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p-Adic and p-Cotorsion Completions of Nilpotent Groups

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The theory of cotorsion Abelian groups is extended to the category of nilpotent groups, and applications are given to the theory of p-adic completions and exactness properties of the p-adic completion functor. The p-cotorsion completion functor was first extended from Abelian to nilpotent groups by Bousfield and Kan, who defined and studied it using topological methods. We develop the theory group-theoretically, and additional results and applications.

An important role is played in some parts of Abelian group theory by the class of *cotorsion* groups, introduced in the late 1950s in three independent papers by Harrison [8], Nunke [12] and Fuchs [4]. These groups are precisely the reduced Abelian groups which can be realized as values of the functor Ext. They can be characterized as those Abelian groups G for which $\mathrm{Hom}(\mathbb{Q}, G) = \mathrm{Ext}(\mathbb{Q}, G) = 0$. There is a natural completion functor c associating to each group G a cotorsion completion $c(G) = \mathrm{Ext}(\mathbb{Q}/\mathbb{Z}, G)$.

Since $\mathbb{Q}/\mathbb{Z}$ is the direct sum of the groups $\mathbb{Z}(p^\infty)$ for all primes p, we also have $c(G) = \prod_p c_p(G)$, where $c_p(G) = \mathrm{Ext}(\mathbb{Z}(p^\infty), G)$, so the entire theory could have been developed locally, using the p-cotorsion completion functors c_p. We will follow this approach here, even though it is not traditional in Abelian group theory.

Cotorsion groups are closely related to groups which are complete in their p-adic or $\mathbb{Z}$-adic topologies, and several points in the theory of complete Abelian groups are most naturally done using cotorsion groups as a tool. In [16], much of the theory of p-adically complete groups and p-adic completions was extended from the Abelian to the nilpotent case. Several points were left groups would fill in the gaps. This was one of the motivations of the present study.

In [3], Bousfield and Kan extended the idea of cotorsion groups from

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Abelian to nilpotent groups, and defined for any nilpotent groups G and "Ext-p-completion" which they again denoted $\mathrm{Ext}(\mathbb{Z}(p^\infty), G)$. The definition of this operation and the development of its properties depended on ideas from semi-simplicial topology.

The first goal of this paper is to develop the group theoretic machinery necessary in order to define and study p-cotorsion nilpotent groups group-theoretically. To this end, we develop some general ideas about nilpotent groups and nilpotent actions in the first section, while in the second section we study completion functors in general and the exactness properties of the p-adic completion functor on nilpotent groups in particular. The main result for what follows is that if N is a normal subgroup of a nilpotent group G, then the image of the natural map of p-adic completions $\hat{N}_p \to \hat{G}_p$ is normal in $\hat{G}_p$. This enables us to define in Section 3 the p-cotorsion completion of a group G, $c_p(G)$, which can be described as follows: we choose a presentation $R \rightarrowtail F \twoheadrightarrow G$ with F and R torsion-free and let $c_p(G)$ be the cokernel of the induced homomorphism $\hat{R}_p \to \hat{F}_p$. This definition is motivated by the fact that in the Abelian case, the functor c_p can be viewed as the zeroth derived functor of the p-adic completion functor. The basic properties of this functor are studied and a long exact sequence is obtained in Section 4.

In Sections 5 and 6 we develop the structure theory of p-cotorsion nilpotent groups, obtaining the results parallel to the known structure theory in the Abelian case. In particular, in Section 6 we obtain the analogue of Harrison's results on adjusted cotorsion groups. In Section 7 we return to the theory of p-adic completions, and use the machinery developed in the previous sections to obtain criteria for a group to be complete. We also develop the nilpotent analogue of the theory of torsion-complete Abelian p-groups. The final section is somewhat peripheral and studies the connection between $c_p(G)$ and extensions of G by $\mathbb{Z}(p^\infty)$. For certain nilpotent groups G there is a one-to-one correspondence between equivalence classes of extensions of G by $\mathbb{Z}(p^\infty)$ and the elements of a certain subgroup of $c_p(G)$.

Though everything we do is local (i.e., concerned with a particular prime p), the theory can be developed globally as well. Thus, instead of the p-adic completion, we could consider the completion $\varprojlim G/nG$, where the limit is taken over all positive integers n. This completion, however, is just the product of the p-adic completions, and its exactness properties, for example, can be studied by an easy reduction to the local case. Similarly, if we consider the "zeroth derived functor" of this completion and get a cotorsion completion $c(G)$, then we will again have $c(G) = \prod_p c_p(G)$. It is therefore usually obvious what the global form of a particular local theorem is, and we do not state it. The same holds for the remarks (before 2.2 and Corollary 2.10) concerning p-pro-finite completions. Globalized, these show that if $\mathscr{F}$ is the class of reduced nilpotent groups G such that G/G^n is finite for all integers n, then the pro-finite completion gives an exact completion functor $\mathscr{F} \to \mathscr{F}$. (The sepcial case of this which asserts the exactness of the pro-finite completion functor on finitely generated nilpotent groups is contained in Bousfield and Kan [3].)

This work was clearly strongly influenced by the work of Bousfeld and Kan, especially in that the existence of their work showed that it should be possible to carry out this program. To see that our functor c_p is actually the same as the "Ext-p-completion" of Bousfield and Kan, one notices that they agree with the p-adic completion (which is the same as their "$\mathbb{Z}_p$-completion") on torsion-free groups and are both right exact. Since there is only one such functor (up to natural equivalence), these two functors are the same. Our "p-cotorsion groups' are defined differently from their "Ext-p-complete groups," but the notions coincide because of Proposition 4.7 below. Given these equivalences, part of our Theorem 4.10 and all of our Proposition 4.8 and Theorem 4.9 and 5.5 are contained in the work of Bousfield and Kan, and the proofs we give of Proposition 4.8 and Theorems 4.9 and 4.10 are partly adapted from theirs.

We will use standard notations from [5] and [6] for Abelian groups, and from [16] for nilpotent groups We let $\mathbb{Z}[1/p]$ denote the subring of the ring of rational numbers consisting of the fractions of the form $a/p^n (a \in \mathbb{Z})$, and $\mathbb{Z}(p^\infty) = \mathbb{Z}[1/p]/\mathbb{Z}$. We also use the arrows $\rightarrowtail$ and $\twoheadrightarrow$ for monomorphisms and epimorphisms, respectively.

1. Nilpotent Groups, Nilpotent Actions, and Radicability

We begin this section be reviewing some basic ideas. If a and b are elements of a group, then we let $[a, b] = a^{-1}b^{-1}ab$. If A and B are subgroups of a group, then $[A, B]$ is the subgroup generated by elements of the form $[a, b]$, $a \in A$, $b \in B$. If G is a group, then the *lower central series* of G is the series of subgroups $\Gamma_i(G)$ (or simply Γ_i if there is no possibility of ambiguity), defined inductively by $\Gamma_1(G) = G$ and $\Gamma_{i+1}(G) = [G, \Gamma_i(G)]$. Similarly, the *upper central series* of G is the series of subgroups $Z_i(G)$ (or simply Z_i, if there is no possibility of ambiguity) defined by letting $Z_1(G)$ be the center of G and defining $Z_{i+1}(G)$ inductively by $Z_{i+1}(G)/Z_i(G) = Z_1(G/Z_i(G))$. The *class* of the nilpotent group G is the smallest integer c such that $\Gamma_c(G) = \{1\}$, or equivalently, the smallest integer c such that $Z_c(G) = G$.

We let $\mathscr{N}$ be the category of nilpotent groups, and $\mathscr{N}_c$ the category of nilpotent groups of class at most c. It is not hard to see that there is no such thing as a "free nilpotent groups" on more than one generator. However, if F is the free group on a set X, then $F/\Gamma_c(F)$ behaves the way a free object should in the cateogry $\mathscr{N}_c$ and is usually called the *free nilpotent group of class c on the set X*. We similarly have difficulty taking coproducts (free products) in the category $\mathcal{N}$ but in the category $\mathcal{N}_c$ this is again very natural. If A and B are objects in $\mathcal{N}_c$, then the free product in $\mathcal{N}_c$ of A and B, written $A *_c B$, is defined to be $(A * B)/\Gamma_c(A * B)$, where $A * B$ is the ordinary free product (in the category of groups.)

1.1. LEMMA. *If F is a free group, then $F/\Gamma_c F$ is torsion-free, for any positive integer c. If A and B are free nilpotent groups of class c (i.e., free objects in the category $\mathcal{N}_c$), then their coproduct in $\mathcal{N}_c$, $A *_c B$, is again free nilpotent of class c and, in particular, torsion-free.*

Proof. In the special case in which F is finitely generated, the first statement is a consequence of Witt's result (see [11, 5.12]) that the factors $\Gamma_i(F)/\Gamma_{i+1}(F)$ are all finitely generated torsion-free Abelian groups. If F is a free group on an arbitrary set X and G is the (free) subgroup of F generated by a subset of X, then there is a retraction $F \to G$ taking $\Gamma_{i+1}F$ onto $\Gamma_{i+1}G$. We conclude from this that the induced homomorphism $\Gamma_i G/\Gamma_{i+1}G \to \Gamma_i F/\Gamma_{i+1}F$ is injective. Since any element of $\Gamma_i F/\Gamma_{i+1}F$ is in the image of such a map from some finitely generated free subgroup G, it follows that $\Gamma_i F/\Gamma_{i+1}F$ is torsion-free for any free group F, and hence so is $F/\Gamma_c F$. The other conclusions of the lemma follow immediately, since coproducts of free objects are free.

If G is a nilpotent group, the elements in G of finite order form a subgroup, the torsion subgroup, denoted $t(G)$. Similarly, for any prime p, the elements of order a power of p form a subgroup, the p-torsion subgroup, denoted $t_p(G)$. (We refer to [16, 4.2] for these facts, first proved by K. Hirsch.) A group G is said to be *p-radicable* (where p is a prime) if for every $x \in G$, there is a $y \in G$ with $y^p = x$. If G is p-radicable for all primes p, G is said to be *radicable.* It is proved in [16, 4.7] that if G is a nilpotent group, G has a unique maximal radicable subgroup, denoted $\rho(G)$, and for each prime p, a unique maximal p-radicable subgroup, $\rho_p(G)$. As the Abelian case, $\rho(G/\rho(G)) = \{1\}$ and $\rho_p(G/\rho_p(G)) = \{1\}$. G is said to be *reduced* if $\rho(G) = \{1\}$ and *p-reduced* if $\rho_p(G) = \{1\}$.

1.2. LEMMA. *If G is a nilpotent group and Z_i $(i = 1,\dots, c)$ are the terms of the upper central series of G, then $\rho_p(G) = \{1\}$ if and only if $\rho_p(Z_1) = \{1\}$, and if G is p-reduced, then so are the groups G/Z_i. Similarly, $\rho(t_p(G)) = \{1\}$ if and only if $\rho(t_p(Z_1)) = \{1\}$, and if $\rho(t_p(G)) = \{1\}$, then for all i, $\rho(t_p(G/Z_i)) = \{1\}$.*

Proof. The operations assigning to G the subgroups $\rho_p(G)$ and $\rho(t_p(G))$ are both idempotent radicals on the category of nilpotent groups in the sense of [16, Chap. 4], generated by the classes of p-divisible Abelian groups and p-divisible Abelian p-groups, respectively. We use $r(G)$ to denote either of these radicals, and we prove the lemma for both cases simultaneously. We infer from [16, 4.1] that $r(G)=\{1\}$ if and only if $r(Z_1)=\{1\}$. If $r(Z_1)=\{1\}$, then since homomorphisms from Z_{i+1}/Z_i to Z_1 separate points in Z_{i+1}/Z_i ([16, 2.1]) and a homomorphic image of a p-radicable group is p-radicable, we conclude that $r(Z_{i+1}/Z_i)=\{1\}$. From this it follows (again by [16, 4.1]) that $r(G/Z_i)=\{1\}$, as required.

The question of when an extension of one nilpotent group by another is again nilpotent leads us naturally to the notion of a nilpotent action of one group on another.

DEFINITION. If N is a group, a *flag* in N of length k is a family of normal subgroups N_i, where $N=N_1 \geqslant N_2 \geqslant \cdots \geqslant N_k \geqslant N_{k+1}=\{1\}$. If G is a group which acts on N, then the flag is said to be *G-invariant* if each N_i is G-invariant, and the action is said to be *nilpotent* with *respect* to this flag if the induced action of G on the groups N_i/N_{i+1} is trivial.

There is a discussion of this notion in [16, Chap. 9]. An important, but easy, consequence of the Jordan–Hölder–Schreier theorem is that if G is a group and N a normal subgroup such that G/N are both nilpotent, then G is nilpotent if and only if the action of G on N is nilpotent with respect to a suitable flag (cf. [16, 9.3]).

If N is a group and F is a flag in N, we denote by $\mathrm{Nil}_F(N)$ the group of all automorphisms of N which are nilpotent with respect to this flag. We say that F is a *central* flag if for each i, N_i/N_{i+1} is in the center of N/N_{i+1}.

1.3. LEMMA. *Let N be a nilpotent group and F a central flag of length k of normal subgroups of N. Then* (i) $\mathrm{Nil}_F(N)$ *is nilpotent of class at most* $k-1$, (ii) *if N is p-reduced so is* $\mathrm{Nil}_F(N)$, *and* (iii) *if N is finitely generated, so is* $\mathrm{Nil}_F(N)$.

Proof. Statement (i) is an immediate consequence of [16, 9.2] or [7, 3.5]. For statement (ii), we must show that $\mathrm{Nil}_F(N)$ cannot have a nontrivial p-radicable subgroup. We may suppose, then, that G is a p-radicable subgroup of $\mathrm{Nil}_F(N)$ and show that G is trivial.

We suppose that $F=\{N_i\}$, $1 \leqslant i \leqslant k+1$, with $N_1=N$ and $N_{k+1}=\{1\}$, and we proceed by induction on k. We may therefore assume that the induces action of G on N_1/N_k is trivial. It follows that if $\phi \in G$, $\phi(x)=xg_\phi(x)$, $g_\phi(x) \in N_k$. It is easy to check that $g_\phi \in \mathrm{Hom}(N_1/N_k, N_k)$ and that the correspondence taking ϕ to g_ϕ is a homomorphism from G to the group $\mathrm{Hom}(N_1/N_k, N_k)$. (Note that N_k is abelian, so this set of homomorphisms is, indeed, a group.) Since N_k is p-reduced, so is $\mathrm{Hom}(N_1/N_k, N_k)$, and since G is p-radicable, the homomorphism taking ϕ to g_ϕ must be trivial. Hence, the action of G is trivial, so $G=\{1\}$. Finally, to prove statement (iii), we proceed similarly, and assume by induction that the image of $\mathrm{Nil}_F(N)$ in $\mathrm{Aut}(N/N_k)$ is finitely generated. The kernel of the natural map $\mathrm{Nil}_F(N) \to \mathrm{Aut}(N/N_k)$ can be identified with $\mathrm{Hom}(N/N_k, N_k)$, which is a finitely generated Abelian group. As an extension of two finitely generated groups, $\mathrm{Nil}_F(N)$ is again finitely generated.

If G is a group and n a positive integer, we define G^n to be the subgroup of G generated by elements of the form x^n, $x \in G$. It may not be true that every element of G^n is of the form x^n. If p is a prime, we define $G^{p^\omega} = \bigcap_{n=1}^{\infty} G^{p^n}$. It is clear that $G^{p^\omega} \supseteq \rho_p(G)$, but well-known Abelian examples show that these subgroups are not equal in general. It is in fact true that if $x \in G^{p^\omega}$ and $n > 0$, there is a $y \in G$ with $x = y^{p^n}$. This is the first of many consequences of the following useful lemma.

1.4. LEMMA. *If p is a prime and c a positive integer, there are positive integers d and k, depending on p and c, such that for any positive integer n,* (a) *if G is a nilpotent group of class at most c and $x \in G^{p^{n+d}}$, then for some $y \in G$, $x = y^{p^n}$, and* (b) *if G is nilpotent of class at most c and N is a normal subgroup of G, and $x \in G$, $y \in N$, then $x^{p^{n+k}} y^{p^{n+k}} = (xy)^{p^{n+k}} h$, where $h \in N^{p^n}$.*

Proof. Both of these statements follow from the Hall-Petresco commutator formula. The first (observed by Balckburn) is in [16, 6.4], and the second follows from the Hall–Petresco formula [16, 6.1] and the observation that if A is the subgroup generated by x and y, then $[A, A] \subseteq N$.

We now turn to some commutativity questions about subgroups of nilpotent groups. The first results of this sort are Chernikov's theorems that a radical p-group is Abelian and that in any radicable group, the p-torsion subgroup is in the center [16, 4.11, 4.12]. More generally, it is shown in [16, 6.13] that for any nilpotent group G, $[t_p(G), G^{p^\omega}] = \{1\}$. (Hence, for example, if G is a p-group, then G^{p^ω} is in the center of G.) In general, G^{p^ω} need not be Abelian, and the torsion-free Malcev groups (as in [16, Chap. 12]) are groups of arbitrary class which are p-radicable. However, in cases where $G^{p^\omega} \neq \rho_p(G)$, there is more to be said, as the following result shows.

1.5. THEOREM. *For any nilpotent group G, $[G^{p^\omega}, G^{p^\omega}] \subseteq \rho_p(G)$.*

Proof. By factoring out $\rho_p(G)$, we reduce to the case in which $\rho_p(G) = \{1\}$, and we must show then that $[G^{p^\omega}, G^{p^\omega}] = \{1\}$. We let x and y be elements of G^{p^ω}, and we show that $[x, y] = 1$.

Lemma 1.4 implies that for each positive integer n, there is a $z_n \in G$ with $y = z_n^{p^n}$. We let W be the group generated by the elements z_n ($n \geqslant 1$) and we show that $[x, W] = 1$. If this were not so, then there would be an integer $n \geqslant 0$ such that $[x, W] \subseteq Z_{n+1}$, but $[x, W] \not\subseteq Z_n$. By the Hall formulae [16, 1.4, 1.5], the map taking w to $[x, w]$ defines a homomorphism $W \to Z_{n+1}/Z_n$. Further, according to [16, 6.13] (cited above), since $x \in G^{p^\omega}$, the p-torsion subgroup of W is in the kernel of this homomorphism. Because of the uniqueness of roots in $W/t_p(W)$ [16, 4.10], and the fact that $(z_{n+1}^p)^{p^n} = y = z_n^{p^n}$, it follows that $z_{n+1}^p = z_n t$ for some $t \in t_p(W)$. Therefore, $W/t_p(W)$ is a homomorphic image of $\mathbb{Z}[1/p]$, and hence p-radicable. However, Lemma 1.2 implies that Z_{n+1}/Z_n is p-reduced, so $\mathrm{Hom}(W/t_p(W), Z_{n+1}/Z_n) = 0$. It follows that $[x, W] \subseteq Z_n$. This contradicts our previous hypothesis, and completes the proof of the theorem.

An obvious question that arises from this result is whether we can do better. If G is p-torsion, then G^{p^ω} is in the center of G. We have shown that if $\rho_p(G) = \{1\}$, then G^{p^ω} is Abelian. Are there reasonable hypotheses which imply that G^{p^ω} is in the center? In general, what is the relation (if any) between the class of $G/\rho_p(G)$ and the class of G/G^{p^ω}? The following example shows that if G is not p-torsion, then even if $\rho_p(G) = \{1\}$, the Abelian subgroup G^{p^ω} can be very far from the center, and the class of G/G^{p^ω} can be much smaller than that of G.

1.6. PROPOSITION. *For every positive integer n, there is a nilpotent group G of class $n+1$ such that $\rho_p(G) = \{1\}$, $G/t(G)$ is cyclic, and G/G^{p^ω} is Abelian.*

Proof. We construct our example by finding an Abelian p-group N and an action of the group $\mathbb{Z}$ of integers on N and taking G to be the semidirect product $N \rtimes \mathbb{Z}$ (with respect to this action). From our discussion earlier in this section, or [16, Chap. 9], we know that G will be nilpotent if this action is nilpotent.

We remind the reader that if N is an Abelian group (written additively), we can define $p^\omega N = \bigcap_{n=1}^{\infty} p^n N$, and, inductively, $p^{\omega(k+1)}N = p^\omega(p^{\omega k}N)$. We let $B = \bigoplus_{i=1}^{\infty} \mathbb{Z}/p^i\mathbb{Z}$. According to Zippin's existence theorem [6, 76.2], there is a p-group N such that $N/p^\omega N \cong B$ and for all k, $k \leqslant n$, $p^{\omega k}N/p^{\omega(k+1)}N) \cong B$, while $p^{\omega(n+1)}N = 0$. An application of Ulm's theorem [6, 77.3] shows that $p^\omega N \cong N/p^{\omega n}N$, so that there is a nilpotent endomorphism $\phi\colon N \to N$ such that $\phi(N) = p^\omega N$ and $\phi(p^{\omega k}N) = p^{\omega(k+1)}N$ $(k \leqslant n)$. We now let α be the automorphism $1 + \phi$, and we use this automorphism to construct the semidirect product of N with the additive group $\mathbb{Z}$ of integers. If $G = N \rtimes \mathbb{Z}$, then according to our previous remarks, G is a nilpotent group. We easily verify that $G^{p^\omega} = p^\omega N$ (still using additive notation for N), and that G is p-reduced. It is also easy to identify the center of G with $p^{\omega n}N$ and, more generally, $Z_i(G) = p^{\omega(n-i+1)}N$ $(i \leqslant n)$, so that the class of G is exactly $n+1$. Finally, $G/G^{p^\omega} \cong \mathbb{Z} \oplus N/p^\omega N$. (In fact, $[G, G] = G^{p^\omega}$.)

2. The p-Adic Completion Functor on Nilpotent Groups

In this section we make some remarks on completion functors in general, and then study the p-adic completion functor on nilpotent groups in some detail. This can be regarded as a continuation of the study in Chapter 7 of [16]. We are particularly interested in the extent to which the p-adic completion functor fails to be exact.

Definition. If $\mathscr{A}$ is a category and $\mathscr{C}$ a subcategory, then $\mathscr{C}$ is a *full* subcategory if for any two objects X and Y of $\mathscr{C}$, the set of morphisms between X and Y in $\mathscr{A}$ is the same as the set of morphisms between X and Y in $\mathscr{C}$. We call $\mathscr{C}$ a *reflexive* subcategory if there is a functor $c: \mathscr{A} \to \mathscr{C}$ and a natural transformation $\tau: Id_{\mathscr{A}} \to c$ satisfying the following universal property: for any X in $\mathscr{A}$, any Y in $\mathscr{C}$, and any morphism $f: X \to Y$, there is a unique morphism ϕ (in $\mathscr{C}$), $\phi: c(X) \to Y$, such that $f = \phi\tau_X$. In this case, the functor c is called the *reflector* of $\mathscr{A}$ into $\mathscr{C}$, and $c(X)$ is the reflection of X.

Many natural operations in algebra can be described as reflections. It will be convenient for us to have another description of this situation (of a full, reflexive subcategory), in terms of the reflector.

2.1. Theorem. *Let $\mathscr{A}$ be a category and $F: \mathscr{A} \to \mathscr{A}$ a functor such that there is a natural transformation $\tau: Id_{\mathscr{A}} \to F$. Call an object X of $\mathscr{A}$ F-*complete *if the morphism $\tau_X: X \to F(X)$ is an isomorphism. The following properties of F are equivalent*:

(i) *For all X in $\mathscr{A}$, the morphisms $F(\tau_X)$ and $\tau_{F(X)}$ are isomorphisms.*

(ii) *The full subcategory of F-complete objects is a reflexive subcategory with reflector F.*

This implies that for any functor F satisfying condition (i), *we have the following universal property*: *for every X in $\mathscr{A}$, every F-complete object Y, and every morphism $f: X \to Y$, there is a unique morphism $\phi: F(X) \to Y$ such that $f = \phi\tau_X$. In particular, $\tau_{F(X)} = F(\tau_X)$ for any object X.*

Definition. We will call any functor F satisfying the conditions of the previous theorem a *completion functor.*

Remark. A *monad* (or *triple*) on $\mathscr{A}$ consists of a functor $F: \mathscr{A} \to \mathscr{A}$ and natural transformations $\tau: Id_{\mathscr{A}} \to F$ and $\kappa: FF \to F$ satisfying certain natural condition (cf. [13, Sects. 2.3, 2.4].) We note that a completion functor F gives us a monad, where κ can be taken to be the identity. This special kind of monad is sometimes called an *idempotent monad* (e.g., [2, p. 56]).

Proof of Theorem 2.1. We suppose that condition (i) holds, that X is an object of $\mathscr{A}$ and Y an F-complete object. We consider the commutative square

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle \tau_X}\downarrow & & \downarrow{\scriptstyle \tau_Y} \\ F(X) & \xrightarrow[F(f)]{} & F(Y) \end{array}$$

Since the vertical arrow on the right is an isomorphism, we conclude that there is a morphism $\phi = \tau_Y^{-1} F(f)\colon F(X) \to Y$ such that $f = \phi\tau_X$, as desired. To complete the proof of property (ii), we must show that this ϕ is unique.

We first consider a commutative triangle

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle \tau_X}\downarrow & \nearrow{\scriptstyle g} & \\ F(X) & & \end{array}$$

and we apply the functor F to the entire diagram, obtaining a commutative triangle

$$\begin{array}{ccc} F(X) & \xrightarrow{F(f)} & F(Y) \\ {\scriptstyle F(\tau_X)}\downarrow & \nearrow{\scriptstyle F(g)} & \\ F(F(X)) & & \end{array}$$

Since the vertical arrow here is an isomorphism, we conclude that $F(g) = F(f)\, F(\tau_X)^{-1}$, so that at least $F(g)$ is uniquely determined. We now consider the commutative square

$$\begin{array}{ccc} F(X) & \xrightarrow{g} & Y \\ {\scriptstyle \tau_{F(X)}}\downarrow & & \downarrow{\scriptstyle \tau_Y} \\ F(F(X)) & \xrightarrow{F(g)} & F(Y) \end{array}$$

Since both vertical arrows here are isomorphisms, it follows that g is determined by $F(g)$. Since $F(g) = F(\phi)$, it follows that $g = \phi$. This completes the proof of property (ii).

The proof that (ii) implies (i) is rudimentary, and we omit it.

EXAMPLE 1. In the category of metric spaces with uniformly continuous maps as morphisms, it is well known that the functor associating to each space its completion satisfies the above conditions and is a completion functor. For our applications, it is more natural to consider pseudo-metric spaces (that is, spaces equipped with a distance function d satisfying all of the usual conditions except that $d(x, y) = 0$ need not imply that $x = y$). On the category of these spaces (again with uniformly continuous maps) the completion is again a "completion functor," though this time the natural transformation taking a space to its completion is not generally an imbedding. The correct general context for this is the category pf uniform spaces and uniformly continuous maps, and we refer to [1, Chap. 2, Sect. 3, No. 7, Theorem 3].

EXAMPLE 2. Just to remind the reader of non-topological examples, we recall that the operation associating to a module M over a commutative ring its *localization* M_P at a prime ideal P is another example of what we have called a completion functor.

EXAMPLE 3. Two completions frequently used which are not completion functors on the category of groups are the pro-finite and p-pro-finite completions. These are the completions with respect to the uniform structures defined by taking respectively the normal subgroups of finite index or the normal subgroups of index a power of p as neighborhoods of the identity. For example, if G is the direct sum of an infinite number of cyclic groups of order p (for some fixed prime p), then the pro-finite and p-pro-finite completions of G are the same, and this completion is not complete in its own pro-finite topology. However, if we restrict our attention to the category $\mathscr{F}_p$ of nilpotent groups G such that G/G^{p^n} is finite for all positive integers n, then the p-pro-finite topology agrees with the p-adic topology dicussed below. It is a consequence of our next theorem that the p-pro-finite completion restricts to a completion functor $\mathscr{F}_p \to \mathscr{F}_p$.

DEFINITION. If G is a nilpotent group, the *p-adic topology* on G is defined by taking the subgroups G^{p^n} (defined in the previous section) as neighborhoods of the identity. The *p-adic completion* of G, written $\hat{G}_p$, is the completion of G with respect to this pseudo-metric, or, equivalently, $\varprojlim G/G^{p^n}$.

2.2. THEOREM. *The functor associating to each nilpotent group G its p-adic completion, together with the natural map $\kappa_G : G \to \hat{G}_p$ (taking an element of G to the constant Cauchy sequence), is a completion functor, with the following additional properties*: (a) *the p-adic metric on $\hat{G}_p$ coincides with the metric induced from the p-adic pseudo-metric on G*, (b) *if $f: G \to H$ is an* *epimorphism, then so is* $\hat{f}_p: \hat{G}_p \to \hat{H}_p$, *and* (c) *the kernel of* κ_G *is the subgroup* G^{p^ω}.

Proof. Statement (a) is just [16, 7.10]. This fact, together with the standard facts about pseudo-metric spaces cited above (in Example 1), show that the p-adic completion is a completion functor. Statement (b) follows from the fact that $H^{p^n} = f(G^{p^n})$ for all n, from which one can see easily that every Cauchy sequence in H is the image of a Cauchy sequence in G. Statement (c) similarly follows from the fact that $\hat{G}_p = \varprojlim G/G^{p^n}$, since $G^{p^\omega} = \bigcap_{n=1}^{\infty} G^{p^n}$.

Remark. We note that the completion of a group is always Hausdorff (separated) in its p-adic topology. Since, in keeping with the terminology at the beginning of this section, we wish the "complete" groups to be precisely the values of the completion functor, a complete group will be just a group which can be identified with its own completion. Equivalently, a complete group is one in which every Cauchy sequence converges and which is Hausdorff in its p-adic topology.

The p-adic completion of G can be regarded as a universal object with respect to homomorphisms from G to nilpotent groups K of exponent a power of p, i.e., such that for some n, $K^{p^n} = \{1\}$. Several similar completions appear in the literature. In particular, in [3], Bousfield and Kan define the "$\mathbb{Z}_p$-completion" of a group as the universal object with respect to homomorphisms of G to groups K such that K has a finite central series whose factors are all p-elementary (i.e., are Abelian with every element other than the identity of order p.) That this is the same as our p-adic completion is a consequence of the following routine lemma. We recall that the subgroups $\Gamma_n^p(G)$ of a group G are defined inductively, where $\Gamma_{n+1}^p(G)$ is generated by elements of the form $xyx^{-1}y^{-1}z^p$, $x \in G$, $y, z \in \Gamma_n^p(G)$. (This is clearly the fastest descending central series with p-elementary factors.)

2.3. LEMMA. *The following properties of a group G are equivalent*: (i) *For some integer n*, $n \geqslant 1$, $\Gamma_n^p(G) = \{1\}$; (ii) *G has a finite central series whose factors are all p-elementary*, (iii) *G is nilpotent and for some* $n \geqslant 1$, $G^{p^n} = \{1\}$.

Proof. If (i) holds, then the series $\Gamma_n^p(G)$ is a series satisfying the conditions of (ii), so (ii) holds. If (ii) holds with a series of length n, then for any $x \in G$, $x^{p^n} = 1$, and G is clearly nilpotent, so (iii) holds. If (iii) holds, then since G is nilpotent, it has a finite central series with factors which are Abelian groups in which every element has order dividing p^n. Every such Abelian group has a finite normal series (of length at most n) for which the factors are p-elementary, so that we can obtain a refinement of our original series to obtain a finite central series satisfying the conditions of (ii). Finally, to show that (ii) implies (i), we note that if $G = G_1 \supset G_2 \supset \cdots \supset G_k = \{1\}$ is any series satisfying the conditions of (ii), then by a routine induction, $G_i \supseteq \Gamma_i^p(G)$, so that (i) holds.

We now turn briefly to the relationship between the class of a nilpotent group and the class of its completion. It is clear that the groups of arbitrarily large class can have trivial completion, as we see, for example, by taking finite q-groups for some q, $q \neq p$, or by taking torsion-free radicable groups (as in [16, Chap. 11]) whose completions at every prime are trivial. It is less clear what happens for p-reduced groups, which have a particular importance in the rest of this paper.

2.4. THEOREM. *If G is a nilpotent group and $\hat{G}_p$ its p-adic completion, then the class of $\hat{G}_p$ is the same as the class of G/G^{p^ω}. If G is p-reduced, then the kernel of the natural map from G to $\hat{G}_p$ is Abelian.*

Remark. Example 1.6 is a p-reduced group of arbitrarily large class whose completion (according to the above result) is Abelian.

Proof. If c is the class of G, then the class of G/G^{p^n} is at most c for all $n \geqslant 1$. If c' is the least upper bound of the classes of the groups G/G^{p^n}, then the product of the groups G/G^{p^n} has class c'. G/G^{p^ω} and $\hat{G}_p$ are both subgroups of this product, and so have class at most c', and both have G/G^{p^n} has a homomorphic image (for all n) so they both have class exactly c'. The second statement follows from Theorems 2.2(b) and 1.5.

If N is a subgroup of a nilpotent group G, and $\phi: N \to G$ is the inclusion map, then there are three naturally defined subgroups of $\hat{G}_p$ arising from N. Two of these are the image of N itself, $\kappa_G \phi(N)$, and the closure of this subgroup in the p-adic topology. The third is the image of $\hat{N}_p$ under the map $\hat{\phi}_p : \hat{N}_p \to \hat{G}_p$. We want to consider these subgroups in more detail and to see what we can conclude if N is normal in G. For the first of these subgroups, it is easy to see that the image of N in $\hat{G}_p$ need not be normal even if N is normal in G. We show, in particular, that the image of G in $\hat{G}_p$ need not be normal.

EXAMPLE. There is a finitely generated torsion-free nilpotent group G which is not normal in its completion. We let $\mathbb{Z}$ act on $\mathbb{Z} \oplus \mathbb{Z}$ by the action taking n to right multiplication by the matrix $\left(\begin{smallmatrix} 1 & n \\ 0 & 1 \end{smallmatrix}\right)$. If G is the semi-direct product $\mathbb{Z} \ltimes \mathbb{Z} \oplus \mathbb{Z}$ with respect to this action, then one easily identifies $\hat{G}_p$ with the semidirect product $\hat{\mathbb{Z}}_p \ltimes \hat{\mathbb{Z}}_p \oplus \hat{\mathbb{Z}}_p$ with respect to the action taking r to the automorphism $\left(\begin{smallmatrix} 1 & r \\ 0 & 1 \end{smallmatrix}\right)$. We compute, for example, that if $x = (0, (1, 0))$ and $y = (r, (0, 0))$, then $x^{-1}y^{-1}xy = (0, (0, r))$. Hence any normal subgroup containing x also contains a copy of the p-adic integers, $(0, (0, \hat{\mathbb{Z}}_p))$.

Our next results concern the closure of $\kappa_G(N)$ and its relation to the image of $\hat{N}_p$ in $\hat{G}_p$. This leads to results (in Theorem 2.6 and Corollary 2.9) on the exactness of the p-adic completion functor. Recall that if $N \rightarrowtail^{f} G \twoheadrightarrow^{g} Q$ is a short exact sequence of nilpotent groups, then it is immediate from Theorem 2.2 that $\hat{g}_p \hat{f}_p$ is trivial, and that $\hat{g}_p$ is an epimorphism. The only exactness questions that arise, therefore, are whether $\operatorname{Ker}(\hat{g}_p) = \operatorname{Im}(\hat{f}_p)$ and whether $\hat{f}_p$ is injective.

2.5. Lemma. *If $f\colon G \to H$ is an epimorphism of nilpotent groups, and K is the kernel of f, then $\bar{K}$, the closure of K in the p-adic topology, is precisely $f^{-1}(H^{p^\omega})$.*

Proof. We note that $KG^{p^n} = f^{-1}(H^{p^n})$, and that $\bar{K} = \bigcap_{n=1}^{\infty} KG^{p^n}$. It follows immediately that $\bar{K} = f^{-1}(H^{p^\omega})$.

2.6. Theorem. *If G is a nilpotent group, N a normal subgroup, $\kappa_G\colon G \to \hat{G}_p$ the natural transformation, and $L = \kappa_G(N)$, then the closure, $\bar{L}$, is a normal subgroup of $\hat{G}_p$. Further, if $f\colon G \to G/N$ is the natural map, then $\bar{L} = \operatorname{Ker} \hat{f}_p$, so that $\hat{G}_p/\bar{L} \cong (G/N)\hat{}_p$.* In particular, if $N \rightarrowtail G \twoheadrightarrow Q$ *is a short exact sequence of nilpotent groups, then the induced sequence $\hat{N}_p \to \hat{G}_p \to \hat{Q}_p \to 1$ is exact if and only if the image of $\hat{N}_p$ in $\hat{G}_p$ is closed.*

Proof. If $x \in \bar{L}$, and $y \in \hat{G}_p$, then for any positive integer n, we can write $x = hz^{p^n}$, $y = gw^{p^n}$, with $h \in L$, $g \in \kappa_G(G)$, and z and w in $\hat{G}_p$. Clearly, $y^{-1}xy$ and $g^{-1}hg$ are congruent modulo $(\hat{G}_p)^{p^n}$. Since $g^{-1}hg \in L$, it follows that $y^{-1}xy \in L(\hat{G}_p)^{p^n}$ for all n, so $y^{-1}xy \in \bar{L}$. Hence $\bar{L}$ is normal.

For the second point, we let $f: G \to G/N$ be the natural map and show that $\hat{f}_p\colon \hat{G}_p \to (G/N)\hat{}_p$ is an epimorphism with kernel $\bar{L}$. That $\hat{f}_p$ is an epimorphism follows from Theorem 2.2(b) above. Lemma 2.5 and the fact that $(G/N)\hat{}_p$ is Hausdorff in its p-adic topology shows that $\bar{L} \subseteq \operatorname{Ker} \hat{f}_p$. We thus have an induced epimorphism $g\colon \hat{G}_p/L \to (G/N)\hat{}_p$. Lemma 2.5 implies that $\hat{G}_p/\bar{L}$ is Hausdorff, and hence $\hat{G}_p/\bar{L}$ is p-adically complete group. Since H is in the kernel of the natural map $G \to \hat{G}_p/\bar{L}$, the universal mapping property (2.1) of the p-adic completion shows that the epimorphism $g\colon \hat{G}_p/\bar{L} \to (G/N)\hat{}_p$ splits, and it σ is the splitting, then the following diagram commutes:

$$
\begin{array}{ccc}
G & \longrightarrow & \hat{G}_p/\bar{L} \\
& \searrow & g \downarrow \uparrow \sigma \\
& & (G/N)\hat{}_p
\end{array}
$$

Since the image of G in each of these groups is dense and the groups are Hausdorff in the p-adic topology, it follows that g is an isomorphism with inverse σ.

2.7. Proposition. *If G is a nilpotent group and H a subgroup, and if* *the topology on H induced by the p-adic topology on G is the same as the p-adic topology on H, then the natural map $\hat{H}_p \to \hat{G}_p$ is injective and its image is closed. Conversely, if H contains a countable subset which is dense in H with respect to the p-adic topology on H, and if the natural map $\hat{H}_p \to \hat{G}_p$ is injective and its image is closed, then the p-adic topology on H agrees with the topology induced by the p-adic topology on G.*

Proof. If the two topologies on H are the same, then a sequence of elements of H which converges to 1 with respect to the topology of G also does so with respect to the topology of H. This shows that the induced map $\hat{H}_p \to \hat{G}_p$ is injective. It is easy to see that if there is an element in the closure of the image of $\hat{H}_p$ in $\hat{G}_p$ which is not actually in the image of $\hat{H}_p$, then there must be a sequence of elements of H which is a Cauchy sequence with respect to the uniform structure on G, but not Cauchy with respect to the uniform structure on H. However, if $\{x_i\}$ is this sequence, and $y_i = x_i^{-1}x_{i+1}$, then $y_i \in H$ and $y_i \to 1$ in the topology of G. Hence, $y_i \to 1$ in the topology of H also, which means that $\{x_i\}$ is Cauchy with respect to the topology of H.

For the second point, we note that an embedding $H \rightarrowtail G$ induces for each positive integer n an embedding $HG^{p^n}/G^{p^n} \rightarrowtail G/G^{p^n}$. If we adopt the notation of Theorem 2.6 and call L the image of H in $\hat{G}_p$ and $\bar{L}$ its closure, then we can identify $\bar{L}$ with $\varprojlim HG^{p^n}/G^{p^n}$. Our condition, therefore, is that the induced map

$$\delta\colon \varprojlim H/H^{p^n} \to \varprojlim HG^{p^n}/G^{p^n}$$

is an isomorphism. The exact sequence of $\varprojlim$ yields

$$\{1\} \to \varprojlim (H \cap G^{p^n})/H^{p^n} \to \hat{H}_p \to \varprojlim HG^{p^n}/G^{p^n}$$
$$\to \varprojlim{}^1 (H \cap G^{p^n})/H^{p^n} \to *$$

(where we recall that $\varprojlim^1$ is not always a group) [14]. Since δ is an isomorphism, we conclude in particular that $\varprojlim^1 (H \cap G^{p^n})/H^{p^n} = \{1\}$. Since $H \cap G^{p^n}/H^{p^n}$ is countable, we infer from [14, Proposition 6] that the system $H \cap G^{p^n}/H^{p^n}$ satisfies the Mittag–Leffler condition. Upon examination, this condition and the fact that $\varprojlim (H \cap G^{p^n})/H^{p^n} = \{1\}$ say that for any n, there is a k such that the induced map

$$(H \cap G^{p^{n+k}})/H^{p^{n+k}} \to (H \cap G^{p^n})/H^{p^n}$$

is trivial. This says precisely that $H \cap G^{p^{n+k}} \subseteq H^{p^n}$, so the topologies do agree.

2.8. PROPOSITION. *Let G be a nilpotent group and H a normal subgroup and $\phi\colon G \to G/H$ the natural map. Suppose that there is a fixed integer*

$m \geqslant 0$, and for every $x \in t_p(G/H)$ there is a $y \in G$ such that $\phi(y) = x^{p^m}$ and y has the same order as x^{p^m}. Then the induced map $\hat{H}_p \to \hat{G}_p$ is injective, and its image is normal and closed.

Proof. According to Theorem 2.6 and Proposition 2.7, it will suffice to show that for any positive integer n, there is a positive integer r such that $H^{p^n} \supseteq H \cap G^{p^{n+r}}$. In the notation of Lemma 1.4, we let c be the class of G and choose $r = m + k + d$ where k and d are the integers depending only on c which appear in Lemma 1.4.

Let $x \in G \cap H^{p^{n+r}}$. We prove the result by showing $x \in H^{p^n}$. According to Lemma 1.4, there is a $y \in G$ with $x = y^{p^{n+m+k}}$. By hypothesis, there is a $z \in G$ such that $z^{-1}y^{p^m} \in H$ and $z^{p^{n+k}} = 1$. We let $w = z^{-1}y^{p^m}$, and notice that by Lemma 1.4 again,

$$z^{p^{n+k}} w^{p^{n+k}} = (zw)^{p^{n+k}} h$$

with $h \in H^{p^n}$. Since $z^{p^{n+k}} = 1$ and $zw = y^{p^m}$, we conclude that

$$w^{p^{n+k}} = xh,$$

and since $w \in H$, we see that $x \in H^{p^n}$ as required.

2.9. Corollary. *Let $N \rightarrowtail G \twoheadrightarrow Q$ be a short exact sequence of nilpotent groups such that $t_p(Q)$ is bounded (i.e., for some positive integer n, $t_p(Q)^{p^n} = \{1\}$.) Then the induced sequence*

$$1 \to \hat{N}_p \to \hat{G}_p \to \hat{Q}_p \to 1$$

is exact.

2.10. Corollary. *Let $\mathscr{RF}_p$ be the category of nilpotent groups G such that $\rho_p(G) = \{1\}$ and $G/\Gamma_2^p(G)$ is finite. Then the p-pro-finite completion functor restricts to an exact completion functor $\mathscr{RF}_p \to \mathscr{RF}_p$.*

Proof. By [16, 6.10 and 1.9], if $G \in \mathscr{RF}_p$, then G/G^{p^n} is finite for all integers n. It follows that the p-pro-finite and p-adic topologies agree, and we infer from Theorem 2.2 that the p-pro-finite completion gives a completion functor taking $\mathscr{RF}_p$ into itself. From [16, 6.11] we conclude that the p-torsion subgroup of a group in $\mathscr{RF}_p$ is finite, and we infer the exactness of the p-pro-finite completion from Corollary 2.9.

2.11. Theorem. *If G is a nilpotent group and N a normal subgroup, with inclusion map $\phi: N \to G$, and if $\hat{\phi}_p : \hat{N}_p \to \hat{G}_p$ is the induced map, then* $\operatorname{Im}(\hat{\phi}_p)$ *is a normal subgroup of $\hat{G}_p$.*

Proof. Use the action of G on N given by conjugation to form the semidirect product $L = N \rtimes G$. We let $\psi: N \to L$ be defined by $\psi(h) = (h, 1)$ and $f: L \to G$ be defined by $f((h, g)) = hg$. Note that f is an epimorphism and $f\psi = \phi$. The subgroup $\psi(N)$ of L clearly satisfies the conditions of Proposition 2.8, from which we infer that $\hat{\psi}_p: \hat{N}_p \to \hat{L}_p$ is injective and $\mathrm{Im}(\hat{\psi}_p)$ is a normal subgroup of $\hat{L}_p$. The epimorphism f induces an epimorphism $\hat{f}_p: \hat{L}_p \to \hat{G}_p$. Since the image of a normal subgroup is normal, we conclude that $\mathrm{Im}(\hat{f}\hat{\psi}_p)$ is normal in $\hat{G}_p$. Since $\widehat{f\psi}_p = \hat{\phi}_p$, this proves Theorem 2.11.

This result is fundamental for all that follows. In the category of Abelian groups, the p-adic completion functor is neither right nor left exact, and its zeroth derived functor is the p-cotorsion completion c_p. The theorem above enables us to extend this functor to the category of nilpotent groups, and this extended functor is the main object of study in the rest of this paper. This theorem also enables us to say something about the homology of the completion functor in the following sense.

If $N \rightarrowtail G \twoheadrightarrow Q$ is a short exact sequence of nilpotent groups, then as in the proof of Proposition 2.7, the homology of the sequence $\hat{N}_p \to \hat{G}_p \to \hat{Q}_p$ is precisely $\varprojlim{}^1 (N \cap G^{p^n})/N^{p^n}$, which we now know (because of Theorem 2.11) is a group. We will see later that this is an Abelian group, and we can say more about it in certain cases.

3. Definition of the p-Cotorsion Completion

In this section we define the p-cotorsion completion of a nilpotent group. We begin by restricting our attention to groups in $\mathcal{N}_c$. Let G be a nilpotent group of class $\leqslant c$ and let $\mathbf{F}: R \overset{\mu}{\rightarrowtail} F \twoheadrightarrow G$ be a *free presentation* of G in $\mathcal{N}_c$, i.e., a short exact sequence where F is a free object in $\mathcal{N}_c$. By Theorem 2.11 the image of the induced homomorphism $\hat{\mu}_p: \hat{R}_p \to \hat{F}_p$ is normal in $\hat{F}_p$. This enables us to define

$$c_p(\mathbf{F}) = \hat{F}_p / \mathrm{Im}\, \hat{\mu}_p \qquad \text{and} \qquad c_p^1(\mathbf{F}) = \mathrm{Ker}\, \hat{\mu}_p.$$

3.1. Proposition. (1) *$c_p(\mathbf{F})$ and $c_p^1(\mathbf{F})$ do not depend on the presentation* $\mathbf{F}$, *but only on G (hence we can write $c_p(G)$ and $c_p^1(G)$, respectively).* (2) *For a given homomorphism $f: G \to H$ there are induced homomorphisms $c_p(f)$ and $c_p^1(f)$ making $c_p(-)$ and $c_p^1(-)$ into functors $\mathcal{N}_c \to \mathcal{N}$.*

Proof. We proceed exactly as in the definition of the left derived functors of an additive functor (cf., e.g., [10, pp. 130–132]). The only difficulty arises in the proof of the subsequent lemma.

Let $f: G \to H$ be a morphism in $\mathcal{N}_c$ and let $\mathbf{E}: R \overset{\mu}{\rightarrowtail} E \twoheadrightarrow G$ and $\mathbf{F}: S \overset{\nu}{\rightarrowtail} F \twoheadrightarrow H$ be free presentations in $\mathcal{N}_c$. Then there are homomorphisms $\phi: E \to F$ and $\psi: R \to S$ such that the diagram

$$\begin{array}{ccccc} R & \overset{\mu}{\rightarrowtail} & E & \twoheadrightarrow & G \\ \downarrow{\scriptstyle\psi} & & \downarrow{\scriptstyle\phi} & & \downarrow{\scriptstyle f} \\ S & \overset{\nu}{\rightarrowtail} & F & \twoheadrightarrow & H \end{array}$$

is commutative. The diagram of p-adic completions corresponding to the left-hand square

$$\begin{array}{ccc} \hat{R}_p & \xrightarrow{\hat{\mu}_p} & \hat{E}_p \\ \downarrow{\scriptstyle\hat{\psi}_p} & & \downarrow{\scriptstyle\hat{\phi}_p} \\ \hat{S}_p & \xrightarrow{\hat{\nu}_p} & \hat{F}_p \end{array}$$

is likewise commutative and induces well-defined homomorphism $f_* : c_p(\mathbf{E}) \to c_p(\mathbf{F})$ and $f^1_* : c^1_p(\mathbf{E}) \to c^1_p(\mathbf{F})$.

3.2. LEMMA. *The homomorphisms f_* and f^1_* do not depend on the choice of ϕ.*

Proof. Assume that both ϕ, $\phi' : E \to F$ induce f and denote their restrictions by ψ, $\psi' : R \to S$, respectively. We consider the map $u: E \to F$ which is given by

$$u(x) = \phi'(x)\,\phi(x^{-1}).$$

In general u is not a homomorphism, but it is uniformly continuous w.r.t. the p-adic pseudo-metrics on E and F. Indeed it follows from Lemma 1.4(a) that for any $x \in E$ and $n \geqslant 0$, $u(xE^{p^{n+d}}) \subseteq u(x)\,F^{p^n}$, where d is as in Lemma 1.4.

Since both ϕ and ϕ' induce f, it follows that $\operatorname{Im} u \subseteq \operatorname{Im} \nu$, hence there is a map $v: E \to S$ such that $u = \nu v$. We claim that v is uniformly continuous w.r.t. the p-adic pseudo-metrics on E and S. Let $r = d + k$ where d, k are as in Lemma 1.4, let $x \in E$ and suppose that $z \in v(xE^{p^{n+r}})$. Then there is $y \in E$ such that $z = v(xy^{p^{n+k}})$. By the definition of v we obtain

$$z = \phi'(x)\,\phi'(y)^{p^{n+k}}\phi(y^{-1})^{p^{n+k}}\phi(x^{-1}),$$

where, from now on, S and $\nu(S)$ are identified. Since $\phi'(y)\,\phi(y^{-1}) \in S$, we infer from Lemma 1.4(b) that there is an $h \in S^{p^n}$ such that

$$\phi'(y^{-1})^{p^{n+k}}(\phi'(y)\,\phi(y^{-1}))^{p^{n+k}} = \phi(y^{-1})^{p^{n+k}}h.$$

It follows that

$$\phi'(y)^{p^{n+k}}\phi(y^{-1})^{p^{n+k}} = (\phi'(y)\,\phi(y^{-1}))^{p^{n+k}}h^{-1} \in S^{p^n}.$$

Therefore, since S^{p^n} is normal in F, there is an $h' \in S^{p^n}$ such that

$$z = v(xy^{p^{n+k}}) = \phi'(x)\,\phi(x^{-1})\,h' = v(x)\,h';$$

hence $v(xE^{p^{n+r}}) \subseteq v(x)\,S^{p^n}$. This proves our claim.

Now by Example 1 after Theorem 2.1 there are unique uniformly continuous maps $\hat{u}_p: \hat{E}_p \to \hat{F}_p$ and $\hat{v}_p: \hat{E}_p \to \hat{S}_p$ such that $\hat{u}_p \kappa_E = \kappa_F u$ and $\hat{v}_p \kappa_E = \kappa_S v$, where κ_E, κ_F and κ_S denote the respective completion maps. On the other hand, the map $\bar{u}: \hat{E}_p \to \hat{F}_p$ which is given by $\bar{u}(x) = \hat{\phi}'_p(x)\,\hat{\phi}_p(x^{-1})$ is likewise uniformly continuous, and an easy computation shows that $\bar{u}\kappa_E = \kappa_F u$. Therefore we have $\bar{u} = \hat{u}_p$ and, since $\hat{u}_p = \hat{v}_p \hat{v}_p$, we conclude that

(i) $\hat{v}_p(\hat{v}_p(x)) = \hat{\phi}'_p(x)\,\hat{\phi}_p(x^{-1})$ for all $x \in \hat{E}_p$.

Similarly, since for $x \in R$, $v(\mu(x)) = \psi'(x)\,\psi(x^{-1})$, we infer that

(ii) $\hat{v}_p(\hat{\mu}_p(x)) = \hat{\psi}'_p(x)\,\hat{\psi}_p(x^{-1})$ for all $x \in \hat{R}_p$.

From (i) and (ii) it follows easily that ϕ and ϕ' induce the same maps $f_*: c_p(\mathbf{E}) \to c_p(\mathbf{F})$ and $f^1_*: c^1_p(\mathbf{E}) \to c^1_p(\mathbf{F})$. This completes the proof of the lemma.

The remaining steps of the proof of Proposition 3.1 can easily be translated from [10].

Next we will show that $c_p(G)$ and $c^1_p(G)$ do not depend on the class c.

3.3. Proposition. *Let $c \leqslant d$ be positive integers. The functors c_p and c^1_p defined on $\mathcal{N}_c$ are naturally equivalent to the respective restrictions to $\mathcal{N}_c$ of c_p and c^1_p defined on $\mathcal{N}_d$.*

Proof. Let G be a group in $\mathcal{N}_c$. For the moment let $c_p(G, c)$ and $c^1_p(G, c)$ denote the respective values of the functors c_p and c^1_p at G defined on $\mathcal{N}_c$. Suppose that $\mathbf{E}: R \rightarrowtail E \twoheadrightarrow G$ is a free presentation of G in $\mathcal{N}_d$. Then $F = E/\Gamma_c(E)$ is free nilpotent of class c and maps onto G. Therefore we obtain a free presentation $\mathbf{F}: S \rightarrowtail F \twoheadrightarrow G$ of G in $\mathcal{N}_c$ and a commutative diagram

$$\begin{array}{ccccc} R & \overset{\mu}{\rightarrowtail} & E & \twoheadrightarrow & G \\ \downarrow{\scriptstyle\psi} & & \downarrow{\scriptstyle\phi} & & \| \\ S & \overset{v}{\rightarrowtail} & F & \twoheadrightarrow & G \end{array}$$

where ϕ is the natural projection. So ϕ and ψ are both epimorphisms and have the same kernel, say K. Now the commutative diagram of p-adic completions

$$\begin{array}{ccc} \hat{R}_p & \xrightarrow{\hat{\mu}_p} & \hat{E}_p \\ \downarrow{\scriptstyle \hat{\psi}_p} & & \downarrow{\scriptstyle \hat{\phi}_p} \\ \hat{S}_p & \xrightarrow{\hat{\nu}_p} & \hat{F}_p \end{array}$$

gives rise to homomorphism $\tau: c_p(G, d) \to c_p(G, c)$ and $\tau^1: c_p^1(G, d) \to c_p^1(G, c)$. It is easy to see that this definition does not depend on the choice of E and is natural. By Lemma 1.1, F (and hence S also) is torsion-free; thus we infer from Corollary 2.9 that $\hat{\phi}_p$ and $\hat{\psi}_p$ have the same kernel, namely, $\hat{K}_p$. It follows that τ and τ^1 are isomorphisms. This completes our proof.

The above proposition allows us to extend c_p and c_p^1 to functors $\mathscr{N} \to \mathscr{N}$.

DEFINITION. We call $c_p(G)$ the *p-cotorsion completion* of G.

Remark. On the category of Abelian groups the functors c_p and c_p^1 agree with $\operatorname{Ext}(\mathbb{Z}(p^\infty), -)$ and $\operatorname{Hom}(\mathbb{Z}(p^\infty), -)$, respectively. This can be verified as follows. Let $R \rightarrowtail F \twoheadrightarrow G$ be a free Abelian presentation of the Abelian group G. Then the induced long exact sequence for $\operatorname{Hom}(\mathbb{Z}(p^\infty), -)$ reduces to

$$\begin{aligned} 0 \to \operatorname{Hom}(\mathbb{Z}(p^\infty), G) \to \operatorname{Ext}(\mathbb{Z}(p^\infty), R) &\to \operatorname{Ext}(\mathbb{Z}(p^\infty), F) \\ &\to \operatorname{Ext}(\mathbb{Z}(p^\infty), G) \to 0. \end{aligned}$$

Now by [12, Theorem 6.3] there is a natural transformation σ: $\operatorname{Ext}(\mathbb{Z}(p^\infty), -) \to (-)\hat{}_p$, which is an equivalence on torsion-free groups (cf. [3, p. 166]). Therefore the middle terms of the above sequence can be naturally identified with $\hat{R}_p$ and $\hat{F}_p$, respectively, and hence by definition we have

$$c_p(G) \cong \operatorname{Ext}(\mathbb{Z}(p^\infty), G) \qquad \text{and} \qquad c_p^1(G) \cong \operatorname{Hom}(\mathbb{Z}(p^\infty), G).$$

The last result of this section says that $c_p(G)$ and $c_p^1(G)$ may be computed by means of any *torsion-free nilpotent presentation* of G. By the latter we mean a short exact sequence $S \rightarrowtail F \twoheadrightarrow G$, where F and S are torsion-free nilpotent groups.

3.4. PROPOSITION. *Let G be a nilpotent group and let $S \overset{\nu}{\rightarrowtail} F \overset{\epsilon}{\twoheadrightarrow} G$ be any torsion-free nilpotent presentation of G. Then there is a natural exact sequence*

$$\{1\} \to c_p^1(G) \to \hat{S}_p \xrightarrow{\hat{\nu}_p} \hat{F}_p \to c_p(G) \to \{1\}.$$

Proof. Let E be a free nilpotent group such that there is an epimorphism $\phi: E \twoheadrightarrow F$, and let R be the kernel of $\epsilon\phi$. Then there is a commutative diagram

$$\begin{array}{ccccc} R & \stackrel{\mu}{\rightarrowtail} & E & \twoheadrightarrow & G \\ \downarrow{\scriptstyle\psi} & & \downarrow{\scriptstyle\phi} & & \| \\ S & \stackrel{\nu}{\rightarrowtail} & F & \stackrel{\varepsilon}{\twoheadrightarrow} & G \end{array}$$

with exact rows and epimorphic columns. Now as in the proof of Proposition 3.3 we obtain natural isomorphisms $\operatorname{Ker}\hat{\mu}_p \cong \operatorname{Ker}\hat{\nu}_p$ and $\hat{E}_p/\operatorname{Im}\hat{\mu}_p \cong \hat{F}_p/\operatorname{Im}\hat{\nu}_p$ which provide the natural maps $c_p^1(G) \to \hat{S}_p$ and $\hat{F}_p \to c_p(G)$ of the sequence. This completes our proof.

4. Basic Properties of the p-Cotorsion Completion

We start our discussion of the functor c_p by showing that it is really a completion functor (in the sense of Theorem 2.1).

4.1. Theorem. *There is a natural transformation $\gamma: Id_{\mathcal{N}} \to c_p$ making c_p into a completion functor.*

Proof. Let G be a nilpotent group and let $R \rightarrowtail^{\mu} F \twoheadrightarrow G$ be a free presentation in $\mathcal{N}_c$ (for an appropriate c). The commutative diagram

$$\begin{array}{ccc} R & \stackrel{\mu}{\rightarrowtail} & F \\ \downarrow{\scriptstyle\kappa_R} & & \downarrow{\scriptstyle\kappa_F} \\ \hat{R}_p & \xrightarrow{\hat{\mu}_p} & \hat{F}_p \end{array}$$

induces a homomorphism $\gamma_G: G \to c_p(G)$. Clearly the definition of γ_G does not depend on the choice of F and is natural. Now let $S \rightarrowtail^{\nu} E \twoheadrightarrow G$ be any torsion-free nilpotent presentation of G. By Proposition 3.4 the cokernel of $\hat{\nu}_p: \hat{S}_p \to \hat{E}_p$ is $c_p(G)$, and the same argument as in the proof of Proposition 3.4 shows that the homomorphism $\beta: G \to c_p(G)$ which is induced by the commutative diagram

$$\begin{array}{ccc} S & \rightarrowtail & E \\ \downarrow{\scriptstyle\kappa_S} & & \downarrow{\scriptstyle\kappa_E} \\ \hat{S}_p & \xrightarrow{\hat{\nu}_p} & \hat{E}_p \end{array}$$

coincides with γ_G.

It remains to verify that both $c_p(\gamma_G)$ and $\gamma_{c_p}(G)$ are isomorphisms. For this purpose we consider the exact sequence $K \rightarrowtail^{\phi} \hat{F}_p \twoheadrightarrow^{\pi} c_p(G)$ where F is as above and $K = \operatorname{Im} \hat{\mu}_p$. By Lemma 1.1 and [16, 7.5] this is a torsion-free nilpotent presentation of $c_p(G)$. Therefore by Proposition 3.4 and the first part there is a commutative diagram

$$\begin{array}{ccccc} K & \overset{\phi}{\rightarrowtail} & \hat{F}_p & \overset{\pi}{\twoheadrightarrow} & c_p(G) \\ \downarrow{\scriptstyle \kappa_K} & & \downarrow{\scriptstyle \kappa_{\hat{F}_p}} & & \downarrow{\scriptstyle \gamma_{c_p(G)}} \\ \hat{K}_p & \longrightarrow & (\hat{F}_p)\hat{}_p & \twoheadrightarrow & c_p(c_p(G)) \end{array}$$

with exact rows. Furthermore, we observe that K is complete in its own p-adic topology. Therefore κ_K and $\kappa_{\hat{F}_p}$ are isomorphisms, and hence so is $\gamma_{c_p}(G)$. Now by Theorem 2.1, $\kappa_{\hat{F}_p}$ agrees with $(\kappa_F)\hat{}_p$, and by Proposition 3.4 we have $c_p(\gamma_G)\pi = \eta(\kappa_F)\hat{}_p$. Thus $c_p(\gamma_G)$ agrees with $\gamma_{c_p}(G)$, hence $c_p(\gamma_G)$ is likewise an isomorphism.

DEFINITION. We say that a nilpotent group G is *p-cotorsion* if $\gamma_G: G \to c_p(G)$ is an isomorphism. (According to Theorem 2.1 such a G should be called c_p-complete; it seemed to us more natural, however, to emphasize the connection with cotorsion Abelian groups).

4.2. COROLLARY. *If G is a nilpotent group, then*

(1) *$c_p(G)$ is p-cotorsion, and*

(2) *for any homomorphism $f: G \to H$ into a p-cotorsion nilpotent group H there is a unique homomorphism $\phi: c_p(G) \to H$ such that $f = \phi\gamma_G$.*

Remark. If G is an Abelian group, γ_G agrees with the connecting homomorphism $\delta_G: G \cong \operatorname{Hom}(\mathbb{Z}, G) \to \operatorname{Ext}(\mathbb{Z}(p^\infty), G)$ of the long exact sequence for $\operatorname{Hom}(-, G)$ associated with the short exact sequence $\mathbb{Z} \rightarrowtail \mathbb{Z}[1/p] \twoheadrightarrow \mathbb{Z}(p^\infty)$. This follows from the fact that $\operatorname{Ext}(\mathbb{Z}(p^\infty), -)$ together with δ is a completion functor with the same complete objects as c_p.

The next result will enable us to investigate the precise relationship between p-adic and p-cotorsion completion.

4.3. THEOREM. *For any nilpotent group G there is a natural epimorphism $\tau_G: c_p(G) \twoheadrightarrow \hat{G}_p$ such that* (1) $\tau_G \gamma_G = \kappa_G$; (2) $\operatorname{Ker} \tau_G = c_p(G)^{p^\omega}$.

Proof. (1) Let $R \overset{\mu}{\rightarrowtail} F \overset{\pi}{\twoheadrightarrow} G$ be a free nilpotent presentation of G and let η denote the projection $\hat{F}_p \twoheadrightarrow c_p(G)$. Then by definition we have $\operatorname{Ker} \eta = \operatorname{Im} \hat{\mu}_p$; on the other hand, we know that $\operatorname{Im} \hat{\mu}_p \leqslant \operatorname{Ker} \hat{\pi}_p$. It follows that there is an epimorphism $\tau_G: c_p(G) \twoheadrightarrow \hat{G}_p$ such that $\hat{\pi}_p = \tau_G \eta$. Therefore, by commutativity of the diagrams

$$\begin{array}{ccc} F & \xrightarrow{\pi} & G \\ \downarrow{\scriptstyle \kappa_F} & & \downarrow{\scriptstyle \gamma_G} \\ \hat{F}_p & \xrightarrow{\eta} & c_p(G) \end{array} \qquad \begin{array}{ccc} F & \xrightarrow{\pi} & G \\ \downarrow{\scriptstyle \kappa_F} & & \downarrow{\scriptstyle \kappa_G} \\ \hat{F}_p & \xrightarrow{\hat{\pi}_p} & \hat{G}_p \end{array}$$

we obtain $\kappa_G \pi = \hat{\pi}_p \kappa_F = \tau_G \eta \kappa_F = \tau_G \gamma_G \pi$, hence τ_G satisfies condition (1). Clearly, τ_G does not depend on the choice of F and is natural.

(2) It follows from the definition of η and τ_G that $\eta^{-1}(\operatorname{Ker} \tau_G) = \operatorname{Ker} \hat{\pi}_p$. By Theorem 2.6, $\operatorname{Ker} \hat{\pi}_p$ is the p-adic closure of $\operatorname{Im} \hat{\mu}_p = \operatorname{Ker} \eta$. Thus we infer from Lemma 2.5 that $\operatorname{Ker} \tau_G = c_p(G)^{p^\omega}$.

Remark. In the Abelian case the kernel of τ_G can be identified with $\operatorname{Pext}(\mathbb{Z}(p^\infty), G)$, the subgroup of $\operatorname{Ext}(\mathbb{Z}(p^\infty), G)$ which classifies the pure extensions of G by $\mathbb{Z}(p^\infty)$. This follows from Theorem 4.3, [5, Theorem 53.3] and the fact that $\operatorname{Ext}(\mathbb{Z}(p^\infty), G)$ is divisible by each prime $\neq p$.

4.4. COROLLARY. *For any nilpotent group G we have $c_p(G)\hat{}_p \cong \hat{G}_p$.*

4.5. PROPOSITION. *Let G be a nilpotent group such that $t_p(G)$ is bounded. Then $\tau_G: c_p(G) \to \hat{G}_p$ is an isomorphism, and $c_p^1(G) = \{1\}$.*

Proof. Let $R \overset{\mu}{\rightarrowtail} F \overset{\pi}{\twoheadrightarrow} G$ be a free nilpotent presentation of G. Then by Corollary 2.9 the sequence of p-adic completions

$$\{1\} \to \hat{R}_p \xrightarrow{\hat{\mu}_p} \hat{F}_p \xrightarrow{\hat{\pi}_p} \hat{G}_p \to \{1\}$$

is exact. It follows that $\operatorname{Ker} \hat{\pi}_p = \operatorname{Ker} \eta$, where η denotes the projection $\hat{F}_p \to c_p(G)$. Therefore by definition (cf. proof of Theorem 4.3) τ_G is an isomorphism. Moreover, since $\hat{\mu}_p$ is monomorphic, it follows that $c_p^1(G) = \{1\}$.

A more general result will be proved (by different methods) in Section 7. As an important tool we now introduce a long exact sequence for the p-cotorsion completion c_p and its "first derived" c_p^1. In view of Theorem 4.10 below this sequence generalizes that of $\operatorname{Hom}(\mathbb{Z}(p^\infty), -)$ for Abelian groups.

4.6. THEOREM. *Let $N \overset{f}{\rightarrowtail} G \overset{g}{\twoheadrightarrow} Q$ be a short exact sequence of nilpotent groups. Then there is a natural connecting homomorphism $\delta: c_p^1(Q) \to c_p(N)$ such that the sequence*

$$\{1\} \longrightarrow c_p^1(N) \xrightarrow{c_p^1(f)} c_p^1(G) \xrightarrow{c_p^1(g)} c_p^1(Q) \xrightarrow{\delta} c_p(N)$$

$$\xrightarrow{c_p(f)} c_p(G) \xrightarrow{c_p(g)} c_p(Q) \longrightarrow \{1\}$$

is exact. Moreover, if $f\colon H \rightarrowtail G$ is any monomorphism of nilpotent groups, then $c_p^1(f)\colon c_p^1(H) \to c_p^1(G)$ is likewise a monomorphism.

Proof. Suppose that the class of G is $\leqslant c$, and let $R \rightarrowtail E \twoheadrightarrow N$ and $S \rightarrowtail F \twoheadrightarrow Q$ be free presentations of N and Q, respectively, in $\mathscr{N}_c$. Let $E *_c F$ be the coproduct of E and F in $\mathscr{N}_c$; by Lemma 1.1 $E *_c F$ is free nilpotent. It is easily seen that there is a commutative diagram

$$\begin{array}{ccccc} E & \xrightarrow{h} & E *_c F & \xrightarrow{k} & F \\ \downarrow & & \downarrow \pi & & \downarrow \eta \\ N & \xrightarrow{f} & G & \xrightarrow{g} & Q \end{array}$$

where h is the natural injection and k the natural projection. The first row is not exact in general, but the kernel K of k contains $\operatorname{Im} h$ and is torsion-free. Therefore we obtain a commutative diagram

$$\begin{array}{ccccc} U & \rightarrowtail & T & \twoheadrightarrow & S \\ \downarrow \mu & & \downarrow \nu & & \downarrow \phi \\ K & \rightarrowtail & E *_c F & \xrightarrow{k} & F \\ \downarrow \epsilon & & \downarrow \pi & & \downarrow \eta \\ N & \xrightarrow{f} & G & \xrightarrow{g} & Q \end{array}$$

where $U = \operatorname{Ker} \varepsilon$ and $T = \operatorname{Ker} \pi$. Since F is torsion-free, by Corollary 2.9 the upper rectangle induces a commutative diagram of p-adic completions

$$\begin{array}{ccccc} \hat{U}_p & \rightarrowtail & \hat{T}_p & \twoheadrightarrow & \hat{S}_p \\ \downarrow \hat{\mu}_p & & \downarrow \hat{\nu}_p & & \downarrow \hat{\phi}_p \\ \hat{K}_p & \rightarrowtail & (E *_c F)\hat{}_p & \twoheadrightarrow & \hat{F}_p \end{array}$$

with exact rows. Thus the first part of the theorem follows from the "snake lemma" [10, III.5.1] which of course is also valid in our situation. Note that we apply Proposition 3.4 in order to identify $c_p(N)$ with $\hat{K}_p/\operatorname{Im} \hat{\mu}_p$ and $c_p^1(N)$ with $\operatorname{Ker} \hat{\mu}_p$, respectively.

Now let $f\colon H \rightarrowtail G$ be a monomorphism of nilpotent groups of class $\leqslant c$. Let $K \rightarrowtail^{\mu} F \twoheadrightarrow^{\pi} G$ be a free presentation of G in $\mathscr{N}_c$, and let $E = \pi^{-1}(\operatorname{Im} f)$. Then there is a commutative diagram

$$\begin{array}{ccccc} K & \rightarrowtail & E & \twoheadrightarrow & H \\ \| & & \downarrow & & \downarrow{\scriptstyle f} \\ K & \overset{\mu}{\rightarrowtail} & F & \twoheadrightarrow & G \end{array}$$

with exact rows. Since E is torsion-free, Proposition 3.4 provides a commutative diagram

$$\begin{array}{ccccc} c_p^1(H) & \rightarrowtail & \hat{K}_p & \xrightarrow{\hat{\nu}_p} & \hat{E}_p \\ \downarrow{\scriptstyle c_p^1(f)} & & \| & & \downarrow \\ c_p^1(G) & \rightarrowtail & \hat{K}_p & \xrightarrow{\hat{\mu}_p} & \hat{F}_p \end{array}$$

We conclude that $c_p^1(f)$ is a monomorphism, and this completes our proof.

4.7. Proposition. *If G is a p-cotorsion nilpotent group, then $c_p^1(G)=\{1\}$.*

Proof. Let $R \rightarrowtail^{\mu} F \twoheadrightarrow G$ be a free nilpotent presentation of G. If G is p-cotorsion, then there is an exact sequence $\hat{R}_p \to^{\hat{\mu}_p} \hat{F}_p \twoheadrightarrow G$. Therefore, in order to determine $c_p^1(G)$ we may as well use the torsion-free presentation $K \rightarrowtail \hat{F}_p \twoheadrightarrow G$ where $K = \operatorname{Im} \hat{\mu}_p$. Since κ_K and $\kappa_{\hat{F}_p}$ are both isomorphisms, it follows that $c_p^1(G)=\{1\}$.

We turn to study the kernel of $\gamma_G : G \to c_p(G)$. For this we need the following proposition.

4.8. Proposition. *For any nilpotent group G, $c_p(G)=\{1\}$ if and only if G is p-radicable.*

Proof. If $c_p(G)=\{1\}$, then by Theorem 4.3, $\hat{G}_p=\{1\}$. We infer from Theorem 2.2 that $G^{p^\omega}=G$, hence, by the comment preceding Lemma 1.4, G is p-radicable. Conversely, suppose that G is p-radicable. Then for any $x \in G$ there is a homomorphism $f\colon \mathbb{Z}[1/p] \to G$ such that $x \in \operatorname{Im} f$. We consider the commutative diagram

$$\begin{array}{ccc} \mathbb{Z}[1/p] & \xrightarrow{f} & G \\ \downarrow{\scriptstyle \gamma_{\mathbb{Z}[1/p]}} & & \downarrow{\scriptstyle \gamma_G} \\ c_p(\mathbb{Z}[1/p]) & \xrightarrow{c_p(f)} & c_p(G) \end{array}$$

It is easy to see that $c_p(\mathbb{Z}[1/p]) \cong \operatorname{Ext}(\mathbb{Z}(p^\infty), \mathbb{Z}[1/p]) = \{1\}$, hence $\gamma_G f$ is the trivial map. Since every $x \in G$ lies in the image of such an f, it follows that γ_G is trivial. Therefore we have $c_p(G)=\{1\}$ by Corollary 4.2.

4.9. THEOREM. *For any nilpotent group G,* Ker $\gamma_G = \rho_p(G)$, *the maximal p-radicable subgroup of G.*

Proof. Let K be the kernel γ_G and let ϕ denote the inclusion $K \to G$. As G/K maps monomorphically into $c_p(G)$, we infer from Theorem 4.6 and Proposition 4.7 that $c_p^1(G/K) = \{1\}$. Therefore by Theorem 4.6, $c_p(\phi)$ is a monomorphism. Since by hypothesis $\gamma_G \phi$ is the trivial map, commutativity of the diagram

$$\begin{array}{ccc} K & \stackrel{\phi}{\rightarrowtail} & G \\ \downarrow{\scriptstyle \gamma_K} & & \downarrow{\scriptstyle \gamma_G} \\ c_p(K) & \stackrel{c_p(\phi)}{\rightarrowtail} & c_p(G) \end{array}$$

yields that γ_K is likewise trivial. Therefore $c_p(K) = \{1\}$ and hence K is p-radicable by Proposition 4.8. On the other hand, we see from the commutative diagram

$$\begin{array}{ccc} \rho_p(G) & \rightarrowtail & G \\ \downarrow{\scriptstyle \gamma_{\rho_p(G)}} & & \downarrow{\scriptstyle \gamma_G} \\ c_p(\rho_p(G)) & \longrightarrow & c_p(G) \end{array}$$

that $\rho_p(G)$ is contained in K, since by Proposition 4.8, $c_p(\rho_p(G)) = \{1\}$. This completes the proof of Theorem 4.9.

Remark. By definition of $c_p(G)$ and by Theorem 2.4 the class of $c_p(G)$ is always less or equal than the class of G. If G is p-reduced, Theorem 4.9 implies that the two classes are equal.

We conclude this section by showing that the functor c_p^1 is naturally equivalent to $\mathrm{Hom}_{\mathcal{N}}(\mathbb{Z}(p^\infty), -)$ which is in fact a functor $\mathcal{N} \to \mathcal{A}$ (the category of Abelian groups).

4.10. THEOREM. *For any nilpotent group G,*

(1) $\mathrm{Hom}_{\mathcal{N}}(\mathbb{Z}(p^\infty), G)$ *is a p-adically complete torsion-free Abelian group;*

(2) $c_p^1(G)$ *is naturally isomorphic to* $\mathrm{Hom}_{\mathcal{N}}(\mathbb{Z}(p^\infty), G)$, *in particular, $c_p^1(G) = \{1\}$ if and only if $\rho(t_p(G)) = \{1\}$.*

Proof. (1) Let $K = \rho(t_p(G))$, so that K is a radicable p-group and hence Abelian by [16, 4.11]. Since every homomorphism f: $\mathbb{Z}(p^\infty) \to G$ lands in K, we may identify $\mathrm{Hom}_{\mathcal{N}}(\mathbb{Z}(p^\infty), G)$ with $\mathrm{Hom}(\mathbb{Z}(p^\infty), K)$, which clearly is a torsion-free Abelian group. To verify that $\operatorname{Hom}(\mathbb{Z}(p^\infty), K)$ is p-adically complete, we use the isomorphism

$$\operatorname{Hom}(\mathbb{Z}(p^\infty), K) \cong \varprojlim \operatorname{Hom}(\mathbb{Z}/p^n\mathbb{Z}, K),$$

where the maps $\mathbb{Z}/p^n\mathbb{Z} \to \mathbb{Z}/p^{n+1}\mathbb{Z}$ of the direct system $\{\mathbb{Z}/p^n\mathbb{Z}\}_{n<\omega}$ are the multiplication by p.

(2) By Proposition 4.5 we have $c_p^1(G/t_p(G)) = \{1\}$. Therefore Theorem 4.6 implies that $c_p^1(G) \cong c_p^1(t_p(G))$. Furthermore, we infer from Theorem 4.9, Proposition 4.7 and Theorem 4.6 that $c_p^1(t_p(G)/K) = \{1\}$, where K is as above. It follows that $c_p^1(t_p(G)) \cong c_p^1(K)$ and hence, using the remark before Proposition 3.4, we obtain

$$c_p^1(G) \cong c_p^1(K) \cong \operatorname{Hom}(\mathbb{Z}(p^\infty), K) = \operatorname{Hom}_{\mathcal{N}}(\mathbb{Z}(p^\infty), G),$$

where all the isomorphisms are natural. It follows, in particular, that $c_p^1(G) = \{1\}$ if and only if $K = \{1\}$. This completes our proof.

5. p-Cotorsion Nilpotent Groups

In this section we give a characterization of p-cotorsion groups in terms of their upper central series and study the relationship between p-cotorsion and p-adically complete groups. We begin by listing some elementary results on p-cotorsion groups.

5.1. Proposition. *Every p-cotorsion nilpotent group is p-reduced and uniquely q-radicable for every prime $q \neq p$.*

Proof. Let G be p-cotorsion. Then by definition $\operatorname{Ker} \gamma_G = \{1\}$, hence G is p-reduced by Theorem 4.9. To verify the second assertion we consider a free nilpotent presentation $R \overset{\mu}{\rightarrowtail} F \twoheadrightarrow G$. By hypothesis we have $G \cong \hat{F}_p/K$, where $K = \operatorname{Im} \hat{\mu}_p$. Since $\hat{R}_p$ is q-radicable, so is K. Since $\hat{F}_p$ is uniquely q-radicable, it follows that G is q-radicable and has no q-torsion, and hence is uniquely q-radicable for every prime $q \neq p$.

5.2. Proposition. *Let $N \overset{f}{\rightarrowtail} G \overset{g}{\twoheadrightarrow} Q$ be a short exact sequence of nilpotent groups, where G is p-cotorsion. Then the following conditions are equivalent:*

(1) *N is p-cotorsion;*

(2) *Q is p-reduced;*

(3) *Q is p-cotorsion.*

Proof. By Theorems 4.1 and 4.6 there is a commutative diagram

$$\begin{array}{ccccccc} & & N & \xrightarrow{f} & G & \xrightarrow{g} & Q \\ & & \downarrow{\scriptstyle \gamma_N} & & \downarrow{\scriptstyle \gamma_G} & & \downarrow{\scriptstyle \gamma_Q} \\ c_p^1(Q) & \longrightarrow & c_p(N) & \xrightarrow{c_p(f)} & c_p(G) & \xrightarrow{c_p(g)} & c_p(Q) \end{array}$$

with exact rows, where by hypothesis γ_G is an isomorphism. If N is p-cotorsion, in particular γ_N is epimorphic and hence γ_Q is monomorphic. Therefore by Theorem 4.9 Q is p-reduced. If Q is p-reduced, then it is p-cotorsion, because γ_Q is epimorphic anyway. Finally, suppose that Q is p-cotorsion. Then $c_p^1(Q) = \{1\}$ by Proposition 4.7, and hence γ_N is an isomorphism.

5.3. PROPOSITION. *Let $N \rightarrowtail G \twoheadrightarrow Q$ be a short exact sequence of nilpotent groups. If N and Q are p-cotorsion, then so is G.*

Proof. We consider the diagram in the previous proof. By hypothesis γ_N and γ_Q are isomorphisms, hence $c_p^1(Q) = \{1\}$ by Proposition 4.7. It follows that γ_G is likewise an isomorphism.

5.4. PROPOSITION. *If G and H are p-cotorsion nilpotent groups, then for every homomorphism $f: G \to H$,* Ker f *and* Im f *are p-cotorsion.*

Proof. By Proposition 5.1, H is p-reduced, hence Im f is p-reduced. But then Proposition 5.2 implies that Im f is p-cotorsion. Therefore, again by Proposition 5.2, Ker f is p-cotorsion as well.

5.5. THEOREM. *A nilpotent group is p-cotorsion if and only if each of the factors of its upper central series is p-cotorsion.*

Proof. For Abelian groups the theorem holds trivially. Let G be a nilpotent groups of class $c > 1$, and assume that the result is true for nilpotent groups of class $<c$. If each factor of the upper central series of G is p-cotorsion, then the center Z_1 of G is p-cotorsion, and so is G/Z_1 by the induction hypothesis. Hence G is p-cotorsion by Proposition 5.3. Conversely, suppose that G is a p-cotorsion. Then G is p-reduced, and hence by Lemma 1.2, G/Z_1 is p-reduced. Thus it follows from Proposition 5.2 that both Z_1 and G/Z_1 are p-cotorsion. Hence by the induction hypothesis each of the factors of the upper central series of G is p-cotorsion. This completes our proof.

The further results of this section concern the relation between p-cotorsion and p-adically complete groups. They correspond to results in [5, Sect. 54] but are mostly proven differently. Following [16] a nilpotent group G is called *residually p-bounded* if $G^{p^\omega} = \{1\}$.

5.6. PROPOSITION. *A p-cotorsion group is p-adically complete if and only if it is residually p-bounded.*

Proof. If G is p-cotorsion, then $\kappa_G: G \to \hat{G}_p$ is epimorphic by Theorem 4.3. Therefore, G is p-adically complete if and only if κ_G is a monomorphism, which means the same as $G^{p^\omega} = \{1\}$.

5.7. PROPOSITION. *Every p-adically complete nilpotent group is p-cotorsion.*

Proof. Let G be p-adically complete, and let K denote the kernel of τ_G. Then by Theorem 4.3(1) $\tau_G \gamma_G$ is an isomorphism. Therefore so is $c_p(\tau_G)$, because $c_p(\gamma_G)$ is an isomorphism anyway (by Theorem 4.1). We infer from Theorem 4.6 that the induced map $c_p(K) \to c_p(c_p(G))$ is trivial. Therefore by commutativity of the diagram

$$\begin{array}{ccc} K & \rightarrowtail & c_p(G) \\ \downarrow{\scriptstyle \gamma_K} & & \downarrow{\scriptstyle \gamma_{c_p(G)}} \\ c_p(K) & \longrightarrow & c_p(c_p(G)) \end{array}$$

and Theorem 4.1 we obtain $K = \{1\}$, and hence both τ_G and γ_G are isomorphisms.

5.8. PROPOSITION. *A nilpotent group is p-cotorsion if and only if it is p-reduced and a homomorphic image of a p-adically complete group.*

Proof. Let G be p-cotorsion, and let $R \rightarrowtail F \twoheadrightarrow G$ be a free nilpotent presentation. Then there is an epimorphism $\hat{F}_p \twoheadrightarrow G$, and G is p-reduced by Proposition 5.1. Conversely, suppose that G is p-reduced and there is an epimorphism $H \twoheadrightarrow G$ where H is p-adically complete. Then G is p-cotorsion by Propositions 5.7 and 5.2.

5.9. THEOREM. *If G is p-cotorsion nilpotent group, then G^{p^ω} is a p-cotorsion Abelian group.*

Remark. By Theorem 4.3(2) this means, in particular, that for any G, Ker τ_G is Abelian (cf. [3, p. 169]).

Proof. Since G^{p^ω} is the kernel of $\kappa_G: G \to \hat{G}_p$, G^{p^ω} is p-cotorsion by Propositions 5.7 and 5.4. Furthermore, as G is p-reduced, it follows from Theorem 2.4(ii) that G^{p^ω} is Abelian.

5.10. Proposition. *For a nilpotent torsion group G the following conditions are equivalent*:

(1) *G is p-adically complete*;

(2) *G is p-cotorsion*;

(3) *G is a bounded p-group.*

Proof. (1) $\Rightarrow$ (2) is a special case of Proposition 5.7, while (3) $\Rightarrow$ (1) is trivial. So it remains to prove (2) $\Rightarrow$ (3). Suppose first that G is a p-cotorsion Abelian group. Then by Proposition 5.6, $G/p^{\omega}G$ is a p-adically complete torsion groups and hence a bounded p-group (cf., e.g. [5, Sect. 40]). It follows that for some positive integer n, $p^nG = p^{\omega}G$. Therefore p^nG is p-radicable, and hence $p^nG = \{1\}$ by Proposition 5.1. Suppose now that G is a p-cotorsion nilpotent torsion group of class $c > 1$. Then by Theorem 5.5 each of the factors of its upper central series is p-cotorsion and torsion, hence a bounded p-group. Therefore G itself is a bounded p-group.

Combining this with Proposition 4.5 we obtain the following immediate consequence.

5.11. Corollary. *Let G be either a p-group or p-torsion-free. Then G is p-cotorsion if and only if it is p-adically complete.*

6. The Structure of p-Cotorsion Nilpotent Groups

In this section we are mainly interested in the extent to which Harrison's theory of cotorsion Abelian groups [8] carries over to our case.

6.1. Proposition. *Let G be nilpotent and let K be a normal subgroup of $c_p(G)$ such that* $\operatorname{Im} \gamma_G \leqslant K$. *Then $c_p(G)/K$ is radicable.*

Proof. Let $G' = G/\rho_p(G)$ and let $\pi\colon G \to G'$ denote the projection. In the commutative diagram

$$\begin{array}{ccc} G & \xrightarrow{\pi} & G' \\ \downarrow{\scriptstyle \gamma_G} & & \downarrow{\scriptstyle \gamma_{G'}} \\ c_p(G) & \xrightarrow{c_p(\pi)} & c_p(G') \end{array}$$

by Theorem 4.6 and Proposition 4.8, the bottom map is an isomorphism. Hence it suffices to prove the proposition for p-reduced G. By Proposition 5.1, $c_p(G)$ is q-radicable for all primes $q \neq p$. Thus it remains to show that $c_p(G)/K$ is p-radicable. Let L be the full preimage of $\rho_p(c_p(G)/K)$

in $c_p(G)$, so that $c_p(G)/L$ is p-reduced. We claim that $L = c_p(G)$. From Lemma 1.2 we know that G/Z_1 is p-reduced, and Theorem 4.6 yields an exact sequence

$$c_p^1(G/Z_1) \to c_p(Z_1) \to c_p(G) \to c_p(G/Z_1) \to \{1\},$$

where by Theorem 4.10 the first term is trivial. We further know that $Z_1 \leqslant L$, and by the remark after Corollary 4.2 we have $c_p(Z_1)/Z_1 \cong \mathrm{Ext}(\mathbb{Z}[1/p], Z_1)$, hence $c_p(Z_1)/Z_1$ is p-radicable. Since $c_p(G)/L$ has no p-radicable subgroups other than $\{1\}$, it follows that $c_p(Z_1) \subseteq L$. We conclude that $c_p(G)/L \cong c_p(G/Z_1)/L'$, where L' is a normal subgroup of $c_p(G/Z_1)$ containing G/Z_1. By induction on the class of G, the result is proved.

6.2. THEOREM. *Let G be a nilpotent group such that $\rho(t_p(G)) = \{1\}$. Then the groups $c_p(Z_i(G))$ can be identified with subgroups of $c_p(G)$, and as such they form a central series for $c_p(G)$. In particular, this is true if G is p-reduced or finitely generated.*

Remark. This series need not be the upper central series for $c_p(G)$, even in the torsion-free case (cf. [16, 7.8]).

Proof. We note that by Lemma 1.2, $\rho(t_p(G/Z_i)) = \{1\}$. Therefore the long exact sequence of c_p (Theorem 4.6) shows that the maps $c_p(Z_i) \to c_p(G)$ are monomorphisms, and that $c_p(Z_{i+1})/c_p(Z_i)$ can be naturally identified with $c_p(Z_{i+1}/Z_i)$. Since this is a normal series for $c_p(G)$, the result will be proved if we can show that the action of $c_p(G)$ on $c_p(Z_{i+1}/Z_i)$ is trivial. We know that the action of G on Z_{i+1}/Z_i is trivial, and hence by Corollary 4.2 we infer that the action of G on $c_p(Z_{i+1}/Z_i)$ is trivial. Therefore the kernel K of the action of $c_p(G)$ on $c_p(Z_{i+1}/Z_i)$ contains G. Since $c_p(Z_{i+1}/Z_i)$ is p-reduced and the action is nilpotent, it follows from Lemma 1.3(ii) that $c_p(G)/K$ is p-reduced. However, as $G \subseteq K$, Proposition 6.1 implies that $c_p(G)/K$ is p-radicable, from which we conclude that $K = c_p(G)$. Hence the action of $c_p(G)$ on $c_p(Z_{i+1}/Z_i)$ is trivial, as desired.

6.3. THEOREM. *If G is a p-reduced nilpotent group, which we identify with the image of γ_G, then $t(c_p(G)) = t(G)$.*

Proof. If G is p-reduced Abelian, there is an exact sequence

$$0 \to G \xrightarrow{\delta_G} \mathrm{Ext}(\mathbb{Z}(p^\infty), G) \to \mathrm{Ext}(\mathbb{Z}[1/p], G) \to 0.$$

As δ_G agrees with γ_G (cf. the remark after Corollary 4.2), it follows that $c_p(G)/G$ is p-torsion-free. Since for every prime $q \neq p$, $c_p(G)$ has no q-torsion, we infer that $t(c_p(G)) = t(G)$.

Now suppose that G is nilpotent of class $c > 1$ and assume that the theorem holds for all groups of class $<c$. By Lemma 1.2 and induction hypothesis we conclude that $t(c_p(G/Z_1)) = t(G/Z_1)$. Since we can identify $c_p(G/Z_1)$ with $c_p(G)/c_p(Z_1)$, it follows that every $x \in t(c_p(G))$ may be written as $x = yz$, where $y \in c_p(Z_1)$, $z \in G$ such that for some positive integer n, $z^{p^n} \in Z_1$. (Here we make use of the fact that for every prime $q \neq p$, G has no q-torsion.) As $c_p(Z_1)$ is central (by Theorem 6.2), we see that $x^{p^n} = y^{p^n} z^{p^n}$ where $x^{p^n} \in t(c_p(Z_1))$. It follows that x^{p^n} and z^{p^n} are in Z_1, whence $y^{p^n} \in Z_1$. Since $c_p(Z_1)/Z_1$ is p-torsion-free and $y \in c_p(Z_1)$, we conclude that $y \in Z_1$, which implies that $x \in G$. This completes our proof.

DEFINITION. A p-cotorsion group G is called *adjusted* if $G/t(G)$ is radicable (cf. Harrison's definition and Proposition 2.2 in [8]).

6.4. PROPOSITION. *For any nilpotent p-group G*, $t(c_p(G)) = \operatorname{Im} \gamma_G$, *and thus* $c_p(G)$ *is adjusted.*

Proof. By the same argument as in the proof of Proposition 6.1 we may assume that G is p-reduced. Therefore by Theorem 6.3 the image of γ_G is the torsion subgroup of $c_p(G)$. Hence by Proposition 6.1, $c_p(G)$ is adjusted. (Note that in this situation $\operatorname{Im} \gamma_G$ is normal in $c_p(G)$.)

We are now in the position to prove the main results of this section which concern the structure of p-cotorsion groups. They correspond to Propositions 2.2 and 2.3 of [8] (see also [5, Theorems 55.5 and 55.6]).

6.5. THEOREM. *Every p-cotorsion nilpotent group G contains a uniquely determined adjusted p-cotorsion normal subgroup A* ($\cong c_p(tG)$), *such that G/A is torsion-free and p-adically complete. If G is Abelian, then A is a direct summand of G.*

Proof. If G is p-cotorsion, then by Theorems 4.6 and 4.10 there is an exact sequence

$$\{1\} \to c_p(t(G)) \to G \to c_p(G/t(G)) \to \{1\}.$$

Therefore, by Proposition 6.4, $A = \operatorname{Im} h$ is an adjusted p-cotorsion normal subgroup of G. We further see from Proposition 4.5 that G/A is isomorphic to $(G/t(G))\hat{}_p$ and hence p-adically complete and torsion-free by [16, 7.5]. If A' is any subgroup of G satisfying the conclusions of the theorem, then, necessarily, A' contains $t(G)$, $A'/t(G)$ is radicable, and G/A' is reduced. This shows that $A' = A$, so A is uniquely determined. Now suppose that G is Abelian. Since every p-cotorsion Abelian group is of the form $\operatorname{Ext}(\mathbb{Z}(p^\infty), H)$ (cf. the remark after Propositions 3.3), it follows that $\operatorname{Ext}(G/A, A) = 0$ (by [5, Theorem 54.6]). Hence A is a direct summand.

EXAMPLE. It is easy to give an example of a 2-cotorsion nilpotent group G of class 2 which is a non-splitting extension of an adjusted 2-cotorsion group by a 2-adically complete torsion-free group.

Let $F\colon (\hat{\mathbb{Z}}_2 \oplus \hat{\mathbb{Z}}_2) \times (\hat{\mathbb{Z}}_2 \oplus \hat{\mathbb{Z}}_2) \to \mathbb{Z}/2\mathbb{Z}$ be the alternating bilinear form which can be given by the matrix $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Let G be the central extension of $\mathbb{Z}/2\mathbb{Z}$ by $\hat{\mathbb{Z}}_2 \oplus \hat{\mathbb{Z}}_2$ which corresponds to F (via [16, 5.4]; note that there is exactly one element in $H^2(\hat{\mathbb{Z}}_2 \oplus \hat{\mathbb{Z}}_2, \mathbb{Z}/2\mathbb{Z})$ arising from F, because $\mathrm{Ext}(\hat{\mathbb{Z}}_2 \oplus \hat{\mathbb{Z}}_2, \mathbb{Z}/2\mathbb{Z}) = 0$). Since $\mathbb{Z}/2\mathbb{Z}$ agrees with the commutator subgroup of G, the extension cannot split.

6.6. THEOREM. *The functors $c_p(-)$ and $t(-)$ are adjoint equivalences between the category of reduced nilpotent p-groups and the category of adjusted p-cotorsion nilpotent groups.*

Proof. Let G be a reduced p-group, Then by Proposition 6.4, $c_p(G)$ is adjusted and its torsion part is isomorphic to G. On the other hand, if G is adjusted p-cotorsion, then $t(G)$ is p-reduced, hence a reduced p-group. Moreover, the subgroup A in Theorem 6.5 agrees with G; therefore $c_p(t(G))$ is isomorphic to G. Finally, it is easy to verify, using Corollary 4.2, that c_p is left adjoint to t.

7. APPLICATIONS

The aim of this section is to apply the theory of p-cotorsion groups to the p-adic completion of nilpotent groups. We first come back to our exactness study (cf. Section 2). Given a sequence of groups and homomorphisms $G \to^f H \to^g K$ where gf is trivial, the set of left cosets $\mathrm{Ker}\, g/\mathrm{Im}\, f$ will be called the *homology* of this sequence.

7.1. PROPOSITION. *Let $N \rightarrowtail^f G \twoheadrightarrow^g Q$ be a short exact sequence of nilpotent groups. Then the homology of the induced sequence $\hat{N}_p \to^{\hat{f}_p} \hat{G}_p \to^{\hat{g}_p} \hat{Q}_p$ is a group which is isomorphic to the cokernel of the induced map $g_* : c_p(G)^{p^\omega} \to c_p(Q)^{p^\omega}$ and thus Abelian. In particular, the sequence $\hat{N}_p \to^{\hat{f}_p} \hat{G}_p \to^{\hat{g}_p} \hat{Q}_p \to \{1\}$ is exact if and only if g_* is an epimorphism.*

Proof. By Corollary 2.10, $\mathrm{Im}\, \hat{f}_p$ is a normal subgroup of $\hat{G}_p$, hence the homology of the induced sequence $\hat{N}_p \to \hat{G}_p \to \hat{Q}_p$ is a group. Now we consider the commutative diagram

$$\begin{array}{ccccccc}
c_p(N) & \xrightarrow{c_p(f)} & c_p(G) & \xrightarrow{c_p(g)} & c_p(Q) & \longrightarrow & \{1\} \\
\downarrow{\scriptstyle \tau_N} & & \downarrow{\scriptstyle \tau_G} & & \downarrow{\scriptstyle \tau_Q} & & \\
\hat{N}_p & \xrightarrow{\hat{f}_p} & \hat{G}_p & \xrightarrow{\hat{g}_p} & \hat{Q}_p & &
\end{array}$$

By Theorem 4.3 the column maps are epimorphisms, and by Theorem 4.6 the top row is exact. Therefore τ_G maps $\operatorname{Ker} c_p(g)$ onto $\operatorname{Im} \hat{f}_p$. Hence by Theorem 4.3 and the non-commutative version of the "snake lemma," $\operatorname{Ker} \hat{g}_p/\operatorname{Im} \hat{f}_p \cong \operatorname{Coker}(g_* : c_p(G)^{p^\omega} \to c_p(Q)^{p^\omega})$ which is Abelian by Theorem 5.9.

The above proposition leads us to the characterization of those nilpotent groups G for which $\tau_G ; c_p(G) \to \hat{G}_p$ is an isomorphism. For this purpose we will need the following:

DEFINITION. We call a nilpotent p-group *torsion-complete* if its is the torsion subgroup of a p-adically complete group. In particular, every bounded nilpotent p-group is torsion complete, and every torsion-complete nilpotent p-group is reduced.

7.2. PROPOSITION. *For a reduced nilpotent p-group G the following conditions are equivalent*:

(1) *G is torsion-complete*;

(2) $c_p(G)^{p^\omega} = \{1\}$;

(3) $G \cong t(\hat{G}_p)$.

Proof. (1) ⇒ (2) Let G be the torsion subgroup of the p-adically complete group H. By Theorem 4.10 the top map in the commutative square

$$\begin{array}{ccc} c_p(G) & \longrightarrow & c_p(H) \\ \downarrow{\scriptstyle \tau_G} & & \downarrow{\scriptstyle \tau_H} \\ \hat{G}_p & \longrightarrow & \hat{H}_p \end{array}$$

is a monomorphism. Since by Proposition 5.7 every p-adically complete group is p-cotorsion, it follows that τ_H is an isomorphism. Therefore τ_G is likewise an isomorphism, and hence by Theorem 4.3, $c_p(G)^{p^\omega} = \{1\}$.

(2) ⇒ (3) By Theorem 6.3 the map γ_G takes G isomorphically onto $t(c_p(G))$. As condition (2) means that τ_G is an isomorphism, it follows that $G \cong t(\hat{G}_p)$. The implication (3) ⇒ (1) holds trivially.

7.3. THEOREM. *For any nilpotent group G, the following conditions are equivalent*:

(a) *The map* $t_G : c_p(G) \to \hat{G}_p$ *is an isomorphism*;

(b) $c_p(G)^{p^\omega} = \{1\}$;

(c) $t_p(G)/\rho(t_p(G))$ *is torsion-complete.*

Proof. The equivalence of (a) and (b) is an immediate consequence of Theorem 4.3.

(b)$\Leftrightarrow$(c): By Corollary 2.10 and Theorem 4.6 the rows of the commutative diagram

$$\begin{array}{ccccccccc} \{1\} & \longrightarrow & c_p(t_p(G)) & \longrightarrow & c_p(G) & \longrightarrow & c_p(G/t_p(G)) & \longrightarrow & \{1\} \\ & & \downarrow{\scriptstyle \tau_{t_pG}} & & \downarrow{\scriptstyle \tau_G} & & \downarrow{\scriptstyle \cong} & & \\ \{1\} & \longrightarrow & t_p(G)\hat{}_p & \longrightarrow & \hat{G}_p & \longrightarrow & (G/t_p(G))\hat{}_p & \longrightarrow & \{1\} \end{array}$$

are exact, and by Proposition 4.5 the right-hand column map is an isomorphism. Thus we infer from Theorem 4.3 that $c_p(t_p(G))^{p^\omega} \cong c_p(G)^{p^\omega}$. Since, by Theorem 4.6 and Proposition 4.8, $c_p(t_p(G)) \cong c_p(t_p(G)/\rho(t_p G))$, we conclude that $c_p(G)^{p^\omega} \cong c_p(t_p(G)/\rho(t_p G))^{p^\omega}$. The result follows now from Proposition 7.2.

The following are now easy consequences of Proposition 7.1 and Theorem 7.3.

7.4. COROLLARY. *Let $N \rightarrowtail G \twoheadrightarrow Q$ be a short exact sequence of nilpotent groups, and suppose that $t_p(G)/\rho(t_p(G))$ is torsion-complete. Then the homology of the induced sequence $\hat{N}_p \to \hat{G}_p \to \hat{Q}_p$ is isomorphic to $c_p(Q)^{p^\omega}$.*

7.5. COROLLARY. *If Q is a nilpotent group such that $t_p(Q)/\rho(t_p(Q))$ is torsion-complete, then for any short exact sequence $N \rightarrowtail G \twoheadrightarrow Q$ the induced sequence $\hat{N}_p \to \hat{G}_p \to \hat{Q}_p \to \{1\}$ is exact.*

Remark. One might expect that in Corollary 2.9 the hypothesis on $t_p(Q)$ could be weakened. However, even in the Abelian case, torsion-completeness of $t_p(Q)$ does not imply exactness of $\{1\} \to \hat{N}_p \to \hat{G}_p \to \hat{Q}_p \to \{1\}$, as is shown by the following example.

EXAMPLE. Let $Q = \prod_{n=1}^{\infty} \mathbb{Z}/p^n\mathbb{Z}$; so that $t_p(Q)$ is torsion-complete (but not bounded). Let G be an Abelian group s.t. $p^\omega G \cong Q$ and $G/p^\omega G \cong Q$ (such a group G exists, e.g., by [15, Theorem 2.6]). Thus we have a short exact sequence

$$Q \overset{\mu}{\rightarrowtail} G \twoheadrightarrow Q,$$

where $\operatorname{Im} \mu = p^\omega G$. Then, of course, in the induced sequence

$$\hat{Q}_p \xrightarrow{\hat{\mu}_p} \hat{G}_p \longrightarrow \hat{Q}_p,$$

$\hat{\mu}_p$ is the trivial map, but $\hat{Q}_p \cong Q \neq 0$.

Furthermore, Theorem 7.3 enables us to generalize Theorem 6.6 and the p-version of 6.9 of [16].

7.6. COROLLARY. *If G is a nilpotent group such that $t_p(G)/\rho(t_p(G))$ is torsion-complete, then $G^{p^\omega} = \rho_p(G)$.*

7.7. COROLLARY. *Let G be a nilpotent group such that $t_p(G)$ is torsion-complete. Then G is residually p-bounded if and only if its center is residually p-bounded.*

Proof. If G is residually p-bounded, then clearly $Z_1(G)$ is residually p-bounded. On the other hand, suppose that $Z_1(G)$ is residually p-bounded. Then $Z_1(G)$ is p-reduced and therefore, by Lemma 1.2, G is p-reduced. Since $t_p(G)$ is torsion-complete, we infer from Corollary 7.6 that G is residually p-bounded.

Finally, we apply the p-cotorsion completion in order to improve part of 7.4 and 7.6 of [16].

7.8. THEOREM. *Let G be a nilpotent group s.t. $\rho(t_p(G)) = \{1\}$. Then the images $\tilde{Z}_i$ of the natural maps $\zeta_i : Z_i(G)\hat{}_p \to \hat{G}_p$ form a central series for $\hat{G}_p$. If in particular $t_p(G)$ is torsion-complete, then the maps ζ_i are monomorphisms; thus $\tilde{Z}_i$ can be identified with $Z_i(G)\hat{}_p$.*

Proof. By Theorem 6.2 the natural maps $c_p(Z_i(G)) \to c_p(G)$ are monomorphisms and their images C_i form a central series for $c_p(G)$. Therefore, $\tau_G : c_p(G) \to \hat{G}_p$ being an epimorphism, the groups $\tau_G(C_i)$ form a central series for $\hat{G}_p$. But by commutativity of the diagram

$$\begin{array}{ccc} c_p(Z_i(G)) & \longrightarrow & c_p(G) \\ \downarrow{\scriptstyle \tau_{Z_i(G)}} & & \downarrow{\scriptstyle \tau_G} \\ Z_i(G)\hat{}_p & \xrightarrow{\zeta_i} & \hat{G}_p \end{array}$$

$\tau_G(C_i)$ coincides with $\tilde{Z}_i$. This proves the first part. Now suppose, in particular, that $t_p(G)$ is torsion-complete. In this case $c_p(G)^{p^\omega} = \{1\}$ by Theorem 7.3, and hence $c_p(Z_i(G))^{p^\omega} = \{1\}$. Therefore both τ_G and $\tau_{Z_i(G)}$ are isomorphisms, hence the ζ_i's are monomorphisms. This completes our proof.

We remark that Theorem 7.8 applies in particular to finitely generated nilpotent groups.

7.9. THEOREM. *Let G be a nilpotent group such that either G is residually p-bounded or $t_p(G)$ is torsion-complete. Then G is p-adically* *complete if and only if each of the factors of its upper central series is p-adically complete.*

Proof. Let G be a p-adically complete. Then G is residually p-bounded and p-cotorsion, the latter by Proposition 5.7. Therefore, $Z_1(G)$ is residually p-bounded and hence, by [16, 2.1], each of the factors $Z_{i+1}(G)/Z_i(G)$ is residually p-bounded. Since by Theorem 5.5 these factors are also p-cotorsion, Proposition 5.6 implies that they are p-adically complete. Conversely, suppose that the factors $Z_{i+1}(G)/Z_i(G)$ are p-adically complete. Then by Proposition 5.7 they are p-cotorsion, and hence G is p-cotorsion by Theorem 5.5. Now, if $G^{p^\omega} = \{1\}$, then the assertion follows from Proposition 5.6. On the other hand, if $t_p(G)$ is torsion-complete, then the result is a consequence of Theorem 7.3.

8. Extensions by $\mathbb{Z}(p^\infty)$

If A is an Abelian group, we can make the identification $c_p(A) = \mathrm{Ext}(\mathbb{Z}(p^\infty), A)$. Bousfield and Kan use the notation $\mathrm{Ext}(\mathbb{Z}(p^\infty), G)$ for what we have written $c_p(G)$, but we have chosen not to do so, partly because in the nilpotent case the elements of $c_p(G)$ do not describe extensions of G by $\mathbb{Z}(p^\infty)$, as we shall see. There is, however, a connection between extensions and elements of $c_p(G)$, which will be described in this section. We begin with a general discussion of extensions of a group G by an Abelian group A which are obtained by extending the center of G.

If N is a nilpotent group, we let $\mathscr{N}(A, N)$ be the set of equivalence classes of nilpotent extensions of N by A (in the sense of [10, p. 84 or p. 206].) Here, A will be Abelian group in all of the instances of interest to us. We first construct a function

$$\varepsilon: \mathrm{Ext}(A, Z_1(N)) \to \mathscr{N}(A, N)$$

as follows. Given an $e \in \mathrm{Ext}(A, Z_1(N))$, corresponding to the sequence

$$Z_1(N) \overset{\alpha}{\rightarrowtail} B \twoheadrightarrow A,$$

we let $\varepsilon(e)$ be the sequence

$$N \rightarrowtail G \twoheadrightarrow A,$$

where $G = (N \times B)/D$, $D = \{(\sigma(z), \alpha(z): z \in Z_1(N)\}$ and $\sigma: Z_1(N) \to N$ is the natural map. (We should remark that despite appearances, this construction does not give us a push-out in the category $\mathscr{N}$.)

8.1. LEMMA. *The map* $\varepsilon\colon \operatorname{Ext}(A, Z_1(N)) \to \mathscr{N}(A, N)$ *just described is injective, and its image is precisely the set of equivalence classes of extensions*

$$N \rightarrowtail G \twoheadrightarrow A$$

such that the induced homomorphism $Z_1(G) \to A$ *is surjective.*

Proof. We first note that we have an extension

$$N \rightarrowtail G \twoheadrightarrow A$$

which arises from the construction outlined above, then we can recover the original extension as $Z_1(N) \rightarrowtail Z_1(G) \twoheadrightarrow A$. Hence the map ε is injective. If we have an extension such that the map $Z_1(G) \to A$ is surjective, then there is an epimorphism $Z_1(G) \times N \twoheadrightarrow G$, which shows that the extension is indeed in the image of the map ε.

Remark. As we shall see, though this map is injective, it may happen that the image of a non-split extension is split. The point is that $\mathscr{N}(A, N)$ may contain many distinct equivalence classes of split extensions.

We now specialize in the case in which $A = \mathbb{Z}(p^\infty)$. If we have an extension $E\colon N \rightarrowtail G \twoheadrightarrow^{f} \mathbb{Z}(p^\infty)$ where G is nilpotent, then the long exact sequence of the functor c_p yields

$$c_p^1 N \rightarrowtail c_p^1 G \to c_p^1 \mathbb{Z}(p^\infty) \xrightarrow{\delta} c_p N \twoheadrightarrow c_p G$$

where we note that $c_p^1 \mathbb{Z}(p^\infty) = \operatorname{Hom}(\mathbb{Z}(p^\infty), \mathbb{Z}(p^\infty)) \cong \hat{\mathbb{Z}}_p$ (Theorem 4.10). If $\mathbf{1}$ is the identity map, $\mathbb{Z}(p^\infty) \to \mathbb{Z}(p^\infty)$, then the correspondence taking E to $\delta(\mathbf{1})$ gives us a map

$$\Delta\colon \mathscr{N}(\mathbb{Z}(p^\infty), N) \to c_p(N).$$

The extension E splits if and only if $\mathbf{1} \in \operatorname{Im}(c_p^1(f))$, which happens if and only if $\delta(\mathbf{1})$ is trivial. For example, this shows that all such extensions split if $c_p(N) = 1$ (a fact which could easily have been derived differently).

To give a complete description of equivalence classes of extensions of N by $\mathbb{Z}(p^\infty)$ in terms of elements of $c_p(N)$ would presumably be the same as saying that the map Δ was injective. Even if N is Abelian, this may fail since if there is a nontrivial nilpotent action of $\mathbb{Z}(p^\infty)$ on N (as there is if $N = \mathbb{Z} \oplus \mathbb{Z}(p^\infty)$ for example), then there will be at least two non-equivalent split extensions. However, in special cases we get strong results.

8.2. THEOREM. *If N is a finitely generated nilpotent group, then every* *group extension of N by $\mathbb{Z}(p^\infty)$ is nilpotent and arises from an Abelian extension of the center $Z_1(N)$ by $\mathbb{Z}(p^\infty)$, so that the map*

$$\varepsilon: c_p(Z_1(N)) \to \mathcal{N}(\mathbb{Z}(p^\infty), N)$$

is bijective, and the map

$$\Delta: \mathcal{N}(\mathbb{Z}(p^\infty), N) \to c_p(N)$$

is injective.

Proof. We first suppose that we have an extension

$$N \rightarrowtail G \twoheadrightarrow \mathbb{Z}(p^\infty).$$

We will use the symbols Z_i for the terms of the upper central series of N (not G). Since the Z_i are characteristic subgroups of the normal subgroup N, it follows that they are normal in G. The induced action of G on Z_{i+1}/Z_i has N in its kernel, and hence is an action of $\mathbb{Z}(p^\infty)$ on Z_{i+1}/Z_i. Since $\mathbb{Z}(p^\infty)$ has no non-trivial finite-dimensional rational representations, we conclude that this action is trivial, and that therefore the subgroups Z_i form a central series for G. G is therefore a nilpotent group.

We notice that $\rho(t_p(N)) = 1$ (or, equivalently, $c_p^1(N) = 1$) so that we may infer from Theorem 6.2 that the induced map $\zeta: c_p(Z_1(N)) \to c_p(N)$ is injective. The naturality of the connecting homomorphism (Theorem 4.6) implies that $\zeta = \Delta\varepsilon$. We already know from Lemma 8.1 that ε is injective, so the conclusions of the theorem will be established if we show that ε is surjective, which we will now do.

We again let $N \rightarrowtail G \twoheadrightarrow \mathbb{Z}(p^\infty)$ be any (nilpotent) extension and we let $C_G(N)$ be the centralizer in G of the subgroup N. Since $G/C_G(N)$ is a subgroup of $\mathrm{Nil}_F(N)$, where F is the flag consisting of the subgroups Z_i, $G/C_G(N)$ is finitely generated (by Lemma 1.3). It follows that $G/NC_G(N)$ is a finitely generated homomorphic image of $\mathbb{Z}(p^\infty)$, and therefore is trivial, so that $G = NC_G(N)$. We conclude that the induced map $C_G(N) \to \mathbb{Z}(p^\infty)$ is surjective, so that $C_G(N)$ is an extension of Z_1 by $\mathbb{Z}(p^\infty)$. According to Lemma 8.1, the result will be proved if we can show that $Z_1(G) = C_G(N)$. Since $G = C_G(N)N$ and $[N, C_G(N)] = 1$, it will suffice to show that $C_G(N)$ is Abelian. If D is the center of $C_G(N)$, then D contains $Z_1(N)$. If D were not all of $C_G(N)$, D would be finitely generated and $C_G(N)/D$ would be isomorphic to $\mathbb{Z}(p^\infty)$. This would contradict Lemma 1.2, since D is the center of $C_G(N)$ and $\rho(t_p(D)) = \{1\}$.

We now will give three examples to illustrate what changes can occur if we change our conditions on N. The first is a group N such that $c_p(N) \neq 1$ but such that there is (up to equivalence) only one extension of N by $\mathbb{Z}(p^\infty)$. The second is a group N such that $c_p(Z_1(N)) \cong \hat{\mathbb{Z}}_p$ but the map

$c_p(Z_1(N)) \to c_p(N)$ is trivial, thus producing an uncountable family of inequivalent split extensions. In both of these cases, N is (necessarily) not p-reduced. The third example is a p-reduced p-group N with extensions by $\mathbb{Z}(p^\infty)$ which are not in the image of the map ε and thus do not arise from an extension of the center.

EXAMPLE 1. There is a nilpotent group N such that $c_p(N) \neq 1$ but such that every extension of N by $\mathbb{Z}(p^\infty)$ splits and is isomorphic to $N \times \mathbb{Z}(p^\infty)$.

We choose N to have center isomorphic to $\mathbb{Z}(p^\infty)$ and central quotient $\mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z}$, which we easily can do using the techniques of [16, Chap. 5]. Let C be the center of N. If G is an extension of N by $\mathbb{Z}(p^\infty)$, then C is a normal subgroup of G, and since it is the center of N, one gets an action of $\mathbb{Z}(p^\infty)$ on C. Since $C \cong \mathbb{Z}(p^\infty)$ and $\mathrm{Aut}(\mathbb{Z}(p^\infty)$ is Abelian with finite torsion subgroup [6, 127.5], it is easy to see that this action is trivial. Hence C is in the center of G. If we factor out C, we get an extension of $\mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z}$ by $\mathbb{Z}(p^\infty)$, and since $\mathrm{Aut}(\mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z})$ is finite, the action is again trivial, so G is nilpotent of class at most three. It follows from a theorem of Baer's [16, 3.24] that the center Z_1 of G has finite index, and hence the image of Z_1 in $\mathbb{Z}(p^\infty)$ is all of $\mathbb{Z}(p^\infty)$. We infer from Lemma 8.1 that the map $\varepsilon\colon \mathrm{Ext}(\mathbb{Z}(p^\infty), C) \to \mathcal{N}(\mathbb{Z}(p^\infty), N)$ is surjective. Since $\mathrm{Ext}(\mathbb{Z}(p^\infty), C) = 0$ and since we have previously checked that all extensions of N by $\mathbb{Z}(p^\infty)$ are nilpotent, we conclude that up to equivalence there is exactly one extension of N by $\mathbb{Z}(p^\infty)$.

EXAMPLE 2. There is a nilpotent group N such that $c_p(Z_1(N)) \cong \hat{\mathbb{Z}}_p$, and the induced homomorphism $c_p(Z_1(N)) \to c_p(N)$ is trivial, so that the elements of $c_p(Z_1(N))$ index an uncountable family of split extensions of N by $\mathbb{Z}(p^\infty)$. For this N there are also non-split extensions of N by $\mathbb{Z}(p^\infty)$.

We use the matrix $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ to define an alternating form $A \times A \to \mathbb{Z}[1/p]$, where $A = \mathbb{Z}[1/p] \oplus \mathbb{Z}[1/p]$. We let B be the subgroup of A given by $B = \mathbb{Z}[1/p] \oplus \mathbb{Z}$. Following this form by a homomorphism, we obtain alternating forms $A \times A \to \mathbb{Z}[1/p]/\mathbb{Z} = \mathbb{Z}(p^\infty)$ and $B \times B \to \mathbb{Z}(p^\infty)$. We use these as in [16, Chap. 5] to construct central extensions $\mathbb{Z}(p^\infty) \rightarrowtail G \twoheadrightarrow A$ and $\mathbb{Z}(p^\infty) \rightarrowtail N \twoheadrightarrow B$. We note that we have an extension $N \rightarrowtail G \twoheadrightarrow \mathbb{Z}(p^\infty)$, so N has non-split extensions by $\mathbb{Z}(p^\infty)$. If $\phi\colon N \twoheadrightarrow B$ is the natural map, then one sees easily that $Z_1(N) = \phi^{-1}(\mathbb{Z} \oplus 0)$ and $Z_1(N) \cong \mathbb{Z}(p^\infty) \oplus \mathbb{Z}$. Since $\rho_p(N) = \phi^{-1}(\mathbb{Z}[1/p] \oplus 0)$, we see that $Z_1(N) \leqslant \rho_p(N)$, so the induced map $c_p(Z_1(N)) \to c_p(N)$ is trivial, even though $c_p(Z_1(N)) \cong \hat{\mathbb{Z}}_p$ as claimed.

EXAMPLE 3. There is a countable nilpotent p-reduced p-group N and an extension of N by $\mathbb{Z}(p^\infty)$ which does not arise from an extension of the center of N.

Let $A = \bigoplus_{n=1}^{\infty} \mathbb{Z}(p^n)$ and let the generators of the cyclic summands be g_n, $n \geqslant 1$. Let e_n be the endomorphism of A defined by $e_n(g_m) = \delta_{nm} g_n$. The subgroup of H of $\operatorname{Hom}(A, A)$ generated by these elements e_n is isomorphic to A. The homomorphism sending e_n to $[p^{-n}]$ is an epimorphism of H onto $\mathbb{Z}(p^\infty)$, whose kernel we will denote H'. We let H act on $A \oplus A$ by setting $(x, y)^h = (x, y + h(x))$. This is clearly a nilpotent action, and we let G and N be the semidirect products

$$G = H \ltimes (A \oplus A), \qquad N = H' \ltimes (A \oplus A)$$

corresponding to this action. We regard N as a subgroup of G and note that $G/N \cong \mathbb{Z}(p^\infty)$. This is the extension we are interested in and we must show it does not arise from an extension of the center of N. We first show that $Z_1(G) = Z_1(N) = (0, (0, A))$ (as a subgroup of $H \ltimes (A \oplus A)$). It is clear that the center of G is the set of elements (x, y) of the normal subgroup $A \oplus A$ such that $h(x) = 0$ for all $h \in H$. By construction, this means $x = 0$, so $Z_1(G) = (0, (0, A))$. To show that this subgroup is also the center of N, it suffices to show that if $x \in A$ and $h(x) = 0$ for all $h \in H'$, then $x = 0$. For such an x, the map $h \to h(x)$ would yield a homomorphism $H/H' \to A$. Since H/H' is divisible and A is reduced, this map would be trivial, so $h(x) = 0$ for all $h \in H$, which (as we noted before) implies $x = 0$. Hence $Z_1(N) = Z_1(G)$.

This clearly shows that this extension does not arise from an Abelian extension of the center in the sense of Lemma 8.1. In fact, it also does not arise in any other sense from an extension of the center, since if we factor out the center, we still have a non-split extension.

8.3. Proposition. *If A is a p-reduced Abelian group, then every nilpotent extension of A by $\mathbb{Z}(p^\infty)$ is Abelian, so that the set of equivalence classes of such extensions is in one-to-one correspondence with the elements of $c_p A$.*

Remark. It is easy to see that for suitable A (e.g., free of countably infinite rank), non-nilpotent extensions may exist, and also that for suitable A (not p-reduced) non-Abelian nilpotent extensions may exist.

Proof. If A is p-reduced, then we infer from Lemma 1.3 that there is no non-trivial nilpotent action of $\mathbb{Z}(p^\infty)$ on A. It follows that an extension is central. If such an extension is

$$A \rightarrowtail B \twoheadrightarrow \mathbb{Z}(p^\infty)$$

and if $Z_1(B) \neq B$, then $Z_1(B)/A$ is finite, so $Z_1(B)$ is p-reduced, while $B/Z_1(B) \cong \mathbb{Z}(p^\infty)$ (just as in the proof of Theorem 8.2). This contradicts Lemma 1.2.

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