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Non-structural controllability of linear elastic systems with structural damping

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Abstract

This paper proves that any initial condition in the energy space for the plate equation with square root damping $\ddot{\zeta} - \rho\Delta\dot{\zeta} + \Delta^2\zeta = u$ on a smooth bounded domain, with hinged boundary conditions $\zeta = \Delta\zeta = 0$, can be steered to zero by a square integrable input function u supported in arbitrarily small time interval $[0, T]$ and subdomain. As T tends to zero, for initial states with unit energy norm, the norm of this u grows at most like $\exp(C_p/T^p)$ for any real $p > 1$ and some $C_p > 0$. Indeed, this fast controllability cost estimate is proved for more general linear elastic systems with structural damping and non-structural controls satisfying a spectral observability condition. Moreover, under some geometric optics condition on the subdomain allowing to apply the control transmutation method, this estimate is improved into $p = 1$ and the dependence of C_p on the subdomain is made explicit. These results are analogous to the optimal ones known for the heat flow.

Keywords: Controllability; Observability; Control cost; Plates; Structural damping; Transmutation

A wide variety of dissipative linear elastic control systems may be represented by a second-order differential equation in a Hilbert space:

$$\ddot{\zeta}(t) + D\dot{\zeta}(t) + S\zeta(t) = Bu(t), \quad t \in \mathbb{R}_+, \tag{1}$$

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where each dot denotes a derivative with respect to the time variable t and the function ζ represents the evolution of the system under the action of the input function u. The *structural vibration modes* of the conservative system represented by (1) with $B = D = 0$ are prescribed by the positive self-adjoint operator S. This ideal system is perturbed by a *dissipative mechanism* prescribed by the positive self-adjoint operator D. The system is actuated through a *control mechanism* prescribed by the operator B (possibly unbounded to take into account trace operators prescribing the boundary value of distributed states). Throughout this paper, *controllability* will always mean the ability of steering any initial state $(z(0), \dot{z}(0))$ to zero over a finite time by some appropriate input function u (i.e. exact controllability to zero or null controllability).

This paper concerns the specific dissipative mechanism $D = S^\alpha$ with $\alpha \in (0, 1)$ called *structural damping*, which generalizes the square root damping model $\alpha = 1/2$ introduced in [6]: "The basic property of structural damping, which is said to be consistent with empirical studies, is that the amplitudes of the normal modes of vibration are attenuated at rates which are proportional to the oscillation frequencies." This model was also studied under the name "proportional damping" (cf. [3]). The quite different case $\alpha = 1$ is known as "Kelvin–Voigt" damping. When B is the identity and $\alpha \in (0, 1]$, this is the first class of parabolic-like control models considered in [13,24] with the extra assumption that S has compact resolvent, dispensed with in [2].

This paper focuses on the cost of fast controls as in [2,24]. The controllability results known for these systems hold for a control time which can be chosen as small as wished. This asymptotic is referred to as *fast control*. The *cost* over a given time is the supremum over every initial state with unit energy norm of the smallest norm of an input function which steers it to zero over the given time. The study of the cost of fast controls was initiated by Seidman (cf. references in [17]) and recently revived by Da Prato who connected it to some properties of stochastic differential equations (cf. references in [2]).

The earlier results restricted to elementary forms of control operators (mainly B is the identity or has rank one, cf. [2,11,13,14,21,23,24]). A key point in their proofs is (loosely speaking) the existence of a common eigenbasis for the three operators S, D and B, modeling respectively the structure, the damping and the control. On the contrary, the controllability results of this paper apply to *non-structural controls*, e.g. locally distributed control.

The main application is to the plate equation with square root damping on a smooth bounded domain M of $\mathbb{R}^d$ with hinged boundary conditions:

$$\ddot{\zeta} - \rho\Delta\dot{\zeta} + \Delta^2\zeta = u \quad \text{on } \mathbb{R}_+ \times M, \qquad \zeta = \Delta\zeta = 0 \quad \text{on } \mathbb{R}_+ \times \partial M, \tag{2}$$

where $\rho > 0$ and the input function u is supported on a non-empty subdomain Ω (cf. Theorem 2). Fast controllability is proved to hold for any control region Ω. As the control time T tends to zero, the cost is proved to grow at most like $\exp(C_\beta/T^\beta)$ for any $\beta > 1$ and some $C_\beta > 0$. If the length L_Ω of the longest generalized ray of geometrical optics in M which does not intersect Ω is finite (this is the condition of [5]) and $\rho < 2$, then the cost is proved to grow at most like $\exp(CL^2/T)$ for all $L > L_\Omega$ and some positive C which does not depend on Ω. These results are analogous to the optimal fast controllability cost known for the heat flow (cf. [10,17]). They confirm the formal analogy: $\partial_t^2 - 2\Delta\partial_t + \Delta^2 = (\partial_t - \Delta)^2$. On the contrary, when $\Omega = M$ (i.e. B is the identity), [2,24] prove that the fast controllability cost grows like $1/T^\beta$ for some $\beta \geqslant 1/2$, as in finite-dimensional systems (cf. [22]).

Earlier methods to estimate the cost of fast controls were global parabolic Carleman estimates (cf. [2,10]), the Fourier transform method for constructing functions bi-orthogonal to exponential series (cf. [17,23] and references therein) and the transmutation control method (cf. [17,19]). The last two are combined in Section 2 to take the geometry of the control region into account and improve the cost estimate for (2) as stated above and more precisely in Theorem 2.

The proof of the abstract result, Theorem 1 of Section 1, applies a new method using the control strategy of Lebeau and Robbiano in [15] as implemented in [16] (the companion paper [20] applies this method to a simpler model: the holomorphic semigroup generated by $\exp(-tS^{\beta})$, $\beta > 0$). The key assumption is an observability condition on the spectral subspaces of S with respect to B stated in Definition 1. It is an abstract version of a result on sums of eigenfunctions of the Dirichlet Laplacian proved in [12,16] by local elliptic Carleman estimates and extended to non-compact manifolds in [18]. Therefore the abstract result applies to (2) (such concrete models with other forms of controls are considered e.g. in [14, Chapter 3]) even if, e.g., M is unbounded, the Dirichlet Laplacian is positive with non-compact resolvent and Ω is the exterior of a compact subdomain.

It is clearly desirable to study plate equations with other boundary conditions or with locally distributed controls on the boundary instead of the interior. Other open problems are mentioned in the remarks of Section 2.1.

1. A structurally damped linear elastic control system

Before stating the abstract model and the theorem precisely, we need to introduce a few notations.

Let H_0 and U be Hilbert spaces with respective norms $\|\cdot\|_0$ and $\|\cdot\|$. Let A be a self-adjoint, positive and boundedly invertible unbounded operator on H_0 with domain $D(A)$. We introduce the Sobolev scale of spaces based on A. For any positive integer p, let H_p denote the Hilbert space $D(A^{p/2})$ with the norm $\|x\|_p = \|A^{p/2}x\|_0$ (which is equivalent to the graph norm $\|x\|_0 + \|A^{p/2}x\|_0$). We identify H_0 and U with their duals. Let H_{-p} denote the dual of H_p. Since H_p is densely continuously embedded in H_0, the pivot space H_0 is densely continuously embedded in H_{-p}, and H_{-p} is the completion of H_0 with respect to the norm $\|x\|_{-p} = \|A^{-p/2}x\|_0$.

Let the observation operator C be in $\mathcal{L}(H_2, U)$, which denotes bounded operators from H_2 to U, and let $B \in \mathcal{L}(U, H_{-2})$ denote the dual of C.

Let $\alpha \in (0, 1)$ denote the structural dissipation power, and let $\rho > 0$ denote the dissipativity constant. With the structural operator A and the control operator B, they define the second-order Cauchy problem with input function u:

$$\ddot{\zeta}(t) + \rho A^{2\alpha}\dot{\zeta}(t) + A^2\zeta(t) = Bu(t),$$
$$\zeta(0) = \zeta_0 \in H_2, \quad \dot{\zeta}(0) = \zeta_1 \in H_0, \quad u \in L^2_{\mathrm{loc}}(\mathbb{R}; U). \tag{3}$$

In order to define the (mild) solution of this problem, we assume that $B \in \mathcal{L}(U, H_0)$ (which is enough for the application in Section 2) or, more generally, that B is admissible in a sense specified later in (9).

To state the "observability condition" on the spectral subspaces of A with respect to C of the main theorem, we first introduce our spectral notations. Given $\gamma > 0$ and $\mu > 1$, applying the functional calculus for self-adjoint operators to the positive operator A^{γ} and the bounded function on $\mathbb{R}^+$ defined by $\mathbf{1}_{\lambda \leqslant \mu} = 1$ if $\lambda \leqslant \mu$ and $\mathbf{1}_{\lambda \leqslant \mu} = 0$, otherwise, yields the spectral projector $\mathbf{1}_{A^{\gamma} \leqslant \mu}$. The image of H_0 under this projection operator is just the *spectral subspace* $\mathbf{1}_{A^{\gamma} \leqslant \mu} H_0$ of A^{γ}.

Definition 1. Let $\gamma > 0$. The *observability of low modes of A^γ through C at exponential cost* holds if there are positive constants D_0 and D_1 such that

$$\forall \mu > 1,\ \forall v \in \mathbf{1}_{A^\gamma \leqslant \mu} H_0, \quad \|v\|_0 \leqslant D_0 e^{D_1 \mu} \|Cv\|. \tag{4}$$

This abstract condition is satisfies in some concrete applications given in the next section. As illustrated in the proof of the following main theorem (cf. Section 1.3), it allows to compare the free dissipation of high modes to the cost of controlling low modes.

Theorem 1. *Assume that observability of low modes of A^γ through C at exponential cost holds for some $\gamma \in (0,1)$ (cf. Definition 1). For all $\rho > 0$ and $\alpha \in (\gamma/2, 1-\gamma/2)$, for all $\beta > (2\min\{\alpha, 1-\alpha\}/\gamma - 1)^{-1}$, there are positive constants C_1 and C_2 such that for all $T \in (0,1]$, for all ζ_0 and ζ_1, there is an input function u such that the solution ζ of (3) satisfies $\zeta(T) = \dot{\zeta}(T) = 0$ with the cost estimate*:

$$\int_0^T \|u(t)\|^2\,dt \leqslant C_2 \exp\left(\frac{C_1}{T^\beta}\right)\left(\|\zeta_0\|_2^2 + \|\zeta_1\|_0^2\right).$$

1.1. The duality between observation and control

The proof of Theorem 1 uses the well-known equivalence between controllability and observability (cf. [9]). In this section, we clarify in what sense the dual of the control problem (3) is the observation of the following Cauchy problem (without input):

$$\ddot{z}(t) + \rho A^{2\alpha}\dot{z}(t) + A^2 z(t) = 0, \qquad z(0) = z_0 \in H_2, \qquad \dot{z}(0) = z_1 \in H_0. \tag{5}$$

The second-order differential equations (3) and (5) may be restated as first-order systems by setting $\xi(t) = (\zeta(t), \dot{\zeta}(t))$ and $x(t) = (z(t), \dot{z}(t))$:

$$\dot{\xi}(t) - \mathcal{A}\xi(t) = \mathcal{B}u(t), \quad \xi(0) = \xi_0 \in X, \quad u \in L^2_{\mathrm{loc}}(\mathbb{R}; U), \tag{6}$$

$$\dot{x}(t) = \mathcal{A}x(t), \quad x(0) = x_0 \in X. \tag{7}$$

The state space is $X = H_2 \times H_0$. The semigroup generator $\mathcal{A}$ of (7) is defined by

$$\mathcal{A} = \begin{pmatrix} 0 & I \\ -A^2 & -\rho A^{2\alpha} \end{pmatrix}, \quad D(\mathcal{A}) = \left\{(z_0, z_1) \in H_2 \times H_2 \mid A^2 z_0 + \rho A^{2\alpha} z_1 \in H_0\right\}.$$

It inherits from $-A$ the necessary and sufficient properties of Lumer–Phillips for generating a contraction semigroup.

The control operator is $\mathcal{B} = \Pi B$, where $\Pi : X \to H_0$ is defined by $\Pi(z_0, z_1) = z_1$. If $B \in \mathcal{L}(U, H_0)$ (as in the application in Section 2) then $\mathcal{B} \in \mathcal{L}(U, X)$. Indeed Theorem 1 is valid in the following more general (canonical) setting introduced by Weiss in [25]. Let X_1 be $D(\mathcal{A})$ with the norm $\|x\|_1 = \|\mathcal{A}x\|$ and let X_{-1} be the completion of X with respect to the norm $\|x\|_{-1} = \|\mathcal{A}^{-1}x\|$. At first, we only assume $\mathcal{B} \in \mathcal{L}(U, X_{-1})$. In order to define the unique (mild) solution $\xi \in C(\mathbb{R}_+; X)$ of (6) by the integral formula

$$\xi(t) = e^{t\mathcal{A}}\xi(0) + \int_0^t e^{(t-s)\mathcal{A}}\mathcal{B}u(s)\,ds, \tag{8}$$

we also make the *admissibility assumption*: for some $T > 0$ (hence for all $T > 0$) there is a positive constant K_T such that

$$\forall u \in L^2_{\mathrm{loc}}(\mathbb{R}; U), \quad \left\| \int_0^T e^{t\mathcal{A}}\mathcal{B}u(t)\,dt \right\|^2 \leqslant K_T \int_0^T \|u(t)\|^2\,dt. \tag{9}$$

We define the *duality pairing* on X by

$$\big\langle (\zeta_0, \zeta_1), (z_0, z_1) \big\rangle = \langle A\zeta_0, Az_0\rangle_0 - \langle \zeta_1, z_1\rangle_0.$$

With respect to this pairing, X and $\mathcal{A}$ are their own dual, X_{-1} is the dual of X_1 and $\mathcal{B}$ is the dual of the observation operator $\mathcal{C} = C\Pi$. The assumptions on $\mathcal{B}$ are equivalent to $\mathcal{C} \in \mathcal{L}(X_1, U)$ and, for all $x_0 \in D(\mathcal{A})$,

$$\int_0^T \left\|\mathcal{C}e^{t\mathcal{A}}x_0\right\|^2 dt \leqslant K_T\|x_0\|^2.$$

Therefore the output map $x_0 \mapsto \mathcal{C}e^{t\mathcal{A}}x_0$ from $D(\mathcal{A})$ to $L^2([0, T]; U)$ has a continuous extension to X. N.b. if $\alpha \leqslant 1/2$, then $X_1 = H_4 \times H_2$ and $X_{-1} = H_0 \times H_{-2}$.

We recall the duality between controllability and observability (cf. [9]).

Lemma 1. *Let $T > 0$ and $C_T > 0$. The following properties are equivalent*:

(i) *For all initial state $\xi_0 \in X$, there is an input function $u \in L^2_{\mathrm{loc}}(\mathbb{R}; U)$ such that the solution $\xi \in C(\mathbb{R}_+; X)$ of* (6) *satisfies $\xi(T) = 0$ and $\|u\|_{L^2(0,T;U)} \leqslant C_T\|\xi_0\|$.*
(ii) *For all initial state $x_0 \in X$, the solution $x(t) = e^{t\mathcal{A}}x_0$ of* (7) *satisfies the observation inequality $\|x(T)\| \leqslant C_T\|\mathcal{C}x(t)\|_{L^2(0,T;U)}$.*

N.b. the smallest constant C_T such that these properties hold is the *controllability cost* mentioned in the introduction. The estimate in Theorem 1 writes $C_T^2 \leqslant C_2 \exp(C_1/T^\beta)$. The contractivity of $e^{t\mathcal{A}}$, (8) and (9) imply the estimates:

$$\left\|\xi(T)\right\|^2 \leqslant 2\big(1 + K_T C_T^2\big)\left\|\xi(0)\right\|^2, \tag{10}$$

$$\int_0^T \left\|\xi(t)\right\|^2 dt \leqslant 2T\big(1 + K_T C_T^2\big)\left\|\xi(0)\right\|^2, \tag{11}$$

since K_t and $\int_0^t \|u(s)\|^2\,ds$ are nondecreasing, although C_t is nonincreasing.

1.2. Spectral and growth bounds

The proof of Theorem 1 relies on a spectral decomposition of the problem. We extend the action of the spectral projector $\mathbf{1}_{A^\gamma\leqslant\mu}$ to X according to $\mathbf{1}_{A^\gamma\leqslant\mu}(z_0,z_1)=(\mathbf{1}_{A^\gamma\leqslant\mu}z_0,\mathbf{1}_{A^\gamma\leqslant\mu}z_1)$. It commutes with $\mathcal{A}$ and the generated semigroup. Therefore X is the orthogonal sum of the invariant subspaces $\mathbf{1}_{A^\gamma\leqslant\mu}X$ (low modes) and $\mathbf{1}_{A^\gamma>\mu}X$ (high modes).

The restriction of $e^{t\mathcal{A}}$ to $\mathbf{1}_{A^\gamma>\mu}X$ satisfies the following exponential decay bound.

Proposition 1. *Let $\gamma'=\gamma/(2\min\{\alpha,1-\alpha\})$. There is an $r>0$ such that*

$$\forall\mu\geqslant 1,\ \forall x\in\mathbf{1}_{A^\gamma>\mu}X,\ \forall t\geqslant 0,\quad \left\|e^{t\mathcal{A}}x\right\|\leqslant\exp\bigl(-r\mu^{1/\gamma'}t\bigr)\|x\|.$$

This reduces to a spectral bound thanks to the results of Chen and Triggiani on the differentiability of $e^{t\mathcal{A}}$ (cf. [7,8]). We first prove two spectral lemmas.

Lemma 2. *The spectrum of $\mathcal{A}$ relates to the spectrum of A according to*

$$\sigma(-\mathcal{A})\subset\bigl\{\lambda\in\mathbb{C}\mid\exists\mu\in\sigma(A),\ P_\mu(\lambda)=0\bigr\}\quad\text{with } P_\mu(\lambda)=\lambda^2-\rho\mu^{2\alpha}\lambda+\mu^2.$$

Proof. Let $\lambda\notin\{\lambda\in\mathbb{C}\mid\exists\mu\in\sigma(A),\ P_\mu(\lambda)=0\}$. The function $\mu\mapsto P_\mu(\lambda)/\mu^2$ is continuous on $(0,+\infty)\supset\sigma(A)$, it tends to 1 as μ tends to infinity and it does not vanish on the closed set $\sigma(A)$. Hence, there is an $\varepsilon>0$ such that, for all $\mu\in\sigma(A)$, $|P_\mu(\lambda)/\mu^2|>\varepsilon$. Since $\mu\mapsto|\mu^2P_\mu(\lambda)^{-1}|$ is bounded on $\sigma(A)$ (by ε^{-1}), we have $A^2P_A(\lambda)^{-1}\in\mathcal{L}(H_0)\subset\mathcal{L}(H_2,H_0)$, $P_A(\lambda)^{-1}\in\mathcal{L}(H_0,H_4)\subset\mathcal{L}(H_0,H_2)$ and $(\lambda I-\rho A^{2\alpha})P_A(\lambda)^{-1}\in\mathcal{L}(H_2)$. Therefore the operator $M(\lambda)$ defined by

$$M(\lambda)=\begin{pmatrix}(\lambda I-\rho A^{2\alpha})P_A(\lambda)^{-1} & -P_A(\lambda)^{-1}\\ A^2P_A(\lambda)^{-1} & \lambda P_A(\lambda)^{-1}\end{pmatrix}$$

is bounded on X. But $M(\lambda)(\lambda I+\mathcal{A})=(\lambda I+\mathcal{A})M(\lambda)=I$, so that $M(\lambda)$ is the bounded inverse of $\lambda I+\mathcal{A}$. Hence $\lambda\notin\sigma(-\mathcal{A})$. □

Lemma 3. *The roots $\lambda_\pm=(1\pm\sqrt{1-(2\mu^{1-2\alpha}/\rho)^2})\rho\mu^{2\alpha}/2$ of $P_\mu(\lambda)$ satisfy*:

$$\forall\mu\geqslant 1,\quad\min\{\operatorname{Re}\lambda_+,\operatorname{Re}\lambda_-\}\geqslant r\mu^{2\min\{\alpha,1-\alpha\}},\quad\text{with } r=\min\{\rho/2,1/\rho\}.$$

Proof. Let $x=2\mu^{1-2\alpha}/\rho$. If $x\geqslant 1$, then $\operatorname{Re}\lambda_+=\operatorname{Re}\lambda_-=\mu^{2\alpha}\rho/2$. Otherwise $\lambda_\pm\in\mathbb{R}$, $\lambda_+=(1+\sqrt{\dots})\mu^{2\alpha}\rho/2\geqslant\mu^{2\alpha}\rho/2$, and $\lambda_-=(1-\sqrt{1-x^2})\mu^{2\alpha}\rho/2\geqslant x^2\mu^{2\alpha}\rho/4=\mu^{2(1-\alpha)}/\rho$. Since $\min\{\mu^{2\alpha},\mu^{2(1-\alpha)}\}=\mu^{2\min\{\alpha,1-\alpha\}}$ for $\mu\geqslant 1$, gathering these lower bounds yields the lemma. □

Proof of Proposition 1. Since $e^{t\mathcal{A}}$ is a differentiable semigroup for $\alpha\in(0,1]$ ([8] proves that this semigroup is of Gevrey class and that it is analytic if and only if $\alpha\in[1/2,1]$), it is eventually continuous for the operator norm topology. The semigroup generated by the restriction $\mathcal{A}_\mu$ of $\mathcal{A}$ to $\mathbf{1}_{A^\gamma>\mu}X$ inherits this property. But Lemma 3 and the proof of Lemma 2 imply $\sigma(-\mathcal{A}_\mu)\subset\{\lambda\in\mathbb{C}\mid\operatorname{Re}\lambda\geqslant r\mu^{1/\gamma'}\}$ with $r=\min\{\rho/2,1/\rho\}$. Therefore the growth bound in Proposition 1 holds. □

1.3. Proof of Theorem 1

The last ingredient of this proof is the following cost estimate corresponding to the control operator $B = I$ proved in [2].

Proposition 2. (Avalos–Lasiecka, 2003) *For all $\rho > 0$, for all $\alpha \in (0,1)$, there are positive constants c_1 and c_2 such that for all $T \in (0,1]$ the solutions of* (5) *satisfy*:

$$\forall z_0 \in H_2,\ \forall z_1 \in H_0, \quad \|z(T)\|_2^2 + \|\dot z(T)\|_0^2 \leqslant \frac{c_2}{T^{c_1}} \int_0^T \|\dot z(t)\|^2\, dt. \tag{12}$$

(Indeed, [2] specifies how the power c_1 depends on α.)

In a first step, from the stationary condition in Definition 1, Proposition 2 and the duality in Lemma 1, we deduce the "controllability of low modes at exponential cost" in the corresponding dynamics. In a second step, combining it with the decay bound in Proposition 1 according to the iterative control strategy introduced by Lebeau and Robbiano in [15], we prove the controllability of all modes. We estimate the controllability cost as the control time tends to zero, like in [20], in the last step.

First step. With the notations introduced in Section 1.1, the observation inequality (12) in Proposition 2 writes:

$$\forall x_0 \in X, \quad \|e^{T\mathcal{A}} x_0\|^2 \leqslant \frac{c_2}{T^{c_1}} \int_0^T \|\Pi e^{t\mathcal{A}} x_0\|^2\, dt. \tag{13}$$

Let $\tau \in (0,1]$, $\mu \geqslant 1$ and $x_0 \in \mathbf{1}_{A^\gamma \leqslant \mu} X$. For all $t \in [0,\tau]$, we may apply (4) to $\Pi e^{t\mathcal{A}} x_0$ since it is in $\mathbf{1}_{A^\gamma \leqslant \mu} H_0$:

$$\|\Pi e^{t\mathcal{A}} x_0\|_0^2 \leqslant D_0^2 e^{2D_1\mu} \|C\Pi e^{t\mathcal{A}} x_0\|^2.$$

First integrating on $[0,\tau]$, then using (13) yields:

$$\|e^{T\mathcal{A}} x_0\|^2 \leqslant D_0^2 e^{2D_1\mu} \frac{c_2}{\tau^{c_1}} \int_0^\tau \|\mathcal{C} e^{t\mathcal{A}} x_0\|^2\, dt.$$

This "low modes fast observability for $e^{t\mathcal{A}}$ at exponential cost" is equivalent, by the same duality as in Lemma 1, to the controllability property: for all $\tau \in (0,1]$ and $\mu > 1$, there is a bounded operator $S_\mu^\tau : X \to L^2(0,\tau;U)$ such that, for all $\xi_0 \in \mathbf{1}_{A^\gamma \leqslant \mu} X$, the solution $\xi \in C(\mathbb{R}_+, X)$ of (6) with input function $u = S_\mu^\tau \xi_0$ satisfies $\mathbf{1}_{A^\gamma \leqslant \mu} \xi(\tau) = 0$, and $\exists d_3 > 0$, $\|S_\mu^\tau\| \leqslant (d_3/\tau^{c_1/2}) e^{D_1\mu}$ (cost estimate).

Second step. The hypothesis on α implies that the γ' of Proposition 1 is lower than 1. We introduce a dyadic scale of modes $\mu_k = 2^k$ ($k \in \mathbb{N}$) and a sequence of time intervals $\tau_k = \sigma_\delta T/\mu_k^\delta$

where $\delta \in (0, \gamma'^{-1} - 1)$ and $\sigma_\delta = (2\sum_{k\in\mathbb{N}} 2^{-k\delta})^{-1} > 0$, so that the sequence of times defined recursively by $T_0 = 0$ and $T_{k+1} = T_k + 2\tau_k$ converges to T. The strategy consists in steering the initial state ξ_0 to 0, through the sequence of states $\xi_k = \xi(T_k) \in \mathbf{1}_{A^\gamma > \mu_{k-1}} X$ composed of ever higher modes, by applying recursively the input function $u_k = S_{\mu_k}^{\tau_k} \xi_k$ to ξ_k during a time τ_k and no input during a time τ_k. Introducing the notations

$$\varepsilon_k = \|\xi_k\|, \qquad C_k = D_2 e^{D_1\mu_k} / \tau_k^{c_1/2}, \quad \text{and} \quad \rho_k = \left(\frac{C_{k+1}\varepsilon_{k+1}}{C_k\varepsilon_k}\right)^2, \tag{14}$$

the cost estimate of the previous step writes $\|S_{\mu_k}^{\tau_k}\| \leqslant C_k$ and implies:

$$\|u\|^2_{L^2(0,T;U)} = \sum_{k\in\mathbb{N}} \|u_k\|^2_{L^2(0,\tau_k;U)} \leqslant \sum_{k\in\mathbb{N}} C_k^2 \varepsilon_k^2. \tag{15}$$

Since $\tau_k \leqslant T \leqslant 1$, the estimate (10) between the times T_k and $T_k + \tau_k$ implies

$$\left\|\xi(T_k + \tau_k)\right\|^2 \leqslant 2\left(1 + K_1 C_k^2\right)\varepsilon_k^2.$$

Since $\mathbf{1}_{A^\gamma \leqslant \mu_k} \xi(T_k + \tau_k) = 0$ and Proposition 1 imply $\varepsilon_{k+1} \leqslant e^{-r\mu_k^{1/\gamma'}\tau_k} \|\xi(T_k + \tau_k)\|$, we deduce

$$\varepsilon_{k+1}^2 \leqslant 2e^{-2r\tau_k\mu_k^{1/\gamma'}} \left(1 + K_1 C_k^2\right)\varepsilon_k^2.$$

Since $C_{k+1}/C_k = 2^{\delta c_1/2} e^{D_1\mu_k}$, we deduce that, for any $D_3 > 4D_1$, there is a $D_4 > 0$ such that

$$\rho_k \leqslant 2^{1+\delta c_1}\left(e^{-2D_1\mu_k} + \frac{K_1 D_2^2}{\tau_k^{c_1}}\right) e^{4D_1\mu_k - 2r\tau_k\mu_k^{1/\gamma'}} \leqslant \frac{D_4}{T^{c_1}} e^{D_3\mu_k - 2r\sigma_\delta T\mu_k^{\gamma'^{-1}-\delta}}. \tag{16}$$

Since $\gamma'^{-1} - \delta > 1$, this implies:

$$\forall \rho \in (0,1),\ \exists N \in \mathbb{N}, \quad k \geqslant N \quad \Rightarrow \quad \rho_k \leqslant \rho.$$

Therefore $\lim_k \varepsilon_k = 0$ and the last series in (15) converges. This completes the proof of the controllability in Theorem 1.

Third step. The controllability cost C_T, formally defined after Lemma 1, satisfies:

$$C_T^2 \leqslant C_0^2\left(1 + \sum_{l\geqslant 1} \prod_{0\leqslant k\leqslant l-1} \rho_k\right). \tag{17}$$

Since

$$l \leqslant \mu_l, \qquad \sum_{0\leqslant k\leqslant l-1} \mu_k \leqslant \mu_l \quad \text{and} \quad \sum_{0\leqslant k\leqslant l-1} \mu_k^{\gamma'^{-1}-\delta} \geqslant \mu_{l-1}^{\gamma'^{-1}-\delta}/2,$$

(16) implies

$$\prod_{0\leqslant k\leqslant l-1} \rho_k \leqslant \exp\bigl(\bigl(D_3 + \ln\bigl(D_4/T^{c_1}\bigr)\bigr)\mu_l - r\sigma_\delta T \mu_{l-1}^{\gamma'^{-1}-\delta}\bigr).$$

Hence, setting $q = 2^{\gamma'^{-1}-\delta}$ and $T' = r\sigma_\delta T/q$ we have

$$\forall l \geqslant 1, \quad \prod_{0\leqslant k\leqslant l-1} \rho_k \leqslant \exp\bigl(D_{T'}2^l - T'q^l\bigr) \quad \text{with } D_{T'} \underset{T'\to 0}{\sim} c_1 \ln(1/T').$$

As in [20], plugging this in (17) yields the cost estimate:

$$\forall \beta > \beta_q,\ \exists D_6 > 0,\ \exists D_7 > 0, \quad C_T^2 \leqslant D_6 \exp\left(\frac{D_7}{T'^\beta}\right) \quad \text{with } \beta_q = \left(\frac{\ln q}{\ln 2} - 1\right)^{-1}.$$

Since T' is proportional to T and β_q decreases to $(\gamma'^{-1} - 1)^{-1} = (2\min\{\alpha, 1-\alpha\}/\gamma - 1)^{-1}$ as δ decreases to 0, this proves the estimate in Theorem 1 restated after Lemma 1.

2. Interior controllability of structurally damped plates

This section concerns concrete applications of the abstract model studied in the previous section. The main application is to the plate equation with square root damping and interior control in Ω with hinged boundary conditions:

$$\begin{gathered}\ddot{\zeta} - \rho\Delta\dot{\zeta} + \Delta^2\zeta = \chi_\Omega u \quad \text{on } \mathbb{R}_+ \times M, \qquad \zeta = \Delta\zeta = 0 \quad \text{on } \mathbb{R}_+ \times \partial M, \\ \zeta(0) = \zeta_0 \in H^2(M) \cap H_0^1(M), \quad \dot{\zeta}(0) = \zeta_1 \in L^2(M), \quad u \in L^2_{\text{loc}}(\mathbb{R}_+ \times M). \end{gathered} \tag{18}$$

In this section, M is a smooth connected complete d-dimensional Riemannian manifold with metric g and non-empty boundary ∂M, M denotes the interior and $\overline{M} = M \cup \partial M$. Let Δ denote the Dirichlet Laplacian on $L^2(M)$ with domain $D(\Delta) = H_0^1(M) \cap H^2(M)$ (thus Δ denotes a negative differential operator with variable coefficients depending on the metric g). N.b. the results are already interesting when (M, g) is a smooth domain of the Euclidean space $\mathbb{R}^d$, so that $\Delta = \partial^2/\partial x_1^2 + \cdots + \partial^2/\partial x_d^2$. Let χ_Ω denote the multiplication by the characteristic function of an open subset $\Omega \neq \emptyset$ of $\overline{M}$.

For simplicity, in the following theorem proved in Section 2.4, we assume that M is compact. The second part of this theorem makes a geometric assumption on Ω due to Bardos–Lebeau–Rauch based on *generalized geodesics*. In this context, the generalized geodesics are continuous trajectories $t \mapsto x(t)$ in $\overline{M}$ which follow geodesic curves at unit speed in M (so that on these intervals $t \mapsto \dot{x}(t)$ is continuous); if they hit ∂M transversely at time t_0, then they reflect as light rays or billiard balls (and $t \mapsto \dot{x}(t)$ is discontinuous at t_0); if they hit ∂M tangentially then either there exists a geodesic in M which continues $t \mapsto (x(t), \dot{x}(t))$ continuously and they branch onto it, or there is no such geodesic curve in M and then they glide at unit speed along the geodesic of ∂M which continues $t \mapsto (x(t), \dot{x}(t))$ continuously until they may branch onto a geodesic in M. For this result and whenever generalized geodesics are mentioned, we make the additional assumption that they can be uniquely continued at the boundary ∂M (as in [5], to ensure this, we may assume either that ∂M has no contacts of infinite order with its tangents, or that g and ∂M are real analytic).

Let L_Ω denote the length of the longest generalized geodesic in $\overline{M}$ which does not intersect Ω. For instance, we recall that $L_\Omega < \infty$ if Ω is a neigbourhood of the boundary of a smooth domain M of $\mathbb{R}^d$ (in that case L_Ω is the length of the longest segment in $M \setminus \Omega$) and that $L_\Omega < 2D$ if Ω is a neighborhood of a hemisphere of the boundary of a Euclidean ball M of diameter D.

Theorem 2. *For all $\rho > 0$ and $\Omega \neq \emptyset$, for all $\beta > 1$, there are $C_1 > 0$ and $C_2 > 0$ such that, for all $T \in (0,1]$, for all ζ_0 and ζ_1, there is an input function u such that the solution ζ of* (18) *satisfies $\zeta(T) = \dot\zeta(T) = 0$ and the cost estimate*:

$$\int_0^T \int_M |u|^2 \, dx \, dt \leqslant C_2 \exp\bigl(C_1/T^\beta\bigr) \int_M |\Delta\zeta_0|^2 + |\zeta_1|^2 \, dx.$$

For all $\rho \in (0,2)$ and $L > L_\Omega$, this result holds with this estimate improved by replacing $\exp(C_1/T^\beta)$ *with* $\exp(C_\rho L^2/T)$ *where C_ρ does not depend on Ω.*

2.1. Application of Theorem 1

Indeed, Theorem 1 applies to structurally damped plates with interior control in Ω more general than (18):

$$\ddot\zeta + \rho(-\Delta)^{2\alpha}\dot\zeta + \Delta^2\zeta = \chi_\Omega u \quad \text{on } \mathbb{R}_+ \times M,$$
$$\zeta(0) = \zeta_0 \in H^2(M) \cap H_0^1(M), \quad \dot\zeta(0) = \zeta_1 \in L^2(M), \quad u \in L^2\bigl([0,T] \times M\bigr). \tag{19}$$

This is the abstract system (6) where the generator is $A = -\Delta$, the state and input space is $H_0 = U = L^2(M)$, and the control and observation operator is $B = C = \chi_\Omega$. N.b. for square root damping ($\alpha = 1/2$), $X_1 = H_4 \times H_2 = \{(z_0, z_1) \in H^4(M) \times H^2(M) \mid z_1 = z_0 = \Delta z_0 = 0 \text{ on } \partial M\}$ and the solution of (6) satisfy the boundary conditions $\zeta = \Delta\zeta = 0$ on $\mathbb{R}_+ \times \partial M$ in a generalized sense.

If M is not compact, assume that Ω is the exterior of a compact set K such that $K \cap \overline{\Omega} \cap \partial M = \emptyset$ and that $0 \notin \sigma(\Delta)$. In this setting, the observability of low modes of $(-\Delta)^{1/2}$ through C at exponential cost holds (cf. Definition 1). When M is compact this is an inequality on sums of eigenfunctions proved as Theorem 3 in [16] and Theorem 14.6 in [12]. This was generalized to non-compact M in [18]. Applying Theorem 1 with $\gamma = 1/2$ yields:

Corollary 1. *For all $\rho > 0$, $\alpha \in (1/4, 3/4)$, and $\beta > \min\{4\alpha - 1, 3 - 4\alpha\}^{-1}$, there are $C_1 > 0$ and $C_2 > 0$ such that, for all $T \in (0,1]$, for all ζ_0 and ζ_1, there is an input function u such that the solution ζ of* (19) *satisfies $\zeta(T) = \dot\zeta(T) = 0$ and the cost estimate*:

$$\|u\|_{L^2}^2 \leqslant C_2 \exp\bigl(C_1/T^\beta\bigr)\bigl(\|\zeta_0\|_{H^2}^2 + \|\zeta_1\|_{L^2}^2\bigr).$$

Remark 1. Note that Proposition 2 proves controllability from $\Omega = M$ when $\alpha \in [0,1)$. It results from [2] that controllability does not hold in Corollary 1 for $\alpha = 1$. For $\alpha = 0$, it can be proved by the transmutation control method that the controllability for $\rho = 0$ (which holds if Ω satisfies the condition $L_\Omega < \infty$ of Theorem 2) implies the controllability over the same time for $\rho > 0$. The case $\alpha \in (0, 1/4] \cup [3/4, 1)$ with $\Omega \neq M$ is still open.

Remark 2. More generally, Theorem 1 applies to $A = (-\Delta)^{1/(2\gamma)}$ with $\gamma \in (0, 1)$. It does not apply to the wave equation which would correspond to $\gamma = 1$. The wave equation with square root damping ($\alpha = 1/2$) is just out of reach and seems to us an interesting open problem (the appendix in [20] proves that it is not controllable by a one-dimensional input). It results from [1] that the wave equation ($\gamma = 1$) with Kelvin–Voigt damping ($\alpha = 1$) is not controllable from any $\Omega \neq M$. It results from [5] that the damped wave equation ($\gamma = 1$, $\alpha = 0$), is controllable from Ω or not depending on whether the control time is greater or lower than L_Ω (defined before Theorem 2).

2.2. *Smoothing*

The control transmutation method of Section 2.4 applies to initial data smoother than in Theorem 2. This drawback is easily overcome by a general abstract remark, made here, concerning the null-controllability of analytic semigroups: in the smoothness scale of Sobolev spaces defined by the generator, if fast controllability holds for initial data in some space, then it holds for initial data in any less smooth space; moreover, the same statement holds for fast controllability at exponential cost.

We recall the setting of Section 1.1. $\mathcal{A}$ is the boundedly invertible generator of a bounded analytic semigroup on the Hilbert space X. For any $p > 0$, let X_p denote the Hilbert space $D((-\mathcal{A})^p)$ with the norm $\|x\|_p = \|(-\mathcal{A})^p x\|$ (which is equivalent to the graph norm) and let X_{-p} be the completion of X with respect to the norm $\|x\|_{-p} = \|(-\mathcal{A})^{-p} x\|$. There is a duality pairing on X such that X and $\mathcal{A}$ are their own dual. For this duality pairing, X_{-p} is the dual of X_p.

For any $p \in \mathbb{R}$, the control operator $\mathcal{B}$ is said admissible in X_p and fast controllability is said to hold in X_p if $\mathcal{B}$ satisfies (9) and the property (i) of Lemma 1 holds for all positive T, respectively, with X and its norm replaced by X_p and its norm. In this case, the admissibility constant K_T and the cost C_T are denoted by $K_{p,T}$ and $C_{p,T}$.

Proposition 3. *For all real numbers p and p' such that $p' \leqslant p$:*

- *Admissibility in X_p implies admissibility in $X_{p'}$. Conversely, if $\mathcal{B} \in \mathcal{L}(U, X_{p'})$ and $p' > p - 1/2$ then $\mathcal{B}$ is admissible in X_p.*
- *Fast controllability in X_p implies fast controllability in $X_{p'}$. Moreover, if there are positive constants β, C_1 and C_2 such that $C_{p,T} \leqslant C_2 \exp(C_1/T^\beta)$, then there are positive constants C'_1 and C'_2 such that $C_{p',T} \leqslant C'_2 \exp(C'_1/T^\beta)$.*

Proof. Since $\mathcal{A}^{-1} \in \mathcal{L}(X)$, $X_p \subset X_{p'}$ continuously which proves the first statement.

Since $e^{t\mathcal{A}}$ is an analytic semigroup, it satisfies the smoothing property: $\forall n > 0$, $\forall m \in \mathbb{R}$, $S_n := \sup_{t>0} \|t^n \mathcal{A}^n e^{t\mathcal{A}}\|_{\mathcal{L}(X_m)} < \infty$.

By duality, $K_{p,T}$ is also defined by

$$\forall x \in X_{-p'}, \quad \int_0^T \left\| \mathcal{C} e^{t\mathcal{A}} x \right\|^2 dt \leqslant K_{p,T} \|x\|_{-p}^2.$$

Therefore

$$K_{p,T} \leqslant \|\mathcal{C}\|^2_{\mathcal{L}(X_{-p'},U)} S^2_{p-p'} \int_0^T t^{2(p'-p)}\, dt$$

is finite for $2(p'-p) > -1$.

By duality, $C_{p,T}$ is also defined by the observability inequality:

$$\forall x \in X_{-p}, \quad \left\| e^{T\mathcal{A}} x \right\|_{-p} \leqslant C_{p,T} \left\| \mathcal{C} e^{t\mathcal{A}} \right\|_{L^2(0,T;U)}.$$

Therefore $C_{p',2T} \leqslant S_{p-p'} C_{p,T} / T^{p-p'}$. □

2.3. Boundary controllability

This section concerns the following boundary control version of the plate equation with square root damping (18):

$$\begin{aligned} &\ddot{\zeta} - \rho\Delta\dot{\zeta} + \Delta^2\zeta = 0 \quad \text{on } \mathbb{R}_+ \times M, \qquad \zeta = 0, \quad \Delta\zeta = \chi_\Gamma u \quad \text{on } \mathbb{R}_+ \times \partial M, \\ &\zeta(0) = \zeta_0 \in H_0^1(M), \quad \dot{\zeta}(0) = \zeta_1 \in H^{-1}(M), \quad u \in L^2_{\mathrm{loc}}(\mathbb{R}_+ \times M), \end{aligned} \tag{20}$$

where χ_Γ denotes the restriction to the boundary followed by the multiplication by the characteristic function of an open subset $\Gamma \neq \emptyset$ of ∂M. A key ingredient of the control transmutation method of Section 2.4 is the so-called "fundamental controlled solution" for (18). It is constructed in Corollary 2 of Theorem 3 in this section which applies [23] to estimate the cost of fast boundary controls for (20) when M is a (Euclidean) segment.

We first adapt the abstract duality framework of Section 1.1 to this boundary control system. Here $A = -\Delta$, $H_0 = L^2(M)$, $U = L^2(\Gamma)$ and $\alpha = 1/2$. It is convenient to use the state space $X = H_1 \times H_{-1}$ with the duality pairing

$$\left\langle (\zeta_0, \zeta_1), (z_0, z_1) \right\rangle = \langle \zeta_1, z_0 \rangle_0 + \langle \zeta_0, z_1 \rangle_0 + \rho \left\langle A^\alpha \zeta_0, A^\alpha z_0 \right\rangle_0.$$

With respect to this pairing, X and $\mathcal{A}$ are their own dual, $X_{-1} = H_{-1} \times H_{-3}$ is the dual of $X_1 = H_3 \times H_1$. Multiplying by $\zeta(T-t)$ the dual homogeneous equation

$$\ddot{z} - \rho\Delta\dot{z} + \Delta^2 z = 0 \quad \text{on } \mathbb{R}_+ \times M, \qquad z = \Delta z = 0 \quad \text{on } \mathbb{R}_+ \times \partial M, \tag{21}$$

and integrating by parts on $(0,T) \times M$, yields that the control operator $\mathcal{B}$ (arising when rewriting the second-order system (20) as the first-order system (6) on $\xi(t) = (\zeta(t), \dot{\zeta}(t))$) is the dual (with respect to the new pairing) of the Neumann observation operator $\mathcal{C} \in \mathcal{L}(X_1, U)$ defined by

$\mathcal{C}(z_0, z_1) = \chi_\Gamma \partial_\nu z_0$, where ∂_ν is a vector field normal to ∂M. As in the proof of Proposition 3, the admissibility of $\mathcal{B}$ results from the analyticity of $e^{t\mathcal{A}}$ and $\mathcal{C} \in \mathcal{L}(X_p; U)$ with $p \in (1/4, 1/2)$. (N.b. $\mathcal{C} \in \mathcal{L}(X_p, U)$ for $p > 1/4$ since $X_p = H_{2p+1} \times H_{2p-1}$ and $\chi_{\partial M}\partial_\nu \in \mathcal{L}(H_s; U)$ for $s > 3/2$.)

Theorem 3. *For all $\rho \in (0, 2)$, there are $C_1 > 0$ and $C_2 > 0$ such that for all $L > 0$ and $T \in]0, \inf(\pi/2, L)^2]$, for all ζ_0 and ζ_1, there is an input function u such that the solution ζ of* (20) *with $M = (-L, L)$ and $\Gamma = \{L\}$ satisfies $\zeta(T) = \dot{\zeta}(T) = 0$ and the cost estimate*

$$\int_0^T \|u\|_{L^2}^2 \, dt \leqslant C_2 \exp\big(C_1 L^2/T\big)\big(\|\zeta_0\|_{H^1}^2 + \|\zeta_1\|_{H^{-1}}^2\big).$$

Proof. By Lemma 1, it is enough to prove that the solution of (21) for any initial data $z(0) \in H_0^1(-L, L)$ and $\dot{z}(0) \in H^{-1}(-L, L)$ satisfies the observation inequality

$$\big\|z(T)\big\|_{H^1}^2 + \big\|\dot{z}(T)\big\|_{H^{-1}}^2 \leqslant C_2 \exp(C_1/T) \int_0^T \big|\partial_s z(t, L)\big|^2 \, ds.$$

The scaling $(t, x) \mapsto (\sigma^2 t, \sigma x)$ reduces the problem to the case $L = \pi$. Using the explicit eigenvalues and eigenfunctions of Δ on $M = (-\pi, \pi)$, this inequality becomes a "window problem" for series of complex exponentials which is almost the one-dimensional setting of "vibrational control with structural damping" considered in [23, Section 6], indeed simpler because more explicit. Therefore [23, Theorem 1] applies and completes the proof of Theorem 3. □

Corollary 2. *For all $\rho \in (0, 2)$ there are positive constants C_ρ and C'_ρ such that, $\forall L > 0$, $\forall T \in (0, 1]$, there is a "fundamental controlled solution" k in $C^0([0, T]; H^{-1}(]-L, L[)) \cap C^1([0, T]; H^{-3}(]-L, L[))$ satisfying*:

$$\partial_t^2 k - \rho \partial_s^2 \partial_t k + \partial_s^4 k = 0 \quad \textit{in } \mathcal{D}'\big(]0, T[\times]-L, L[\big),$$
$$k_{\rceil t=0} = \delta, \qquad \partial_t k_{\rceil t=0} = \delta' \quad \textit{and} \quad k_{\rceil t=T} = \partial_t k_{\rceil t=T} = 0,$$
$$\int_0^T \big(\big\|k(t, \cdot)\big\|_{H^{-1}(]-L,L[)}^2 + \big\|\partial_t k(t, \cdot)\big\|_{H^{-3}(]-L,L[)}^2\big) \, dt \leqslant C'_\rho e^{C_\rho L^2/T}.$$

Proof. The fast controllability in $X = H_0^1(-L, L) \times H^{-1}(-L, L)$ stated in Theorem 3 implies, by Proposition 3, the fast controllability in $X_{-1} = H^{-1}(-L, L) \times H^{-3}(-L, L)$ with the same form of cost estimate. Since the Dirac mass at the origin δ is in $H^s(\mathbb{R})$ for all $s < -1/2$, we may consider the controlled solution k obtained by applying this result to the initial data $(k, \partial_t k)_{\rceil t=0} = (\delta, \delta') \in X_{-1}$. The cost estimate and (11) on X_{-1} imply the estimate in Corollary 2. □

Although not needed in the proof of Theorem 2, we state another corollary of Theorem 3 for its own sake. It is based on the following idea: the controllability cost of a system is not increased by taking its tensor product with a contraction semigroup (it was proved in [19] that the controllability cost of a system is not changed by taking its tensor product with a unitary group). It applies in particular when M is a rectangle or an infinite strip in the plane controlled from one side (this controllability problem in a rectangle with other boundary conditions was solved in [11] without the cost estimate, which was added later at the end of [23]). N.b. in this example, the condition $L_\Gamma < \infty$ of [5] required in Theorem 2 is not satisfied.

Corollary 3. *Let $\tilde{M}$ denote another smooth complete Riemannian manifold. For all $\rho \in (0,2)$, there are $C_1 > 0$ and $C_2 > 0$ such that, for all $L > 0$ and $T \in (0,1]$ for all ζ_0 and ζ_1, there is an input function u such that the solution ζ of* (20) *with $M = (-L, L) \times \tilde{M}$ and $\Gamma = \{L\} \times \partial\tilde{M}$ satisfies $\zeta(T) = \dot{\zeta}(T) = 0$ and the cost estimate*:

$$\int_0^T \|u\|_{L^2}^2 \, dt \leqslant C_2 \exp\bigl(C_1 L^2/T\bigr)\bigl(\|\zeta_0\|_{H^1}^2 + \|\zeta_1\|_{H^{-1}}^2\bigr).$$

Proof. Let (s, y) denote the variable on $M = (-L, L) \times \tilde{M}$. Denoting respectively by Δ_s and Δ_y the Dirichlet Laplacians on the segment $(-L, L)$ and on $\tilde{M}$, we have $\Delta = \Delta_s + \Delta_y$. Since Δ is boundedly invertible, (20) may also be restated as a first-order system on $X = H^{-1}(M) \times H^{-1}(M)$ by setting $\xi(t) = (\Delta\zeta(t), \dot{\zeta}(t))$. Then the semigroup generator $\mathcal{A}$ of the dual homogeneous system (7) becomes:

$$\mathcal{A} = \Delta R \quad \text{with } R = \begin{pmatrix} 0 & 1 \\ -1 & -\rho \end{pmatrix}, \quad \text{and} \quad e^{t\mathcal{A}} = e^{t\Delta_s R} e^{t\Delta_y R} = e^{t\Delta_y R} e^{t\Delta_s R}.$$

The observation operator $\mathcal{C}_s$ defined by $\mathcal{C}_s x = \chi_\Gamma \partial_\nu z = \partial_s z_{\rceil s=L}$ commutes with $e^{t\Delta_y R}$. We shall estimate the cost by the duality in Lemma 1. Fix the initial state $x_0 \in X$ and $T > 0$. Applying to $s \mapsto (e^{T\Delta_y R} x_0)(s, y)$ for fixed y the observability inequality corresponding to Theorem 3 yields, with $C_T' := C_2 \exp(C_1 L^2/T)$:

$$\int_0^L \bigl|e^{T\Delta_s R} e^{T\Delta_y R} x_0\bigr|^2 \, ds \leqslant C_T' \int_0^T \int_0^L \bigl|\mathcal{C}_s e^{t\Delta_s R} e^{T\Delta_y R} x_0\bigr|^2 \, ds \, dt.$$

Integrating this inequality over $\tilde{M}$ yields (the first and last step use Fubini's theorem and the commutation of operators acting separately on s and y, the second step uses that $e^{t\Delta_y R}$ is a contraction):

$$\begin{aligned} \iint_M \bigl|e^{T\mathcal{A}} x_0\bigr|^2 \, ds \, dy &\leqslant C_T' \int_0^T \int_0^L \int_{\tilde{M}} \bigl|e^{T\Delta_y R} \mathcal{C}_s e^{t\Delta_s R} x_0\bigr|^2 \, dy \, ds \, dt \\ &\leqslant C_T' \int_0^T \int_0^L \int_{\tilde{M}} \bigl|e^{t\Delta_y R} \mathcal{C}_s e^{t\Delta_s R} x_0\bigr|^2 \, dy \, ds \, dt = C_T' \int_0^T \iint_M \bigl|\mathcal{C}_s e^{t\mathcal{A}} x_0\bigr|^2 \, ds \, dy \, dt. \end{aligned}$$

This is the observability inequality corresponding to Corollary 3. □

2.4. Proof of Theorem 2

The first part of Theorem 2 is Corollary 1 for $\alpha = 1/2$. We shall now prove the second part of Theorem 2 by the transmutation control method (cf. [17,19]). According to Proposition 3, it is sufficient to consider initial data in the space $X_2 = H_6 \times H_4$ which is smoother than the energy space claimed in Theorem 2.

It results from the work of Bardos–Lebeau–Rauch that (n.b. the control time and the time variable are denoted by L and s here):

Theorem 4. [4,5] *Let $L > L_\Omega$. For all (w_0, w_1) and (w_2, w_3) in $H^4(M) \cap H_0^1(M) \times H^3(M) \cap H_0^1(M)$ there is an input function $v \in H^3(]0, L[\times M)$ supported in $]0, L[\times \Omega$ such that the solution $w \in \bigcap_{n\in\mathbb{N}} C^n([0, L]; H^{4-n}(M))$ of*

$$\partial_s^2 w - \Delta w = v \quad \text{in }]0, L[\times M, \qquad w = 0 \quad \text{on }]0, L[\times \partial M,$$

with Cauchy data $(w, \partial_s w) = (w_0, w_1)$ at $s = 0$, satisfies $(w, \partial_s w) = (w_2, w_3)$ at $s = L$. Moreover, the operator S_W defined by $S_W((w_0, w_1), (w_2, w_3)) = v$ is continuous in the corresponding norms.

Let $T \in (0, 1]$ and $L > L_\Omega$ be fixed from now on.

Let $(\zeta_0, \zeta_1) \in X_2 = H_6 \times H_4$ be an initial data for the plate equation (18). Let $v_\pm$ and $w_\pm$ be the input function and solution for the wave equation obtained from Theorem 4 with $w_0 = \zeta_0$, $w_1 = \pm\zeta_1$ and $w_2 = w_3 = 0$. Let $w(\pm s, \cdot) = w_\pm(s, \cdot)$ and $v(\pm s, \cdot) = v_\pm(s, \cdot)$ for $s \in [0, L]$. We define $\underline{w} \in \bigcap_{n\in\mathbb{N}} C^n(\mathbb{R}; H^{4-n}(M))$ and $\underline{v} \in H^3(\mathbb{R} \times M)$ as the extensions of w and v by zero outside $[-L, L] \times M$. They inherit from $w_\pm$ and $v_\pm$ the properties:

$$\partial_s^2 \underline{w} - \Delta \underline{w} = \underline{v} \quad \text{in } \mathcal{D}'(\mathbb{R} \times M), \qquad \underline{w} = 0 \quad \text{on } \mathbb{R} \times \partial M,$$
$$(\underline{w}, \partial_s \underline{w})_{\rceil s=0} = (\zeta_0, \zeta_1) \quad \text{and} \quad (\underline{w}, \partial_s \underline{w})_{\rceil s=\pm L} = (0, 0). \tag{22}$$

Let k, C_ρ and C'_ρ be the fundamental controlled solution and corresponding constants given by Corollary 2. We define $\underline{k}$ as the extension of k by zero outside $]0, T[\times]-L, L[$. It inherits from k the following properties:

$$\partial_t^2 \underline{k} - \rho \partial_s^2 \partial_t \underline{k} + \partial_s^4 \underline{k} = 0 \quad \text{in } \mathcal{D}'\big(]0, T[\times]-L, L[\big), \tag{23}$$
$$\underline{k}_{\rceil t=0} = \delta, \qquad \partial_t \underline{k}_{\rceil t=0} = \delta' \quad \text{and} \quad \underline{k}_{\rceil t=T} = \partial_t \underline{k}_{\rceil t=T} = 0,$$
$$\int_0^T \big(\|\underline{k}(t, \cdot)\|^2_{H^{-1}(\mathbb{R})} + \|\partial_t \underline{k}(t, \cdot)\|^2_{H^{-3}(\mathbb{R})}\big)\, dt \leqslant C'_\rho e^{C_\rho L^2/T}. \tag{24}$$

The principle of the control transmutation method is to use $\underline{k}$ as a kernel to transmute $\underline{w}$ and $\underline{v}$ into a solution ζ and an input function u for (18). Since $\underline{k} \in C^0(\mathbb{R}_+; H^{-1}(\mathbb{R})) \cap C^1(\mathbb{R}_+; H^{-3}(\mathbb{R}))$, $\underline{w} \in H^1(\mathbb{R}; H^3(M)) \cap H^3(\mathbb{R}; H^1(M))$ and $\underline{v} \in H^1(\mathbb{R}; H^2(M)) \cap H^3(\mathbb{R}; L^2(M))$, the transmutation formulas

$$\zeta(t,x)=\int_{\mathbb{R}}\underline{k}(t,s)\underline{w}(s,x)\,ds,$$

$$u(t,x)=\int_{\mathbb{R}}\big(\rho\partial_t\underline{k}(t,s)\underline{v}(s,x)+\underline{k}(t,s)\big(\partial_s^2+\Delta\big)\underline{v}(s,x)\big)\,ds$$

define functions $\zeta\in C^0(\mathbb{R}_+;H^3(M))\cap C^1(\mathbb{R}_+;H^1(M))$ and $u\in L^2(\mathbb{R}_+\times M)$. This ζ satisfies the required initial conditions: $\underline{k}_{\rceil t=0}=\delta$ and $\underline{w}_{\rceil s=0}=\zeta_0$ imply $\zeta_{\rceil t=0}=\zeta_0$; $\partial_t\underline{k}_{\rceil t=0}=\partial_s\delta$ and $\partial_s\underline{w}_{\rceil s=0}=\zeta_1$ imply $\partial_t\zeta_{\rceil t=0}=\zeta_1$ by integrating by parts. This ζ satisfies the required final conditions: $\underline{k}_{\rceil t=T}=\partial_t\underline{k}_{\rceil t=T}=0$ implies $\zeta_{\rceil t=T}=\partial_t\zeta_{\rceil t=T}=0$. This ζ satisfies the required boundary conditions: $\underline{w}_{\rceil\partial M}=0$ implies $\zeta_{\rceil\partial M}=0$ and $\Delta\underline{w}_{\rceil\partial M}=\partial_s^2\underline{w}_{\rceil\partial M}=0$ implies $\Delta\zeta_{\rceil\partial M}=0$. The input u is supported in $[0,T]\times\Omega$ since $\underline{k}$ is supported in $[0,T]\times(-L,L)$ and $\underline{v}$ is supported in $(-L,L)\times\Omega$. These ζ and u satisfy the plate equation (18): using (22) in the second step, integration by parts in the third, and (23) in the fourth,

$$\begin{aligned}&\ddot{\zeta}-\rho\Delta\dot{\zeta}+\Delta^2\zeta\\&=\int\partial_t^2\underline{k}\,\underline{w}-\rho\partial_t\underline{k}\Delta\underline{w}+\underline{k}\Delta^2\underline{w}\\&=\int\partial_t^2\underline{k}\,\underline{w}-\rho\partial_t\underline{k}\big(\partial_s^2\underline{w}-v\big)+\underline{k}\big(\partial_s^2\big(\partial_s^2\underline{w}-v\big)-\Delta v\big)\\&=\int\big(\partial_t^2\underline{k}-\rho\partial_s^2\partial_t\underline{k}+\partial_s^4\underline{k}\big)\underline{w}+\big(\rho\partial_t\underline{k}+\underline{k}\big(\partial_s^2+\Delta\big)\big)\underline{v}=u=\chi_\Omega u.\end{aligned}$$

Finally, the cost estimate

$$\|u\|^2_{L^2(\mathbb{R}\times M)}\leqslant C_2\exp\big(C_\rho L^2/T\big)\big(\|\zeta_0\|^2_{H^6}+\|\zeta_1\|^2_{H^4}\big)$$

results from (24),

$$\|\underline{v}\|^2_{H^3(\mathbb{R}\times M)}\leqslant 2\|S_W\|^2\big(\|\zeta_0\|^2_{H^4(M)}+\|\zeta_1\|^2_{H^3(M)}\big)\quad\text{and}$$
$$\|u\|_{L^2(\mathbb{R}\times M)}\leqslant\rho\|\partial_t\underline{k}\|_{L^2(\mathbb{R};H^{-3}(\mathbb{R}))}\|\underline{v}\|_{H^3(\mathbb{R};L^2(M))}+\|\underline{k}\|_{L^2(\mathbb{R};H^{-1}(\mathbb{R}))}\|\underline{v}\|_{H^1(\mathbb{R};H^2(M))}.$$

Note added in proof

After our article was accepted, we became aware of a paper to appear in Asymptotic Analysis: "Internal null-controllability for a structurally damped beam equation" by Julian Edward and Louis Tebou. This paper concerns (18) when M is a segment and focuses on the limit $\rho\to 0$. We claim that our Theorem 2 generalizes the main result of this paper (n.b. $L_\Omega<\infty$ always hold when the dimension of M is one). Since Theorem 1 of this paper says that the cost does not depend on $\rho<2$ and our Theorem 2 estimates the cost by $C_2\exp(C_\rho L^2/T)$, we need only explain why the constants C_2 and C_ρ do not depend on ρ. By the control transmutation method used in Section 2.4, this reduces to proving that the constants C_1 and C_2 in our Theorem 3 do not depend on ρ. But these constants come from applying Theorem 1 of [23], so that these constants only depend on the function denoted ν there. Moreover, Section 5.2 of [23] proves that ν depends only on $|c|$ when $\lambda_k=a+ck^2$. But the formula $\lambda_\pm=(\rho\pm i\sqrt{2^2-\rho^2})\mu/2$ (Lemma 3 with $\alpha = 1/2$) implies $|\lambda_\pm| = \mu$ so that $|c|$ does not depend on ρ here, which completes the proof of our claim.

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