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Common fixed point theorems for ordered contractions and quasicontractions in ordered cone metric spaces

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ABSTRACT

In the first part of this paper we generalize results on common fixed points in ordered cone metric spaces obtained by I. Altun and G. Durmaz [I. Altun, G. Durmaz, Some fixed point theorems on ordered cone metric spaces, Rend. Circ. Mat. Palermo, 58 (2009) 319–325] and I. Altun, B. Damnjanović and D. Djorić [I. Altun, B. Damnjanović, D. Djorić, Fixed point and common fixed point theorems on ordered cone metric spaces, Appl. Math. Lett. (2009) doi:10.1016/j.aml.2009.09.016] by weakening the respective contractive condition. Then, the notions of quasicontraction and g-quasicontraction are introduced in the setting of ordered cone metric spaces and respective (common) fixed point theorems are proved. In such a way, known results on quasicontractions and g-quasicontractions in metric spaces and cone metric spaces are extended to the setting of ordered cone metric spaces. Examples show that there are cases when new results can be applied, while old ones cannot.

1. Introduction

Ordered normed spaces and cones have applications in applied mathematics, for instance, in using Newton's approximation method [1–3] and in optimization theory [4]. Numerous generalizations of the Banach Contraction Principle in the setting of metric spaces were given by many authors. Abstract (cone) metric spaces were studied by Huang and Zhang [5]. They proved the basic versions of the fixed point theorem, which were later generalized by several authors.

The existence of fixed points in partially ordered sets was investigated, e.g., by Ran and Reurings [6], and then by Nieto and Lopez [7]. The following two versions of the fixed point theorem were proved, among others, in these papers.

Theorem 1.1 ([6,7]). *Let $(X, \sqsubseteq)$ be a partially ordered set and let d be a metric on X such that (X, d) is a complete metric space. Let $f : X \to X$ be a nondecreasing map w.r.t. $\sqsubseteq$. Suppose that the following conditions hold:*

(i) *there exists $k \in (0, 1)$ such that $d(fx, fy) \le kd(x, y)$ for all $x, y \in X$ with $y \sqsubseteq x$;*
(ii) *there exists $x_0 \in X$ such that $x_0 \sqsubseteq fx_0$;*
(iii) *f is continuous, or*
(iii′) *if a nondecreasing sequence $\{x_n\}$ converges to $x \in X$, then $x_n \sqsubseteq x$ for all n.*

Then f has a fixed point $x^* \in X$.

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Fixed point results in ordered metric spaces were investigated by many authors (see, e.g., [8–14]).

Fixed point results in ordered cone metric spaces were obtained by Altun and Durmaz [15], as well as by Altun, Damnjanović and Djorić [16]. We state the basic result from [15].

Theorem 1.2 ([15]). *Let $(X, \sqsubseteq)$ be a partially ordered set and let d be a cone metric on X (defined over a normal cone P with the normal constant K) such that (X, d) is a complete cone metric space. Let $f : X \to X$ be a continuous and nondecreasing map w.r.t. $\sqsubseteq$. Suppose that the following conditions hold:*

(i) *there exists $k \in (0, 1)$ such that $d(fx, fy) \preceq kd(x, y)$ for all $x, y \in X$ with $y \sqsubseteq x$;*
(ii) *there exists $x_0 \in X$ such that $x_0 \sqsubseteq fx_0$.*

Then f has a fixed point $x^* \in X$.

In [16], some generalizations of the previous theorem were proved, including the case when the underlying cone P is not normal. Also, some common fixed point theorems were obtained. We state the following theorem which is an "ordered" variant of a result of Abbas and Rhoades [17].

Theorem 1.3 ([16]). *Let $(X, \sqsubseteq)$ be a partially ordered set and let d be a cone metric on X (defined over a cone P with $\operatorname{int} P \neq \emptyset$) such that (X, d) is a complete cone metric space. Let $f, g : X \to X$ be self-maps such that (f, g) is a weakly increasing pair w.r.t. $\sqsubseteq$. Suppose that the following conditions hold:*

(i) *there exist $\alpha, \beta, \gamma \geq 0$ such that $\alpha + 2\beta + 2\gamma < 1$ and*

$$d(fx, gy) \preceq \alpha d(x, y) + \beta[d(x, fx) + d(y, gy)] + \gamma[d(x, gy) + d(y, fx)] \tag{1.1}$$

for all comparable $x, y \in X$;
(ii) *f or g is continuous, or*
(ii′) *if a nondecreasing sequence $\{x_n\}$ converges to $x \in X$, then $x_n \sqsubseteq x$ for all n.*

Then f and g have a common fixed point $x^* \in X$.

Note that a pair (f, g) of self-maps on a partially ordered set $(X, \sqsubseteq)$ is said to be *weakly increasing* if $fx \sqsubseteq gfx$ and $gx \sqsubseteq fgx$ for all $x \in X$. There are examples (see [16]) when neither of such mappings f, g is nondecreasing w.r.t. $\sqsubseteq$. In particular, the pair (f, i_X) (i_X—the identity function) is weakly increasing if and only if $x \sqsubseteq fx$ for each $x \in X$.

We show by the following simple example that a mapping on an ordered cone metric space can be an "ordered" contraction, while it is not a contraction in the classical sense. (Examples of similar kind were given also in [15,16].)

Example 1.4. Let $X = \{1, 2, 4\}$, $\sqsubseteq = \{(1, 1), (2, 2), (4, 4), (1, 4)\}$; $E = \mathbb{R}^2$, $P = \{(a, b) : a, b \geq 0\}$, $d(x, y) = (|x - y|, 2|x - y|)$, and let $f : X \to X$, $f1 = 2$, $f2 = 1$, $f4 = 1$.

The mapping f is a (Banach-type) contraction in the ordered cone metric space $(X, \sqsubseteq, d)$, i.e.

$$d(fx, fy) \preceq \lambda d(x, y), \quad y \sqsubseteq x, \tag{1.2}$$

for some $\lambda \in [0, 1)$. Indeed, we have only to check validity of (1.2) for $y = 1, x = 4$. But it is equivalent to $|f4 - f1| \leq \lambda |4 - 1|$, i.e., $|1 - 2| \leq \lambda |4 - 1|$, which is satisfied if (and only if) $\lambda \in \left[\frac{1}{3}, 1\right)$.

On the other hand, f is not a contraction in the (non-ordered) cone metric space (X, d). Indeed, for $x = 2, y = 1$ we have that

$$|f2 - f1| \leq \lambda |2 - 1| \Leftrightarrow 1 \leq \lambda \cdot 1 \Leftrightarrow \lambda \geq 1.$$

It also means that f is not a contraction in the metric space (X, d_1) where d_1 is the usual metric $d_1(x, y) = |x - y|$ on $\mathbb{R}$.

In Section 3 of this paper we generalize results from [15,16] by weakening condition (1.1) (see Theorem 3.1). Then, in Section 4, the notions of quasicontraction and g-quasicontraction are introduced in the setting of ordered cone metric spaces and respective (common) fixed point theorems are proved. In such a way, known results on quasicontractions and g-quasicontractions in metric spaces [18,19] and cone metric spaces [20–23] are extended to ordered cone metric spaces. Examples show that there are cases when new results can be applied, while old ones cannot.

2. Preliminaries

We need the following definitions and results, consistent with [4,5].

Let E be a real Banach space. A subset P of E is a *cone* if:

(i) P is closed, nonempty and $P \neq \{0\}$;
(ii) $a, b \in \mathbb{R}$, $a, b \geq 0$, and $x, y \in P$ imply $ax + by \in P$;
(iii) $P \cap (-P) = \{0\}$.

Given a cone $P \subset E$, we define the partial ordering $\preceq$ with respect to P by $x \preceq y$ if and only if $y - x \in P$. We write $x \prec y$ to indicate that $x \preceq y$ but $x \neq y$, while $x \ll y$ stands for $y - x \in \operatorname{int} P$ (the interior of P).

A cone $P \subset E$ is called *normal* if there is a number $K > 0$ such that for all $x, y \in P$,

$$0 \preceq x \preceq y \text{ implies } \|x\| \leq K\|y\| \tag{2.1}$$

or, equivalently, if $x_n \preceq y_n \preceq z_n$ and

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} z_n = x \text{ imply } \lim_{n\to\infty} y_n = x. \tag{2.2}$$

The least positive number K satisfying (2.1) is called the normal constant of P. It is clear that $K \geq 1$. Most of ordered Banach spaces used in applications posses a cone with the normal constant $K = 1$, and if this is the case, proofs of the corresponding results are much alike as in the metric setting. If $K > 1$, this is not the case.

Example 2.1 (*[2]*). Let $E = C^1_{\mathbb{R}}[0, 1]$ with $\|x\| = \|x\|_\infty + \|x'\|_\infty$ and $P = \{x \in E : x(t) \geq 0 \text{ for } t \in [0, 1]\}$. This cone is not normal. Consider, for example, $x_n(t) = \frac{t^n}{n}$, $y_n(t) = \frac{1}{n}$. Then $0 \preceq x_n \preceq y_n$ and $\lim_{n\to\infty} y_n = 0$, but

$$\|x_n\| = \max_{t\in[0,1]} \left|\frac{t^n}{n}\right| + \max_{t\in[0,1]} \left|t^{n-1}\right| = \frac{1}{n} + 1 > 1;$$

hence (x_n) does not converge to zero. It follows by (1.2) that P is a nonnormal cone.

Definition 2.2 (*[5]*). Let X be a nonempty set and P a cone in a Banach space E. Suppose that a mapping $d : X \times X \to E$ satisfies:

(d1) $0 \preceq d(x, y)$ for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$;
(d2) $d(x, y) = d(y, x)$ for all $x, y \in X$;
(d3) $d(x, y) \preceq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

Then d is called a *cone metric* on X and (X, d) is called a *cone metric space*.

The concept of a cone metric space is more general than that of a metric space, because each metric space is a cone metric space where $E = \mathbb{R}$ and $P = [0, +\infty)$.

The following remark will be useful in the sequel.

Remark 2.3. (1) If $u \preceq v$ and $v \ll w$, then $u \ll w$.
(2) If $0 \preceq u \ll c$ for each $c \in \operatorname{int} P$, then $u = 0$.
(3) If $a \preceq b + c$ for each $c \in \operatorname{int} P$ then $a \preceq b$.
(4) If $0 \preceq x \preceq y$, and $0 \leq a$, then $0 \preceq ax \preceq ay$.
(5) If $0 \preceq x_n \preceq y_n$ for each $n \in \mathbb{N}$, and $\lim_{n\to\infty} x_n = x$, $\lim_{n\to\infty} y_n = y$, then $0 \preceq x \preceq y$.
(6) If $0 \preceq d(x_n, x) \preceq b_n$ and $b_n \to 0$, then $d(x_n, x) \ll c$ where x_n, x are, respectively, a sequence and a given point in X.
(7) If E is a real Banach space with a cone P and if $a \preceq \lambda a$ where $a \in P$ and $0 < \lambda < 1$, then $a = 0$.
(8) If $c \in \operatorname{int} P$, $0 \preceq a_n$ and $a_n \to 0$, then there exists n_0 such that for all $n > n_0$ we have $a_n \ll c$.

For other basic properties of cone metric spaces we refer to [5].

3. Common fixed points of weakly increasing mappings

In the rest of the paper $(X, \sqsubseteq, d)$ will always be an ordered cone metric space, i.e., $\sqsubseteq$ will be a partial order on the set X, and d will be a cone metric on X with the underlying cone P such that $\operatorname{int} P \neq \emptyset$ (such a cone will be called *solid*). Normality of the cone is not assumed.

In our first result we shall use conditions similar to those used in [17] in non-ordered case.

Theorem 3.1. *Let $(X, \sqsubseteq, d)$ be an ordered complete cone metric space. Let (f, g) be a weakly increasing pair of self-maps on X w.r.t. $\sqsubseteq$. Suppose that the following conditions hold:*

(i) *there exist $p, q, r, s, t \geq 0$ satisfying $p + q + r + s + t < 1$ and $q = r$ or $s = t$, such that*

$$d(fx, gy) \preceq pd(x, y) + qd(x, fx) + rd(y, gy) + sd(x, gy) + td(y, fx) \tag{3.1}$$

for all comparable $x, y \in X$;
(ii) *f or g is continuous, or*
(ii′) *if a nondecreasing sequence $\{x_n\}$ converges to $x \in X$, then $x_n \sqsubseteq x$ for all $n \in \mathbb{N}$.*

Then f and g have a common fixed point $x^* \in X$.

Proof. Let $x_0 \in X$ be arbitrary and define a sequence $\{x_n\}$ by $x_{2n+1} = fx_{2n}$ and $x_{2n+2} = gx_{2n+1}$ for all $n \in \mathbb{N}_0$. Using that the pair of mappings (f, g) is weakly increasing, it can be easily shown that the sequence $\{x_n\}$ is nondecreasing w.r.t. $\sqsubseteq$, i.e., $x_0 \sqsubseteq x_1 \cdots \sqsubseteq x_n \sqsubseteq x_{n+1} \sqsubseteq \cdots$. In particular, x_{2n} and x_{2n+1} are comparable, so we can apply relation (3.1) to obtain

$$\begin{aligned} d(x_{2n+1}, x_{2n+2}) &= d(fx_{2n}, gx_{2n+1}) \\ &\preceq pd(x_{2n}, x_{2n+1}) + qd(x_{2n}, x_{2n+1}) + rd(x_{2n+1}, x_{2n+2}) + sd(x_{2n}, x_{2n+2}) + td(x_{2n+1}, x_{2n+1}) \\ &\preceq pd(x_{2n}, x_{2n+1}) + qd(x_{2n}, x_{2n+1}) + rd(x_{2n+1}, x_{2n+2}) + s[d(x_{2n}, x_{2n+1}) + d(x_{2n+1}, x_{2n+2})]. \end{aligned}$$

It follows that

$$(1 - r - s)d(x_{2n+1}, x_{2n+2}) \preceq (p + q + s)d(x_{2n}, x_{2n+1}),$$

i.e.

$$d(x_{2n+1}, x_{2n+2}) \preceq \frac{p+q+s}{1-(r+s)} d(x_{2n}, x_{2n+1}). \tag{3.2}$$

In a similar way one obtains that

$$d(x_{2n+2}, x_{2n+3}) \preceq \frac{p+q+t}{1-(q+t)} \cdot \frac{p+q+s}{1-(r+s)} d(x_{2n}, x_{2n+1}). \tag{3.3}$$

Now, from (3.2) and (3.3), by induction, we obtain that

$$\begin{aligned} d(x_{2n+1}, x_{2n+2}) &\preceq \frac{p+q+s}{1-(r+s)} d(x_{2n}, x_{2n+1}) \\ &\preceq \frac{p+q+s}{1-(r+s)} \cdot \frac{p+r+s}{1-(q+t)} d(x_{2n-1}, x_{2n}) \\ &\preceq \frac{p+q+s}{1-(r+s)} \cdot \frac{p+r+s}{1-(q+t)} \cdot \frac{p+q+s}{1-(r+s)} d(x_{2n-2}, x_{2n-1}) \\ &\preceq \cdots \preceq \frac{p+q+s}{1-(r+s)} \left(\frac{p+r+t}{1-(q+t)} \cdot \frac{p+q+s}{1-(r+s)} \right)^n d(x_{0,} x_1), \end{aligned}$$

and

$$\begin{aligned} d(x_{2n+2}, x_{2n+3}) &\preceq \frac{p+r+t}{1-(q+t)} d(x_{2n+1}, x_{2n+2}) \\ &\preceq \cdots \preceq \left(\frac{p+r+t}{1-(q+t)} \cdot \frac{p+q+s}{1-(r+s)} \right)^{n+1} d(x_0, x_1). \end{aligned}$$

Let

$$A = \frac{p+q+s}{1-(r+s)}, \qquad B = \frac{p+r+t}{1-(q+t)}.$$

In the case $q = r$,

$$AB = \frac{p+q+s}{1-(q+s)} \cdot \frac{p+r+t}{1-(q+t)} = \frac{p+q+s}{1-(q+t)} \cdot \frac{p+r+t}{1-(r+s)} < 1 \cdot 1 = 1,$$

and if $s = t$,

$$AB = \frac{p+q+s}{1-(r+s)} \cdot \frac{p+r+s}{1-(q+t)} < 1 \cdot 1 = 1.$$

Now, for $n < m$ we have

$$\begin{aligned} d(x_{2n+1}, x_{2m+1}) &\preceq d(x_{2n+1}, x_{2n+2}) + \cdots + d(x_{2n}, x_{2m+1}) \\ &\preceq \left(A \sum_{i=n}^{m-1} (AB)^i + \sum_{i=n+1}^{m} (AB)^i \right) d(x_0, x_1) \\ &\preceq \left(\frac{A(AB)^n}{1-AB} + \frac{(AB)^{n+1}}{1-AB} \right) d(x_0, x_1) \\ &= (1+B) \frac{A(AB)^n}{1-AB} d(x_0, x_1). \end{aligned}$$

Similarly, we obtain

$$d(x_{2n}, x_{2m+1}) \preceq (1+A)\frac{(AB)^n}{1-AB}d(x_0, x_1),$$
$$d(x_{2n}, x_{2m}) \preceq (1+A)\frac{(AB)^n}{1-AB}d(x_0, x_1)$$

and

$$d(x_{2n+1}, x_{2m}) \preceq (1+B)\frac{A(AB)^n}{1-AB}.$$

Hence, for $n < m$

$$d(x_n, x_m) \preceq \max\left\{(1+B)\frac{A(AB)^n}{1-AB}, (1+A)\frac{(AB)^n}{1-AB}\right\}d(x_0, x_1) = \lambda_n d(x_0, x_1),$$

where $\lambda_n \to 0$, as $n \to \infty$.

Now, using (8) and (1) of Remark 2.3 and only the assumption that the underlying cone is solid, we conclude that $\{x_n\}$ is a Cauchy sequence. Since the space (X, d) is complete, there exists $x^* \in X$ such that $x_n \to x^*$ $(n \to \infty)$.

Suppose that, for example, f is a continuous mapping, then we have that $fx_n \to fx^*$, which (taking n even) implies that $fx^* = x^*$. Now, since $x^* \sqsubseteq x^*$, taking $x = y = x^*$ in relation (3.1), we obtain that

$$d(fx^*, gx^*) \preceq pd(x^*, x^*) + qd(x^*, fx^*) + rd(x^*, gx^*) + sd(x^*, gx^*) + td(x^*, fx^*),$$

i.e., since $fx^* = x^*$,

$$d(x^*, gx^*) \preceq (r+s)d(x^*, gx^*).$$

Since $r + s < 1$, using Remark 2.3(7), it follows that $gx^* = x^*$, and x^* is a common fixed point of f and g.

The proof is similar when g is a continuous mapping.

Consider now the case when condition (ii′) is satisfied. For the sequence $\{x_n\}$ we have $x_n \to x^* \in X$ $(n \to \infty)$ and $x_n \sqsubseteq x^*$ $(n \in \mathbb{N})$. By the construction, $fx_n \to x^*$ and $gx_n \to x^*$ $(n \to \infty)$. Let us prove that x^* is a common fixed point of f and g.

Putting $x = x^*$ and $y = x_n$ in (3.1) (since they are comparable) we get

$$d(fx^*, gx_n) \preceq pd(x^*, x_n) + qd(x^*, fx^*) + rd(x_n, gx_n) + sd(x^*, gx_n) + td(x_n, fx^*).$$

For the first and fourth term on the right-hand side we have $d(x_n, x^*) \ll c$ and $d(x^*, gx_n) \ll c$ (for $c \in \operatorname{int} P$ arbitrary and $n \geq n_0$). For the second term, $d(x^*, fx^*) \preceq d(x^*, x_n) + d(x_n, gx_n) + d(gx_n, fx^*)$ (again the first term on the right can be neglected), and for the fifth term $d(x_n, fx^*) \preceq d(x_n, gx_n) + d(gx_n, fx^*)$. It follows that

$$(1-q-t)d(fx^*, gx_n) \preceq (q+r+t)d(x_n, gx_n).$$

But, $x_n \to x^*$ and $gx_n \to x^*$ implies that $d(x_n, gx_n) \ll c$, which means that also $d(fx^*, gx_n) \ll c$, i.e. $gx_n \to fx^*$. It follows that $fx^* = x^*$ and, in a symmetric way (using that $x^* \sqsubseteq x^*$), $gx^* = x^*$. □

Remark 3.2. Theorem 1.3 is a special case of Theorem 3.1, obtained for $\alpha = p\beta = q = r$ and $\gamma = s = t$.

Now, adapting an example from [24], we give an example of the situation when Theorem 3.1 can be applied, while Theorem 1.3 cannot.

Example 3.3. Let $X = \{1, 2, 3\}$, $\sqsubseteq = \{(1, 1), (2, 2), (3, 3), (2, 3), (3, 1), (2, 1)\}$, and $d : X \times X \to C^1_{\mathbb{R}}[0, 1]$ be defined by $d(x, y)(t) = 0$ for $x = y$ and

$$d(1,2)(t) = d(2,1)(t) = 6e^t, \qquad d(1,3)(t) = d(3,1)(t) = \frac{30}{7}e^t, \qquad d(2,3) = d(3,2) = \frac{24}{7}e^t.$$

Further, let $fx = 1$, $x \in X$, and $g1 = g3 = 1$, $g2 = 3$.

We have that $d(f3, g2)(t) = d(1, 3)(t) = \frac{30}{7}e^t$. But, the right-hand side of (1.1) for $x = 3$, $y = 2$ has the form

$$\begin{aligned}&\alpha d(3,2) + \beta[d(3,f3) + d(2,g2)] + \gamma[d(3,g2) + d(2,f3)]\\ &\quad = \alpha\frac{24}{7}e^t + \beta\left(\frac{30}{7}e^t + \frac{24}{7}e^t\right) + \gamma(0 + 6e^t) = \frac{24\alpha}{7}e^t + \frac{54\beta}{7}e^t + 6\gamma e^t,\end{aligned}$$

which is less than $\frac{30}{7}e^t$ for arbitrary α, β, γ satisfying the condition $\alpha + 2\beta + 2\gamma < 1$. Indeed, $\frac{24}{7}\alpha + \frac{54}{7}\beta + 6\gamma < \frac{30}{7}$ follows from $\frac{24}{30}\alpha + \frac{54}{30}\beta + \frac{42}{30}\gamma < \alpha + 2\beta + 2\gamma < 1$.

Hence, the conditions of Theorem 1.3 are not fulfilled and this theorem cannot be used to conclude that f and g have a common fixed point.

On the other hand, taking $p = q = r = s = 0, t = \frac{5}{7}$ all the conditions of Theorem 3.1 are fulfilled. Indeed, since $f1 = g1 = f3 = g3 = 1$, we have only to check that

$$d(f3, g2)(t) \preceq 0 \cdot d(3, 2)(t) + 0 \cdot d(3, f3)(t) + 0 \cdot d(2, g2)(t) + 0 \cdot d(3, g2)(t) + \frac{5}{7} d(2, f3)(t),$$

which is equivalent to

$$\frac{30}{7}\mathrm{e}^t \preceq \frac{5}{7} d(2, f3)(t) = \frac{5}{7} d(2, 1)(t) = \frac{5}{7} \cdot 6\mathrm{e}^t = \frac{30}{7}\mathrm{e}^t.$$

Thus, we can apply Theorem 3.1 and conclude that the mappings f and g have a (unique) common fixed point $u = 1$.

The next example (where the idea is taken from [25]) shows that the condition $p+q+r+s+t < 1$ alone is not sufficient to obtain the conclusion of Theorem 3.1. We shall stay in the setting of metric spaces—it would be easy to adapt it to the setting of ordered cone metric spaces.

Example 3.4. Let $X = \{x, y, u, v\}$, where $x = (0, 0, 0), y = (4, 0, 0), u = (2, 2, 0), v = (2, -2, 1)$, and let d be the Euclidean metric in $\mathbb{R}^3$. Consider the mappings

$$f = \begin{pmatrix} x & y & u & v \\ u & v & v & u \end{pmatrix}, \qquad g = \begin{pmatrix} x & y & u & v \\ y & x & y & x \end{pmatrix}.$$

By a careful computation it is easy to obtain that

$$d(fa, gb) \le \frac{3}{4} \max\{d(a, b), d(a, fa), d(b, gb), d(a, gb), d(b, fa)\}, \tag{3.4}$$

for all $a, b \in X$. We shall show that f and g satisfy the following contractive condition: there exist $p, q, r, s, t \ge 0$ with $p + q + r + s + t < 1$ and $q \ne r, s \ne t$ such that

$$d(fa, gb) \le pd(a, b) + qd(a, fa) + rd(b, gb) + sd(a, gb) + td(b, fa) \tag{3.5}$$

holds true for all $a, b \in X$. Obviously, f and g do not have a common fixed point.

Taking (3.4) into account, we have to consider the following cases:

1° $d(fa, gb) \le \frac{3}{4} d(a, b)$. Then (3.5) holds for $p = \frac{3}{4}, r = t = 0$ and $q = s = \frac{1}{9}$.
2° $d(fa, gb) \le \frac{3}{4} d(a, fa)$. Then (3.5) holds for $q = \frac{3}{4}, p = r = t = 0$ and $s = \frac{1}{5}$.
3° $d(fa, gb) \le \frac{3}{4} d(b, gb)$. Then (3.5) holds for $r = \frac{3}{4}, p = q = t = 0$ and $s = \frac{1}{5}$.
4° $d(fa, gb) \le \frac{3}{4} d(a, gb)$. Then (3.5) holds for $s = \frac{3}{4}, p = r = t = 0$ and $q = \frac{1}{5}$.
5° $d(fa, gb) \le \frac{3}{4} d(b, fa)$. Then (3.5) holds for $t = \frac{3}{4}, p = r = s = 0$ and $q = \frac{1}{5}$.

Corollary 3.5. *Let $(X, \sqsubseteq, d)$ be an ordered complete cone metric space. Let $f : X \to X$ be a self-map such that $x \sqsubseteq fx$ for all $x \in X$. Suppose that the following conditions hold:*

(i) *there exist $p, q, r, s, t \ge 0$ satisfying $p + q + r + s + t < 1$ and $q = r$ or $s = t$, such that*

$$d(f^m x, f^n y) \preceq pd(x, y) + qd(x, f^m x) + rd(y, f^n y) + sd(x, f^n y) + td(y, f^m x)$$

for some $m, n \in \mathbb{N}$, $m \le n$ and all comparable $x, y \in X$;
(ii) *f is continuous.*

Then f has a fixed point $x^* \in X$.

Proof. Follows from Theorem 3.1 by putting $f^m \equiv f, f^n \equiv g$. □

Taking $m = n = 1$ in the previous corollary, one obtains

Corollary 3.6. *Let $(X, \sqsubseteq, d)$ be an ordered complete cone metric space. Let $f : X \to X$ be a self-map such that $x \sqsubseteq fx$ for all $x \in X$. Suppose that the following conditions hold:*

(i) *there exist $p, q, r, s, t \ge 0$ such that $p + q + r + s + t < 1$ and*

$$d(fx, fy) \preceq pd(x, y) + qd(x, fx) + rd(y, y) + sd(x, fy) + td(y, fx) \tag{3.6}$$

for all comparable $x, y \in X$;
(ii) *f is continuous.*

Then f has a fixed point $x^* \in X$.

Note that here (when just one function f is considered) there was no need for additional assumptions on coefficients p, q, r, s, t.

Remark 3.7. In the case, as in the previous corollary, when just one function f is considered, it can be easily shown that conditions (3.6) and (1.1) (in a special case when $f = g$) are equivalent. Indeed, it is enough to change places for x and y in (3.6) and after adding up both sides of two inequalities, denote $\alpha = \frac{p}{2}$, $\beta = \frac{q+r}{2}$, $\gamma = \frac{s+t}{2}$ to obtain condition (1.1). When two functions f and g enter these conditions, this procedure cannot be applied.

Remark 3.8. One possible condition that can guarantee the uniqueness of fixed point (or common fixed point) was given in [6,7]. This condition is: "every pair of elements in $(X, \sqsubseteq)$ has a lower bound and an upper bound".

4. Fixed points of quasicontractions on ordered cone metric spaces

The notion of a quasicontraction in a metric space was first used by Ćirić [18] and Das and Naik [19]. Cone metric version of this notion was considered by Ilić and Rakočević [20], as well as Kadelburg, Radenović and Rakočević [21] and Pathak and Shahzad [22]. Generalized g-quasicontractions in cone metric spaces were investigated in [23]. We shall introduce here the notion of an ordered g-quasicontraction in an ordered cone metric space and prove the respective common fixed point theorem.

Let (f, g) be a pair of self-maps on an ordered cone metric space $(X, \sqsubseteq, d)$ such that $f(X) \subset g(X)$. Let the mapping f be *g-nondecreasing*, i.e., let for each $x, y \in X$, $gx \sqsubseteq gy$ implies $fx \sqsubseteq fy$. Suppose also that there is a point $x_0 \in X$ such that $gx_0 \sqsubseteq fx_0$. Then it is possible to construct a so called *Jungck sequence* in the following way: starting with the given x_0, choose $x_1 \in X$ such that $fx_0 = gx_1$ (which is possible since $fX \subset gX$). Now it is $gx_0 \sqsubseteq gx_1$ which implies that $fx_0 \sqsubseteq fx_1$. Then there exists $x_2 \in X$ such that $fx_1 = gx_2$, and again $fx_0 \sqsubseteq fx_1$ implies that $gx_1 \sqsubseteq gx_2$ and $fx_1 \sqsubseteq fx_2$. Continuing this procedure, we obtain:

$$fx_0 \sqsubseteq fx_1 \sqsubseteq fx_2 \sqsubseteq \cdots \sqsubseteq fx_n \sqsubseteq fx_{n+1} \sqsubseteq \cdots$$

and

$$gx_1 \sqsubseteq gx_2 \sqsubseteq \cdots \sqsubseteq gx_{n+1} \sqsubseteq gx_{n+2} \sqsubseteq \cdots .$$

Definition 4.1. The mapping f is called an *ordered g-quasicontraction* if there exists $\lambda \in [0, 1/2)$ such that for each $x, y \in X$ satisfying $gy \sqsubseteq gx$, there exists

$$u \in M_0^{f,g}(x, y) = \{d(gx, gy), d(gx, fx), d(gy, fy), d(gx, fy), d(gy, fx)\}, \tag{4.1}$$

such that $d(fx, fy) \preceq \lambda \cdot u$ holds.

Theorem 4.2. *Let (f, g) be a pair of self-maps on a complete ordered cone metric space $(X, \sqsubseteq, d)$ such that $f(X) \subset g(X)$ and such that there is a point $x_0 \in X$ with $gx_0 \sqsubseteq fx_0$. Suppose that*

(i) *f is an ordered g-quasicontraction;*
(ii) *$g(X)$ is closed in X;*
(iii) *f is g-nondecreasing;*
(iv) *if $\{g(x_n)\} \subset X$ is a nondecreasing sequence, converging to some gz, then $gx_n \sqsubseteq gz$ and $gz \sqsubseteq ggz$.*

Then f and g have a coincidence point, i.e., there exists $z \in X$ such that $fz = gz$.

If, further, f and g are weakly compatible, then they have a common fixed point.

Recall (see [26,27]) that the mappings f and g are said to be *weakly compatible* if, for each $x \in X$, $fx = gx$ implies $fgx = gfx$.

Proof. Starting with the given x_0 construct the Jungck sequence $fx_{n-1} = gx_n$ of the pair (f, g), with the initial point x_0. We shall prove that it is a Cauchy sequence in X.

Let us prove first that

$$d(fx_n, fx_{n+1}) \preceq \frac{\lambda}{1-\lambda} d(fx_{n-1}, fx_n) \tag{4.2}$$

for all $n \geq 1$. Indeed, since $gx_n \sqsubseteq gx_{n+1}$, we can apply condition (i) to obtain

$$d(fx_n, fx_{n+1}) \preceq \lambda u_n, \tag{4.3}$$

where

$$\begin{aligned} u_n &\in \{d(gx_n, gx_{n+1}), d(gx_n, fx_n), d(gx_{n+1}, fx_{n+1}), d(gx_n, fx_{n+1}), d(gx_{n+1}, fx_n)\} \\ &= \{d(fx_{n-1}, fx_n), d(fx_n, fx_{n+1}), d(fx_{n-1}, fx_{n+1}), 0\} . \end{aligned}$$

There are four possible cases:

1° $d(fx_n, fx_{n+1}) \preceq \lambda d(fx_{n-1}, fx_n) \preceq \frac{\lambda}{1-\lambda} d(fx_{n-1}, fx_n)$ since $\lambda \leq \frac{\lambda}{1-\lambda}$;
2° $d(fx_n, fx_{n+1}) \preceq \lambda d(fx_n, fx_{n+1})$; it follows that $d(fx_n, fx_{n+1}) = 0$. Hence, (4.2) holds true;
3° $d(fx_n, fx_{n+1}) \preceq \lambda d(fx_{n-1}, fx_{n+1}) \preceq \lambda d(fx_{n-1}, fx_n) + \lambda d(fx_n, fx_{n+1})$; hence, (4.2) holds true;
4° $d(fx_n, fx_{n+1}) \preceq \lambda \cdot 0 = 0$ and so $d(fx_n, fx_{n+1}) = 0$ and again (4.2) holds.

Put $h = \frac{\lambda}{1-\lambda}$. Then it follows from (4.2) that

$$d(fx_n, fx_{n+1}) \preceq hd(fx_{n-1}, fx_n) \preceq \cdots \preceq h^n d(fx_0, fx_1),$$

for all $n \geq 1$. Now we have for $m, n \in \mathbb{N}$, $n > m$ that

$$\begin{aligned} d(fx_n, fx_m) &\preceq d(fx_n, fx_{n-1}) + d(fx_{n-1}, fx_{n-2}) + \cdots + d(fx_{m+1}, fx_m) \\ &\preceq (h^{n-1} + h^{n-2} + \cdots + h^m) d(fx_0, fx_1) \\ &\preceq \frac{h^m}{1-h} d(fx_0, fx_1) \to 0 \quad \text{as } m \to \infty. \end{aligned}$$

According to Remark 2.3, (1) and (8), $\{fx_n\}$, i.e., $\{gx_n\}$ is a Cauchy sequence and, since X is complete and gX is closed, there exists $z \in X$ such that

$$gx_n \to gz \quad \text{i.e., } fx_n \to gz \quad \text{as } n \to \infty.$$

We will prove that $fz = gz$.

Since $gx_n \sqsubseteq gz$ (condition (iv)) putting $x = x_n$, $y = z$ in (4.1), we get

$$d(fx_n, fz) \preceq \lambda \cdot u_n \tag{4.4}$$

where $u_n \in \{d(gx_n, gz), d(gx_n, fx_n), \underline{d(gz, fz)}, d(gz, fx_n), \underline{d(gx_n, fz)}\}$. Observe that $\underline{d(gz, fz)} \preceq d(gz, fx_n) + d(fx_n, fz)$ and $\underline{d(gx_n, fz)} \preceq d(gx_n, fx_n) + d(fx_n, fz)$. Now let $0 \ll c$ be given. In all of the possible five cases there exists $n_0 \in \mathbb{N}$ such that (using (4.4)) one obtains that $d(fx_n, fz) \ll c$:

1° $d(fx_n, fz) \preceq \lambda \cdot d(gx_n, gz) \ll \lambda \frac{c}{\lambda} = c$;
2° $d(fx_n, fz) \preceq \lambda \cdot d(gx_n, fx_n) \ll \lambda \frac{c}{\lambda} = c$;
3° $d(fx_n, fz) \preceq \lambda \cdot d(gz, fz) \preceq \lambda d(gz, fx_n) + \lambda d(fx_n, fz)$; it follows that $d(fx_n, fz) \preceq \frac{\lambda}{1-\lambda} d(gz, fx_n) \ll \frac{\lambda}{1-\lambda} \frac{(1-\lambda)c}{\lambda} = c$;
4° $d(fx_n, fz) \preceq \lambda \cdot d(gz, fx_n) \ll \lambda \frac{c}{\lambda} = c$;
5° $d(fx_n, fz) \preceq \lambda \cdot d(gx_n, fz) \preceq \lambda d(gx_n, fx_n) + \lambda d(fx_n, fz)$; it follows that $d(fx_n, fz) \preceq \frac{\lambda}{1-\lambda} d(gx_n, fx_n) \ll \frac{\lambda}{1-\lambda} \frac{(1-\lambda)c}{\lambda} = c$.

It follows that $fx_n \to fz$ ($n \to \infty$). The uniqueness of limit in a cone metric space implies that $fz = gz = t$. Thus, in the terminology of [28], z is a coincidence point of the pair (f, g), and t is a point of coincidence.

Suppose now that f and g are weakly compatible. By the assumption (iv), $gz \sqsubseteq ggz$ and hence we obtain that

$$fgz = gfz = ffz = ggz.$$

Suppose that it is not $fz = ffz$. Then, the contractibility condition (4.1) for $x = z$, $y = fz$ implies that

$$d(fx, fy) = d(fz, ffz) \preceq \lambda u,$$

where

$$\begin{aligned} u &\in \{d(gz, gfz), d(gz, fz), d(gfz, ffz), d(gfz, fz), d(gz, ffz)\} \\ &= \{d(fz, ffz), 0, d(ffz, ffz), d(ffz, fz), d(fz, ffz)\} \\ &= \{0, d(fz, ffz)\}, \end{aligned}$$

so we have only two possibilities:

1° $d(fz, ffz) \preceq \lambda \cdot 0 = 0 \Rightarrow d(fz, ffz) = 0 \Rightarrow fz = ffz$;
2° $d(fz, ffz) \preceq \lambda d(fz, ffz) \Rightarrow$ (by Remark 2.3) $d(fz, ffz) = 0$, i.e., $fz = ffz$.

In other words, $fz = gz$ is a common fixed point of the mappings f and g. □

Taking $g = i_X$ (the identity function) in Theorem 4.2 we obtain a result for ordered quasicontractions in ordered cone metric spaces.

Corollary 4.3. *Let f be a self-map on a complete ordered cone metric space $(X, \sqsubseteq, d)$ such that there is a point $x_0 \in X$ with $x_0 \sqsubseteq fx_0$. Suppose that*

(i) *f is an ordered quasicontraction, i.e., there exists $\lambda \in [0, 1/2)$ such that for each $x, y \in X$ satisfying $y \sqsubseteq x$, there exists*

$$u \in \{d(x, y), d(x, fx), d(y, fy), d(x, fy), d(y, fx)\}, \tag{4.5}$$

such that $d(fx, fy) \preceq \lambda \cdot u$ holds;

(ii) f *is nondecreasing;*
(iii) *if* $\{x_n\} \subset X$ *is a nondecreasing sequence, converging to some* z*, then* $x_n \sqsubseteq z$.

Then f *has a fixed point in* X.

Remark 4.4. If, in the Definition 4.1 of an ordered g-quasicontraction, we use the set

$$\{d(gx, gy), d(gx, fx), d(gy, fy)\},$$

instead of $M_0^{f,g}(x, y)$, then it can be proved in a similar way that Theorem 4.2 holds even with $\lambda \in [0, 1)$.

If we further reduce this set to $\{d(gx, fx), d(gy, fy)\}$, then an ordered version of the known Bianchini's result [29, (5)] is obtained.

Finally, if we take a singleton $\{d(gx, gy)\}$, we obtain an ordered version of a result of Jungck which is a direct generalization of the Banach's principle.

In the sequel, we shall modify condition of ordered g-quasicontraction by considering, together with $M_0^{f,g}(x, y)$ (see (4.1)) the following sets:

$$M_1^{f,g}(x, y) = \left\{d(gx, gy), d(gx, fx), d(gy, fy), \frac{d(gx, fy) + d(gy, fx)}{2}\right\},$$
$$M_2^{f,g}(x, y) = \left\{d(gx, gy), \frac{d(gx, fx) + d(gy, fy)}{2}, \frac{d(gx, fy) + d(gy, fx)}{2}\right\}.$$

Sets of this kind were used in several papers to introduce contractive-type conditions. In the setting of cone metric spaces, they were used, for example, in [30,31] (where non-self-mappings were considered) and in [32] (when considering strict contractive conditions). We shall prove here two related results in the setting of ordered cone metric spaces.

Theorem 4.5. *Let* (f, g) *be a pair of self-maps on a complete ordered cone metric space* $(X, \sqsubseteq, d)$ *such that* $f(X) \subset g(X)$ *and such that there is a point* $x_0 \in X$ *with* $gx_0 \sqsubseteq fx_0$. *Suppose that*

(i) *there exists* $\lambda \in [0, 1)$ *such that for each* $x, y \in X$ *satisfying* $gy \sqsubseteq gx$*, there exists*

$$u \in M_1^{f,g}(x, y),$$

such that $d(fx, fy) \preceq \lambda \cdot u$ *holds.*
(ii) $g(X)$ *is closed in* X;
(iii) f *is* g*-nondecreasing;*
(iv) *if* $\{g(x_n)\} \subset X$ *is a nondecreasing sequence, converging to some* gz*, then* $gx_n \sqsubseteq gz$ *and* $gz \sqsubseteq ggz$.

Then f *and* g *have a coincidence point.*

Moreover, if f *and* g *are weakly compatible, then they have a common fixed point.*

Proof. Starting from the given x_0, construct the Jungck sequence as in the proof of Theorem 4.2:

$$fx_0 \sqsubseteq fx_1 \sqsubseteq fx_2 \sqsubseteq \cdots \sqsubseteq fx_n \sqsubseteq fx_{n+1} \sqsubseteq \cdots,$$
$$gx_1 \sqsubseteq gx_2 \sqsubseteq \cdots \sqsubseteq gx_{n+1} \sqsubseteq gx_{n+2} \sqsubseteq \cdots.$$

First we prove that

$$d(fx_n, fx_{n+1}) \preceq \lambda d(fx_{n-1}, fx_n) \quad \text{for } n \geq 1. \tag{4.6}$$

Since $gx_n \sqsubseteq gx_{n+1}$, it is

$$d(fx_n, fx_{n+1}) \preceq \lambda \cdot u,$$

where

$$u \in \left\{d(gx_n, gx_{n+1}), d(fx_n, gx_n), d(fx_{n+1}, gx_{n+1}), \frac{d(fx_n, gx_{n+1}) + d(fx_{n+1}, gx_n)}{2}\right\}$$
$$= \left\{d(fx_{n-1}, fx_n), d(fx_n, fx_{n+1}), \frac{d(fx_{n-1}, fx_{n+1})}{2}\right\}.$$

Now we have to consider the following three cases.

1° If $u = d(fx_{n-1}, fx_n)$ then clearly (4.6) holds.
2° If $u = d(fx_n, fx_{n+1})$ then according to Remark 2.3(7) $d(fx_n, fx_{n+1}) = 0$, and (4.6) is immediate.

3° Finally, suppose $u = \frac{d(fx_{n-1},fx_{n+1})}{2}$. Now

$$d(fx_n, fx_{n+1}) \preceq \lambda \frac{d(fx_{n-1}, fx_{n+1})}{2} \preceq \frac{\lambda}{2} d(fx_{n-1}, fx_n) + \frac{1}{2}(fx_n, fx_{n+1}).$$

Hence $d(fx_n, fx_{n+1}) \preceq \lambda d(fx_{n-1}, fx_n)$, and we have proved (4.6).

Now, we have

$$d(f_n, fx_{n+1}) \preceq \lambda^n d(fx_0, fx_1).$$

We shall show that $\{f_n\}$ is a Cauchy sequence. For $m, n \in \mathbb{N}$, $n > m$ we have

$$d(fx_n, fx_m) \preceq d(fx_n, fx_{n-1}) + d(fx_{n-1}, fx_{n-2}) + \cdots + d(fx_{m+1}, fx_m),$$

and we obtain

$$\begin{aligned} d(fx_n, fx_m) &\preceq \left(\lambda^{n-1} + \lambda^{n-2} + \cdots + \lambda^m\right) d(fx_0, fx_0) \\ &\preceq \frac{\lambda^m}{1-\lambda} d(fx_0, fx_1) \to 0 \text{ as } m \to \infty. \end{aligned}$$

From Remark 2.3(8) it follows that for $0 \ll c$ and m sufficiently large, $\lambda^m(1-\lambda)^{-1} d(fx_0, fx_1) \ll c$; then also $d(fx_n, fx_m) \ll c$. Hence, $\{fx_n\}$ is a Cauchy sequence.

Since $f(X) \subset g(X)$, $g(X)$ is closed, and X is complete, there exists $u \in g(X)$ such that $gx_n \to u$ as $n \to \infty$. Consequently, we can find $z \in X$ such that $gz = u$.

Let us show that $fz = u$. For this we have (because of $gx_n \sqsubseteq gz$)

$$d(fz, u) \preceq d(fz, fx_n) + d(fx_n, u) \preceq \lambda \cdot u_n + d(fx_n, u),$$

where

$$u_n \in \left\{ d(gx_n, gz), d(fx_n, gx_n), d(fz, gz), \frac{d(fx_n, gz) + d(fz, gz)}{2} \right\}.$$

Let $0 \ll c$ be given. Since $gx_n \to gz$, in each of the following cases there exists n_0 such that for $n \geq n_0$ we have $d(fz, u) \ll c$.

1° $d(fz, u) \preceq \lambda \cdot d(gx_n, gz) + d(fx_n, u) \ll \lambda \cdot \frac{c}{2\lambda} + \frac{c}{2} = c$.

2° $d(fz, u) \preceq \lambda \cdot d(fx_n, gx_n) + d(fx_n, u) \preceq \lambda \cdot d(fx_n, u) + \lambda \cdot d(u, gx_n) + d(fx_n, u) = (\lambda + 1) \cdot d(fx_n, u) + \lambda \cdot d(u, gx_n) \ll (\lambda + 1) \cdot \frac{c}{2(\lambda+1)} + \lambda \cdot \frac{c}{2\lambda} = c$.

3° $d(fz, u) \preceq \lambda \cdot d(fz, u) + d(fx_n, u)$, i.e., $d(fz, u) \ll \frac{1}{1-\lambda} \cdot (1 - \lambda)c = c$.

4° $d(fz, u) \preceq \lambda \cdot \frac{d(fx_n,gz)+d(fz,gz)}{2} + d(fx_n, u) \preceq \frac{\lambda d(fx_n,gz)}{2} + \frac{1}{2} d(fz, gz) + d(fx_n, u)$, i.e., $d(fz, u) \preceq (\lambda + 2) d(fx_n, u) \ll (\lambda + 2) \frac{c}{(\lambda+2)} = c$.

Using Remark 2.3(2) we conclude that $d(fz, u) = 0$, i.e., $fz = u$.

Hence, we have proved that f and g have a coincidence point $z \in X$ and a point of coincidence $u \in X$ such that $u = f(z) = g(z)$. If they are weakly compatible, then

$$ggz = gfz = fgz = ffz.$$

We shall prove that $fz = gz$ is a common fixed point of the mappings f and g. Using $gz \sqsubseteq ggz$ (condition (iv)), we obtain from condition (i) that

$$d(fz, ffz) \preceq \lambda \cdot u,$$

where

$$\begin{aligned} u \in &\left\{ d(gz, gfz), d(fz, gz), d(ffz, gfz), \frac{d(fz, gfz) + d(ffz, gz)}{2} \right\} \\ = &\left\{ d(fz, ffz), 0, \frac{d(fz, ffz) + d(ffz, fz)}{2} \right\} = \{0, d(fz, ffz)\}. \end{aligned}$$

Hence, by Remark 2.3, $d(fz, ffz) = 0$, i.e., $fz = ffz$. Similarly, $gz = ggz$ and the theorem is proved. □

Theorem 4.6. *Let (f, g) be a pair of self-maps on a complete ordered cone metric space $(X, \sqsubseteq, d)$ such that $f(X) \subset g(X)$ and such that there is a point $x_0 \in X$ with $gx_0 \sqsubseteq fx_0$. Suppose that*

(i) *there exists $\lambda \in [0, 1)$ such that for each $x, y \in X$ satisfying $gy \sqsubseteq gx$, there exists*

$$u \in M_2^{f,g}(x, y),$$

such that $d(fx, fy) \preceq \lambda \cdot u$ holds.

(ii) *$g(X)$ is closed in X;*
(iii) *f is g-nondecreasing;*
(iv) *if $\{g(x_n)\} \subset X$ is a nondecreasing sequence, converging to some gz, then $gx_n \sqsubseteq gz$ and $gz \sqsubseteq ggz$.*

Then f and g have a coincidence point.

Moreover, if f and g are weakly compatible, then they have a common fixed point.

The proof is similar, and so is omitted.

Note that conditions (i) of Theorems 4.2, 4.5 and 4.6 are incomparable in the cone metric settings (to the contrary with the situation in metric settings), since for $a, b \in P$, if a and b are incomparable, then also $\frac{a+b}{2}$ is incomparable, both with a and with b.

Remark 4.7. Putting $E = \mathbb{R}$, $P = [0, +\infty)$ in Theorems 4.5 and 4.6, one obtains the respective common fixed point theorems in ordered metric spaces (we could not find explicit formulations for some of these assertions in literature).

For example, taking $u = d(gx, gy)$, $g = i_X$, a result of Abbas and Jungck from [28] is obtained; then, taking $E = \mathbb{R}$, $P = [0, +\infty)$ the respective result in the setting of ordered metric spaces follows. If we take $u = \frac{1}{2}(d(gx, fx) + d(gy, fy))$, $g = i_X$, we obtain an ordered cone metric version of Kannan's Theorem [29, (4)]; again, ordered metric version of this theorem follows immediately. The same applies for the known Zamfirescu's result [29, (19)].

We conclude with an example showing that our Theorems 4.2, 4.5 and 4.6 are proper extensions of the respective results from the setting of cone metric spaces. Namely, we shall construct an example of a mapping which is an ordered g-quasicontraction (wherefrom the existence of common fixed point of f and g follows), while it is not a g-quasicontraction in cone metric sense. Similar conclusion then applies for relationship between contractive conditions in ordered metric spaces and simple metric spaces.

Example 4.8. Let $X = [0, +\infty)$ and let order relation $\sqsubseteq$ be defined by

$$x \sqsubseteq y \Leftrightarrow \{(x = y) \text{ or } (x, y \in [0, 1] \text{ with } x \le y)\}.$$

Let $E = C^1_{\mathbb{R}}[0, 1]$ with $\|x\| = \|x\|_\infty + \|x'\|_\infty$ and $P = \{x \in E : x(t) \ge 0 \text{ on } [0, 1]\}$ (this cone is not normal). Define $d : X \times X \to E$ by $d(x, y) = |x - y|\,\varphi$ where $\varphi : [0, 1] \to \mathbb{R}$ such that $\varphi(t) = e^t$. It is easy to see that d is a cone metric on X. Consider the mappings

$$fx = \begin{cases} \dfrac{x}{4}, & 0 \le x \le 1, \\ 4x - \dfrac{15}{4}, & x > 1; \end{cases} \qquad gx = \begin{cases} x, & 0 \le x \le 1, \\ \dfrac{3}{4}x, & x > 1. \end{cases}$$

Then, for $y \sqsubseteq x$ we have that

$$d(fx, fy)(t) = |fx - fy|\,e^t = \frac{1}{4}|x - y|\,e^t \le \lambda\,|x - y|\,e^t, \quad \forall t \in [0, 1] \Leftrightarrow \lambda \in \left[\frac{1}{4}, 1\right),$$

while for $x, y > 1$

$$d(fx, fy)(t) = |fx - fy|\,e^t = 4\,|x - y|\,e^t \le \lambda\frac{3}{4}\,|x - y|\,e^t, \quad \forall t \in [0, 1] \Leftrightarrow \lambda \in \left[\frac{16}{3}, +\infty\right),$$

and, checking all other conditions, one concludes that f is an ordered g-quasicontraction, while it is not a g-quasicontraction in a (non-ordered) cone metric sense. Obviously, $f(0) = g(0) = 0$.

Similar conclusions apply to conditions of Theorems 4.5 and 4.6.

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