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The local Hölder function of a continuous function

Stéphane Seuret and Jacques Lévy Véhel

Projet Fractales, INRIA Rocquencourt, BP 105, 78153 Le Chesnay cedex, France

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Abstract

This work focuses on the local Hölder exponent as a measure of the regularity of a function around a given point. We investigate in detail the structure and the main properties of the local Hölder function (i.e., the function that associates to each point its local Hölder exponent). We prove that it is possible to construct a continuous function with prescribed local *and* pointwise Hölder functions outside a set of Hausdorff dimension 0.

1. Introduction

There exist various ways to measure the regularity of a function around a given point. The most popular one is to use the pointwise Hölder exponent (hereafter denoted α_p), but other characterizations of local regularity exist. These include the local Hölder exponent, the chirp and oscillation exponents, the local box and Hausdorff dimensions and the degree of fractional differentiability. We shall mainly be concerned in this paper with the study of the local Hölder exponent and the local Hölder function, i.e., the function that associates to each point its local Hölder exponent.

There are several motivations for investigating the local Hölder exponent. First, this exponent is computed through a localization of the global Hölder exponent, and is thus perhaps the most natural exponent in the list above.

Another obvious reason for introducing regularity exponents other than α_p is that the knowledge of the sole pointwise Hölder exponent does not provide a full description of the regularity of a function. For instance the *cusp* function $x \to |x|^\gamma$ and the *chirp* function $x \to |x|^\gamma \sin(1/|x|^\beta)$, where γ and β are positive reals, have the same pointwise Hölder exponent at 0, namely γ. However, they have strongly different behaviours around 0. In these cases, the local Hölder exponents α_l are respectively γ and $\gamma/(1+\beta)$. The lower value of α_l for the chirp function gives a clue about the oscillatory behaviour of the function around 0.

A further advantage of the local Hölder exponent over the pointwise exponent is that α_l is stable through fractional integro-differentiation, while α_p is not. This means for instance that the following equality always holds: $\alpha_l^F = \alpha_l^f + 1$, where α_l^F is the local exponent of a primitive F of f. In contrast, one can only ensure in general that $\alpha_p^F \geqslant \alpha_p^f + 1$.

E-mail addresses: stephane.seuret@inria.fr (S. Seuret), jacques.levy_vehel@inria.fr (J. Lévy Véhel).

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From a practical point of view, most methods for estimating α_{p} make implicitly or explicitly the assumption that $\alpha_{\mathrm{p}} = \alpha_{\mathrm{l}}$. It is thus of interest to investigate the domain of validity of this equality.

Finally, in many application, the local Hölder exponent and its evolution in "time" are a relevant tool for characterizing or processing signals (see, for instance, [8]).

While the main properties of the pointwise Hölder function have already been investigated, no such study has been conducted yet for the local one. We prove in this paper that the class of local Hölder functions of continuous functions is exactly the one of non-negative lower semicontinuous functions. The next natural question consists in determining the exact links between the two Hölder-based regularity characterizations, i.e., the pointwise and local one. In other words, we want to answer the following question: to what extent can one prescribe independently the pointwise and local Hölder functions of a continuous function? We show that any couple of functions (f, g) such that $f \leqslant g$, and f (respectively g) belongs to the class of local (respectively pointwise) Hölder functions can be jointly the local and pointwise Hölder functions of a continuous function except on a set of Hausdorff dimension 0 (see Theorem 4.1 for a precise statement).

In Section 2, we recall the definition and main properties of the pointwise exponent, and we start studying the local one. In Section 3, we give the structure of local Hölder functions. We provide various comparisons between the exponents in Section 4. Section 5 is devoted to the construction of a continuous function with prescribed local and pointwise Hölder functions.

2. Definitions of the exponents

We recall in this section the definitions of the two regularity exponents we are interested in. The first one, the pointwise Hölder exponent, is well-known. The second one is the local Hölder exponent. We give a slightly enhanced definition of this exponent (as compared to the one in [4]), and investigate its basic properties.

2.1. Pointwise Hölder exponent

Definition 2.1. Let $f:\mathbb{R} \to \mathbb{R}$ be a function, $s > 0$, $s \notin \mathbb{N}$, and $x_0 \in \mathbb{R}$. Then $f \in C^s(x_0)$ if and only if there exists a real $\eta > 0$, a polynomial P with degree less than $[s]$ and a constant C such that

$$\forall x \in B(x_0, \eta), \quad \bigl|f(x) - P(x - x_0)\bigr| \leqslant C|x - x_0|^s. \tag{1}$$

By definition, the *pointwise Hölder* exponent of f at x_0, denoted by $\alpha_{\mathrm{p}}(x_0)$, is $\sup\{s\colon\ f \in C^s(x_0)\}$.

The following wavelet characterization of this exponent, due to Jaffard [7], will be useful in the sequel by the following proposition.

Proposition 2.1. *Assume that $f \in C^{\alpha}(x_0)$. If $|k2^{-j} - x_0| \leqslant 1/2$, then*

$$|d_{j,k}| \leqslant C2^{-\alpha j}\bigl(1 + 2^j\bigl|k2^{-j} - x_0\bigr|\bigr)^{\alpha}. \tag{2}$$

Conversely, if (2) holds for all (j, k)'s such that $|k2^{-j} - x_0| \leqslant 2^{-j/(\log j)^2}$, and if $f \in C^{\log}$, then there exist a constant C and a polynomial P of degree at most $[\alpha]$ such that

$$\bigl|f(x) - P(x - x_0)\bigr| \leqslant C|x - x_0|^{\alpha}\bigl(\log\bigl(|x - x_0|\bigr)\bigr)^2. \tag{3}$$

$C^{\log}$ is the class of functions f whose wavelet coefficients verify

$$|d_{j,k}| \leqslant C2^{-j/\log j}.$$

This regularity condition is stronger than uniform continuity, but does not imply a uniform Hölder continuity.

2.2. Local Hölder exponent

Let $f:\Omega\to\mathbb{R}$ be a function, where $\Omega\subset\mathbb{R}$ an open set. One classically says that $f\in C_1^s(\Omega)$ where $0<s<1$ if there exists a constant C such that, for all x, y in Ω,

$$\bigl|f(x)-f(y)\bigr|\leqslant C|x-y|^s. \tag{4}$$

If $m<s<m+1$ ($m\in\mathbb{N}$), then $f\in C_1^s(\Omega)$ means that there exists a constant C such that, for all x, y in Ω,

$$\bigl|\partial^m f(x)-\partial^m f(y)\bigr|\leqslant C|x-y|^{s-m}.$$

Set now $\alpha_1(\Omega)=\sup\{s\colon\ f\in C_1^s(\Omega)\}$. Remark that, if $\Omega'\subset\Omega$, $\alpha_1(\Omega')\geqslant\alpha_1(\Omega)$. We will use the following lemma to define the local Hölder exponent.

Lemma 2.1. *Let $(O_i)_{i\in I}$ be a family of decreasing open sets (i.e., $O_i\subset O_j$ if $i>j$), such that*

$$\bigcap_i O_i=\{x_0\}.$$

Set

$$\alpha_1(x_0)=\sup\bigl\{\alpha_1(O_i)\colon\ i\in I\bigr\}. \tag{5}$$

Then $\alpha_1(x_0)$ does not depend on the choice of the family $(O_i)_{i\in I}$.

Proof. Let $(O_i)_{i\in I}$ and $(\widetilde{O}_i)_{i\in I}$ be two families of sets satisfying the above conditions, and let us define the two corresponding exponents

$$\alpha_1(x_0)=\sup\bigl\{\alpha_1(O_i)\colon\ i\in I\bigr\},\qquad \tilde\alpha_1(x_0)=\sup\bigl\{\alpha_1(\widetilde{O}_i)\colon\ i\in I\bigr\}.$$

Assume that, for example, $\alpha_1(x_0)>\tilde\alpha_1(x_0)$. Then there exists an integer i_0 such that $\alpha_1(O_i)>\tilde\alpha_1(x_0)$. Since the $(\widetilde{O}_i)_{i\in I}$ are decreasing, and using that $\bigcap_i\widetilde{O}_i=\{x_0\}$, there exists another integer $i_1>i_0$ such that $\widetilde{O}_{i_1}\subset O_{i_0}$.

Then $\tilde\alpha_1(x_0)>\alpha_1(\widetilde{O}_{i_1})\geqslant\alpha_1(\widetilde{O}_{i_0})$, which gives a contradiction. □

Since α_1 is independent of the choice of the family $\{O_i\}_i$, we shall define the local Hölder exponent using a sequence of intervals containing x_0.

Definition 2.2. Let f be a function defined on a neighborhood of x_0. Let $\{I_n\}_{n\in\mathbb{N}}$ be a sequence of open decreasing intervals converging to x_0. The local Hölder exponent of the function f at x_0, denoted by $\alpha_1(x_0)$, is

$$\alpha_1(x_0)=\sup_{n\in\mathbb{N}}\alpha_1(I_n)=\lim_{n\to+\infty}\alpha_1(I_n). \tag{6}$$

It is straightforward to prove that one always has $\alpha_1(x_0)\leqslant\alpha_p(x_0)$.

It is also easy to obtain a wavelet characterization of $\alpha_1(x)$, which will be a simple consequence of the following classical proposition [10].

Proposition 2.2. *Let $x_0\in\mathbb{R}$ and $\eta>0$. Then $f\in C_1^s(B(x_0,\eta))$ if and only if there exists a constant C, such that for all (j,k) such that $k2^{-j}\in B(x_0,\eta)$, one has $|d_{j,k}|\leqslant C2^{-sj}$.*

The last proposition leads to the following characterization.

Proposition 2.3.

$$\alpha_1(x_0)=\lim_{n\to 0}\bigl(\sup\bigl\{s\colon\ \exists C,\ k2^{-j}\in B(x_0,\eta)\Rightarrow|d_{j,k}|\leqslant C2^{-sj}\bigr\}\bigr). \tag{7}$$

Proof. The proof is straightforward using the characterization provided by Proposition 2.2. □

Remark 2.1. When dealing with compactly supported functions, one can assume that compactly supported wavelet, like the Daubechies ones, for example, [2], are used.

3. The structure of Hölder functions

One can associate to each x its pointwise Hölder exponent $\alpha_{\mathrm{p}}(x)$. This defines a function $x \to \alpha_{\mathrm{p}}(x)$, called the pointwise Hölder function of f. A natural question is to investigate the structure of the functions $\alpha_{\mathrm{p}}(x)$ when f spans the set of continuous functions. The answer is given by the following theorem [1].

Theorem 3.1. *Let $g:\mathbb{R} \to \mathbb{R}^+$ be a function. The two following properties are equivalent*:

- *g is a liminf of a sequence of continuous functions*;
- *There exists a continuous function f such that the pointwise Hölder function of f, $\alpha_{\mathrm{p}}(x)$ satisfies $\alpha_{\mathrm{p}}(x) = g(x)$, $\forall x$.*

As in the case of the pointwise exponent, one can associate to each x the local exponent of f at x. This defines a local Hölder function $x \to \alpha_{\mathrm{l}}(x)$. The structure of local Hölder functions is more constrained than the one of pointwise Hölder functions, since the former must be lower semicontinuous functions [4]. More precisely, we have the following theorem.

Theorem 3.2. *Let $g:\mathbb{R} \to \mathbb{R}^+$ be a function. The two following properties are equivalent*:

- *g is a non-negative lower semicontinuous* (*lsc*) *function.*
- *There exists a continuous function f such that the local Hölder function of f, $\alpha_{\mathrm{l}}(x)$, satisfies $\alpha_{\mathrm{l}}(x) = g(x)$, $\forall x$.*

Proof. From the definition of $\alpha_{\mathrm{l}}(x_0)$, for all $\epsilon > 0$, there exists an interval I_ϵ containing x_0 such that

$$\alpha_{\mathrm{l}}(I_\epsilon) > \alpha_{\mathrm{l}}(x_0) - \epsilon.$$

Then, using the definition of $\alpha_{\mathrm{l}}(y)$ for every $y \in I_\epsilon$, one concludes that

$$\forall y \in I_\epsilon, \quad \alpha_{\mathrm{l}}(y) \geqslant \alpha_{\mathrm{l}}(I_\epsilon) \geqslant \alpha_{\mathrm{l}}(x_0) - \epsilon.$$

This exactly shows that $x \to \alpha_{\mathrm{l}}(x)$ is an lsc function. Obviously, the continuity of f entails $\alpha_{\mathrm{l}} \geqslant 0$.

That the converse property holds, i.e., any non-negative lsc function is the local Hölder function of a continuous function $f:\mathbb{R} \to \mathbb{R}$, will be a consequence of Theorem 4.1. □

Now that we have discussed the structures of both α_{l} and α_{p}, we proceed to examine the relation between them.

4. Relations between α_{l} and α_{p}

We start with two simple general bounds.

Proposition 4.1. *Let $f:I \to \mathbb{R}$ be a continuous function* (*I is an interval of $\mathbb{R}$*). *Let α_{p} and α_{l} be respectively its pointwise and local Hölder functions. Then, $\forall x \in I$,*

$$\alpha_{\mathrm{l}}(x) \leqslant \min\Big(\alpha_{\mathrm{p}}(x), \liminf_{t\to x} \alpha_{\mathrm{p}}(x)\Big). \tag{8}$$

Proof. We give the proof in the case $\alpha_{\mathrm{p}} < 1$.

By definition, $\forall \epsilon$, there exists a constant C such that, for t close enough to x, $|f(t) - f(x)| \leqslant C|t - x|^{\alpha_{\mathrm{p}}(x)-\epsilon}$. Comparing this to the definition of $\alpha_{\mathrm{p}}(x)$, one deduces that $\alpha_{\mathrm{l}}(x) \leqslant \alpha_{\mathrm{l}}(x) - \epsilon$, $\forall \epsilon$, hence $\alpha_{\mathrm{l}}(x) \leqslant \alpha_{\mathrm{p}}(x)$.

On the other hand, for every $\eta > 0$, $\forall y \in B(x,\eta)$, one has $\alpha_{\mathrm{l}}(B(x,\eta)) \leqslant \alpha_{\mathrm{p}}(y)$. Combining this with the fact that $\alpha_{\mathrm{l}}(x) = \lim_{\eta\to 0} \alpha_{\mathrm{l}}(B(x,\eta))$, one obtains that $\alpha_{\mathrm{l}}(x) \leqslant \liminf_{t\to x} \alpha_{\mathrm{p}}(t)$. □

Proposition 4.2. *Let $f : I \to \mathbb{R}$ be a continuous function (I is an interval of $\mathbb{R}$). If there exists α such that $\{x\colon \alpha_{\mathrm{p}}(x) = \alpha\}$ is dense around x_0, then $\alpha_{\mathrm{l}}(x_0) \leqslant \alpha$.*

Proof. The proof is straightforward using Proposition 4.1. □

This proposition has an important consequence in multifractal analysis: "multifractal" functions, as IFS (see below and [1]) or repartition functions of multinomial measures [3], usually have the property that, for all α, $E_\alpha = \{x\colon \alpha_{\mathrm{p}}(x) = \alpha\}$ is either dense on the support of the function or empty. For functions of this kind, α_{l} is constant. A consequence is that it is not interesting in general to base a multifractal analysis on the local Hölder exponent, since the corresponding spectrum would be degenerate.

Let us now make a few remarks that go against some common thoughts about the relation between local and pointwise Hölder exponents.

- $x \to \alpha_{\mathrm{p}}(x)$ is a continuous function does not imply that $\alpha_{\mathrm{l}}(x) = \alpha_{\mathrm{p}}(x)$ for every x. For a counter-example, consider the sum of a Weierstrass function with pointwise exponent α and a chirp (α, β) at 0, where $\beta < \alpha$. Then $\alpha_{\mathrm{l}}(x) = \alpha_{\mathrm{p}}(x) = \alpha$ for all $x \neq 0$, and $a_{\mathrm{p}}(0) = \alpha$ while $\alpha_{\mathrm{l}}(0) = \beta < \alpha$.
- The converse proposition is also false: $x \to \alpha_{\mathrm{l}}(x)$ is a continuous function does not imply that $\alpha_{\mathrm{l}}(x) = \alpha_{\mathrm{p}}(x)$ for every x: Any well-chosen IFS has a constant local Hölder exponent while $x \to \alpha_{\mathrm{p}}(x)$ is everywhere discontinuous.

We now move to a different kind of relation between α_{p} and α_{l}. The following proposition assesses that the two exponents can not differ everywhere.

Proposition 4.3. *Let $f : I \to \mathbb{R}$ be a continuous function, where I is an interval of $\mathbb{R}$. Assume that there exists $\gamma > 0$ such that $f \in C^\gamma(I)$. Then there exists a subset D of I such that*:

- *D is dense, uncountable and has Hausdorff dimension 0.*
- *$\forall x \in D$, $\alpha_{\mathrm{p}}(x) = \alpha_{\mathrm{l}}(x)$.*

Furthermore, this result is optimal, i.e., there exist functions with global Hölder regularity $\gamma > 0$ such that $\alpha_{\mathrm{p}}(x) \neq \alpha_{\mathrm{l}}(x)$ for all x outside a set of Hausdorff dimension 0.

Proof. We give the proof of the last Proposition in the case $\forall x$, $\alpha_{\mathrm{p}}(x) \leqslant 1$. The general result follows with similar arguments.

Let us consider a ball $B(x_0, \eta_0) \subset I$. We construct three sequences of points $\{x_n\}_n$, $\{y_n\}_n$, $\{z_n\}_n$ by the following method.

Let $\{\epsilon_n\}_n$ be a positive sequence converging to 0 when $n \to +\infty$. Let us denote by β_0 the real number $\alpha_{\mathrm{l}}(B(x_0, \eta_0/2))$. By definition of α_{l}, there exist two real numbers y_1 and z_1 such that

$$\begin{aligned}
&y_1 \in B(x_0, \eta_0/2), \qquad z_1 \in B(x_0, \eta_0/2),\\
&y_1 < z_1 \quad \text{and} \quad \bigl|f(y_1) - f(z_1)\bigr| > |y_1 - z_1|^{\beta_0 + \epsilon_0}.
\end{aligned}$$

Let us now denote by x_1 the middle point of $[y_1, z_1]$, and by η_1 the number $\min(2^{-1}, |y_1 - z_1|/2)$.

Now consider the smaller ball $B(x_1, \eta_1/2)$, and its associated exponent $\beta_1 = \alpha_{\mathrm{l}}(B(x_1, \eta_1/2))$. There exist two real numbers y_2 and z_2 such that

$$\begin{aligned}
&y_2 \in B(x_1, \eta_1/2), \qquad z_2 \in B(x_1, \eta_1/2),\\
&y_2 < z_2 \quad \text{and} \quad \bigl|f(y_2) - f(z_2)\bigr| > |y_2 - z_2|^{\beta_1 + \epsilon_1}.
\end{aligned}$$

We denote by x_2 the middle point of $[y_2, z_2]$, and by η_2 the real number $\min(2^{-2}, |y_2 - z_2|/2)$.

We iterate this construction scheme, and thus obtain the desired three sequences $\{x_n\}_n$, $\{y_n\}_n$, $\{z_n\}_n$. Now one easily proves that:

- The sequence $\{x_n\}_n$ converges to a real number x.
- The sequences $\{y_n\}_n$ and $\{z_n\}_n$ also converge to x.
- For all n, one has the inequalities

$$\frac{|y_n - z_n|}{4} \leqslant |x - y_n| \leqslant |y_n - z_n|, \qquad \frac{|y_n - z_n|}{4} \leqslant |x - z_n| \leqslant |y_n - z_n|.$$

One can sum up these inequalities by writing

$$\forall n, \quad |x - y_n| \sim |x - z_n| \sim |y_n - z_n|. \tag{9}$$

Let us now study the local and pointwise Hölder exponents of the limit point x respectively denoted by β_x and α_x. Since $f \in C^\gamma([0,1])$, one has $\gamma \leqslant \beta_x \leqslant \alpha_x$.

First remark that the sequence $\{\beta_n\}_n$ is non-decreasing, since the intervals $B(x_n, \eta_n/2)$ are embedded. By Proposition 3.2, one has $\beta_x = \lim_n \beta_n$. Indeed, since one can choose any decreasing sequence of open sets converging to x, one specifically chooses the interval $B(x_n, \eta_n/2)$ (the converge of β_n is ensured by the fact than one always has $\beta_n \leqslant \alpha_x$).

Let us now turn to the pointwise Hölder exponent. For every $\epsilon > 0$, there exist $\eta > 0$ and a constant C such that, $\forall y \in B(x, \eta)$, one has $|f(x) - f(y)| \leqslant C|x - y|^{\alpha_x - \epsilon}$. On the other hand, there exists an infinite number of couples (y_n, z_n) such that $y_n \in B(x, \eta)$ and $z_n \in B(x, \eta)$. For those couples, one can write

$$\big|f(y_n) - f(z_n)\big| \geqslant |y_n - z_n|^{\beta_n + \epsilon_n}$$

and, on the other side,

$$\begin{aligned}\big|f(y_n) - f(z_n)\big| &\leqslant \big|f(y_n) - f(x)\big| + \big|f(x) - f(z_n)\big| \\ &\leqslant C|y_n - x|^{\alpha_x - \epsilon} + C|x - z_n|^{\alpha_x - \epsilon} \\ &\leqslant C|y_n - z_n|^{\alpha_x - \epsilon},\end{aligned}$$

where one has used (9).

Assume now that $\beta_x < \alpha_x$, and let us take $\epsilon < (\alpha_x - \beta_x)/4$. Since $\lim_n \beta_n + \epsilon_n = \beta_x$, there exists N such that $n \geqslant N$ implies $\beta_n + \epsilon_n \leqslant \alpha_x - 2\epsilon$. For such n's, one has

$$\begin{aligned}\forall n \geqslant N, \quad & C|y_n - z_n|^{\alpha_x - 2\epsilon} \leqslant C|y_n - z_n|^{\beta_n + \epsilon_n} \leqslant \big|f(y_n) - f(z_n)\big| \quad \text{and} \\ & \big|f(y_n) - f(z_n)\big| \leqslant C|y_n - z_n|^{\alpha_x - \epsilon},\end{aligned}$$

which gives

$$\forall n \geqslant N, \quad C|y_n - z_n|^{\alpha_x - 2\epsilon} \leqslant C|y_n - z_n|^{\alpha_x - \epsilon}.$$

Since $|y_n - z_n| \to 0$ when n goes to infinity, this is absurd.

One concludes $\alpha_x = \beta_x$ for the x we have found.

A simple modification of the above construction shows that the set $\{x\colon \alpha_{\mathrm{p}}(x) = \alpha_{\mathrm{l}}(x)\}$ is uncountable. Indeed, starting from the interval $I_0 = [y_0, z_0]$, one can split it into five equal parts. Focus now on the second and the forth subintervals, and apply the construction we have described above. One thus obtains two subintervals I_1^1 (the "left" one) and I_1^2 (the "right" one). Iterating this scheme, at each stage n, one obtains 2^n distinct intervals I_n^i, $i \in \{1, 2, \ldots, 2^n\}$. Using this method one constructs a Cantor set C_f. It is easy to see that it is uncountable, and that each point $x \in C_f$ still satisfies $\alpha_{\mathrm{p}}(x) = \alpha_{\mathrm{l}}(x)$.

Finally, both the optimality and the fact that the set where the exponents coincide has Hausdorff dimension 0 are a consequence of Theorem 4.1 below. Alternatively, one may consider the case of an IFS, for which one has $\alpha_{\mathrm{l}}(x) = \alpha_{\mathrm{p}}(x)$ exactly on a dense uncountable set of dimension 0. More precisely, consider an (attractor of an) IFS defined on [0, 1], verifying the functional identity

$$f(x) = c_1 f(2x) + c_2(f)(2x - 1), \tag{10}$$

where $0.5 < |c_1| < |c_2| < 1$. It is known that for such a function, $\alpha_l(t) = -\log_2(|c_2|)$ for all t. Furthermore (see [1]), $\alpha_p(t)$ is everywhere discontinuous, and ranges in the interval $[-\log_2(|c_2|), -\log_2(|c_1|)]$. Finally, for all α in this interval, the set of t for which $\alpha_p(t) = \alpha$ is dense in $[0, 1]$. This is thus an example where the local and pointwise exponents have drastically different behaviors, with a constant α_l and a wildly varying α_p. It is easy to show that the set D on which $\alpha_p(t) = \alpha_l(t) = -\log_2(|c_2|)$ is exactly the set of points for which the proportion of 0 in the dyadic expansion is 1. That this set D is dense, uncountable, and of Hausdorff dimension 0 is a classical result in number theory. □

So far, we have proved that α_l must be not larger than α_p in the sense made precise by Proposition 4.1, and that both exponents must coincide at least on a subset of a certain "size." Are there other constraints that rule the relations between α_l and α_p? The following theorem essentially answers in the negative.

Theorem 4.1. *Let $\gamma > 0$, $f : [0, 1] \to [\gamma, +\infty)$ a liminf of continuous functions, with $\|f\|_\infty < +\infty$, and $g : [0, 1] \to [\gamma, +\infty)$ a lower semicontinuous function. Assume the* compatibility *condition, i.e., $\forall t \in [0, 1]$, $f(t) \geqslant g(t)$. Then there exists a continuous function $F : [0, 1] \to \mathbb{R}$ such that*:

- *for all x, $\alpha_l(x) = g(x)$,*
- *for all x outside a set D of Hausdorff dimension 0, $\alpha_p(x) = f(x)$.*

We prove this theorem in the next section, by explicitly constructing F.

5. Joint prescription of the Hölder functions

5.1. The case where α_l is constant

We are going in this section to present a function whose local Hölder function is constant, and whose pointwise Hölder function is everywhere constant (and thus equal to the local Hölder exponent) except at 0, where $\alpha_p(0) > \alpha_p(x)$, $x \neq 0$. This is the "inverse" case of a cusp or a chirp, where the regularity at a single point is lower than at all the other points.

This construction is paving the way to the more general result we will prove in the next section.

Proposition 5.1. *Let $0 < \beta < \alpha$ a be two real numbers. Then there exists a function $f :]-1, 1[\to \mathbb{R}$ such that $\forall x \neq 0$, $\alpha_p(x) = \beta$ and $\alpha_p(0) = \alpha$. Moreover, one has $\alpha_l(x) = \beta$, $\forall x \in]-1, 1[$.*

Proof. The existence of such a function is obvious: take, for example, the function

$$F_W : x \to |x|^{\alpha-\beta} W_\beta(x),$$

where W_β is the Weierstrass function

$$W_\beta(x) = \sum_{n=1}^{+\infty} 2^{-n\beta} \sin\bigl(2\pi 2^n x\bigr). \tag{11}$$

We will exhibit another function f with the same property. This function is built using a wavelet method that can be generalized to prescribe arbitrary Hölder functions.

First we are going to select some particular couples (j, k) among the whole set of indices $\{(j, k)\}_{j\in\mathbb{N},\, k\in\mathbb{Z}}$. To achieve this, consider the function g defined by

$$g : x \to \begin{cases} e^{-1/x^2}, & \text{if } x \neq 0, \\ 0, & \text{if } x = 0. \end{cases}$$

It is known that this function is infinitely differentiable at 0, and that one $\forall k \in \mathbb{N}$, $g^{(k)}(0) = 0$.

For all $n \in \mathbb{N}^*$, choose one integer $i \in \{1, \ldots, 2^n\}$, and define

$$p_{i,n} = \frac{g(i2^{-n})}{2^n}. \tag{12}$$

Consider the unique integer j such that $1 \leqslant 2^j p_{i,n} < 2$, and define another (unique) integer $k = i2^{j-n}$.

We have thus built a function, which associates with each couple (n, i) (where $n \geqslant 1$ and $i \in \{1, \ldots, 2^n\}$) another couple of indices (j, k). Let us denote by Γ this set of selected indices.

Let us define the following set of wavelet coefficients:

$$\forall j, \quad d_{j,0} = 2^{-j\alpha}, \qquad d_{j,k} = \begin{cases} 2^{-j\beta}, & \text{if } (j,k) \in \Gamma, \\ 0, & \text{everywhere else.} \end{cases}$$

We add, in a uniform manner, some larger coefficients along exponential curves in the time–frequency domain.

We can define a function f by the reconstruction formula

$$f = \sum_j \sum_k d_{j,k} \psi_{j,k}. \tag{13}$$

Let us now prove that this function satisfies the desired properties.

First this function is well defined, since, $\forall (j,k)$, $|d_{j,k}| \leqslant 2^{-j\beta}$. By the theorem of Jaffard, f is at least $C^\beta(x)$ for all $x \in \,]-1, 1[$.

Case $(x \neq 0)$. $\forall j, \forall k$, one has $|d_{j,k}| \leqslant 2^{-j\beta}$. Thus $\alpha_{\mathrm{p}}(x) \geqslant \beta$.

The proof of $\alpha_{\mathrm{p}}(x) \leqslant \beta$ is more delicate. For each integer n, define the unique integer i_n verifying $i_n 2^{-n} \leqslant x < (i_n + 1)2^{-n}$. When $n \to +\infty$, $i_n 2^{-n} \to x$, and, since g is continuous, $g(i_n 2^{-n}) \sim g(x)$. The associated couple (j, k) satisfies

$$k2^{-j} = i_n 2^{-n}, \qquad 1 \leqslant \frac{g(i_n 2^{-n})}{2^n} 2^j < 2.$$

One can rewrite the last inequality in

$$g\bigl(i_n 2^{-n}\bigr) 2^{-n-1} \leqslant 2^{-j} \leqslant g(i_n 2^{-n}) 2^{-n},$$

or equivalently, using that $g(i_n 2^{-n}) \sim g(x)$ when n goes to infinity, and taking the logarithm,

$$n + C_x \leqslant j \leqslant (n+1) + C_x,$$

where C_x is a constant depending only on x.

Now, for the associated couple (j, k), one has

$$2^j \bigl|x - k2^{-j}\bigr| \leqslant C2^{n+1} \bigl|x - k2^{-j}\bigr| \leqslant C2^{n+1} \bigl|x - i_n 2^{-n}\bigr| \leqslant C2,$$

since by construction $|x - i_n 2^{-n}| \leqslant 2^{-n}$. Thus for such couples (j, k), one has exactly

$$d_{j,k} = 2^{-j\beta} \sim 2^{-j\beta} \bigl(1 + 2^j \bigl|x - k2^{-j}\bigr|\bigr)^\beta. \tag{14}$$

Hence, the inequality $\forall j, k$, $|d_{j,k}| \leqslant C2^{-j\beta}(1 + 2^j |x - k2^{-j}|)^\beta$ is optimal, and $\alpha_{\mathrm{p}}(x) \leqslant \beta$. One concludes $\alpha_{\mathrm{p}}(x) = \beta$, since we already showed $\alpha_{\mathrm{p}}(x) \geqslant \beta$.

Case $(x = 0)$. One notices first that, by construction, for $k = 0$, $d_{j,0} = 2^{-j\alpha}$, thus $\alpha_{\mathrm{p}}(0) \leqslant \alpha$.

If $k \neq 0$, $d_{j,k} = 0$, except if there exists an integer $n \geqslant 1$, and an integer $i \in \{1, \ldots, 2^n\}$, such that

$$k2^{-j} = i2^{-n}, \qquad 1 \leqslant 2^j \frac{g(i2^{-n})}{2^n} < 2.$$

Then, for this kind of indices (j,k),

$$|d_{j,k}| = 2^{-j\beta} \leqslant \big(2^{-n} g\big(i2^{-n}\big)\big)^{\beta} \leqslant \big(i2^{-n}\big)^{\beta} \big(g\big(i2^{-n}\big)\big)^{\beta}.$$

But, using the structure of the function g, there exists a constant C (independent of x) such that, $\forall x > 0$, $g(x) \leqslant C|x|^{(\alpha+1)/\beta}$.

Thus

$$\begin{aligned}|d_{j,k}| &= C\big(\big|i2^{-n}\big|\big)^{\beta}\big(\big|i2^{-n}\big|^{(\alpha+1)/\beta}\big)^{\beta} \leqslant C\big|i2^{-n}\big|^{\alpha+\beta+1} \leqslant C\big|k2^{-j}\big|^{\alpha+\beta+1}\\ &\leqslant C2^{-j(\alpha+\beta+1)}\big(1+|k|\big)^{\alpha+\beta+1}.\end{aligned}$$

This proves that these coefficients, which are larger than $2^{-j\alpha}$, are nevertheless seen as very regular ones from the point 0. The main contribution to the pointwise regularity is thus given by the wavelet coefficients that are located at 0, the $d_{j,0}$. One concludes $\alpha_{\mathrm{p}}(0) = \alpha$.

To end the proof, we need to prove that $\alpha_{\mathrm{l}}(x) = \beta$, $\forall x \in\,]-1,1[$. This is easily done. Indeed, using the characterization given by (7), one obtains that $\forall x \neq 0$, $\alpha_{\mathrm{l}}(x) = \beta$. At 0, one can still write $\alpha_{\mathrm{l}}(0) \geqslant \beta$, but on the other hand one uses (8) and concludes that $\alpha_{\mathrm{l}}(0) \leqslant \liminf_{x\to 0} \alpha_{\mathrm{l}}(x) = \beta$. This concludes the proof. □

5.2. The general case

In the last section, we have built a function whose pointwise exponent at 0 was larger than all the other ones. In particular, at 0, we have forced the local exponent to be equal to a given value β, while at the same time the pointwise exponent was forced to be larger than β. The next step is to be able to do this uniformly, on a set of x as large as possible. The purpose of this subsection is to prove the theorem stated in Section 4 that we recall here for convenience.

Theorem 5.1. *Let $0 < \gamma < 1$, $f:[0,1] \to [\gamma, +\infty)$ a liminf of continuous functions, with $\|f\|_\infty < +\infty$, and $g:[0,1] \to [\gamma,+\infty)$ a lower semicontinuous function. Assume the* compatibility *condition, i.e., $\forall t \in [0,1]$, $f(t) \geqslant g(t)$. Then there exists a continuous function $F:[0,1] \to \mathbb{R}$ such that for all x:*

$$\alpha_{\mathrm{l}}(x) = g(x), \tag{15}$$

outside a set D of Hausdorff dimension 0*,*

$$\alpha_{\mathrm{p}}(x) = f(x). \tag{16}$$

Let us make a few remarks:

- The proof is a kind of generalization of the method used in Proposition 5.1. We are going to enlarge some coefficients, but this time we are going to do this "uniformly" and not only around a single point.
- Our construction introduces an asymmetry between the local and the pointwise exponent: one can prescribe *everywhere* the local exponent, while one can not do the same thing at the same time (with this construction) for the pointwise exponent.
- Eventually, we will see that, applying the method we introduce, one can prescribe the pointwise exponent everywhere except on a set of Hausdorff dimension 0. This restriction is weaker that the one which occurs when one wants to prescribe at the same time the *chirp* and the pointwise Hölder exponent: Jaffard [6] has proved that, in this frame, the excluded set is of Lebesgue measure 0 and of Hausdorff dimension 1. Working with the local Hölder exponent thus allows more flexibility.

Proof. We shall one more time construct the function by a wavelet method.

First we are going to construct some specific approximations sequences of continuous functions that will approximate the functions f and g.

By definition, one knows that there exist two sequences of continuous functions $\{f_n^0\}_n$ and $\{g_n^0\}_n$ such that

$$\liminf_n f_n^0 = f, \tag{17}$$

$$\sup_n g_n^0 = g. \tag{18}$$

We will use the two following lemmas, that roughly say that one can *slow down* the speed of convergence of these two sequences.

Lemma 5.1. *Let f be a liminf of continuous functions. Then there exists a sequence of polynomials f_n^1 that verifies*

$$f(t) = \liminf_n f_n^1(t), \quad \forall t \in [0,1],$$
$$\left\| \left(f_n^1\right)'(t) \right\|_{L^\infty} \leqslant \log n, \quad \forall n \geqslant 1 \textit{ and } t \in [0,1].$$

The proof of this fact can be found in [5] or [1].

Lemma 5.2. *Let g be an lsc function. Then there exists a sequence of polynomials g_n^1 that verifies*

$$g(t) = \sup_n g_n^1(t), \quad \forall t \in [0,1],$$
$$\left\| \left(g_n^1\right)'(t) \right\|_{L^\infty} \leqslant \log n, \quad \forall n \geqslant 1 \textit{ and } t \in [0,1].$$

Proof. This is a little bit more complicated. First let us define, for all n and x, $g_n^2(x) = \max_{p \leqslant n}\{g_p(x)\}$. One still has $g(x) = \sup_n g_n^2(x)$. One also has $g(x) = \sup_n g_n^3(x)$ with $g_n^3(x) = g_n^2(x) - 1/n$.

For each $n > 0$, there exists a polynomial P_n such that $\|g_n^3 - P_n\|_{L^\infty} \leqslant 2^{-n}$. One has thus built a sequence of polynomials such that $g = \sup_n P_n$.

One can now, by the same method as in Lemma 5.1, slow down the sequence $\{P_n\}_n$ such that it will satisfy the desired conditions. □

We now set the desired sequences $\{f_n\}_n$ and $\{g_n\}_n$ by

$$g_n(t) = \max_{p \leqslant n}\left(g_p^1(t), \gamma/2\right), \qquad f_n(t) = \max\left(f_n^1(t), g_n(t) + 1/n\right).$$

They verify the following properties:

- They still respectively satisfy (17) and (18).
- For each n, the right and left derivatives of g_n and f_n at each point $x \in [0,1]$ are lower than $\log n$, since they are maxima of a finite number of polynomials of derivative lower than $\log n$.
- g_n is non-decreasing, i.e., $\forall t \in [0,1]$, $\{g_n(t)\}_n$ is an non-decreasing sequence of real numbers.
- One has the inequality $f_n \geqslant g_n$ for all $n \in \mathbb{N}^*$.

We are now going to select some couples of indices, which will be the basis of our construction of a function F satisfying (15) and (16).

For $n \in \{1, 2, 3, \ldots\}$ and $i \in \{1, 2, 3, \ldots, 2^{n-1}\}$, let us define the two integers j_n and $k_{n,i}$ by

$$j_n = 2^n, \qquad k_{n,i} = 2^{j_n} \frac{2i-1}{j_n}.$$

At each n, one obtains 2^{n-1} couples, which are uniformly distributed on $[0,1]$ in the sense that the $x_{n,i} = k_{n,i} 2^{-j_n} = (2i-1)/j_n$ are uniformly distributed on $[0,1]$. We denote by Λ the set of these selected couples $(j_n, k_{n,i})$.

We are now ready to construct the wavelet coefficients of F. We define

$$d_{j,k} = 2^{-jg_j(x_{n,i})} = 2^{-jg_j(k_{n,i}2^{-j_n})}, \quad \text{if } (j,k) \in \Lambda,$$
$$d_{j,k} = 2^{-jf_j(x_{n,i})}, \quad \text{everywhere else.}$$

The operation we are doing is a re-scaling of some coefficients, according to the local regularity.

Remark that for all (j,k), $|d_{j,k}| \leqslant 2^{-j\gamma/2}$, thus

$$F(x) = \sum_j \sum_k d_{j,k} \psi_{j,k}(x)$$

is well defined and is $C^{\gamma/2}([0,1])$.

5.2.1. Local Hölder exponent

Let $x_0 \in [0,1]$ and $\epsilon > 0$. One has $g(x_0) = \sup_n g_n(x_0)$, thus there exists an integer N_1 such that $n \geqslant N_1 \Rightarrow g_n(x_0) > g(x_0) - \epsilon/2$. Let N_2 be an integer such that $\log(N_2)2^{-N_2} \leqslant \epsilon/2$. Define $N = \max(N_1, N_2)$. Then, using the boundedness of the derivatives of g_N, if $\eta = 2^{-N}$, one obtains $\forall y \in B(x_0, \eta)$,

$$\left|g_N(y) - g_N(x_0)\right| \leqslant (\log N)|y - x_0| \leqslant (\log N)2^{-N} \leqslant \epsilon/2,$$

and thus $\forall y \in B(x_0, \eta)$,

$$g_N(y) \geqslant g_N(x_0) - \epsilon/2.$$

One thus has $g_N(y) \geqslant g_N(x_0) - \epsilon/2 \geqslant g(x_0) - \epsilon$, and since the sequence g_n is non-decreasing, the last property is still true for any g_n, $n \geqslant N$. One obtains the key property

$$\forall y \in B(x_0, \eta), \ \forall n \geqslant N, \quad g_n(y) \geqslant g(x_0) - \epsilon. \tag{19}$$

Consider now the wavelet coefficients $d_{j,k}$ such that their support is included in $B(x_0, \eta)$ (these coefficients are the ones one shall focus on to compute $\alpha_l(B(x_0, \eta))$). There are two sorts of such coefficients:

- the "normal" ones, those which do not belong to Λ. One can write for them
$$|d_{j,k}| \leqslant 2^{-jf_j(k2^{-j})} \leqslant 2^{-jg_j(k2^{-j})} \leqslant 2^{-j(g(x_0)-\epsilon)}.$$
- those which belong to Λ. For them,
$$|d_{j,k}| \leqslant 2^{-jg_n(x_{n,i})} \leqslant 2^{-j(g(x_0)-\epsilon)}.$$

Eventually, for all the interesting couples of coefficients (j,k), $|d_{j,k}| \leqslant 2^{-j(g(x_0)-\epsilon)}$. One concludes $\alpha_l(B(x_0, \eta)) \geqslant g(x_0) - \epsilon$. The result is clearly still true on every ball $B(x_0, \eta_1)$ with $\eta_1 \leqslant \eta$, thus one has $\alpha_l(x_0) \geqslant g(x_0) - \epsilon$.

On the other hand, $\forall n > 0$, consider the unique integer i that verifies $x_{n,i} = k_{n,i}2^{j_n} \in [x_0 - j_n^{-1}, x_0 + j_n^{-1}]$. Then, using the boundedness of the derivatives of g_n, one can write

$$\left|g_{j_n}(x_{n,i}) - g_{j_n}(x_0)\right| \leqslant \log(j_n) j_n^{-1} \leqslant n2^{-n}.$$

Let N_3 be such that $N_2 2^{-N_3} \leqslant \epsilon/2$. For $n \geqslant \max(N_3, N)$ (where N has been above defined), one has

$$g_{j_n}(x_{n,i}) \leqslant g_{j_n}(x_0) + \epsilon/2 \leqslant g(x_0) + \epsilon. \tag{20}$$

There is an infinite number of such couples (n,i), whose associated wavelet coefficients satisfy

$$|d_{j,k}| = |d_{j_n,k_{n,i}}| = 2^{-j_n g_{j_n}(x_{n,i})} \geqslant 2^{-j_n(g(x_0)+\epsilon)}. \tag{21}$$

Now, by Proposition 2.2, $\alpha_l(B(x_0, \eta)) \leqslant g(x_0) + \epsilon$. Since, one more time, this is also true for any $\eta_1 \leqslant \eta$, one has $\alpha_l(x_0) \leqslant g(x_0) + \epsilon$.

Eventually, $\alpha_l(x_0) = g(x_0)$.

5.2.2. Pointwise Hölder exponent

The estimation of this exponent is more complicated. Let $x_0 \in [0,1]$ and $\epsilon > 0$.

Without the rescaled coefficients (i.e., if the $d_{j_n,k_{n,i}}$ were all equal to $2^{-j_n f_{j_n}(x_{n,i})}$), it has been proved in [1] that $\forall x$, $\alpha_{\mathrm{p}}(x) = f(x)$. The question is: do we change something when we modify the values of these specific coefficients? The modifications may have big influence on regularity, because the new coefficients are larger than the "normal" ones (indeed, remember that $f(x) \geqslant g(x)$).

We will show that in fact, the rescaled coefficients are not seen by most of the points x. Thus, for such points, one still has $\alpha_{\mathrm{p}}(x) = f(x)$.

Let us define the set E_M by

$$E_M = \left\{x\colon \exists C,\ \exists N_x,\ \forall n \geqslant N_x,\ \forall i,\ \left|x - \frac{2i-1}{2^n}\right| \geqslant C2^{-2^n\gamma/M}\right\}, \tag{22}$$

where M verifies $M \geqslant \|f\|_\infty$. Let x_0 be in E_M. Since $x_{n,i} = (2i-1)/2^n$, one has, for every i and $n \geqslant N_x$,

$$2^{-2^n\gamma/M} \leqslant C|x_0 - x_{n,i}|, \tag{23}$$

or equivalently, replacing j_n and $k_{n,i}$ by their values,

$$2^{-j^n\gamma/M} \leqslant C\left|x_0 - k_{n,i}2^{-j_n}\right|.$$

We know that $\gamma \leqslant g_{j_n}$ and $f(x_0) < M$ by construction, thus $\forall y \in [0,1]$, $g_{j_n}(y)/f(x_0) \geqslant \gamma/M$, and for every i and n,

$$2^{-j_n g_{j_n}(y)/f(x_0)} \leqslant C\left|x_0 - k_{n,i}2^{-j_n}\right|.$$

This is equivalent to

$$2^{-j_n g_{j_n}(x_{n,i})} \leqslant C\left|x_0 - k_{n,i}2^{-j_n}\right|^{f(x_0)},$$

which implies

$$\begin{aligned} 2^{-j_n g_{j_n}(x_{n,i})} &\leqslant C2^{-j_n f(x_0)}\left(2^{j_n}\left|x_0 - k_{n,i}2^{-j_n}\right|\right)^{f(x_0)}, \\ &\leqslant C2^{-j_n f(x_0)}\left(1 + 2^{j_n}\left|x_0 - k_{n,i}2^{-j_n}\right|\right)^{f(x_0)}. \end{aligned}$$

But $d_{j_n,k_{n,i}} = 2^{-j_n g_{j_n}(x_{n,i})}$, hence, for any $x_0 \in E_M$, there exists a constant C such that

$$|d_{j_n,k_{n,i}}| \leqslant C2^{-f(x_0)j_n}\left(1 + 2^{j_n}\left|x_0 - k_{n,i}2^{-j_n}\right|\right)^{f(x_0)}. \tag{24}$$

This shows that, if $x_0 \in E_M \cap [0,1]$, $\forall n \geqslant N_x$, $\forall p$, one has (24), which ensures $\alpha_{\mathrm{p}}(x_0) = f(x_0)$. The large coefficients, those which are rescaled, are not "seen" by the pointwise Hölder exponent at x_0.

To end the proof, it is sufficient to measure the size of E_M. We prove in Section 6 that the complementary set D_M of the set E_M has Hausdorff dimension 0. Moreover, any rational number $x = p/q$ belongs to E_M. □

Remark 5.1. One cannot say anything about the x's that are in $D_M = [0,1]\backslash E_M$, except that for such points x, $g(x) = \alpha_{\mathrm{l}}(x) \leqslant \alpha_{\mathrm{p}}(x)$. Nevertheless some of them must satisfy $\alpha_{\mathrm{p}}(x) = \alpha_{\mathrm{l}}(x)$ even if the functions f and g satisfy $f(y) > g(y)$ for all y in $[0,1]$.

Remark 5.2. Combining the construction we used with the construction due to Jaffard [6], one can certainly prescribe, outside a set of Hausdorff dimension 1 but of Lebesgue measure 0, three different regularity exponents at the same time: the local Hölder exponent, the pointwise Hölder exponent, and the chirp exponent [10]. This is a first step towards a more complete prescription of the regularity of a function. See [9] for more on this topic.

6. Study of the set E_M

We begin by computing the Hausdorff dimension of the complementary set of E_M.

Proposition 6.1. *For all $M > 0$, the Hausdorff dimension of the set D_M defined by*

$$D_M = [0,1]\backslash E_M \tag{25}$$

is 0.

Proof. Let $M > 0$, $C > 0$, and define E_M^C by

$$E_M^C = \left\{x \in [0,1]\colon \exists N_x,\ \forall n \geqslant N_x,\ \forall i,\ \left|x - \frac{2i-1}{2^n}\right| \geqslant C2^{-2^n\gamma/M}\right\}, \tag{26}$$

or equivalently,

$$E_M^C = \left\{x \in [0,1]\colon \exists N_x \in \mathbb{N},\ x \notin \bigcup_{n\geqslant N_x} F_n^C\right\}, \tag{27}$$

where

$$F_n^C = \bigcup_{i=1}^{2^{n-1}} B_{n,i}^C \quad \text{and} \quad B_{n,i}^C = \left]\frac{2i-1}{2^n} - C2^{-2^n\gamma/M}, \frac{2i-1}{2^n} + C2^{-2^n\gamma/M}\right[.$$

Let $D_M^C = [0,1]\backslash E_M^C$. It obviously satisfies

$$D_M^C = \bigcap_{N\in\mathbb{N}} \bigcup_{n\geqslant N} F_n^C.$$

Let $\epsilon > 0$. One has

$$\sum_{n\geqslant N}\sum_{i=1}^{2^{n-1}} \left|B_{n,i}^C\right|^\epsilon \leqslant \sum_{n\geqslant N} 2^{n-1}\left|2C2^{-2n\gamma/M}\right|^\epsilon \leqslant C'2^{-2^{N(\gamma/M)\epsilon+N-1}},$$

which goes to zero when N goes to infinity (C' is a constant independent of N). Since for all N, $\bigcup_{n\geqslant N} F_N^C$ is obviously a cover of D_M^C by balls of size $2^{-2^N\gamma/M}$, one has exactly shown that the ϵ-dimensional Hausdorff measure of D_M^C is 0, $\forall\epsilon > 0$. We conclude that the Hausdorff dimension of D_M^C is 0.

Remark now that $D_M \subset \bigcap_{n\in\mathbb{N}^*} D_M^{1/n}$. D_M is thus also of Hausdorff dimension 0. □

In Theorem 4.1, one may choose, for all x, $f(x) = M > \gamma = g(x) > 0$. Using Proposition 4.3, we deduce that $D_M = [0,1]\backslash E_M$ must be dense and uncountable, otherwise α_{l} would be different from α_{p} on a too large set. This implies

Corollary 6.1. *D_M is uncountable and dense in* $[0,1]$.

We remark finally that our construction also allows to prescribe the pointwise Hölder exponent at any rational point (even at dyadic ones). Indeed,

Proposition 6.2. $\mathbb{Q} \cap [0,1] \subset E_M$.

Proof. Let $x = p/q$ be a rational number.

For every $n \in \mathbb{N}$,

$$\left|x - \frac{2p-1}{2^n}\right| = \left|\frac{p}{q} - \frac{2p-1}{2^n}\right| = \left|\frac{2^np-(2p-1)q}{q2^n}\right|.$$

Let us decompose the integer q as $q = 2^{n_x}q_1$, where q_1 is an odd integer. Thus, for $n \geqslant n_x + 1$,

$$2^np - (2p-1)q = 2^{n_x}\left(2^{n-n_x}p - (2p-1)_{q_1}\right) \neq 0,$$

since $2^{n-n_x}p$ is an even integer and $(2p-1)q_1$ is an odd integer. Consequently, $\forall n$ such that $2^n \geqslant q$,

$$\left|x - \frac{2p-1}{2^n}\right| = \left|\frac{2^np-(2p-1)q}{q2^n}\right| \geqslant \frac{1}{q2^n} \geqslant \left(2^{-n}\right)^2.$$

Thus $x \in E_M$ and Proposition 6.2 is proved. □

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