{360W'(0)}(\gamma_4(360c_2(\gamma_4 - c_4)W'(0) + 180c_3^2(-2W'(0) + W''(0))\\
&\quad - 60c_2^2c_3(3W'(0)^2V''(0) + W'(0)(-120 + G'''(0)) - W''(0)(-30 + G'''(0)))\\
&\quad + c_2^4(-30W'(0)^2V''(0)(-30 + G'''(0)) + 5W''(0)(-30 + G'''(0))^2\\
&\quad + 3W'(0)(-6000 + 200G'''(0) - 10G^{(4)}(0) + 5H^{(4)}(0) + 5W(0)V^{(4)}(0) - G^{(5)}(0))))).
\end{aligned} \tag{13}$$

With the condition $W'(0) \neq 0$, it is clear that $R_8 \neq 0$, thus (6) converge to x^* with eighth-order, and the error equation becomes

$$\begin{aligned}
e_{n+1} &= \frac{1}{2160W'(0)}((6c_2c_3 + c_2^3(-30 + G'''(0)))(-360c_2c_4W'(0) + 180c_3^2(-2W'(0) + W''(0))\\
&\quad - 60c_2^2c_3(3W'(0)^2V''(0) + W'(0)(-114 + G'''(0)) - W''(0)(-30 + G'''(0)))\\
&\quad + c_2^4(-30W'(0)^2V''(0)(-30 + G'''(0)) + 5W''(0)(-30 + G'''(0))^2 + 3W'(0)(-5400 + 180G'''(0)
\end{aligned}$$

$$- 10G^{(4)}(0) + 5H^{(4)}(0) + 5W(0)V^{(4)}(0) - G^{(5)}(0)))))e_n^8 + O(e_n^9). \quad (14)$$

After some simplifications, we can easily obtain the conditions of Theorem 1. This finishes the proof of Theorem 1. □

There are four weight functions in scheme (6). With some choice of weight functions $G(t) = \frac{1-t}{1-2t}$ and $W(t) = t$, the scheme (6) becomes

$$\begin{aligned} y_n &= x_n - \frac{f(x_n)}{f'(x_n)}, \\ z_n &= x_n - \frac{f(x_n)}{f'(x_n)}\frac{f(x_n)-f(y_n)}{f(x_n)-2f(y_n)}, \\ x_{n+1} &= z_n - \frac{f(z_n)}{f'(x_n)}\left[H\left(\frac{f(y_n)}{f(x_n)}\right) + V\left(\frac{f(y_n)}{f(x_n)}\right)\frac{f(z_n)}{f(y_n)}\right]. \end{aligned} \quad (15)$$

Theorem 1 can then be simplified as follows.

Theorem 1′. *Assume that functions H, V, f are sufficiently differentiable and f has a simple zero $x^* \in D$. If the initial point x_0 is sufficiently close to x^*, then the methods defined by (15) converge to x^* with eighth-order under the conditions $H(0) = 1, H'(0) = 2, H''(0) = 10, H'''(0) = 72, V(0) = 1, V'(0) = 4$.*

In what follows, we give some concrete forms of iterative schemes (6) and (15).

Example 1.1. The functions $G(t), H(t), V(t), W(t)$ defined by

$$G(t) = \frac{1-t}{1-2t}, \qquad H(t) = \frac{1}{2}, \qquad V(t) = \frac{5+8t+2t^2}{5-12t}, \qquad W(t) = \frac{1}{2} + t$$

satisfy the conditions of Theorem 1. A new eighth-order method is then obtained

$$\begin{aligned} y_n &= x_n - \frac{f(x_n)}{f'(x_n)}, \\ z_n &= x_n - \frac{f(x_n)}{f'(x_n)}\frac{f(x_n)-f(y_n)}{f(x_n)-2f(y_n)}, \\ x_{n+1} &= z_n - \frac{f(z_n)}{f'(x_n)}\left[\frac{1}{2} + \frac{5f(x_n)^2 + 8f(x_n)f(y_n) + 2f(y_n)^2}{5f(x_n)^2 - 12f(x_n)f(y_n)}\left(\frac{1}{2} + \frac{f(z_n)}{f(y_n)}\right)\right]. \end{aligned} \quad (16)$$

Example 1.2. The functions $H(t), V(t)$ defined by

$$H(t) = \frac{5-2t+t^2}{5-12t}, \qquad V(t) = 1 + 4t$$

satisfy the conditions of Theorem 1′. A new eighth-order method is then obtained

$$\begin{aligned} y_n &= x_n - \frac{f(x_n)}{f'(x_n)}, \\ z_n &= x_n - \frac{f(x_n)}{f'(x_n)}\frac{f(x_n)-f(y_n)}{f(x_n)-2f(y_n)}, \\ x_{n+1} &= z_n - \frac{f(z_n)}{f'(x_n)}\left\{\frac{5f(x_n)^2 - 2f(x_n)f(y_n) + f(y_n)^2}{5f(x_n)^2 - 12f(x_n)f(y_n)} + \left[1 + 4\frac{f(y_n)}{f(x_n)}\right]\frac{f(z_n)}{f(y_n)}\right\}. \end{aligned} \quad (17)$$

Next, we subjoin a combination of the known information $f(z_n), f(x_n)$ and consider the following iteration scheme by using the method of weight functions:

$$\begin{aligned} y_n &= x_n - \frac{f(x_n)}{f'(x_n)}, \\ z_n &= x_n - \frac{f(x_n)}{f'(x_n)}G\left(\frac{f(y_n)}{f(x_n)}\right), \\ x_{n+1} &= z_n - \frac{f(z_n)}{f'(x_n)}H\left(\frac{f(z_n)}{f(x_n)}\right)\left[V\left(\frac{f(y_n)}{f(x_n)}\right) + W\left(\frac{f(z_n)}{f(y_n)}\right)\right], \end{aligned} \quad (18)$$

where $G(t), H(t), V(t)$ and $W(t)$ are real-valued functions. Similar to the proof of Theorem 1, we can make the following conclusion.

Theorem 2. *Assume that functions* G, H, V, W, f *are sufficiently differentiable and* f *has a simple zero* $x^* \in D$. *If the initial point* x_0 *is sufficiently close to* x^*, *then the methods defined by* (18) *converge to* x^* *with eighth-order under the conditions* $G(0) = 1, G'(0) = 1, G''(0) = 4, H(0) = \frac{1}{W'(0)}, H'(0) = \frac{4}{W'(0)}, V(0) = -W(0) + W'(0), V'(0) = 2W'(0), V''(0) = \frac{1}{3}W'(0)(6 + G'''(0)), V'''(0) = \frac{1}{4}W'(0)(-96 + 8G'''(0) + G^{(4)}(0))$ *and* $W'(0) \neq 0$. *The error equation of* (18) *is*

$$\begin{aligned} e_{n+1} = {} & \frac{1}{2160W'(0)}(c_2(6c_3 + c_2^2(-30 + G'''(0)))(-360c_2c_4W'(0) + 180c_3^2(-2W'(0) + W''(0)) \\ & - 60c_2^2c_3(W'(0)(-66 + G'''(0)) - W''(0)(-30 + G'''(0))) + c_2^4(5(W''(0)(-30 + G'''(0))^2 \\ & + 3V^{(4)}(0)) + W'(0)(60G'''(0) - 3(600 + 10G^{(4)}(0) + G^{(5)}(0))))))e_n^8 + O(e_n^9). \end{aligned} \tag{19}$$

Again, with some choice of functions $G(t) = \frac{4-5t-t^2}{4-9t}$ and $V(t) = \frac{8t}{4-11t}$, (18) becomes:

$$\begin{aligned} y_n &= x_n - \frac{f(x_n)}{f'(x_n)}, \\ z_n &= x_n - \frac{f(x_n)}{f'(x_n)} \frac{4f(x_n)^2 - 5f(x_n)f(y_n) - f(y_n)^2}{4f(x_n)^2 - 9f(x_n)f(y_n)}, \\ x_{n+1} &= z_n - \frac{f(z_n)}{f'(x_n)} H\left(\frac{f(y_n)}{f(x_n)}\right)\left[\frac{8f(y_n)}{4f(x_n) - 11f(y_n)} + W\left(\frac{f(z_n)}{f(x_n)}\right)\right]. \end{aligned} \tag{20}$$

Theorem 2 can then be simplified as follows.

Theorem 2′. *Assume that functions* H, W, f *are sufficiently differentiable and* f *has a simple zero* $x^* \in D$. *If the initial point* x_0 *is sufficiently close to* x^*, *then the methods defined by* (20) *converge to* x^* *with eighth-order under the conditions* $H(0) = 1, H'(0) = 4, W(0) = 1, W'(0) = 1$.

In what follows, we give some concrete forms of iterative schemes (18) and (20).

Example 2.1. The functions $G(t), H(t), V(t), W(t)$ defined by

$$G(t) = \frac{1-t}{1-2t}, \qquad H(t) = 1 + \frac{4t}{1+at}, \qquad V(t) = \frac{1}{1-2t-t^2}, \qquad W(t) = t$$

satisfy the conditions of Theorem 2. A new family of one-parameter eighth-order methods is obtained

$$\begin{aligned} y_n &= x_n - \frac{f(x_n)}{f'(x_n)}, \\ z_n &= x_n - \frac{f(x_n)}{f'(x_n)} \frac{f(x_n) - f(y_n)}{f(x_n) - 2f(y_n)}, \\ x_{n+1} &= z_n - \frac{f(z_n)}{f'(x_n)}\left[1 + \frac{4f(z_n)}{f(x_n) + af(z_n)}\right]\left[\frac{f(x_n)^2}{f(x_n)^2 - 2f(x_n)f(y_n) - f(y_n)^2} + \frac{f(z_n)}{f(y_n)}\right], \end{aligned} \tag{21}$$

where a is a constant.

Example 2.2. The functions $H(t), W(t)$ defined by

$$H(t) = 1 + 4t, \qquad W(t) = 1 + t$$

satisfy the conditions of Theorem 2′. A new eighth-order method is then obtained

$$\begin{aligned} y_n &= x_n - \frac{f(x_n)}{f'(x_n)}, \\ z_n &= x_n - \frac{f(x_n)}{f'(x_n)} \frac{4f(x_n)^2 - 5f(x_n)f(y_n) - f(y_n)^2}{4f(x_n)^2 - 9f(x_n)f(y_n)}, \\ x_{n+1} &= z_n - \frac{f(z_n)}{f'(x_n)}\left[1 + 4\frac{f(z_n)}{f(x_n)}\right]\left[\frac{8f(y_n)}{4f(x_n) - 11f(y_n)} + 1 + \frac{f(z_n)}{f(y_n)}\right]. \end{aligned} \tag{22}$$

By Theorems 1 and 2, we can see that by means of the combination of the known information, we can present new families of eighth-order methods. A combination of different forms of the known information may constitute new iteration schemes with high convergence. Finally, we consider the following iterative scheme by using the method of weight functions:

$$
\begin{aligned}
y_n &= x_n - \frac{f(x_n)}{f'(x_n)},\\
z_n &= y_n - \frac{f(y_n)}{f'(x_n)} G\left(\frac{f(y_n)}{f(x_n)}\right),\\
x_{n+1} &= z_n - \frac{f(z_n)}{f'(x_n)} \frac{H\left(\frac{f(z_n)}{f(x_n)}\right)}{U\left(\frac{f(y_n)}{f(x_n)}\right) + V\left(\frac{f(y_n)}{f(x_n)}\right) W\left(\frac{f(z_n)}{f(y_n)}\right)},
\end{aligned}
\tag{23}
$$

where $G(t), H(t), U(t), V(t)$ and $W(t)$ represent real-valued functions. Similar to the proof of Theorem 1, we can make the following conclusion.

Theorem 3. *Assume that functions G, H, U, V, W, f are sufficiently differentiable and f has a simple zero $x^* \in D$. If the initial point x_0 is sufficiently close to x^*, then the methods defined by (23) converge to x^* with eighth-order under the conditions $G(0) = 1, G'(0) = 2, H(0) = -V(0)W'(0), H'(0) = V'(0)W'(0), U(0) = -V(0)(W(0) + W'(0)), U'(0) = -W(0)V'(0) + 2V(0)W'(0), U''(0) = V(0)W'(0)(-6 + G''(0)) - W(0)V''(0), U'''(0) = V(0)W'(0)(-6G''(0) + G'''(0)) - W(0)V'''(0)$ and $V(0)W'(0) \neq 0$. The error equation of (23) is*

$$
\begin{aligned}
e_{n+1} = {} & \frac{1}{48V(0)W'(0)}(c_2(2c_3 + c_2^2(G''(0) - 10))(-24c_2c_4V(0)W'(0) + 12c_3^2V(0)W''(0)\\
& - 12c_2^2c_3(W'(0)(4V'(0) + V''(0)) + V(0)(W'(0)(G''(0) - 6) - (G''(0) - 10)W''(0)))\\
& + c_2^4(-24V'(0)W'(0)(G''(0) - 10) + 60W'(0)V''(0) - 6W'(0)G''(0)V''(0) + V(0)(3(G''(0) - 10)^2W''(0)\\
& + W'(0)(-24 + 12G''(0) + 8G'''(0) - G^{(4)}(0))) + U^{(4)}(0) + W(0)V^{(4)}(0))))e_n^8 + O(e_n^9).
\end{aligned}
\tag{24}
$$

In particular, with

$$
\begin{aligned}
& G(t) = \frac{2-t}{2-5t}, \qquad H(0) = 1, \qquad H'(0) = 2, \qquad U(t) = \frac{1}{G(t)} - \frac{t^2G(t)^2}{(tG(t)+1)^2},\\
& V(t) = -\left[\frac{1}{G(t)} + \frac{t^2G(t)}{(tG(t)+1)^2}\right], \qquad W(t) = t,
\end{aligned}
$$

satisfying the conditions of Theorem 3, scheme (23) becomes (4). Therefore, Bi et al.'s method in [2] is a special case of the presented eighth-order methods.

In particular, with

$$
\begin{aligned}
& G(t) = h(t), \qquad h(0) = 1, \qquad h'(0) = 2, \qquad h''(0) = 10, \qquad H(t) = \frac{1 + (\gamma+2)t}{1+\gamma t},\\
& U(t) = \frac{1}{G(t)} - \frac{[tG(t)]^2}{[tG(t)+1]^2}, \qquad V(t) = -\left[\frac{1}{G(t)} + \frac{t^2G(t)}{(tG(t)+1)^2}\right], \qquad W(t) = t,
\end{aligned}
$$

satisfying the conditions of Theorem 3, scheme (23) becomes (5). Therefore, Bi et al.'s method in [3] is a special case of the presented eighth-order methods.

Similar to Theorems 1 and 2, with the specific forms of the weight functions, $G(t) = \frac{4-t}{4-9t}$, $U(t) = \frac{4-11t}{-4+3t}$, $W(t) = t$, scheme (23) can be simplified as

$$
\begin{aligned}
y_n &= x_n - \frac{f(x_n)}{f'(x_n)},\\
z_n &= y_n - \frac{f(y_n)}{f'(x_n)} \frac{4f(x_n) - f(y_n)}{4f(x_n) - 9f(y_n)},\\
x_{n+1} &= z_n - \frac{f(z_n)}{f'(x_n)} \frac{H\left(\frac{f(z_n)}{f(x_n)}\right)}{\frac{4f(x_n) - 11f(y_n)}{-4f(x_n)+3f(y_n)} + V\left(\frac{f(y_n)}{f(x_n)}\right)\frac{f(z_n)}{f(y_n)}}.
\end{aligned}
\tag{25}
$$

Theorem 3 can then be simplified as follows.

Theorem 3′. *Assume that functions H, V, f are sufficiently differentiable and f has a simple zero $x^* \in D$. If the initial point x_0 is sufficiently close to x^*, then the methods defined by (25) converge to x^* with eighth-order under the conditions $H(0) = -1,\ V(0) = 1,\ H'(0) = V'(0)$.*

In what follows, we give some concrete forms of iterative schemes (23) and (25).

Example 3.1. The functions $G(t), H(t), U(t), V(t), W(t)$ defined by

$$G(t)=\frac{-4+t}{-4+9t},\qquad H(t)=\frac{4-(3+4a)t}{4},\qquad U(t)=\frac{-2+(11+2a)t}{-4+3t},$$
$$V(t)=\frac{2+2at}{4-3t},\qquad W(t)=\frac{1-t}{1+t}$$

satisfy the conditions of Theorem 3. A new family of one-parameter eighth-order methods is obtained

$$\begin{aligned} y_n &= x_n-\frac{f(x_n)}{f'(x_n)},\\ z_n &= y_n-\frac{f(y_n)}{f'(x_n)}\frac{-4f(x_n)+f(y_n)}{-4f(x_n)+9f(y_n)},\\ x_{n+1} &= z_n-\frac{f(z_n)}{f'(x_n)}\frac{\frac{4f(x_n)-(3+4a)f(z_n)}{4f(x_n)}}{\frac{-2f(x_n)+(11+2a)f(y_n)}{-4f(x_n)+3f(y_n)}+\frac{2f(x_n)+2af(y_n)}{4f(x_n)-3f(y_n)}\frac{f(y_n)-f(z_n)}{f(y_n)+f(z_n)}}, \end{aligned} \tag{26}$$

where a is constant.

Example 3.2. The functions $H(t), V(t)$ defined by

$$H(t)=\frac{-1+at}{1+bt},\qquad V(t)=\frac{1+ct}{1-(a-c+b)t}$$

satisfy the conditions of Theorem 3′. A new family of three-parameter eighth-order methods is obtained

$$\begin{aligned} y_n &= x_n-\frac{f(x_n)}{f'(x_n)},\\ z_n &= y_n-\frac{f(y_n)}{f'(x_n)}\frac{-4f(x_n)+f(y_n)}{-4f(x_n)+9f(y_n)},\\ x_{n+1} &= z_n-\frac{f(z_n)}{f'(x_n)}\frac{\frac{-f(x_n)+af(z_n)}{f(x_n)+bf(z_n)}}{\frac{4f(x_n)-11f(y_n)}{-4f(x_n)+3f(y_n)}+\frac{f(x_n)+cf(y_n)}{f(x_n)-(a-c+b)f(y_n)}\frac{f(z_n)}{f(y_n)}}, \end{aligned} \tag{27}$$

where a, b, c are constants.

The developed methods require evaluations of only three functions and one first derivative per iteration. Consider the efficiency index (e.g. [1,10,11]) defined as $p^{\frac{1}{w}}$, where p is the order of the method and w is the number of function evaluations per iteration. Assume that all the derivatives' evaluations have the same cost as the function's evaluation. The efficiency index of the new methods is $8^{\frac{1}{4}}=1.682$, which is better than those in [4–6] that require four function evaluations but have the convergence order less than eight. In addition to the high efficiency index methods provided recently in Bi et al. [2] and [3], this study provides new families of eighth-order methods with efficiency index 1.682.

3. Numerical results and conclusions

In this section, we present the results of numerical simulations to compare the efficiencies of the methods. The considered methods are (1), (2) with $\beta=1$, (3), (4) with $H(\mu_n)=\frac{1+3\mu_n}{1+\mu_n}$, (5) with $h(\mu_n)=(\frac{1}{1-3\mu_n})^{\frac{2}{3}}$ and $\gamma=1$, and new methods (16), (22), (26) with $a=-3$ and (27) with $a=-1, b=1, c=3$. Here, only the NM method is of the second-order, the other methods are of the eighth-order.

Numerical computations reported here have been carried out in a Mathematica 4.0 environment. Table 1 shows the distance between the root x^* and the approximation x_n for test functions $f_i(x)$ $(i=1,2,\ldots,6)$ with initial approximation x_0, where x^* is the exact root computed with 800 significant digits and x_n is calculated by using the same total number of function evaluations (TNFE) for all methods. The absolute values of the function ($|f(x_n)|$) and the computational order of convergence (COC) are also shown in Table 1. Here, COC is defined by [12]

$$\rho\approx\frac{\ln|(x_{n+1}-x^*)/(x_n-x^*)|}{\ln|(x_n-x^*)/(x_{n-1}-x^*)|}.$$

Table 1
Comparison of various iterative methods under the same total number of function evaluations (TNFE = 8).

	$\|x_n - x^*\|$	$\|f(x_n)\|$	COC	$\|x_n - x^*\|$	$\|f(x_n)\|$	COC
	$f_1(x), x_0 = 2.99$			$f_2(x), x_0 = -1.21$		
(1)	2.60388e−20	3.38504e−19	1.99999877	3.90252e−40	7.92501e−39	2.00000000
(2)	2.36876e−28	3.07939e−27	8.17128707	7.63422e−109	1.55031e−107	7.99923240
(3)	1.17270e−72	1.52452e−71	8.01248268	1.31218e−156	2.66471e−155	7.99992918
(4)	1.47916e−72	1.92291e−71	8.01214361	9.14655e−153	1.85743e−151	7.99998145
(5)	3.74294e−75	4.86583e−74	8.00404108	8.89414e−155	1.80617e−153	8.00013679
(16)	6.46826e−84	8.40873e−83	8.02118000	2.68767e−167	5.45797e−166	8.00005452
(22)	1.16646e−80	1.51639e−79	8.02043636	3.67406e−158	7.46107e−157	7.99985324
(26)	4.15202e−81	5.39763e−80	8.01892109	1.03561e−158	2.10305e−157	7.99989013
(27)	1.28261e−86	1.66740e−85	8.00627985	3.17822e−168	6.45415e−167	7.99948255
	$f_3(x), x_0 = 2.15$			$f_4(x), x_0 = 1.39$		
(1)	2.28744e−43	3.18520e−42	1.99999999	1.13930e−31	2.82828e−31	2.00000000
(2)	5.49526e−123	7.65203e−122	8.00354340	2.05523e−111	5.10206e−111	7.99647751
(3)	1.42021e−165	1.97761e−164	8.00035003	6.82250e−117	1.69367e−116	8.00330413
(4)	1.79628e−171	2.50128e−170	8.00047285	6.43975e−127	1.59865e−126	8.00747975
(5)	7.24033e−174	1.00820e−172	8.00029861	1.20198e−129	2.98388e−129	8.00509655
(16)	1.20020e−175	1.67125e−174	7.99988368	7.43869e−125	1.84663e−124	7.99987646
(22)	1.86479e−187	2.59667e−186	8.00137896	2.34793e−130	5.82867e−130	7.99823112
(26)	1.38435e−184	1.92767e−183	8.00068175	1.75694e−132	4.36157e−132	7.99597558
(27)	2.88561e−178	4.01815e−177	8.00000935	1.42231e−126	3.53083e−126	8.00098399
	$f_5(x), x_0 = -0.47$			$f_6(x), x_0 = 2.26$		
(1)	4.21072e−28	6.91485e−28	1.99999989	7.11546e−68	3.38853e−67	2.00000000
(2)	4.96057e−91	8.14626e−91	8.00413154	1.54240e−235	7.34523e−235	7.99997814
(3)	2.81655e−106	4.62535e−106	8.00292959	6.45584e−263	3.07440e−262	7.99999329
(4)	3.69596e−107	6.06951e−107	8.00293553	7.78290e−269	3.70637e−268	7.99999082
(5)	1.90899e−109	3.13495e−109	8.00128633	3.33230e−271	1.58691e−270	7.99999418
(16)	1.19166e−119	1.95695e−119	8.00791568	6.41677e−273	3.05579e−272	8.00000220
(22)	1.78201e−117	2.92642e−117	8.00568178	5.58830e−285	2.66126e−284	7.99997157
(26)	9.05325e−118	1.48673e−117	8.00539382	5.46462e−282	2.60236e−281	7.99998622
(27)	6.59410e−121	1.08288e−120	8.00320978	1.47375e−275	7.01831e−275	7.99999978

The test functions $f_i(x)$ $(i = 1, 2, \ldots, 6)$ are listed as follows

$$
\begin{aligned}
&f_1(x) = e^{x^2+7x-30} - 1, && x^* = 3 \\
&f_2(x) = xe^{x^2} - \sin^2 x + 3\cos x + 5, && x^* \approx -1.207647827130918927 \\
&f_3(x) = x^3 - 10, && x^* \approx 2.154434690031883721 8 \\
&f_4(x) = \sin^2 x - x^2 + 1, && x^* \approx 1.404491648215341226 \\
&f_5(x) = (x+2)e^x - 1, && x^* \approx -0.442854401002388583 \\
&f_6(x) = (x-1)^3 - 2, && x^* \approx 2.259921049894873164 8.
\end{aligned}
$$

From these numerical experiments, the presented methods appear to be more robust and thus more competitive than the other methods compared. Table 1 also reveals that the methods introduced in this study have better performance than the other known methods of the same order.

In the implementation of the iterative methods, the appropriate choice of initial approximation value x_0 is very important since a badly chosen initial approximation produces a bad predictor and, consequently, destroys the rapid convergence. A good study on this topic can be found in [13,14]. The root approximation obtained by using the methods therein is a good candidate for the initial approximation x_0 for the iterative methods in this study. The method in [13] (see (5) therein) is outlined as follows:

$$x^* \approx \frac{1}{2}\left\{a + b + \operatorname{sgn}(f(a)) \cdot \int_a^b \tanh(\beta \cdot f(x))\,dx\right\}, \tag{28}$$

where x^* is a simple root of $f(x) = 0$ on an interval $[a, b]$ with $f(a) \cdot f(b) < 0$, and $\beta > 0$ is a constant. This method (28) is used for the initial approximations x_0 in Table 1. For example, for $f_1(x)$, with the choice of $a = 0$, $b = 5$ and $\beta = 3$, the above method gives $x_0 = 2.98855$ and $|f(x_0)| = 0.138163$. For simplification, we use $x_0 = 2.99$ as the initial approximation.

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